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Article

Rough $\mathcal{I}_2$-Lacunary Statistical Convergence of Double Sequences

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Abstract: In this paper, we introduce and study the notion of rough $\mathcal{I}_2$-lacunary statistical convergence of double sequences in normed linear spaces. We also introduce the notion of rough $\mathcal{I}_2$-lacunary statistical limit set of a double sequence and discuss about some properties of this set.

Keywords: Statistical convergence, $\mathcal{I}$-convergence, rough convergence, lacunary sequences, double sequences.

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1. Introduction

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [13] and Schoenberg [34]. This concept was extended to the double sequences by Mursaleen and Edely [22]. Lacunary statistical convergence was defined by Fridy and Orhan [15]. Çakan and Altay [5] presented multidimensional analogues of the results presented by Fridy and Orhan [15].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [17] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Kostyrko et al. [18] studied the idea of $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in area about statistical convergence, $\mathcal{I}$-convergence and double sequences after the works of [1,11,13,14,21,28–30,34].

The notion of lacunary ideal convergence of real sequences was introduced in [35]. Das et al. [8] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Belen et al. [4] introduced the notion of ideal statistical convergence of double sequences, which is a new generelization of the notions of statistical convergence and usual convergence. Kumar et al. [36] introduced $\mathcal{I}$-lacunary statistical convergence of double sequences. More investigation and applications on this notion can be found in [16].

The idea of rough convergence was first introduced by Phu [25] in finite-dimensional normed spaces. In another paper [26] related to this subject, Phu defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $dim Y < \infty$ and $r > 0$, where X and Y are normed spaces. In [27], Phu extended the results given in [25] to infinite-dimensional normed spaces. Aytar [2] studied the rough statistical convergence. Also, Aytar [3] studied that the rough limit set and the core of a real sequence. Recently, Dündar and Çakan [10,11] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notion of rough convergence and the set of rough limit points of a

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double sequence. Further this notion of rough convergence of double sequence has been extended to rough statistical convergence of double sequence by Malik et. al. [19] using double natural density of $\mathbb{N} \times \mathbb{N}$ in the similar way as the notion of convergence of double sequence in Pringsheim sense was generalized to statistical convergence of double sequence. Also, Dündar [12] investigated rough $\mathcal{I}_2$-convergence of double sequences. The notion of $\mathcal{I}$-statistical convergence of double sequences was introduced by Malik and Ghosh [20] in the theory of rough convergence.

In view of the recent applications of ideals in the theory of convergence of sequences, it seems very natural to extend the interesting concept of rough lacunary statistical convergence further by using ideals which we mainly do here.

So it is quite natural to think, if the new notion of $\mathcal{I}$-lacunary statistical convergence of double sequences can be introduced in the theory of rough convergence.

2. Definitions and notations

In this section, we recall some definitions and notations, which form the base for the present study ([1,2,4,10–12,17,19,20,25,28,36]).

During the paper, let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_i) \subset \mathbb{R}^n$.

The sequence $x = (x_i)$ is said to be r-convergent to x_*, denoted by $x_i \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \exists i_\varepsilon \in \mathbb{N} : \ i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_i \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be r-convergent if $\mathtt{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [25]).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$,
(ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
(iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if
(i) $\emptyset \notin \mathcal{F}$,
(ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
(iii) for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$.

A sequence $x = (x_i)$ is said to be rough $\mathcal{I}$-convergent (r-$\mathcal{I}$-convergent) to x_* with the roughness degree r, denoted by $x_i \xrightarrow{r-\mathcal{I}} x_*$ provided that $\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_i - x_*\| \leq r \tag{1}$$

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is satisfied. In addition, we can write $x_i \xrightarrow{r-\mathcal{I}} x_*$ iff the inequality $\|x_i - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all i.

A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense (shortly, p-convergent to $L \in \mathbb{R}$), if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

We recall that a subset K of $\mathbb{N}\times\mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{m,n\to\infty} \frac{K(m,n)}{m.n},$$

where $K(m,n) = |\{(j,k) \in \mathbb{N}\times\mathbb{N} : j \leq m, k \leq n\}|$.

Throughout the paper we consider a sequence $x = (x_{mn})$ such that $(x_{mn}) \in \mathbb{R}^n$.

