Article

Helicoidal Hypersurfaces of Dini-Type with Spacelike Axis in Minkowski 4-Space

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Abstract: In this paper, we define Ulisse Dini-type helicoidal hypersurface with spacelike axis in Minkowski 4-space $\mathbb{E}_1^4$. We compute the Gaussian and the mean curvature of the hypersurface. Moreover, we obtain some special symmetry to the curvatures when they are flat and maximal.

Keywords: Minkowski space; Dini-type helicoidal hypersurface; Gauss map, spacelike axis

1. Introduction

The notion of finite type immersion of submanifolds of a Euclidean space has been used in classifying and characterizing well known Riemannian submanifolds [4]. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. An Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ [4]. Further, the notion of finite type can be extended to any smooth functions on a submanifold of a Euclidean space or a pseudo-Euclidean space. Then the theory of submanifolds of finite type has been studied by many geometers.

Takahashi [28] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$, where r is the position vector, $\lambda \in \mathbb{R}$. Ferrandez, Garay and Lucas [11] prove that the surfaces of $\mathbb{E}^3$ satisfying $\Delta H = AH$, where H is the mean curvature and $A \in Mat(3,3)$, are either minimal, or an open piece of sphere or of a right circular cylinder. Choi and Kim [7] characterize the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind.

Dillen, Pas and Verstraelen [8] prove that the only surfaces in $\mathbb{E}^3$ satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres and the circular cylinders. Senoussi and Bekkar [27] study helicoidal surfaces M^2 in $\mathbb{E}^3$ which are of finite type in the sense of Chen with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III.

In classical surface geometry in Euclidean space, it is well known that the right helicoid (resp. catenoid) is the only ruled (resp. rotational surface) which is minimal. If we focus on the ruled (helicoid) and rotational characters, we have Bour's theorem in [3].

About helicoidal surfaces in Euclidean 3-space, do Carmo and Dajczer [9] proved that, by using a result of Bour [3], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface.

Lawson [20] give the general definition of the Laplace-Beltrami operator in his lecture notes. Magid, Scharlach and Vrancken [22] introduce to the affine umbilical surfaces in 4-space. Vlachos [30] consider hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Scharlach [26] study on affine geometry of surfaces and hypersurfaces in 4-space. Cheng and Wan [5] consider complete hypersurfaces of 4-space with constant mean curvature. Arslan, Deszcz and Yaprak [1] study on Weyl pseudosymmetric hypersurfaces.

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Arvanitoyeorgos, Kaimakamais and Magid [2] show that if the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M_1^3 has constant mean curvature in Minkowski 4-space $\mathbb{E}_1^4$. This equation is a natural generalization of the biharmonic submanifold equation $\Delta H = 0$.

General rotational surfaces as a source of examples of surfaces in the four dimensional Euclidean space were introduced by Moore [23,24]. Ganchev and Milousheva [12] consider the analogue of these surfaces in the Minkowski 4-space. They classify completely the minimal general rotational surfaces and the general rotational surfaces consisting of parabolic points.

Verstraelen, Valrave and Yaprak [29] study on the minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n.

Kim and Turgay [19] study surfaces with L_1-pointwise 1-type Gauss map in $\mathbb{E}^4$. Moruz and Munteanu [25] consider hypersurfaces in the Euclidean space $\mathbb{E}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes. They call them minimal translation hypersurfaces in $\mathbb{E}^4$ and give a classification of these hypersurfaces. Kim·et al [18] focus on Cheng-Yau operator and Gauss map of surfaces of revolution.

Güler Magid and Yaylı [15] study Laplace Beltrami operator of a helicoidal hypersurface in $\mathbb{E}^4$. Güler, Hacisalihoglu and Kim [13] work on the Gauss map and the third Laplace-Beltrami operator of rotational hypersurface in $\mathbb{E}^4$. Güler, Kaimakamis and Magid [14] introduce the helicoidal hypersurfaces in Minkowski 4-space $\mathbb{E}_1^4$. Güler and Turgay [16] study Cheng-Yau operator and Gauss map of rotational hypersurfaces in $\mathbb{E}^4$. Moreover, Güler, Turgay and Kim [17] consider L_2 operator and Gauss map of rotational hypersurfaces in $\mathbb{E}^5$.

