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Nonlinear vibration of a nonlocal nanobeam resting on fractional-order viscoelastic Pasternak foundations

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Abstract: In the present study, nonlinear vibration of a nanobeam resting on fractional order viscoelastic Winkler-Pasternak foundation is studied using nonlocal elasticity theory. D'Alembert principle is used to derive the governing equation and the associated boundary conditions. The approximate analytical solution is obtained by applying the multiple scales method. Detailled parametric study is conducted, the effects of variation in different parameters belonging to the application problems on the system are calculated numerically and depicted. We remark that the order and the coefficient of the fractional derivative have significant effect on the natural frequency and the amplitude of vibrations.

Keywords: Nanobeam; fractional-order; nonlinearity; Winkler-Pasternak foundation

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1. Introduction

Due to the recent and rapid advances in nanomechanics, nanobeam have become the most important structure used extensively in technology such as those nano-electromechanical systems (NEMs), opto-mechanichal or nanoresonator devices. The exclusive properties of nanoscale beam are due to their size, this size plays an important role in static and in dynamic analysis. In front of the difficulties of classical continuum mechanics to take into account the size effect in modelling the behaviour of this structure kind , various size-dependent continuum theories have been developped. These theories include nonlocal continuum theory, strain gradient theory or a combinaison of both (nonlocal strain gradient theory), modified couple stress theory, micropolar theory and the surface elasticity theory. Among these theories, Eringen's nonlocal elasticty theory [1,2] was utilized by a number of researchers to capture size-efects.

These kinds of structures can be modelled as a beam structure on a viscoelastic foundaton. The beam can be modelled as Timoshenko beam [3,4], or as a Rayleigh beam [5] or as a Euler-bernouilli beam [6] and the foundation as a Winkler model [7-9] or as a Pasternak model or combinaison of both (Winkler-Pasternak model) or as a nonlinear elastic model and fractional order viscoelastic model [10]. The Winkler model is a one parameter model namely Winkler-type elastic foundation, consists of a serie of closely spaced elastic springs. Pasternak model is a two parameters model namely Pasternak-type viscoelastic foundation, consists of a Winkler-type elastic springs and transverse shear deformation. The nonlinear model is a three parameters in which the layer is indicated by linear elastic spring, shear deformation and cubic nonlinearity elastic spring. Fractional order Winkler-Pasternak [10] has been well developped, this fractional order is due to the long memory effects of some kind of viscoelastic materials. In vibration analysis of nanostructures, it is so important to evaluate the impact of surrounding medium on the dynamic of beams. Niknam and Aghdam [11] proposed an analytical approach to study dynamic of nonlocal functionally graded beam resting on nonlinear elastic support. A meshless approach for free transverse vibration of SWCNT was proposed by Kiani [12]. Eringen's nonlocal theory and timoshenko beam theory were used to make a bukling analysis of SWCNT on

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elastic medium [13, 14]. Non−conservative dynamic of nonlocal cantilever CNTs on viscoelastic medium is proposed [15]. Mikhasev[16] researched localized modes of free vibrations of SWCNT. Mustapha and Zhong [17] studied dynamic of non−prismatic SWCNT in viscoelastic medium, Lee and Chang[18] studied dynamic of a viscous−fluid conveying SWCNT, Kiani[19,20] examined elastically restrained DWCNT and SWCNT for delivering nanoparticules, instability analysis of CNT conveying fluid is conducted [21], Yas and Samadi[22] examined CNT−reinforced composite on elastic medium , small scale effect in nonuniform CNT conveying fluid on viscoelastic medium is examined[23], Aydoglu[24] analysed nanorods on an elastic medium, dynamic analysis of nanotubes on elastic matrix is conducted by Wang [25], dynamic of curved SWCNT on a Pasternak elastic foundation is examined [26] . Aydogdu and Arda [27] researched the torsional dynamic of nonlocal DWCNTs. Necla [28] studied nonlinear vibration of a nonlocal nanobeam resting on Winkler-type foundation. The work of Anague [10] is based on dynamics of Rayleigh beams resting on fractional order viscoelastic Pasternak foundation subjected to moving loads.
Many of time−space differential equations are very difficults to solve, sometimes these equations are exactly impossible to solve. In front of these difficulties, it is needed sophisticated analytical and numerical method to find appoximated solutions. Ozturk and Coskun[29] proposed the homotopy pertubation method, multiple scale method is used to analyse nonlinear vibration of CNT [30−33], He′s variational method exhibited more advantages [12, 34−37], the direct iterative method is used in dynamical analysis of DWCNT [38], the finite element method [21,23], and the differential quadrature method [13, 22] also exhibited more advantages.
The above investigations clearly show that most of the studies presented in the litterature are related to the nonlocal and nonlinear structures, but studies on the nonlocal and nonlinear fractional order vibration are very limited. When it is observed in the fields, linear and nonlinear frequencies amplitude of beams are major topics but dynamic analysis of beams embedded in fractional order viscoelastic medium is very rare. The nonlinear free vibration of the nanotube with damping effect was studied by using nonlocal elasticity theory [31]. To our knowledge, there is no published work on a fractional order nonlocal nonlinear vibration of nanobeam resting on viscoelastic foundation. The nonlinearity of the problem is obtained by considering the von karman geometric nonlinearity that introduces a cubic nonlinearity into the equations. In the present paper we analyse the nonlinear vibration of a nanobeam resting on fractional order viscoelastic Winkler-Pasternak foundation using Eringen's nonlocal elasticity. Nonlinear fractional order frequency response and modes shapes are drawn for nanobeam with different end conditions.