Let $x = (x_{mn})$ be a double sequence in a normed space $(X, \|.\|)$ and r be a non negative real number. x is said to be r-statistically convergent to ξ, denoted by $x \xrightarrow{r-st_2} \xi$, if for $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - \xi\| \geq r + \varepsilon\}$. In this case, ξ is called the r-statistical limit of x.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

Let (X, ρ) be a metric space A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})$ is said to be rough convergent (r-convergent) to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r} x_*$ provided that

$$\forall\varepsilon > 0 \;\exists k_\varepsilon \in \mathbb{N} : \;\; m, n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon, \tag{2}$$

or equivalently, if

$$\limsup \|x_{mn} - x_*\| \leq r. \tag{3}$$

A double sequence $x = (x_{mn})$ is said to be r-$\mathcal{I}_2$-convergent to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*$ provided that

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2, \tag{4}$$

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for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I}_2 - \limsup \|x_{mn} - x_*\| \leq r \tag{5}$$

is satisfied. In addition, we can write $x_{mn} \overset{r-\mathcal{I}_2}{\longrightarrow} x_*$ iff the inequality $\|x_{mn} - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all (m, n).

Now, we give the definition of $\mathcal{I}_2$-asymptotic density of $\mathbb{N} \times \mathbb{N}$.

A subset $K \subset \mathbb{N} \times \mathbb{N}$ is said to be have $\mathcal{I}_2$-asymptotic density $d_{\mathcal{I}_2}(K)$ if

$$d_{\mathcal{I}_2}(K) = \mathcal{I}_2 - \lim_{m,n\to\infty} \frac{|K(m,n)|}{m.n},$$

where $K(m,n) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : j \leq m, k \leq n; (j,k) \in K\}$ and $|K(m,n)|$ denotes number of elements of the set $K(m,n)$.

A double sequence $x = \left\{x_{jk}\right\}$ of real numbers is $\mathcal{I}_2$-statistically convergent to ε, and we write $x \overset{\mathcal{I}_2-st}{\rightarrow} \xi$, provied that for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} \left|\left\{(j,k) : \left\|x_{jk} - \xi\right\| \geq \varepsilon, j \leq m, k \leq n\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Let $x = \left\{x_{jk}\right\}$ be a double sequence in a normed linear space $(X, \|.\|)$ and r be a non negative real number. Then x is said to be rough $\mathcal{I}_2$-statistical convergent to ξ or r-$\mathcal{I}_2$-statistical convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} \left|\left\{(j,k), j \leq m, k \leq n : \left\|x_{jk} - \xi\right\| \geq r + \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, ξ is called the rough $\mathcal{I}_2$-statistical limit of $x = \left\{x_{jk}\right\}$ and we denote it by $x \overset{r-\mathcal{I}_2-st}{\longrightarrow} \xi$.

A double sequence $\overline{\theta} = \theta_{us} = \{(k_u, l_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers (k_u) and (l_s) such that

$$k_0 = 0, h_u = k_u - k_{u-1} \to \infty \ \ and \ \ l_0 = 0, \overline{h}_s = l_s - l_{s-1} \to \infty, \quad u, s \to \infty.$$

We will use the following notation $k_{us} := k_u l_s$, $h_{us} := h_u \overline{h}_s$ and θ_{us} is determined by

$$J_{us} := \{(k,l) : k_{u-1} < k \leq k_u \text{ and } l_{s-1} < l \leq l_s\},$$

$$q_u := \frac{k_u}{k_{u-1}}, \overline{q}_s := \frac{l_s}{l_{s-1}} \text{ and } q_{us} := q_u \overline{q}_s.$$

Throughout the paper, by $\theta_2 = \theta_{us} = \{(k_u, l_s)\}$ we will denote a double lacunary sequence of positive real numbers, respectively, unless otherwise stated.

A double sequence $x = \{x_{mn}\}$ of numbers is said to be $\mathcal{I}_2$-lacunary statistical convergent or $S_{\theta_2}(\mathcal{I}_2)$-convergent to L, if for each $\varepsilon > 0$ and $\delta > 0$,

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} |\{(m,n) \in J_{us} : |x_{mn} - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $x_{mn} \to L\left(S_{\theta_2}(\mathcal{I}_2)\right)$ or $S_{\theta_2}(\mathcal{I}_2)$-$\lim_{m,n\to\infty} x_{mn} = L$.

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3. Main results

Definition 1. *Let $x = \{x_{jk}\}$ be a double sequence in a normed linear space $(X, \|.\|)$ and r be a non negative real number. Then x is said to be rough lacunary statistical convergent to ξ or r-lacunary statistical convergent to ξ if for any $\varepsilon > 0$*

$$\lim_{u,s\to\infty} \frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| \geq r + \varepsilon\right\}\right| = 0.$$

In this case ξ is called the rough lacunary statistical limit of $x = \{x_{jk}\}$ and we denote it by $x \overset{r-S_{\theta_2}}{\longrightarrow} \xi$.