In this paper, we study the Ulisse Dini-type helicoidal hypersurface with spacelike axis in Minkowski 4-space $\mathbb{E}_1^4$. We give some basic notions of four dimensional Minkowskian geometry, and define helicoidal hypersurface in section 2. Moreover, we obtain Ulisse Dini-type helicoidal hypersurface, and calculate its curvatures in the last section.

2. Preliminaries

In this section we would like to describe the notaion that we will use in the paper after we give some of basic facts and basic definitions.

Let $\mathbb{E}^m$ denote the Minkowskian m-space with the canonical Euclidean metric tensor given by

$$\widetilde{g} = \langle \, , \, \rangle = \sum_{i=1}^{m-1} dx_i^2 - dx_m^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a coordinate system in $\mathbb{E}_1^m$.

Consider an n-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote Levi-Civita connections of $\mathbb{E}_1^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. We shall use letters $X,\ Y,\ Z,\ W$ (resp., $\xi,\ \eta$) to denote vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \tag{2}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

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The Gauss and Codazzi equations are given, respectively, by

$$\langle R(X,Y,)Z,W\rangle = \langle h(Y,Z),h(X,W)\rangle - \langle h(X,Z),h(Y,W)\rangle, \tag{3}$$
$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z), \tag{4}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, S its shape operator and x its position vector. We consider a local orthonormal frame field $\{e_1,e_2,\ldots,e_n\}$ of consisting of principal direction of M corresponding from the principal curvature k_i for $i = 1,2,\ldots n$. Let the dual basis of this frame field be $\{\theta_1,\theta_2,\ldots,\theta\}$. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i = 1,2,\ldots,n \tag{5}$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. We denote the Levi-Civita connection of M and $\mathbb{E}^{n+1}$ by ∇ and $\widetilde{\nabla}$, respectively. Then, from the Codazzi equation (3) we have

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \tag{6}$$
$$\omega_{ij}(e_l)(k_i - k_j) = \omega_{il}(e_j)(k_i - k_l) \tag{7}$$

for distinct $i,j,l = 1,2,\ldots,n,$.

We put $s_j = \sigma_j(k_1,k_2,\ldots,k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1,a_2,\ldots,a_n) = \sum_{1\le i_1<i_2<\ldots,i_j\le n} a_{i_1}a_{i_2}\ldots a_{i_j}.$$

We also use the following notation

$$r_i^j = \sigma_j(k_1,k_2,\ldots,k_{i-1},k_{i+1},k_{i+2},\ldots,k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$.

On the other hand, we will call the function s_k as the k-th mean curvature of M. We would like to note that functions $H = \frac{1}{n}s_1$ and $K = s_n$ are called the mean curvature and Gauss-Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

2.2. Helicoidal hypersurfaces with spacelike axis in Minkowskian spaces

In this section, we will obtain the helicoidal hypersurfaces in Minkowski 4-space. Before we proceed, we would like to note that the definition of rotational hypersurfaces in Riemannian space forms were defined in [10]. A rotational hypersurface $M \subset \mathbb{E}^n_1$ generated by a curve C around an axis $\mathbf{r}$ that does not meet C is obtained by taking the orbit of C under those orthogonal transformations of $\mathbb{E}^n_1$ that leaves $\mathbf{r}$ pointwise fixed (See [10, Remark 2.3]).

Suppose that when a curve C rotates around the axis $\mathbf{r}$, it simultaneously displaces parallel lines orthogonal to the axis $\mathbf{r}$, so that the speed of displacement is proportional to the speed of rotation. Then the resulting hypersurface is called the *helicoidal hypersurface* with axis $\mathbf{r}$ and pitches $a,b \in \mathbb{R}\backslash\{0\}$.