2. Preliminaries

2.1. *Fractional order viscoelasticity*

Fractional calculus is a part of mathematical analysis that has found many applications in nanomechanics. The role of fractional calculus is studied of an arbitrary real or complex order integrals and derivatives. There are many definitions of fractional order integrals and derivatives that were given by different authors. However, in our study we will consider only the Riemann-Liouville's definition of fractional derivative as follows: If $x(\bullet)$ is an absolutely continuous function in $[a, b]$ and $0 \prec a \prec 1$, then

1 The left Riemann-Liouville fractional derivative of order α is of the form

$$ {}_aD_t^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau, t \in [a, b] \tag{1}$$

2 The right Riemann-Liouville fractional derivative of order α is of the form

$$ {}_aD_t^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dt}\right) \int_a^t \frac{x(\tau)}{(\tau-t)^{\alpha}} d\tau, t \in [a, b] \tag{2}$$

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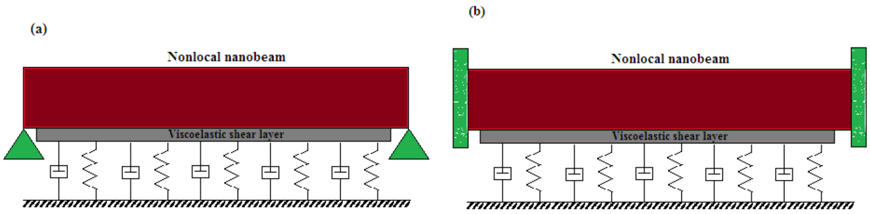

Figure 1. Boundary conditions for different beam support. (a) Simple-Simple case and (b) Clamped-Clamped case.

Fractional derivatives are used in the accurate modelling in rheology as well as structural mechanics to model internal damping. In [39], it was shown that classical viscoelastic models failed to describe damping of viscoelastic solid and that improved fractional derivative based models need to be considered. Such models have few advances. First, they are based on molecular theories [40]. Second, such models are satisfying thermodynamic laws. At least, they require a few parameters to describe viscoelastic behavior.

In follow, we give a constitutive relation of fractional order viscoelastic Winkler-Pasternak foundation beam interaction force (per unit length of the beam's axis) which is obtained including the fractional derivative term as [41]

$$q\left(x,t\right)=kw\left(x,t\right)+c\frac{\partial w\left(x,t\right)}{\partial t}-\left[\mu_e+\mu_v D_t^{\alpha}\right]\frac{\partial^2 w\left(x,t\right)}{\partial x^2}, \tag{3}$$

in which the deformed beam can be described by the tranverse deflection $w\left(x,t\right)$, k and c are foundation stiffness and damping coefficients, μ_e and μ_v are foundation shear elastic and viscosity coefficients. D_t^{α} is the fractional derivative with order α.

2.2. *Nonlocal theory*

In the nonlocal elasticity theory the stress at a point x is a function of the strains at all other points of an elastic body. The integral form of nonlocal constitutive relation for three dimensional structure is

$$\sigma_{ij}\left(x\right)=\int \chi\left(\left|x-x'\right|,\tau\right)t_{ij}\left(x'\right)dV(x'),\forall x\in V, \tag{4}$$

where σ_{ij} is the nonlocal stress tensor, t_{ij} is the local or classical stress tensors at a point x', $\chi\left(\left|x-x'\right|,\tau\right)$ denotes attenuation function which imcorporates nonlocal effects into the constitutive equation, $\left|x-x'\right|$ is a distance in Euclidian norm and $\tau={}^{e_0 a}/_{l}$ is nonlocal parameter where l is the external characteristic length (crack length or wave length), a is internal characteristic length (lattice parameter, granular etc.) and e_0 is a material constant that can be determined from molecular dynamics simulations or by using dispersive curve of the Born-Karman model of lattice dynamics. Later, Eringen [2] proposed a differential form of constitutive relation with an appropriate kernel function as

$$\left(1-\tau^2 l^2\nabla^2\right)\sigma_{ij}=t_{ij}. \tag{5}$$

For one dimensional case, the local stress t_{xx} at a point x' can be explained according to the Hooke's law as

$$t_{xx}\left(x'\right)=E\varepsilon_{xx}\left(x'\right), \tag{6}$$

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where E denotes elastic modulus and ε_{xx} the strain. That yields the following differential form of nonlocal constitutive equation for one dimensional elastic body

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\varepsilon_{xx}, \tag{7}$$

where $\mu = (e_0 a)^2$ is the nonlocal parameter and σ_{xx} is the nonlocal stress.

3. Governing equation of nanobeam resting on fractional order viscoelastic foundation

This study is carried out on the basis of the nonlocal Euler-Bernouilli nanobeam of length L, cross sectional area A, density ρ and tranverse deflection $w(x,t)$ in z direction . Two types of boundary conditions, which are simple-simple and clamped-clamped are considered in this work and shown in Fig. 1. We assume that cross sectional area is constant along the x coordinate and that material of a nanobeam is homogeneous. The nanobeam is resting on a fractional order viscoelastic Winkler-Pasternak foundation in which k and c are stiffness and damping coefficient, μ_e and μ_v are foundation shear elastic and viscosity coefficients. We also consider that the nanobeam is under the influence of time varying axial load. According to Euler-Bernouilli beam theory, the displacement fields at any point of beam can be expressed as

$$u_x(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x}, u_y = 0, u_z = w(x,t), \tag{8}$$

where u and w are the axial and tranverse displacements, respectively. By assuming the von Karman nonlinear strain displacement relation for the given displacement fields, we get

$$\varepsilon_0 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \varepsilon_1 = -z\bar{k}, \bar{k} = \frac{\partial^2 w}{\partial x^2}. \tag{9}$$

where ε_0 is the nonlinear extensional strain and $\bar{k}$ is the bending strain. The von Karman nonlinear normal strain can be expressed as

$$\varepsilon = \varepsilon_0 + \varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}. \tag{10}$$

By applying D'Alembert principle to the infinitesimal element of the nanobeam, equilibrium equation can be obtained as

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial T}{\partial x}, \tag{11a}$$

$$\rho A \frac{\partial^2 w}{\partial t^2} = \frac{\partial Q}{\partial x} + T\frac{\partial^2 w}{\partial x^2} - q(x,t), \tag{11b}$$

$$\rho I \frac{\partial^3 w}{\partial x \partial t^2} = Q - \frac{\partial M}{\partial x}, \tag{11c}$$

in which the stress resultant is defined as

$$(Q, T, M) = \int_0^A (\bar{\tau}_{xz}, \bar{\sigma}_{xx}, z\bar{\sigma}_{xx}) dA, \tag{12}$$

where Q , T, and M are the transversal force, the axial force and the bending moment, respectively. $\bar{\tau}_{xz}$ and $\bar{\sigma}_{xx}$ are shear and normal stress components. The longitudinal inertia $\frac{\partial^2 u}{\partial t^2}$ can be neglected based

