Article

The Third Laplace-Beltrami Operator of the Rotational Hypersurface in 4-Space

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Abstract: We consider rotational hypersurface in the four dimensional Euclidean space. We calculate the mean curvature and the Gaussian curvature, and some relations of the rotational hypersurface. Moreover, we define the third Laplace-Beltrami operator and apply it to the rotational hypersurface.

Keywords: 4-space; the third Laplace-Beltrami operator; rotational hypersurface; Gaussian curvature; mean curvature

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1. Introduction

Chen [4] posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. A Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ. Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space. Then the theory of submanifolds of finite type has been studied by many geometers.

Takahashi [23] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$. Ferrandez, Garay and Lucas [11] prove that the surfaces of $\mathbb{E}^3$ satisfying $\Delta H = AH$, $A \in Mat(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder. Choi and Kim [7] characterize the minimal helicoid in terms of pointwise 1-type Gauss map of the first kind. Chen, Choi and Kim [5] study on surfaces of revolution with pointwise 1-type Gauss map.

Dillen, Pas and Verstraelen [9] prove that the only surfaces in $\mathbb{E}^3$ satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres and the circular cylinders. Senoussi and Bekkar [22] study helicoidal surfaces M^2 in $\mathbb{E}^3$ which are of finite type in the sense of Chen with respect to the fundamental forms I, II and III, i.e., their position vector field $r(u,v)$ satisfies the condition $\Delta^J r = Ar$, $J = I, II, III$, where $A = (a_{ij})$ is a constant 3×3 matrix and Δ^J denotes the Laplace operator with respect to the fundamental forms I, II and III.

When we focus on the ruled (helicoid) and rotational characters, we see Bour's theorem in [3]. About helicoidal surfaces in Euclidean 3-space, do Carmo and Dajczer [10] prove that, by using a result of Bour [3], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface. Güler [13] studies on a helicoidal surface with lightlike profile curve using Bour's theorem in Minkowski geometry. Also, Hieu and Thang [14] study helicoidal surfaces by Bour's theorem in 4-space. Choi et al [8] study on helicoidal surfaces and their Gauss map in Minkowski 3-space. Kim and Turgay [15] classify the helicoidal surfaces with L_1-pointwise 1-type Gauss map.

Lawson [16] gives the general definition of the Laplace-Beltrami operator in his lecture notes. Magid, Scharlach and Vrancken [17] introduce the affine umbilical surfaces in 4-space. Vlachos [25] consider hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. Scharlach [21] studies the affine geometry of surfaces and hypersurfaces in 4-space. Cheng and Wan [6] consider complete hypersurfaces of 4-space with constant mean curvature.

Arvanitoyeorgos, Kaimakamais and Magid [2] show that if the mean curvature vector field of M_1^3 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M_1^3 has constant mean curvature in Minkowski 4-space $\mathbb{E}_1^4$. This equation is a natural generalization of the biharmonic submanifold equation $\Delta H = 0$.

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General rotational surfaces as a source of examples of surfaces in the four dimensional Euclidean space were introduced by Moore [18,19]. Arslan et al [1] study on generalized rotation surfaces in $\mathbb{E}^4$. Ganchev and Milousheva [12] consider the analogue of these surfaces in the Minkowski 4-space. They classify completely the minimal general rotational surfaces and the general rotational surfaces consisting of parabolic points. Moruz and Munteanu [20] consider hypersurfaces in the Euclidean space $\mathbb{E}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes. They call them minimal translation hypersurfaces in $\mathbb{E}^4$ and give a classification of these hypersurfaces. Verstraelen, Walrave and Yaprak [24] study the minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n.

We consider the rotational hypersurfaces in Euclidean 4-space $\mathbb{E}^4$ in this paper. We give some basic notions of the four dimensional Euclidean geometry in Section 2. In Section 3, we give the definition of a rotational hypersurface. We calculate the mean curvature and the Gaussian curvature of the rotational hypersurface. We introduce the third Laplace-Beltrami operator in Section 4. Moreover, we calculate the third Laplace-Beltrami operator of the rotational hypersurface in $\mathbb{E}^4$ in the last section.

2. Preliminaries

In this section, we will introduce the first and second fundamental forms, matrix of the shape operator $\mathbf{S}$, Gaussian curvature K, and the mean curvature H of hypersurface $\mathbf{M} = \mathbf{M}(r, \theta_1, \theta_2)$ in Euclidean 4-space $\mathbb{E}^4$. In the rest of this work, we shall identify a vector $\overrightarrow{\alpha}$ with its transpose.