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Consider the particular case $n = 4$ and let C be the curve parametrized by

$$\gamma(u) = (\varphi(u), f(u), 0, 0,). \tag{8}$$

If $\mathbf{r}$ is the spacelike x_1-axis, then an orthogonal transformations of $\mathbb{E}_1^n$ that leaves $\mathbf{r}$ pointwise fixed has the form $Z(v, w)$ as follows:

$$Z(v,w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh w & 0 & \sinh w \\ 0 & \sinh v \sinh w & \cosh v & \cosh w \sinh v \\ 0 & \cosh v \sinh w & \sinh v & \cosh v \cosh w \end{pmatrix}, \tag{9}$$

where $v, w \in \mathbb{R}$. Therefore, the parametrization of the rotational hypersurface generated by a curve C around an axis $\mathbf{r}$ is

$$\mathbf{H}(u, v, w) = Z(v, w)\gamma(u)^t + (av + bw)(1, 0, 0, 0)^t, \tag{10}$$

where $u \in I$, $v, w \in [0, 2\pi]$, $a, b \in \mathbb{R} \backslash \{0\}$.

Clearly, we write helicoidal hypersurface with spacelike axis as follows:

$$\mathbf{H}(u, v, w) = \begin{pmatrix} \varphi(u) + av + bw \\ u \sinh w \\ u \sinh v \cosh w \\ u \cosh v \cosh w \end{pmatrix}. \tag{11}$$

When $w = 0$, we have helicoidal surface with spacelike axis in $\mathbb{E}_1^4$.

In the rest of this paper, we shall identify a vector (a,b,c,d) with its transpose $(a,b,c,d)^T$. Now we give some basic elements of the Minkowski 4-space $\mathbb{E}_1^4$.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}_1^4$. The inner product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$ is defined as follows:

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x_i y_i - x_4 y_4,$$

and the vector product of $\vec{x} \times \vec{y} \times \vec{z}$ is defined as follows:

$$\det \begin{pmatrix} e_1 & e_2 & e_3 & -e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

For a hypersurface $\mathbf{M}$ in $\mathbb{E}_1^4$, the first fundamental form matrix is as follows:

$$\mathbf{I} = \begin{pmatrix} g_{ij} \end{pmatrix}_{3\times 3},$$

and

$$\det \mathbf{I} = \det \begin{pmatrix} g_{ij} \end{pmatrix},$$

and then, the second fundamental form matrix is as follows:

$$\mathbf{II} = \begin{pmatrix} h_{ij} \end{pmatrix}_{3\times 3},$$

and

$$\det \mathbf{II} = \det \begin{pmatrix} h_{ij} \end{pmatrix},$$

where $1 \leq i, j \leq 5$,

$$g_{11} = \mathbf{M}_u \cdot \mathbf{M}_u,\ g_{12} = \mathbf{M}_u \cdot \mathbf{M}_v, ...,\ g_{55} = \mathbf{M}_w \cdot \mathbf{M}_w,$$

and

$$h_{11} = \mathbf{M}_{uu} \cdot \mathbf{G},\ h_{12} = \mathbf{M}_{uv} \cdot \mathbf{G}, ...,\ h_{55} = \mathbf{M}_{ww} \cdot \mathbf{G},$$

" . " means dot product, and some partial differentials that we represent are $\mathbf{M}_u = \frac{\partial \mathbf{M}}{\partial u}$, $\mathbf{M}_{uw} = \frac{\partial^2 \mathbf{M}}{\partial u \partial w}$,

$$\mathbf{G} = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|}$$

is the Gauss map (i.e. the unit normal vector). The product matrices

$$\left(g_{ij} \right)^{-1} . \left(h_{ij} \right),$$

gives the matrix of the shape operator (i.e. Weingarten map) $\mathbf{S}$ as follows:

$$\mathbf{S} = \frac{1}{\det \mathbf{I}} \left(s_{ij} \right)_{3\times 3}, \tag{12}$$