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on the discussion about the nonlinear vibration of continuous systems [42,43], then the axial normal force T can be represented as

$$T = F + \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{13}$$

Assuming that the axial force is periodic and time-dependent and combining Equation (11) and Equation (7), the nonlinear vibration equation of motion for the nanobeam resting on the fractional order viscoelastic Pasternak-type foundaton in terms of transversal displacements is obtained as follows

$$\begin{aligned}\rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2\partial t^2} - \left(F\cos\Omega t + \frac{EA}{2L}\int\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2} + kw + c\frac{\partial w}{\partial t} + \left(\mu_e + \mu_v D_t^{\alpha}\right)\frac{\partial^2 w}{\partial x^2} + EI\frac{\partial^4 w}{\partial x^4} \\ - \mu\frac{\partial^2}{\partial x^2}\left(\rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2\partial t^2}\right) + \mu\frac{\partial^2}{\partial x^2}\left(\left(F\cos\Omega t + \frac{EA}{2L}\int\left(\frac{\partial w}{\partial x}\right)^2 dx\right)\frac{\partial^2 w}{\partial x^2}\right) \\ - \mu\frac{\partial^2}{\partial x^2}\left(kw + c\frac{\partial w}{\partial t} + \left(\mu_e + \mu_v D_t^{\alpha}\right)\frac{\partial^2 w}{\partial x^2}\right) = 0,\end{aligned} \tag{14}$$

where F is the amplitude of axial load and Ω is the frequency of this load. The following non-dimensional quantities aims to study problem under general form as

$$\bar{x} = \frac{x}{L}, \varpi = \frac{w}{L}, \bar{t} = \frac{t}{L^2}\sqrt{\frac{EI}{\rho A}}, \eta^2 = \frac{\mu}{L^2}, K = \frac{kL^4}{EI}, \epsilon C = c\sqrt{\frac{L^4}{\rho A\left(EI\right)}}, K_P = \frac{\mu_e L^2}{EI}, \epsilon\bar{F} = \frac{FL^2}{EI}$$

$$\epsilon C_P = \frac{\mu_v L^{2(1-\alpha)}}{\left(\rho A\right)^{\frac{1}{2}\alpha}\left(EI\right)^{\frac{1}{2}(2-\alpha)}}, \delta = \frac{I}{AL^2}. \tag{15}$$

In the non-dimensional Equation (14) and Equation (15) can be expressed as

$$\begin{aligned}\frac{\partial^2 \varpi}{\partial \bar{t}^2} - \delta\frac{\partial^4 \varpi}{\partial \bar{x}^2\partial \bar{t}^2} - \left(\epsilon\bar{F}\cos\bar{\Omega}\bar{t} + \frac{1}{2}\epsilon\int\left(\frac{\partial \varpi}{\partial \bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^2 \varpi}{\partial \bar{x}^2} + K\varpi + \epsilon C\frac{\partial \varpi}{\partial \bar{t}} + \left(K_P + \epsilon C_P D_{\bar{t}}^{\alpha}\right)\frac{\partial^2 \varpi}{\partial \bar{x}^2} + \frac{\partial^4 \varpi}{\partial \bar{x}^4} \\ - \eta^2\frac{\partial^2}{\partial \bar{x}^2}\left(\frac{\partial^2 \varpi}{\partial \bar{t}^2} - \delta\frac{\partial^4 \varpi}{\partial \bar{x}^2\partial \bar{t}^2}\right) + \eta^2\frac{\partial^2}{\partial \bar{x}^2}\left(\left(\epsilon\bar{F}\cos\bar{\Omega}\bar{t} + \frac{1}{2}\epsilon\int\left(\frac{\partial \varpi}{\partial \bar{x}}\right)^2 d\bar{x}\right)\frac{\partial^2 \varpi}{\partial \bar{x}^2}\right) \\ - \eta^2\frac{\partial^2}{\partial \bar{x}^2}\left(K\varpi + \epsilon C\frac{\partial \varpi}{\partial \bar{t}} + \left(K_P + \epsilon C_P D_{\bar{t}}^{\alpha}\right)\frac{\partial^2 \varpi}{\partial \bar{x}^2}\right) = 0,\end{aligned} \tag{16}$$

in which K and C denote dimensionless stiffness and viscosity medium, K_P and C_P denote dimensioless shear elastic and viscosity coefficient, $\bar{F}$ represent the dimensionless amplitude of axial load, η, ϖ and $\bar{t}$ denote the nonlocal parameter, transversal displacement and time, respectively, in dimensionless form. The small bookkeeping parameter ϵ is used to emphase the transversal deformatin, viscosity coefficients and tension fluctuation compared to the other terms.
The non-dimensional form of boundary conditions can be expressed as

Simple-Simple case:

$$\varpi\left(0\right) = 0, \varpi\left(1\right) = 0, \varpi''\left(0\right) = 0, \varpi''\left(1\right) = 0; \tag{17}$$

Clamped-Clamped case:

$$\varpi\left(0\right) = 0, \varpi\left(1\right) = 0, \varpi'\left(0\right) = 0, \varpi'\left(1\right) = 0. \tag{18}$$

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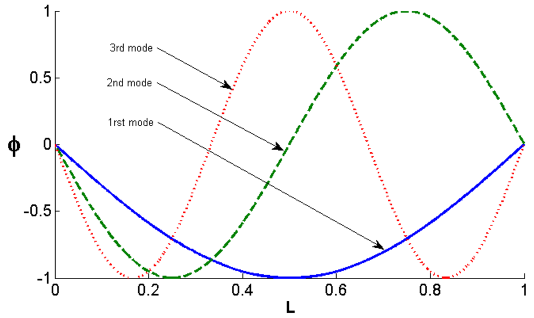

Figure 2. First three vibration modes shape for Simple-Simple case boundary condition.