Definition 2. *Let $x = \{x_{jk}\}$ be a double sequence in a normed linear space $(X, \|.\|)$ and r be a non negative real number. Then, x is said to be rough $\mathcal{I}_2$-lacunary statistical convergent to ξ or r-$\mathcal{I}_2$-lacunary statistical convergent to ξ if for any $\varepsilon > 0$ and $\delta > 0$*

$$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| \geq r + \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, ξ is called the rough $\mathcal{I}_2$-lacunary statistical limit of $x = \{x_{jk}\}$ and we denote it by $x \overset{r-\mathcal{I}_{\theta_2}-st}{\longrightarrow} \xi$.

Remark 1. *Note that if $\mathcal{I}_2$ is the ideal*

$$\mathcal{I}_2^0 = \{A \subset \mathbb{N}\times\mathbb{N} : \exists m(A) \in \mathbb{N} \text{ such that } i,j \geq m(A) \Rightarrow (i,j) \notin A\},$$

then rough $\mathcal{I}_2$-lacunary statistical convergence coincide with rough lacunary statistical convergence.

Here r in the above definition is called the roughness degree of the rough $\mathcal{I}_2$-lacunary statistical convergence. If $r = 0$, we obtain the notion of $\mathcal{I}_2$-lacunary convergence. But our main interest is when $r > 0$. It may happen that a double sequence $x = \{x_{jk}\}$ is not $\mathcal{I}_2$-lacunary statistical convergent in the usual sense, but there exists a double sequence $y = \{y_{jk}\}$, which is $\mathcal{I}_2$-lacunary statistically convergent and satisfying the condition $\left\|x_{jk} - y_{jk}\right\| \leq r$ for all (j,k). Then, x is rough $\mathcal{I}_2$-lacunary statistically convergent to the same limit.

From the above definition it is clear that the rough $\mathcal{I}_2$-lacunary statistical limit of a double sequence is not unique. So we consider the set of rough $\mathcal{I}_2$-lacunary statistical limits of a double sequence x and we use the notation $\mathcal{I}_{\theta_2}$-st-LIM_x^r to denote the set of all rough $\mathcal{I}_2$-lacunary statistical limits of a double sequence x. We say that a double sequence x is rough $\mathcal{I}_2$-lacunary statistically convergent if $\mathcal{I}_{\theta_2}$-st-$LIM_x^r \neq \emptyset$.

Throughout the paper X denotes a normed linear space $(X, \|.\|)$ and x denotes the double sequence $\{x_{jk}\}$ in X.

Now, we discuss some basic properties of rough $\mathcal{I}_2$-lacunary statistically convergence of double sequences.

Theorem 1. *Let $x = \{x_{jk}\}$ be a double sequence and $r \geq 0$. Then, $\mathcal{I}_{\theta_2}$-st-$LIM_x^r \leq 2r$. In particular if x is rough $\mathcal{I}_2$-lacunary statistically convergent to ξ, then*

$$\mathcal{I}_{\theta_2} - st - LIM_x^r = \overline{B_r(\xi)},$$

where $\overline{B_r(\xi)} = \{y \in X : \|y - \xi\| \leq r\}$ and so

$$diam\left(\mathcal{I}_{\theta_2} - st - LIM_x^r\right) = 2r.$$

Proof. Let $diam\left(\mathcal{I}_{\theta_2} - st - LIM_x^r\right) > 2r$. Then, there exist $y, z \in \mathcal{I}_{\theta_2}\text{-}st\text{-}LIM_x^r$ such that $\|y-z\| > 2r$. Now, we select $\varepsilon > 0$ so that $\varepsilon < \frac{\|y-z\|}{2} - r$. Let

$$A = \left\{(j,k) \in J_{us} : \left\|x_{jk} - y\right\| \geq r + \varepsilon\right\} \text{ and } B = \left\{(j,k) \in J_{us} : \left\|x_{jk} - z\right\| \geq r + \varepsilon\right\}.$$

Then,

$$\frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A \cup B\}\right|$$

$$\leq \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A\}\right| + \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in B\}\right|,$$

and so by the property of $\mathcal{I}_2$-convergence

$$\mathcal{I}_2\text{-}\lim_{u,s\to\infty} \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A \cup B\}\right|$$

$$\leq \mathcal{I}_2\text{-}\lim_{u,s\to\infty} \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A\}\right| + \mathcal{I}_2\text{-}\lim_{u,s\to\infty} \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in B\}\right| = 0.$$

Thus,

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A \cup B\}\right| \geq \delta\right\} \in \mathcal{I}_2$$

for all $\delta > 0$. Let

$$H = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A \cup B\}\right| \geq \frac{1}{2}\right\}.$$

Clearly $H \in \mathcal{I}_2$, so choose $(u_0, s_0) \in \mathbb{N} \times \mathbb{N} \setminus H$. Then,

$$\frac{1}{h_{u_0 s_0}} \left|\{(j,k) \in J_{us} : (j,k) \in A \cup B\}\right| < \frac{1}{2}.$$

So, we have

$$\frac{1}{h_{u_0 s_0}} \left|\{(j,k) \in J_{us} : (j,k) \notin A \cup B\}\right| \geq 1 - \frac{1}{2} = \frac{1}{2},$$

131 i.e., $\{(j,k) \in J_{us} : (j,k) \notin A \cup B\}$ is a nonempty set.