So, we get the formulas of the Gaussian curvature and the mean curvature, respectively, as follow:

$$K = \det(\mathbf{S}) = \frac{\det \mathbf{II}}{\det \mathbf{I}}, \tag{13}$$

and

$$H = \frac{1}{5} tr\left(\mathbf{S}\right). \tag{14}$$

93 ## 3. Dini-Type Helicoidal Hypersurface with Spacelike Axis

Now, we consider Dini-type helicoidal hypersurface with spacelike axis in $\mathbb{E}_1^4$, as follows:

$$\mathfrak{D}(u, v, w) = \begin{pmatrix} \varphi(u) + av + bw \\ \sinh u \sinh w \\ \sinh u \sinh v \cosh w \\ \sinh u \cosh v \cosh w \end{pmatrix}. \tag{15}$$

94 where $u \in \mathbb{R} \backslash \{0\}$ and $0 \leq v, w \leq 2\pi$.

Using the first differentials of (15) with respect to u, v, w, we get the first quantities as follow:

$$I = \begin{pmatrix} \varphi'^2 - \cosh^2 u & a\varphi' & b\varphi' \\ a\varphi' & \sinh^2 u \cosh^2 w + a^2 & ab \\ b\varphi' & ab & \sinh^2 u + b^2 \end{pmatrix},$$

and have

$$\det I = \left\{ \varphi'^2 \sinh^2 u \cosh^2 w - \left[a^2 + \left(b^2 + \sinh^2 u \right) \cosh^2 w \right] \cosh^2 u \right\} \sinh^2 u,$$

95 where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$.

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Using the second differentials with respect to u, v, w, we have the second quantities as follow:

$$\begin{aligned}
L &= \frac{\sinh^2 u \cosh w \left(-\varphi'' \cosh u + \varphi' \sinh u\right)}{\sqrt{\|\det I\|}}, \\
M &= \frac{a \sinh u \cosh^2 u \cosh w}{\sqrt{\|\det I\|}}, \\
N &= \frac{\sinh^2 u \cosh^2 w \left(\varphi' \sinh u \cosh w - b \cosh u \sinh w\right)}{\sqrt{\|\det I\|}}, \\
P &= \frac{b \sinh u \cosh^2 u \cosh w}{\sqrt{\|\det I\|}}, \\
T &= \frac{a \sinh^2 u \cosh u \sinh w}{\sqrt{\|\det I\|}}, \\
V &= \frac{\varphi' \sinh^3 u \cosh w}{\sqrt{\|\det I\|}},
\end{aligned}$$

and we get

$$\det II = \frac{\left(\begin{array}{c} -\varphi'^2 \varphi'' \sinh^8 u \cosh u \cosh^5 w + b\varphi' \varphi'' \sinh^7 u \cosh^2 u \sinh w \cosh^4 w \\ +a^2 \varphi'' \sinh^6 u \cosh^3 u \sinh^2 w \cosh w + \varphi'^3 \cosh^5 w \sinh^9 u \\ -b\varphi'^2 \sinh^8 u \cosh u \cosh^4 w \sinh w \\ -\left[a^2 \sinh^2 u \sinh^2 w + \left(a^2 + b^2 \cosh^2 w\right) \cosh^2 u \cosh^2 w\right] \varphi' \sinh^5 u \cosh^2 u \cosh w \\ +b\left(2a^2 + b^2 \cosh^2 w\right) \sinh^4 u \cosh^5 u \sinh w \cosh^2 w \end{array}\right)}{(\det I)^{3/2}}.$$

The Gauss map of the helicoidal hypersurface with spacelike axis is

$$e_{\mathfrak{D}} = \frac{1}{\sqrt{\det I}} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}, \tag{16}$$

where

$e_1 = -\sinh^2 u \cosh u \cosh w$,

$e_2 = \left(-\varphi' \sinh u \sinh w - b \cosh u \cosh w\right) \sinh u \cosh w$,

$e_3 = \left(-\varphi' \sinh u \sinh v \cosh^2 w + a \cosh u \cosh v + b \cosh u \sinh v \sinh w \cosh w\right) \sinh u$,

$e_4 = \left(-\varphi' \sinh u \cosh v \cosh^2 w + a \cosh u \sinh v + b \cosh u \cosh v \sinh w \cosh w\right) \sinh u$.