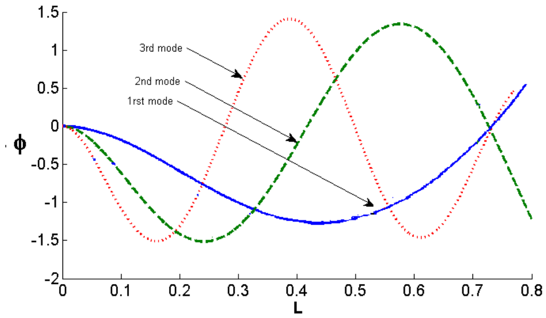

Figure 3. First three vibration modes shape for Clamped-Clamped case boundary condition.

3.1. *Solution of the governing equation*

The dimensionless fractional order nonlinear partial differential equation, Equation (16) describes the transversal vibration of nanobeam resting on fractional order viscoelastic foundation under the influence of periodic axial load. In order to obtain the asymptotic approximate solution in the first order for the problem, the pertubation method of multiple scales will be employed. By applying Garlekin method we assume the asymptotic approximate solution in the following form

$$\bar{w}\left(\bar{x},\bar{t}\right) = q\left(\bar{t}\right)\phi\left(\bar{x}\right), \tag{19}$$

in which $q\left(\bar{t}\right)$ is the unknown time function and $\phi\left(\bar{x}\right)$ is the linear mode shape determined from the boundary conditions. the linear mode shape of Equation (17) and Equation (18) are given by

$$\phi\left(\bar{x}\right) = c_1 \exp i\alpha_1\bar{x} + c_2 \exp i\alpha_2\bar{x} + c_3 \exp i\alpha_3\bar{x} + c_4 \exp i\alpha_4\bar{x}. \tag{20}$$

The boundary conditions are applied, the constants c_i and α_i can be obtained. Mode shape of the linear first frequency are plotted in Figure 2 and Figure 3. By introducing Equation (19) into Equation (16), multiplying the results by the linear mode shape function $\phi\left(\bar{x}\right)$ and then integrating them over the length of the nanobeam, we obtain a fractional order nonlinear ordinary differential equation expressed as

$$\frac{d^2q}{d\bar{t}^2} + \epsilon\tilde{C}\frac{dq}{d\bar{t}} + \left(w_0^2 + \epsilon\gamma\bar{F}\cos\bar{\Omega}\bar{t}\right)q + \frac{1}{4}\epsilon\chi q^3 + \epsilon\tilde{C}_P\frac{d^\alpha q}{d\bar{t}^\alpha} = 0, \tag{21}$$

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where w_0 is the natural frequency for the linear system, $\tilde{C}$ and $\tilde{C}_P$ are normal damping ratio and shear damping ratio, χ is the reduced nonlinear stiffness and γ is the constant

$$w_0^2 = \frac{K\left(a_1 - \eta^2 a_2\right) + K_P\left(-a_2 + \eta^2 a_3\right) + a_3}{\left(a_1 - \eta^2 a_2\right) + \delta\left(-a_2 + \eta^2 a_3\right)}, \tilde{C} = \frac{C\left(a_1 - \eta^2 a_2\right)}{\left(a_1 - \eta^2 a_2\right) + \delta\left(-a_2 + \eta^2 a_3\right)},$$
$$\gamma = \frac{\left(a_2 - \eta^2 a_3\right)}{\left(a_1 - \eta^2 a_2\right) + \delta\left(-a_2 + \eta^2 a_3\right)}, \chi = \frac{2a_4\left(-a_2 + \eta^2 a_3\right)}{\left(a_1 - \eta^2 a_2\right) + \delta\left(-a_2 + \eta^2 a_3\right)},$$
$$\tilde{C}_P = \frac{C_P\left(-a_2 + \eta^2 a_3\right)}{\left(a_1 - \eta^2 a_2\right) + \delta\left(-a_2 + \eta^2 a_3\right)}, \{a_1, a_2, a_3, a_4\} = \int_0^L \left\{\phi^2, \phi\phi', \phi\phi^{IV}, (\phi')^2\right\}. \quad (22)$$

Differential equation Equation (21) is a new form of parametrical excited Duffing differential equation due to the presence of fractional order term. In order to determine the asymptotic approximate solution with combined effects of nonlinearity, parametric excitation and fractional order damping, we will apply the pertubation method of multiple scales. A straightforward asymptotic expansion can be introduced

$$q\left(\bar{t};\epsilon\right) = \epsilon^0 q_0\left(T_0, T_1\right) + \epsilon^1 q_1\left(T_0, T_1\right), \quad (23)$$

where $T_0 = \bar{t}$ and $T_1 = \epsilon\bar{t}$ represent the fast and low timescale. The fast timescale is associated with the linear unpertubed system, while the slow timescale is characterized by modulation of amplitude and phase in the presence of possible resonance. Denoting $D_0 = \partial/\partial T_0$, $D_1 = \partial/\partial T_1$, the ordinary times derivatives can be transformed into partial derivative as

$$\frac{d}{dt} = D_0 + \epsilon D_1 + ..., \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + ..., \left(\frac{d}{dt}\right)^\alpha = D_0^\alpha + \epsilon\alpha D_0^{\alpha-1} D_1 + ..., \quad (24)$$

Inserting Equation (23) and Equation (24) into Equation (21), we obtain the following relation

$$\left(\epsilon^0\right): D_0^2 q_0 + \omega_0^2 q_0 = 0, \quad (25a)$$
$$\left(\epsilon^1\right): D_0^2 q_1 + \omega_0^2 q_1 = -2D_0 D_1 q_0 - \tilde{C}_P D_0^\alpha q_0 - \tilde{C} D_0 q_0 - \frac{1}{4}\chi q_0^3 + \gamma\bar{F}\cos\left(\bar{\Omega}T_0\right), \quad (25b)$$