Let $\mathbf{M} = \mathbf{M}(r, \theta_1, \theta_2)$ be an isometric immersion of a hypersurface M^3 in the $\mathbb{E}^4$.

Definition 1. *The inner product of* $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ *on* $\mathbb{E}^4$ *is defined by as follows:*

$$\overrightarrow{x} \cdot \overrightarrow{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

Definition 2. *The vector product of* $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ *on* $\mathbb{E}^4$ *is defined by as follows:*

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

Definition 3. *For a hypersurface* $\mathbf{M}(r, \theta_1, \theta_2)$ *in 4-space, we have*

$$\det \mathbf{I} = \det \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - A^2 G + 2ABF - B^2 E, \tag{1}$$

$$\det \mathbf{II} = \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = \left(LN - M^2\right) V - P^2 N + 2PTM - T^2 L, \tag{2}$$

and

$$\det \mathbf{III} = \det \begin{pmatrix} X & Y & U \\ Y & Z & J \\ U & J & O \end{pmatrix} = \left(XZ - Y^2\right) O - J^2 X + 2JUY - U^2 Z, \tag{3}$$

where $\mathbf{I}, \mathbf{II}$ *and* $\mathbf{III}$ *are the first, the second and the third fundamental form matrices, respectively,* $A = \mathbf{M}_r \cdot \mathbf{M}_{\theta_2}$, $B = \mathbf{M}_{\theta_1} \cdot \mathbf{M}_{\theta_2}$, $C = \mathbf{M}_{\theta_2} \cdot \mathbf{M}_{\theta_2}$, $P = \mathbf{M}_{r\theta_2} \cdot \mathbf{e}$, $T = \mathbf{M}_{\theta_1\theta_2} \cdot \mathbf{e}$, $V = \mathbf{M}_{\theta_2\theta_2} \cdot \mathbf{e}$, $X = \mathbf{e}_r \cdot \mathbf{e}_r$, $Y = \mathbf{e}_r \cdot \mathbf{e}_{\theta_1}$, $U = \mathbf{e}_r \cdot \mathbf{e}_{\theta_2}$, $Z = \mathbf{e}_{\theta_1} \cdot \mathbf{e}_{\theta_1}$, $J = \mathbf{e}_{\theta_1} \cdot \mathbf{e}_{\theta_2}$, $O = \mathbf{e}_{\theta_2} \cdot \mathbf{e}_{\theta_2}$. *Here,* $\mathbf{e}$ *is the Gauss map (i.e. the unit normal vector) defined by*

$$\mathbf{e} = \frac{\mathbf{M}_r \times \mathbf{M}_{\theta_1} \times \mathbf{M}_{\theta_2}}{\left\| \mathbf{M}_r \times \mathbf{M}_{\theta_1} \times \mathbf{M}_{\theta_2} \right\|}. \tag{4}$$

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Definition 4. *Product matrices*

$$\begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}^{-1} \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}$$

gives the matrix of the shape operator **S** *as follows:*

$$\mathbf{S} = \frac{1}{\det \mathbf{I}} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \tag{5}$$

where

$$\begin{aligned}
s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L, \\
s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M, \\
s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P, \\
s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M, \\
s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N, \\
s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T, \\
s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P, \\
s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T, \\
s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V.
\end{aligned}$$

Definition 5. *The formulas of the Gaussian and the mean curvatures, respectively as follow:*

$$K = \det(\mathbf{S}) = \frac{\det \mathbf{II}}{\det \mathbf{I}}, \tag{6}$$

and

$$H = \frac{1}{3} tr\,(\mathbf{S}), \tag{7}$$

where

$$\begin{aligned} tr\,(\mathbf{S}) \quad &= \quad \frac{1}{\det \mathbf{I}}[(EN + GL - 2FM)C + (EG - F^2)V - A^2N \\ &\quad -B^2L - 2(APG + BTE - ABM - ATF - BPF)]. \end{aligned}$$

A hypersurface **M** *is minimal if $H = 0$ identically on* **M**.

3. Curvatures of a rotational hypersurface in 4-space

We define the rotational hypersurface in $\mathbb{E}^4$. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^4$, and let ℓ be a straight line in Π.