Take $(j_0, k_0) \in J_{us}$ such that $(j_0, k_0) \notin A \cup B$. Then, $(j_0, k_0) \in A^c \cap B^c$ and so $\left\|x_{j_0 k_0} - y\right\| < r + \varepsilon$ and $\left\|x_{j_0 k_0} - z\right\| < r + \varepsilon$. Hence, we have

$$\|y - z\| \leq \left\|x_{j_0 k_0} - y\right\| + \left\|x_{j_0 k_0} - z\right\| \leq 2(r + \varepsilon) \leq \|y - z\|,$$

132 which is absurd. Therefore, $\mathcal{I}_{\theta_2}\text{-}st\text{-}LIM_x^r \leq 2r$.

If $\mathcal{I}_{\theta_2}\text{-}st\text{-}LIM_x^r = \xi$, then we proceed as follows. Let $\varepsilon > 0$ and $\delta > 0$ be given. Then,

$$A = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

Then, for $(u,s) \notin A$ we have

$$\frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| \geq \varepsilon\right\}\right| < \delta,$$

i.e.,

$$\frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| < \varepsilon\right\}\right| \geq 1 - \delta. \tag{1}$$

Now, for each $y \in \overline{B_r(\xi)}$ we have

$$\left\|x_{jk}-y\right\| \leq \left\|x_{jk}-\xi\right\| + \|\xi - y\| \leq \left\|x_{jk}-\xi\right\| + r. \tag{2}$$

Let

$$B_{us} = \left\{(j,k) \in J_{us} : \left\|x_{jk}-\xi\right\| < \varepsilon\right\}.$$

Then, for $(j,k) \in B_{us}$ we have $\left\|x_{jk}-y\right\| < r+\varepsilon$. Hence, we have

$$B_{us} = \left\{(j,k) \in J_{us} : \left\|x_{jk}-y\right\| < r+\varepsilon\right\}.$$

This implies

$$\frac{|B_{us}|}{h_{us}} \leq \frac{1}{h_{us}}\left|\left\{(j,k) \in J_{us} : \left\|x_{jk}-y\right\| < r+\varepsilon\right\}\right|$$

i.e.,

$$\frac{1}{h_{us}}\left|\left\{(j,k) \in J_{us} : \left\|x_{jk}-y\right\| < r+\varepsilon\right\}\right| \geq 1-\delta.$$

Thus, for all $(j,k) \notin A$,

$$\frac{1}{h_{us}}\left|\left\{(j,k) \in J_{us} : \left\|x_{jk}-y\right\| \geq r+\varepsilon\right\}\right| < 1-(1-\delta)$$

and so we have

$$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}}\left|\left\{(j,k) \in J_{us}; \left\|x_{jk}-y\right\| \geq r+\varepsilon\right\}\right| \geq \delta\right\} \subset A.$$

Since $A \in \mathcal{I}_2$ then,

$$\left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}}\left|\left\{(j,k) \in J_{us} : \left\|x_{jk}-y\right\| \geq r+\varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

133 This shows that $y \in \mathcal{I}_{\theta_2}\text{-}st\text{-}LIM_x^r$. Therefore, $\mathcal{I}_{\theta_2}\text{-}st\text{-}LIM_x^r \supset \overline{B_r(\xi)}$.

Conversely, let $y \in \mathcal{I}_{\theta_2}\text{-}st\text{-}LIM_x^r$, $\|y-\xi\| > r$ and $\varepsilon = \frac{\|y-\xi\|-r}{2}$. Now, we take

$$M_1 = \left\{(j,k) \in J_{us} : \left\|x_{jk}-y\right\| \geq r+\varepsilon\right\} \text{ and } M_2 = \left\{(j,k) \in J_{us} : \left\|x_{jk}-\xi\right\| \geq \varepsilon\right\}.$$

Then,

$$\frac{1}{h_{us}}\left|\{(j,k) \in J_{us} : (j,k) \in M_1 \cup M_2\}\right|$$

$$\leq \frac{1}{h_{us}}\left|\{(j,k) \in J_{us} : (j,k) \in M_1\}\right| + \frac{1}{h_{us}}\left|\{(j,k) \in J_{us} : (j,k) \in M_2\}\right|,$$