Fundamental frequencies are obtained by solving the first order of expansion and the solvability condition is obtained by solving the second order of expansion. The solution of first order equation is given as

$$q_0\left(T_0, T_1\right) = A\left(T_1\right)\exp i\omega_0 T_0 + \bar{A}\left(T_1\right)\exp -i\omega_0 T_0, \quad (26)$$

where $i = \sqrt{-1}$, A is complex function of slow timescale and $\bar{A}$ is the complex conjugate. Excitation frequency is assumed to close to one of the natural frequencies of the system, the dimensionless form of this excitation frequency can be written as

$$\bar{\Omega} = \omega_0 + \epsilon\sigma, \quad (27)$$

where σ is a detuning parameter. Substituting Equation (26) in the second order of expansion and using the dimensionless form of excitation frequency, yields

$$D_0^2 q_1 + \omega_0^2 q_1 = -2i\omega_0\left(D_1 A + \frac{1}{2}\tilde{C}A\right)\exp i\omega_0 T_0$$
$$-\left(\frac{3}{4}\chi A^2\bar{A} + (i\omega_0)^\alpha \tilde{C}_P A + \frac{1}{2}\gamma\bar{A}\bar{F}\exp\left(\sigma T_1\right)\right)\exp i\omega_0 T_0 + cc + NST, \quad (28)$$

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where cc and NST represent the complex conjugate and the non-secular term, respectively. The solvability condition for Equation (28) is obtained as follows

$$2i\omega_0\left(D_1A+\frac{1}{2}\tilde{C}A\right)+\frac{3}{4}\chi A^2\bar{A}+(i\omega_0)^{\alpha}\tilde{C}_PA-\frac{1}{2}\gamma\bar{A}\bar{F}\exp\left(\sigma T_1\right)=0. \tag{29}$$

Taking into account the real amplitude a and phase β, the complex amplitude A can be written as

$$A=a\left(T_1\right)\exp i\beta\left(T_1\right). \tag{30}$$

Then amplitude and phase modulation equations are

$$D_1a+\frac{1}{2}\tilde{C}a+\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_Pa\sin\frac{\alpha\pi}{2}+\frac{1}{4}\frac{\gamma a\bar{F}}{\omega_0}\sin\psi=0, \tag{31a}$$

$$D_1\beta-\frac{3\chi}{8\omega_0}a^2-\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}-\frac{1}{4}\frac{\gamma\bar{F}}{\omega_0}\cos\psi, \tag{31b}$$

in which $\psi=\sigma T_1-2\beta$ is the new phase angle. In the steady-case, Equation (31) will be solved in the future section.

4. Numerical Results

Numerical examples of frequencies are presented in this section. The linear fundamental frequencies for different kinds of boundary conditions will be evaluated and the fractional order nonlinear frequencies for free vibrations will also be evaluated in the case of steady-state. To show correctness of presented study, we compared obtained results with results proposed by Mustapha and Zhong [17], Yokoyama [44] and Togun et al [28]. Detailed parametric study will be conducted to investivate the effects of system parameters such as stiffness and damping of viscoelastic foundation, nonlocal parameter and fractional parameter on the dimensional fractional order nonlinear natural frequencies of nanobeam with Simple-Simple boundary conditions and frequency response curve obtained by pertubation method . For free vibraton $\bar{F}=0$, in the case of steady-state, we obtain

$$D_1a=0\Rightarrow a=a_0. \tag{32}$$

By introducing Eq.(32) in Eq.(31b), we get

$$\beta\left(T_1\right)=\left(\frac{3\chi}{8\omega_0}a_0^2+\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}\right)T_1+\beta_0, \tag{33}$$

where a_0 and β_0 are the constants steady-state real amplitude and phase which are determined from the initial conditions. Introducing the obtained results into Eq.(26), gives the first order vibration response

$$q_0\left(T_0,T_1\right)=a_0\exp i\left(\frac{3\chi}{8\omega_0}a_0^2+\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}\right)\epsilon\bar{t}\times\exp i\left(\omega_0\bar{t}+\beta_0\right)+cc, \tag{34}$$

and hence the fractional order nonlinear frequency is

$$\omega_{nl}^{(\alpha)}=\omega_0+D_1\beta=\omega_0+\epsilon\frac{3\chi}{8\omega_0}a_0^2+\epsilon\frac{1}{2}\omega_0^{\alpha-1}\tilde{C}_P\cos\frac{\alpha\pi}{2}, \tag{35}$$

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Table 1. The first five non-dimensional natural frequencies of a local Euler-Bernouilli beam resting on Winkler-Pasternak foundation for the Simple-Simple boundary condition.($\eta = 0$, $\delta = 0$, $K = 25$, $K_p = 25$).

Mode	Present	Ref [28]	Ref [17]	Ref [44]
1	19.2133	19.2133	19.2178	19.21
2	50.7002	50.7002	50.7804	50.71
3	100.6767	100.677	-	-
4	170.0281	170.028	-	-
5	258.9868	258.987	-	-

where $\lambda = \frac{3\chi}{8\omega_0}$ is the nonlinear correction coefficient and the third term is a correction of natural frequency due to the fractional order damping term. At the steady-state $D_1 a = 0$ and $D_1\psi = 0$. The detuning parameter or amplitude-frequency response is as follows

$$\sigma = \frac{3\chi}{4\omega_0}a_0^2 + \omega_0^{\alpha-1}\tilde{C}_P \cos\frac{\alpha\pi}{2} \pm \sqrt{\frac{1}{4}\frac{\gamma^2\bar{F}^2}{\omega_0^2} - \left(\tilde{C} + \omega_0^{\alpha-1}\tilde{C}_P \sin\frac{\alpha\pi}{2}\right)^2}. \tag{36}$$