Finally, we calculate the Gaussian curvature of the helicoidal hypersurface with spacelike axis as follows:

$$K = \frac{\alpha_1 \varphi'^2 \varphi'' + \alpha_2 \varphi' \varphi'' + \alpha_3 \varphi'' + \alpha_4 \varphi'^3 + \alpha_5 \varphi'^2 + \alpha_6 \varphi' + \alpha_7}{(\det I)^{5/2}},$$

where

$\alpha_1 = -\sinh^8 u \cosh u \cosh^5 w$,

$\alpha_2 = b \sinh^7 u \cosh^2 u \sinh w \cosh^4 w$,

$\alpha_3 = a^2 \sinh^6 u \cosh^3 u \sinh^2 w \cosh w$,

$\alpha_4 = \sinh^9 u \cosh^5 w$,

$\alpha_5 = -b \sinh^8 u \cosh u \cosh^4 w \sinh w$,

$\alpha_6 = -\left[a^2 \sinh^7 u \sinh^2 w + \left(a^2 + b^2 \cosh^2 w\right) \sinh^5 u \cosh^2 u \cosh^2 w\right] \cosh^2 u \cosh w$,

$\alpha_7 = b\left(2a^2 + b^2 \cosh^2 w\right) \sinh^4 u \cosh^5 u \sinh w \cosh^2 w$.

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Then we calculate the mean curvature of the helicoidal hypersurface with spacelike axis as follows:

$$H=\frac{\beta_1\varphi''+\beta_2\varphi'^3+\beta_3\varphi'^2+\beta_4\varphi'+\beta_5}{3(\det I)^{3/2}},$$

where

$$\beta_1=-\left[\sinh^6 u\cosh^3 w+\left(a^2+b^2\cosh^2 w\right)\sinh^4 u\cosh w\right]\varphi''\cosh u$$

$$\beta_2=+2\varphi'^3\sinh^5 u\cosh^3 w,$$

$$\beta_3=-b\varphi'^2\sinh^4 u\cosh u\sinh w\cosh^2 w,$$

$$\beta_4=\sinh^7 u\cosh^3 w+\left(a^2+b^2\cosh^2 w\right)\sinh^5 u\cosh w-2\sinh^5 u\cosh^2 u\cosh^3 w$$
$$-3\left(a^2+b^2\cosh^2 w\right)\sinh^3 u\cosh^2 u\cosh w,$$

$$\beta_5=+b\sinh^4 u\cosh^3 u\cosh^2 w\sinh w+b\left(2a^2+b^2\cosh^2 w\right)\sinh^2 u\cosh^3 u\sinh w.$$

Theorem 1. *Let $\mathfrak{D}: M^3\longrightarrow\mathbb{E}^4$ be an isometric immersion given by (15). Then M^3 is flat if and only if*

$$\alpha_1\varphi'^2\varphi''+\alpha_2\varphi'\varphi''+\alpha_3\varphi''+\alpha_4\varphi'^3+\alpha_5\varphi'^2+\alpha_6\varphi'+\alpha_7=0.$$

Theorem 2. *Let $\mathfrak{D}: M^3\longrightarrow\mathbb{E}^4$ be an isometric immersion given by (15). Then M^3 is maximal if and only if*

$$\beta_1\varphi''+\beta_2\varphi'^3+\beta_3\varphi'^2+\beta_4\varphi'+\beta_5=0.$$

Solutions of these two eqs. are attracted problem for us.