134 and by the property of $\mathcal{I}_2$-convergence

$$\begin{aligned}\mathcal{I}_2\text{-}\lim_{u,s\to\infty}\frac{1}{h_{us}}\left|\{(j,k) \in J_{us} : (j,k) \in M_1 \cup M_2\}\right| &= \mathcal{I}_2\text{-}\lim_{u,s\to\infty}\frac{1}{h_{us}}\left|\{(j,k) \in J_{us} : (j,k) \in M_1\}\right| \\ &+ \mathcal{I}_2\text{-}\lim_{u,s\to\infty}\frac{1}{h_{us}}\left|\{(j,k) \in J_{us} : (j,k) \in M_2\}\right| \\ &= 0.\end{aligned}$$

Now, we let

$$M = \left\{(u,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{us}}\left|\{(j,k) : (j,k) \in M_1 \cup M_2\}\right| \geq \frac{1}{2}\right\}.$$

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Clearly $M \in \mathcal{I}_2$ and we choose $(u_0, s_0) \in \mathbb{N} \times \mathbb{N} \setminus M$. Then, we have

$$\frac{1}{h_{u_0 s_0}} |\{(j,k) : (j,k) \in M_1 \cup M_2\}| < \frac{1}{2},$$

and so

$$\frac{1}{h_{u_0 s_0}} |\{(j,k) : (j,k) \notin M_1 \cup M_2\}| \geq 1 - \frac{1}{2} = \frac{1}{2},$$

i.e., $\{(j,k) : (j,k) \notin M_1 \cup M_2\}$ is a nonempty set. Let $(j_0, k_0) \in J_{us}$ such that $(j_0, k_0) \notin M_1 \cup M_2$. Then, $(j_0, k_0) \in M_1^c \cap M_2^c$ and hence $\left\| x_{j_0 k_0} - y \right\| < r + \varepsilon$ and $\left\| x_{j_0 k_0} - \xi \right\| < \varepsilon$. So

$$\|y - \xi\| \leq \left\| x_{j_0 k_0} - y \right\| + \left\| x_{j_0 k_0} - \xi \right\| \leq r + 2\varepsilon \leq \|y - \xi\|$$

which is absurd. Therefore, $\|y - \xi\| \leq r$ and so $y \in \overline{B_r(\xi)}$. Consequently, we have

$$\mathcal{I}_{\theta_2} - st - LIM_x^r = \overline{B_r(\xi)}.$$

□

Theorem 2. *Let $x = \left\{ x_{jk} \right\}$ be a double sequence and $r \geq 0$ be a real number. Then, rough $\mathcal{I}_2$-lacunary statistical limit set of the double sequence x, i.e., the set $\mathcal{I}_{\theta_2}$-st-LIM_x^r is closed.*

Proof. If $\mathcal{I}_{\theta_2}$-st-$LIM_x^r = \emptyset$, then nothing to prove.

Let us assume that $\mathcal{I}_{\theta_2}$-st-$LIM_x^r \neq \emptyset$. Now, consider a double sequence $\left\{ y_{jk} \right\}$ in $\mathcal{I}_{\theta_2}$-st-LIM_x^r with $\lim_{j,k \to \infty} y_{jk} = y$. Choose $\varepsilon > 0$ and $\delta > 0$. Then, there exists $i_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all $j,k \geq i_{\frac{\varepsilon}{2}}$

$$\left\| y_{jk} - y \right\| < \frac{\varepsilon}{2}.$$

Let $j_0, k_0 > i_{\frac{\varepsilon}{2}}$. Then, $y_{j_0 k_0} \in \mathcal{I}_{\theta_2}$-$st$-$LIM_x^r$. Consequently, we have

$$A = \left\{ (u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us}; \left\| x_{jk} - y_{j_0 k_0} \right\| \geq r + \frac{\varepsilon}{2} \right\} \right| \geq \delta \right\} \in \mathcal{I}_2.$$

Clearly $M = \mathbb{N} \times \mathbb{N} \setminus A$ is nonempty, choose $(u,s) \in M$. We have

$$\frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| x_{jk} - y_{j_0 k_0} \right\| \geq r + \frac{\varepsilon}{2} \right\} \right| < \delta$$

and so

$$\frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| x_{jk} - y_{j_0 k_0} \right\| < r + \frac{\varepsilon}{2} \right\} \right| \geq 1 - \delta.$$

Put

$$B_{us} = \left\{ (j,k) \in J_{us} : \left\| x_{jk} - y_{j_0 k_0} \right\| < r + \frac{\varepsilon}{2} \right\}$$

and select $(j,k) \in B_{us}$. Then, we have

$$\left\| x_{jk} - y \right\| \leq \left\| x_{jk} - y_{j_0 k_0} \right\| + \left\| y_{j_0 k_0} - y \right\| < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon.$$

and so

$$B_{us} \subset \left\{ (j,k) \in J_{us} : \left\| x_{jk} - y \right\| < r + \varepsilon \right\},$$