4.1. Validation study

Studies related to the nonlinear nonlocal nanobeam resting on Winkler-Pasternak viscoelastic foundation in the litterature are so limited. In order to validate present analitycal results for amplitude -frequency response of the dynamical fractional order nonlinear nonlocal nano-beam with Simple-Simple boundary condition, we compared obtained results proposed by Mustapha and Zhong [17], Yokoyama [44] and Togun et al [28]. Let us consider the case of free vibration and only the classical damping influence ($\alpha = 1$), thus, the fractonal order correction to the natural frequency is absent in Eq. (35) and Eq. (36) and then, we recognise the common form of nonlinear frequency and detuning parameter

$$\omega_{nl} = \omega_0\left(1 + \epsilon\frac{3\chi}{8\omega_0^2}a_0^2\right), \tag{37}$$

$$\sigma = \frac{3\chi}{4\omega_0}a_0^2 \pm \sqrt{\frac{1}{4}\frac{\gamma^2\bar{F}^2}{\omega_0^2} - \mu^2}, \tag{38}$$

where $\lambda = \frac{3\chi}{8\omega_0}$ is the nonlinear correction coefficient and $\mu = \tilde{C} + \tilde{C}_P$ the damping coefficient. The work of Mustapha and Zhong [17] studies the non-uniform SWCNT depended on a nonlocal Rayleigh beam resting on pasternak-type foundation, Yokoyama [44] studies free transverse vibration of the classical Euler-Bernouilli beam resting on a Winkler-Pasternak foundation and Togun et al [28] studies nonlinear vibration of a nonlocal nanobeam on a Winkler-Pasternak foundation using Euler-Bernouilli beam theory. A comparison study is performed to check the correctness of the present study. For this aim, linear frequency of local case of our nanobeam resting on a Winkler-Pasternak foundation for the Simple-Simple boundary condition are compared with those of the work of Mustapha and Zhong [17], Yokoyama [44] and Togun et al [28]. It can be seen from the Table 1 and Table 2 that there is good harmony between the four results. Figure 4 shows the nonlocal parameter effect on the fractional nonlinear frequency, it can be deduced that the natural frequency decreases when the nonlocal parameter increases. Variation of the fractional nonlinear frequency with amplitude for the first three modes of vibration is shown in Figure 5. It can be seen from Figure 5 that the fractional nonlinear frequencies increases with an increase in the mode number. In Figures 6-8 the fractional nonlinear frequency versus amplitude for different values of system parameter are shown for the first mode of vibration.

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Table 2. The first five non-dimensional natural frequencies of a local Euler-Bernouilli beam resting on Winkler-Pasternak foundation for the Simple-Simple boundary condition.($\eta = 0$, $\delta = 0$, $K = 36$, $K_p = 36$).

Mode	Present	Ref [28]	Ref [17]	Ref [44]
1	22.1069	22.1069	22.1112	-
2	54.9160	54.916	55.1873	-
3	105.4698	105.47	-	-
4	175.0932	175.093	-	-
5	264.1956	264.196	-	-

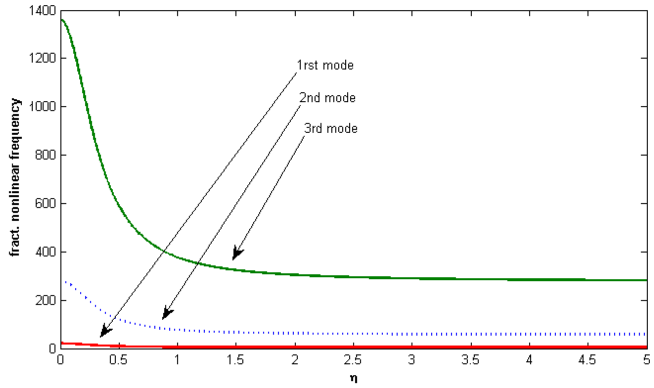

Figure 4. First three mode of fractional nonlinear frequency versus nonlocalty η **(** $\alpha = 0.5$, $K = 5$, $Kp = 2$, $Cp = 0.001$**).**

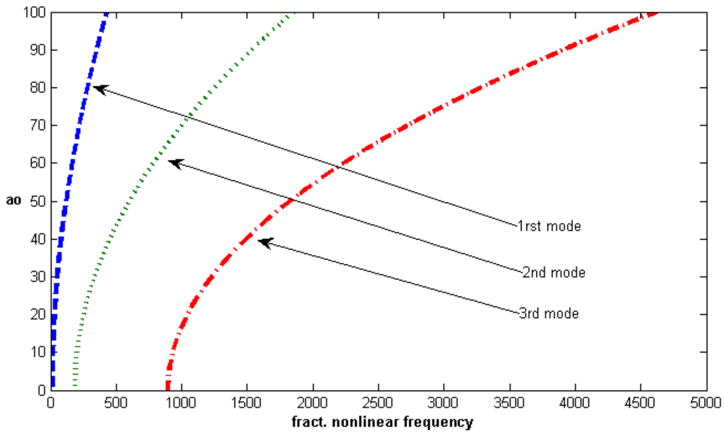

Figure 5. First three mode of fractional nonlinear frequency versus amplitude ($\alpha = 0.5$, $K = 5$, $Kp = 2$, $Cp = 0.001$, $\eta = 0.5$**).**

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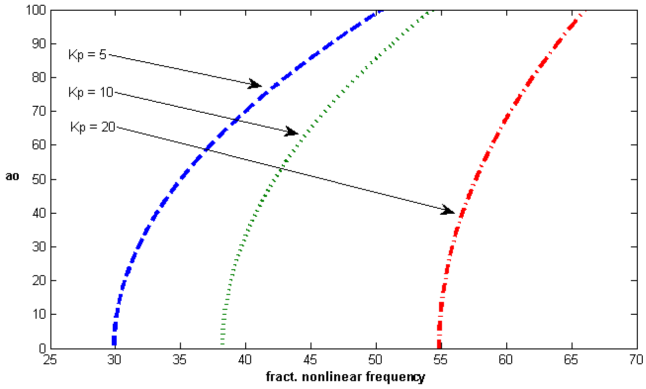

Figure 6. Fractional nonlinear frequency versus amplitude for different value of Kp ($\alpha = 0.5$, $K = 100$, $Cp = 0.001$, $\eta = 0.5$).

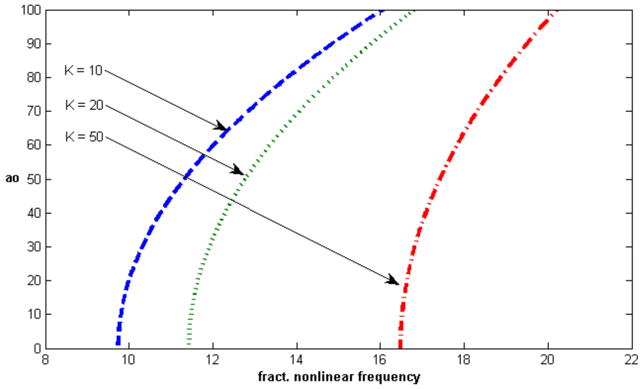

Figure 7. Fractional nonlinear frequency versus amplitude for different value of K ($\alpha = 0.5$, $Kp = 5$, $Cp = 0.001$, $\eta = 0.5$).