Proposition 1. *If $\mathfrak{D}$ is Dini-type maximal helicoidal hypersurface with spacelike axis (i.e. $H=0$) in Minkowski 4-space, taking (as in Dini helicoidal surface in Euclidean 3-space)*

$$\varphi(u)=\cosh u+\log\left(\tanh\frac{u}{2}\right),$$

then we get

$$\sum_{i=0}^{6}A_i\tanh^i\left(\frac{u}{2}\right)=0,$$

where

$$\begin{aligned}
A_6&=-\beta_2,\\
A_5&=2\beta_1+6\beta_2\sinh u+2\beta_3,\\
A_4&=\left(3-12\sinh^2 u\right)\beta_2-8\beta_3\sinh u-4\beta_4,\\
A_3&=8\beta_1\cosh u+\left(8\sinh^3 u-12\sinh u\right)\beta_2\\
&\quad+\left(8\sinh^2 u-4\right)\beta_3+8\beta_4\sinh u+8\beta_5,\\
A_2&=\left(-3+12\sinh^2 u\right)\beta_2+8\beta_3\sinh u+4\beta_4,\\
A_1&=-2\beta_1+6\beta_2\sinh u+2\beta_3,\\
A_0&=\beta_2.
\end{aligned}\tag{17}$$

Proposition 2. *If $\mathfrak{D}$ is Dini-type flat hypersurface with spacelike axis (i.e. $K=0$) in Minkowski 4-space, taking (as in Dini helicoidal surface in Euclidean 3-space)*

$$\varphi(u)=\cosh u+\log\left(\tanh\frac{u}{2}\right),$$

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then we get

$$\sum_{i=0}^{8} B_i \tanh^i \left(\frac{u}{2}\right) = 0,$$

where

$$\begin{aligned}
B_8 &= \alpha_1, \\
B_7 &= -4\alpha_1 \sinh u - 2\alpha_2 - 2\alpha_4, \\
B_6 &= \left(-2 + 4\sinh^2 u + 4\cosh u\right) \alpha_1 + 4\alpha_2 \sinh u + 4\alpha_3 + 12\alpha_4 \sinh u + 4\alpha_5, \\
B_5 &= (4\sinh u - 16\cosh u \sinh u)\, \alpha_1 + (2 - 8\cosh u)\, \alpha_2 + \left(6 - 24\sinh^2 u\right) \alpha_4 \\
&\quad - 16\alpha_5 \sinh u - 8\alpha_6, \\
B_4 &= \left(-8\cosh u + 16\cosh u \sinh^2 u\right) \alpha_1 + 16\alpha_2 \cosh u \sinh u + 16\alpha_3 \cosh u \\
&\quad + \left(16\sinh^3 u - 24\sinh u\right) \alpha_4 + \left(16\sinh^2 u - 8\right) \alpha_5 + 16\alpha_6 \sinh u + 16\alpha_7, \\
B_3 &= (4\sinh u + 16\cosh u \sinh u)\, \alpha_1 + 2\alpha_2 + 8\alpha_2 \cosh u + \left(-6 + 24\sinh^2 u\right) \alpha_4 \\
&\quad + 16\alpha_5 \sinh u + 8\alpha_6, \\
B_2 &= \left(2 - 4\sinh^2 u + 4\cosh u\right) \alpha_1 - 4\alpha_2 \sinh u - 4\alpha_3 + 12\alpha_4 \sinh u + 4\alpha_5, \\
B_1 &= -4\alpha_1 \sinh u - 2\alpha_2 + 2\alpha_4, \\
B_0 &= -\alpha_1.
\end{aligned}$$

Corollary 1. *In Proposition 1, and Proposition 2, we see following special symmetries, respectively:*

$$A_6 = -A_0,\ A_5 \sim A_1,\ A_4 = -A_2,$$

and

$$B_8 = -B_0,\ B_7 \sim B_1,\ B_6 \sim B_2,\ B_5 \sim B_3,$$

where " $\sim$ " means similar ignored sign (or equal without sign of coefficient).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors completed the writing of this paper and read and approved the final manuscript.

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