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which implies
$$1-\delta \leq \frac{|B_{us}|}{h_{us}} \leq \frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-y\right\|<r+\varepsilon\right\}\right|.$$
Therefore,
$$\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-y\right\|\geq r+\varepsilon\right\}\right|<1-(1-\delta)=\delta$$
and so we have
$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-y\right\|\geq r+\varepsilon\right\}\right|\geq\delta\right\}\subset A\in\mathcal{I}_2.$$

This shows that $y\in\mathcal{I}_{\theta_2}$-$st$-$LIM_x^r$. Hence, $\mathcal{I}_{\theta_2}$-st-LIM_x^r is a closed set. □

Theorem 3. *Let $x=\left\{x_{jk}\right\}$ be a double sequence and $r\geq 0$ be a real number. Then, the rough $\mathcal{I}_2$-lacunary statistical limit set $\mathcal{I}_{\theta_2}$-st-LIM_x^r of the double sequence x is a convex set.*

Proof. Let $y_0,y_1\in\mathcal{I}_{\theta_2}$-$st$-$LIM_x^r$ and $\varepsilon>0$ be given. Let
$$A_0=\left\{(j,k)\in J_{us}:\left\|x_{jk}-y_0\right\|\geq r+\varepsilon\right\}\text{ and }A_1=\left\{(j,k)\in J_{us}:\left\|x_{jk}-y_1\right\|\geq r+\varepsilon\right\}.$$
Then by Theorem 1, for $\delta>0$ we have
$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\in A_0\cup A_1\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$
Now, we choose $0<\delta_1<1$ such that $0<1-\delta_1<\delta$ and let
$$A=\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\in A_0\cup A_1\}\right|\geq 1-\delta_1\right\}.$$
Then, $A\in\mathcal{I}_2$. For all $(u,s)\notin A$, we have
$$\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\in A_0\cup A_1\}\right|<1-\delta_1$$
and so
$$\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\notin A_0\cup A_1\}\right|\geq\{1-(1-\delta_1)\}=\delta_1.$$
Therefore, $\{(j,k):(j,k)\notin A_0\cup A_1\}$ is a nonempty set. Let us take $(j_0,k_0)\in A_0^c\cap A_1^c$ and $0\leq\mu\leq 1$. Then,
$$\begin{aligned}\left\|x_{j_0k_0}-[(1-\mu)\,y_0+\mu y_1]\right\| &= \left\|(1-\mu)\,x_{j_0k_0}+\mu x_{j_0k_0}-[(1-\mu)\,y_0+\mu y_1]\right\|\\ &\leq (1-\mu)\left\|x_{j_0k_0}-y_0\right\|+\mu\left\|x_{j_0k_0}-y_1\right\|\\ &< (1-\mu)\,(r+\varepsilon)+\mu\,(r+\varepsilon)=r+\varepsilon.\end{aligned}$$
Let
$$M=\left\{(j,k)\in J_{us}:\left\|x_{jk}-[(1-\mu)\,y_0+\mu y_1]\right\|\geq r+\varepsilon\right\}.$$
Then clearly, $A_0^c\cap A_1^c\subset M^c$. So for $(u,s)\notin A$, we have
$$\delta_1\leq\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\notin A_0\cup A_1\}\right|\leq\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\notin M\}\right|$$

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and so

$$\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\in M\}\right|<1-\delta_1<\delta.$$

Therefore, $A^c\subset\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\in M\}\right|<\delta\right\}$. Since $A^c\in\mathcal{F}(\mathcal{I}_2)$, then we have

$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\{(j,k)\in J_{us}:(j,k)\in M\}\right|<\delta\right\}\in\mathcal{F}(\mathcal{I}_2)$$

and so

$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\{(j,k):(j,k)\in M\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

This completes the proof. □

Theorem 4. *A double sequence $x=\left\{x_{jk}\right\}$ is rough $\mathcal{I}_2$-lacunary statistical convergent to ξ if and only if there exists a double sequence $y=\left\{y_{jk}\right\}$ such that $\mathcal{I}_{\theta_2}$-st-$y=\xi$ and $\left\|x_{jk}-y_{jk}\right\|\leq r$, for all $(j,k)\in\mathbb{N}\times\mathbb{N}$.*