Figure 6 shows the effect of Pasternak parameter Kp on the fractional nonlinear frequency versus amplitude curves. It can be seen in Figure 6 that the fractional nonlinear frequency increases with an increase of Kp. In Figure 7, the fractional nonlinear frequency also increases with an increase of Winkler stiffness parameter K. In Figure 8, the fractional nonlinear frequency versus amplitude for different values of fractional damping coefficient Cp is shown. It can be deduced from Figure 8 that the fractional nonlinear frequency increases slowly when the fractional damping coefficient increases. It is normal because the fractional nonlinear frequency have a direct relation with Cp. Also it can be observed a hardening behavior in Figures 6-8 because the fractional nonlinear frequency increases as the amplitude increases. Frequency response curves are presented in Figure 9 for different values of nondimensional nonlinear coefficient. It can be seen from Figure 9 that our system is really nonlinear. In Figures 10-12, the fractional contribution frequency versus Winkler parameter K and nonlocal parameter η for different values of fractional parameter α are shown. It can be seen from Figures 10-12 that the fractional contribution frequency increases and reaches to the constant maximum value when the nonlocal parameter increases. For the small values of the nonlocal parameter the fractional contribution increases quickly, but for the high values this contribution is constant. In Figures 13-14, the fractional contribution frequency versus Pasternak parameter Kp and nonlocal parameter η curves for different values of fractional parameter α are shown. It is observed that the variation of fractional contribution depend of the interval of variation of the nonlocal parameter η. For the small value of η,

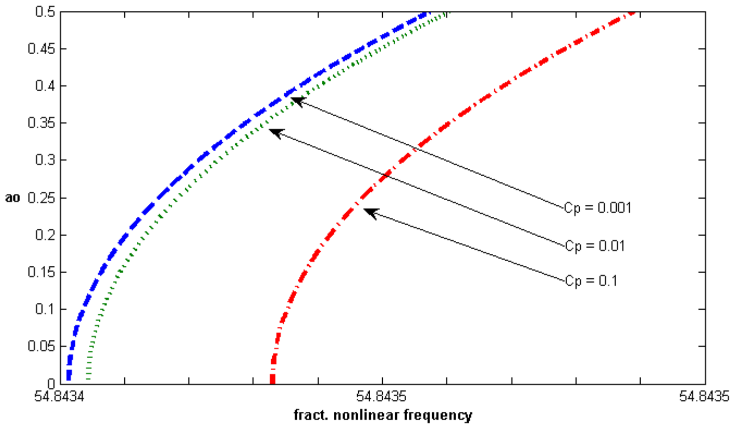

Figure 8. **Fractional nonlinear frequency versus amplitude for different value of Cp ($\alpha = 0.5$, $Kp = 5$, $K = 100$, $\eta = 0.5$).**

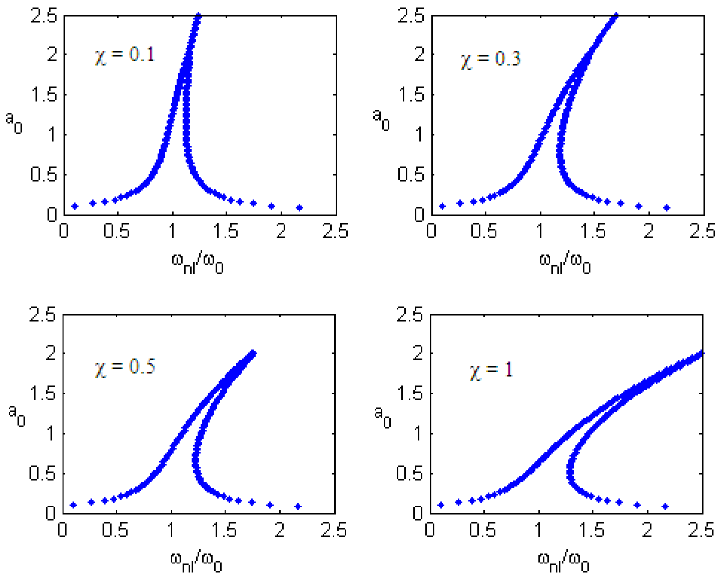

Figure 9. **Frequency-response curves versus amplitude for different value of χ ($\alpha = 1$, $C = 0.025$, $Cp = 0.025$, $wo = 1$, $\bar{F} = 0.2$).**

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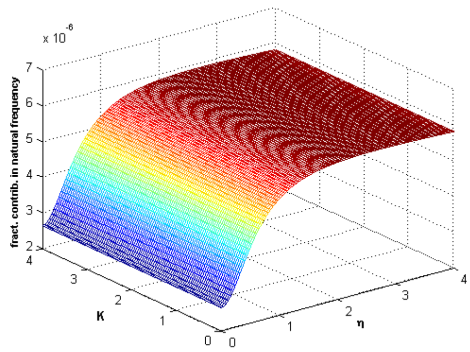

Figure 10. Fractional Contribution frequency versus stiffness K and nonlocalty η ($\alpha = 0.2$).

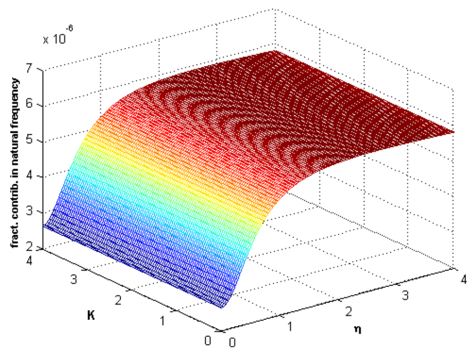

Figure 11. Fractional Contribution frequency versus stiffness K and nonlocalty η ($\alpha = 0.5$).