Proof. Let $y=\left\{y_{jk}\right\}$ be a double sequence in X, which is $\mathcal{I}_2$-lacunary statistically convergent to ξ and $\left\|x_{jk}-y_{jk}\right\|\leq r$, for all $(j,k)\in\mathbb{N}\times\mathbb{N}$. Then, for any $\varepsilon>0$ and $\delta>0$

$$A=\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|y_{jk}-\xi\right\|\geq\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

Let $(u,s)\notin A$. Then, we have

$$\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|y_{jk}-\xi\right\|\geq\varepsilon\right\}\right|<\delta\Rightarrow\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|y_{jk}-\xi\right\|<\varepsilon\right\}\right|\geq1-\delta.$$

Now, we let

$$B_{us}=\left\{(j,k)\in J_{us}:\left\|y_{jk}-\xi\right\|<\varepsilon\right\}.$$

Then, for $(j,k)\in B_{us}$, we have

$$\left\|x_{jk}-\xi\right\|\leq\left\|x_{jk}-y_{jk}\right\|+\left\|y_{jk}-\xi\right\|<r+\varepsilon.$$

and so

$$B_{us}\subset\left\{(j,k)\in J_{us}:\left\|x_{jk}-\xi\right\|<r+\varepsilon\right\}$$

$$\begin{aligned}
&\Rightarrow\frac{|B_{us}|}{h_{us}}\leq\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-\xi\right\|<r+\varepsilon\right\}\right|.\\
&\Rightarrow\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-\xi\right\|<r+\varepsilon\right\}\right|\geq1-\delta.\\
&\Rightarrow\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-\xi\right\|\geq r+\varepsilon\right\}\right|<1-(1-\delta)=\delta.
\end{aligned}$$

Thus, we have

$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-\xi\right\|\geq r+\varepsilon\right\}\right|\geq\delta\right\}\subset A$$

and since $A\in\mathcal{I}_2$, so

$$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\left|\left\{(j,k)\in J_{us}:\left\|x_{jk}-\xi\right\|\geq r+\varepsilon\right\}\right|\geq\delta\right\}\in\mathcal{I}_2.$$

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146 Hence, $\mathcal{I}_{\theta_2}$-st-$y = \xi$.

Conversely, suppose that $\mathcal{I}_{\theta_2}$-st-$y = \xi$. Then, for $\varepsilon > 0$ and $\delta > 0$,

$$A = \left\{ (u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| x_{jk} - \xi \right\| \geq r + \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}_2.$$

Let $(u,s) \notin A$. Then, we have

$$\frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| x_{jk} - \xi \right\| \geq r + \varepsilon \right\} \right| < \delta$$

and so

$$\frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| x_{jk} - \xi \right\| < r + \varepsilon \right\} \right| \geq 1 - \delta.$$

Let

$$B_{us} = \left\{ (j,k) \in J_{us} : \left\| x_{jk} - \xi \right\| < r + \varepsilon \right\}.$$

Now, we define a double sequence $y = \left\{ y_{jk} \right\}$ as follows,

$$y_{jk} = \begin{cases} \xi, & \text{if } \left\| x_{jk} - \xi \right\| \leq r, \\ x_{jk} + r \frac{\xi - x_{jk}}{\left\| x_{jk} - \xi \right\|}, & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} \left\| y_{jk} - \xi \right\| &= \begin{cases} 0, & \text{if } \left\| x_{jk} - \xi \right\| \leq r, \\ \left\| x_{jk} - \xi + r \frac{\xi - x_{jk}}{\left\| x_{jk} - \xi \right\|} \right\|, & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } \left\| x_{jk} - \xi \right\| \leq r, \\ \left\| x_{jk} - \xi \right\| - r, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $(j,k) \in B_{us}$. Then, we have

$$\left\| y_{jk} - \xi \right\| = 0, \text{ if } \left\| x_{jk} - \xi \right\| \leq r \text{ and } \left\| y_{jk} - \xi \right\| < \varepsilon, \text{ if } r < \left\| x_{jk} - \xi \right\| < r + \varepsilon$$

and so

$$B_{us} \subset \left\{ (j,k) \in J_{us} : \left\| y_{jk} - \xi \right\| < \varepsilon \right\}.$$

This implies

$$\frac{|B_{us}|}{h_{us}} \leq \frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| y_{jk} - \xi \right\| < \varepsilon \right\} \right|.$$

Hence, we have

$$\begin{aligned} & \tfrac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| y_{jk} - \xi \right\| < \varepsilon \right\} \right| \geq 1 - \delta \\ & \Rightarrow \tfrac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| y_{jk} - \xi \right\| \geq \varepsilon \right\} \right| < 1 - (1 - \delta) = \delta. \end{aligned}$$

and so

$$\left\{ (u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left| \left\{ (j,k) \in J_{us} : \left\| x_{jk} - \xi \right\| \geq \varepsilon \right\} \right| \geq \delta \right\} \subset A.$$