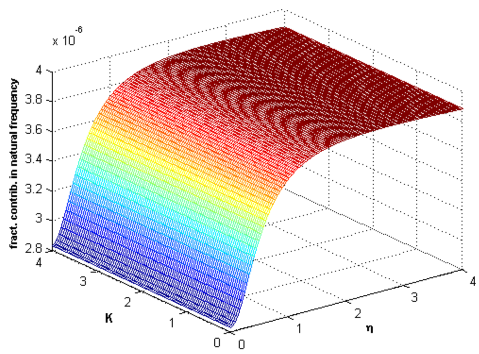

Figure 12. Fractional Contribution frequency versus stiffness K and nonlocalty η ($\alpha = 0.8$).

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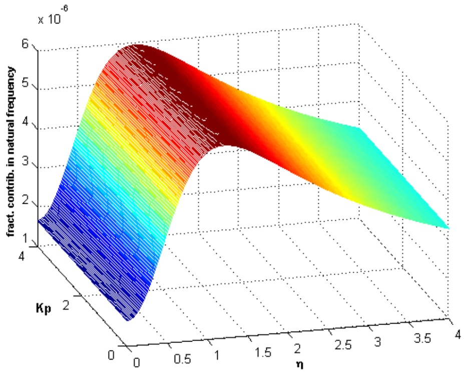

Figure 13. Fractional Contribution frequency versus stiffness Kp and nonlocalty η ($\alpha = 0.2$).

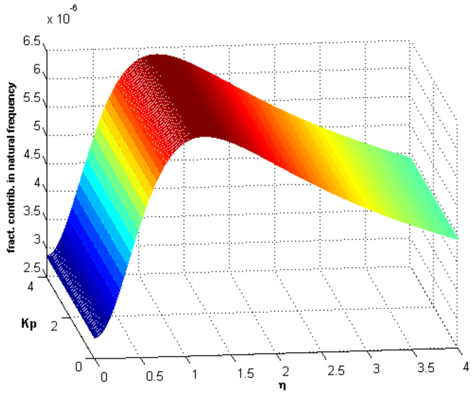

Figure 14. Fractional Contribution frequency versus stiffness Kp and nonlocalty η ($\alpha = 0.5$).

the fractional contribution increases but for the high value of η, this fractional contribution decreases. In Figures 15-17, the fractional contribution frequency versus fractional damping coefficient Cp and nonlocal parameter η curves are shown for different values of fractional parameter α. It can be seen that the fractional contribution increases when Cp increases. In front of these all observations, it is easily normal to say that every system parameter has significant effect on the natural frequency of nanobeam, specially the fractional parameter and fractional damping coefficient.

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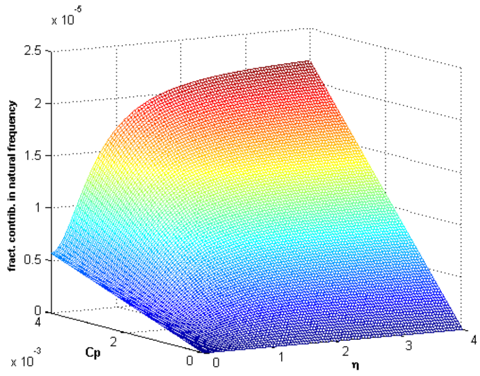

Figure 15. Fractional contribution frequency versus fractional damping coefficient Cp and nonlocalty η ($\alpha = 0.2$).

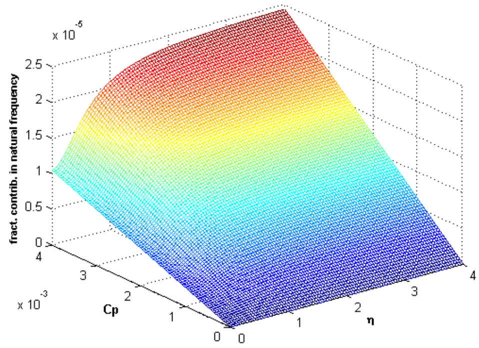

Figure 16. Fractional contribution frequency versus fractional damping coefficient Cp and nonlocalty η ($\alpha = 0.5$).

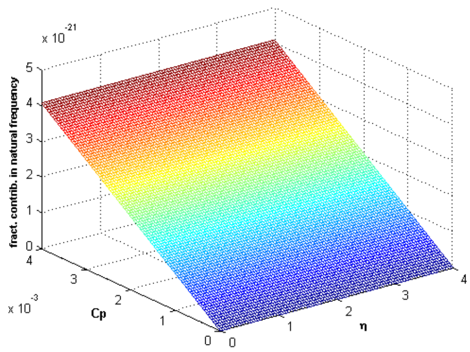

Figure 17. Fractional contribution frequency versus fractional damping coefficient Cp and nonlocalty η ($\alpha = 1$).

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5. Conclusion

In this study, using the concept of fractional derivative, nonlinear vibration of nanobeam resting on fractional order viscoelastic Winkler- Pasternak foundation is studied using nonlocal elasticity theory. For this purpose, Eringen's nonlocal elasticity theory, the von Karman geometric nonlinearity and the Euler-Bernouilli beam theory are employed. D'Alembert principle is used to derive the governing equation and the associated boundary conditions. In the solution procedure, employing the Garlekin scheme, the fractional integro-partial differential governing equation is first simplified into the time-dependant fractional ordinary differential equation. This new equation is known as fractional order nonlinear Duffing equation which is then solved by multiple scales method. Detailled parametric study is conducted to get the effects of system parameter such as Winkler stiffness parameter, Pasternak stiffness parameter, nonlocal parameter, nonlinear coefficient, fractional damping coefficient and fractional parameter on the fractional nonlinear frequency of the nanobeam. It is found that fractional nonlinear frequency decreases when the nonlocal parameter increases. Also this fractional nonlinear frequency increases when Winkler parameter, Pasternak parameter, mode, fractional damping coefficient and amplitude increases. It is further found that every parameter of the system has significant effect on the fractional contribution frequency.

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Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

CNT	Carbon nanutube
SDCNT	Single walled carbon nanotube
DWCNT	Double walled carbon nanotube

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