Since $A \in \mathcal{I}_2$ then, we have

$$\left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

So, $\mathcal{I}_{\theta_2}$-st-$y = \xi$. □

Definition 3. *A double sequence $x = \left\{x_{jk}\right\}$ is said to be $\mathcal{I}_{\theta_2}$-statistically bounded if there exists a positive number K such that for any $\delta > 0$ the set*

$$A = \left\{(u,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left|\left\{(j,k) \in J_{us} : \left\|x_{jk}\right\| \geq K\right\}\right| \geq \delta\right\} \in \mathcal{I}_2.$$

The next result provides a relationship between boundedness and rough $\mathcal{I}_{\theta_2}$-statistical convergence of double sequences.

Theorem 5. *If a double sequence $x = \left\{x_{jk}\right\}$ is bounded then there exists $r \geq 0$ such that $\mathcal{I}_{\theta_2}$-st-$LIM_x^r \neq \emptyset$.*

Proof. Let $x = \left\{x_{jk}\right\}$ be bounded double sequence. There exists a positive real number K such that $\left\|x_{jk}\right\| < K$, for all $(j,k) \in J_{us}$. Let $\varepsilon > 0$ be given. Then,

$$\left\{(j,k) \in J_{us} : \left\|x_{jk} - 0\right\| \geq K + \varepsilon\right\} = \emptyset.$$

Therefore, $0 \in \mathcal{I}_{\theta_2}$-$st$-$LIM_x^K$ and so $\mathcal{I}_{\theta_2}$-st-$LIM_x^K \neq \emptyset$. □

Remark 2. *The converse of the above theorem is not true. For example, let us consider the double sequence $x = \left\{x_{jk}\right\}$ in $\mathbb{R}$ defined by*

$$x_{jk} = \begin{cases} jk, & \text{if } j \text{ and } k \text{ are squares} \\ 5, & \text{otherwise.} \end{cases}$$

Then, $\mathcal{I}_{\theta_2}$-st-$LIM_x^0 = \{5\} \neq \emptyset$ but the double sequence x is unbounded.

Definition 4. *A point $\lambda \in X$ is said to be an $\mathcal{I}_2$-lacunary statistical cluster point of a double sequence $x = \left\{x_{jk}\right\}$ in X if for any $\varepsilon > 0$*

$$d_{\mathcal{I}_2}\left(\left\{(j,k) \in J_{us} : \left\|x_{jk} - \lambda\right\| < \varepsilon\right\}\right) \neq 0,$$

where

$$d_{\mathcal{I}_2}(A) = \mathcal{I}_2\text{-}\lim_{u,s\to\infty} \frac{1}{h_{us}} \left|\{(j,k) \in J_{us} : (j,k) \in A\}\right|,$$

if exists. The set of $\mathcal{I}_2$-lacunary statistical cluster point of x is denoted by $\Lambda_x^{S_{\theta_2}}(\mathcal{I}_2)$.

Theorem 6. *For any arbitrary $\alpha \in \Lambda_x^{S_{\theta_2}}(\mathcal{I}_2)$ of a double sequence $x = \left\{x_{jk}\right\}$ we have $\|\xi - \alpha\| \leq r$, for all $\xi \in \mathcal{I}_{\theta_2}$-$st$-$LIM_x^r$.*

Proof. Assume that there exists a point $\alpha \in \Lambda_x^{S_{\theta_2}}(\mathcal{I}_2)$ and $\xi \in \mathcal{I}_{\theta_2}$-$st$-$LIM_x^r$ such that $\|\xi - \alpha\| > r$. Let $\varepsilon = \frac{\|\xi-\alpha\|-r}{3}$. Then,

$$\left\{(j,k) \in J_{us} : \left\|x_{jk} - \xi\right\| \geq r + \varepsilon\right\} \supset \left\{(j,k) \in J_{us} : \left\|x_{jk} - \alpha\right\| < \varepsilon\right\}. \tag{6}$$

Since $\alpha \in \Lambda_x^{S_{\theta_2}}(\mathcal{I}_2)$ we have $d_{\mathcal{I}_2}\left(\left\{(j,k) \in J_{us} : \left\| x_{jk} - \alpha \right\| < \varepsilon\right\}\right) \neq 0$. Hence by (6) we have

$$d_{\mathcal{I}_2}\left(\left\{(j,k) \in J_{us} : \left\| x_{jk} - \alpha \right\| \geq r + \varepsilon\right\}\right) \neq 0,$$

which contradicts that $\xi \in \mathcal{I}_{\theta_2}$-$st$-$LIM_x^r$. Hence, $\|\xi - \alpha\| \leq r$. □

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