Definition 6. *A rotational hypersurface in $\mathbb{E}^4$ is hypersurface rotating a curve γ around a line ℓ (these are called the profile curve and the axis, respectively).*

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We may suppose that ℓ is the line spanned by the vector $(0,0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$Z(\theta_1,\theta_2) = \begin{pmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1 & -\cos\theta_1\sin\theta_2 & 0 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1 & -\sin\theta_1\sin\theta_2 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{8}$$

where $\theta_1, \theta_2 \in \mathbb{R}$. The matrix Z can be found by solving the following equations simultaneously;

$$Z\ell = \ell, \;\; Z^t Z = ZZ^t = I_4, \quad \det Z = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. Parametrization of the profile curve is given by

$$\gamma(r) = (r, 0, 0, \varphi(r)),$$

where $\varphi(r) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function for all $r \in I$. So, the rotational hypersurface which is spanned by the vector $(0,0,0,1)$, is as follows:

$$\mathbf{R}(r,\theta_1,\theta_2) = Z(\theta_1,\theta_2).\gamma(r)^t \tag{9}$$

in $\mathbb{E}^4$, where $r \in I$, $\theta_1, \theta_2 \in [0, 2\pi]$. When $\theta_2 = 0$, we have rotational surface in $\mathbb{E}^4$.

$$\mathbf{R}(r,\theta_1,\theta_2) = \begin{pmatrix} r\cos\theta_1\cos\theta_2 \\ r\sin\theta_1\cos\theta_2 \\ r\sin\theta_2 \\ \varphi(r) \end{pmatrix}. \tag{10}$$

where $r \in \mathbb{R}\backslash\{0\}$ and $0 \leq \theta_1, \theta_2 \leq 2\pi$.

Next, we obtain the mean curvature and the Gaussian curvature of the rotational hypersurface (10).

The first differentials of (10) with respect to r, θ_1, θ_2, respectively, we get

$$\mathbf{R}_r = \begin{pmatrix} \cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 \\ \sin\theta_2 \\ \varphi' \end{pmatrix}, \; \mathbf{R}_{\theta_1} = \begin{pmatrix} -r\sin\theta_1\cos\theta_2 \\ r\cos\theta_1\cos\theta_2 \\ 0 \\ 0 \end{pmatrix}, \; \mathbf{R}_{\theta_2} = \begin{pmatrix} -r\cos\theta_1\sin\theta_2 \\ -r\sin\theta_1\sin\theta_2 \\ r\cos\theta_2 \\ 0 \end{pmatrix}.$$

The first quantities of (10) are as follow:

$$\mathbf{I} = \begin{pmatrix} 1+\varphi'^2 & 0 & 0 \\ 0 & r^2\cos^2\theta_2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{11}$$

We have

$$\det \mathbf{I} = r^4(1+\varphi'^2)\cos^2\theta_2,$$

where $\varphi = \varphi(r)$, $\varphi' = \frac{d\varphi}{dr}$.

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Using (4) , we get the Gauss map of the rotational hypersurface (10) as follows

$$\mathbf{e_R} = \frac{1}{\sqrt{\det \mathbf{I}}} \begin{pmatrix} r^2\varphi' \cos\theta_1 \cos^2\theta_2 \\ r^2\varphi' \sin\theta_1 \cos^2\theta_2 \\ r^2\varphi' \sin\theta_2 \cos\theta_2 \\ -r^2 \cos\theta_2 \end{pmatrix}. \tag{12}$$

The second differentials of (10) with respect to r, θ_1, θ_2, respectively, we get

$$\mathbf{R}_{rr} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \varphi'' \end{pmatrix}, \mathbf{R}_{\theta_1\theta_1} = \begin{pmatrix} -r\cos\theta_1\cos\theta_2 \\ -r\sin\theta_1\cos\theta_2 \\ 0 \\ 0 \end{pmatrix}, \mathbf{R}_{\theta_2\theta_2} = \begin{pmatrix} -r\cos\theta_1\cos\theta_2 \\ -r\sin\theta_1\cos\theta_2 \\ -r\sin\theta_2 \\ 0 \end{pmatrix}.$$

Using the second differentials above and the Gauss map (12) of the rotational hypersurface (10) , we have the second quantities as follow:

$$\mathbf{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = \begin{pmatrix} -\frac{\varphi''}{\sqrt{(1+\varphi'^2)}} & 0 & 0 \\ 0 & -\frac{r\varphi' \cos^2\theta_2}{\sqrt{(1+\varphi'^2)}} & 0 \\ 0 & 0 & -\frac{r\varphi'}{\sqrt{(1+\varphi'^2)}} \end{pmatrix}. \tag{13}$$

So, we have

$$\det \mathbf{II} = -\frac{r^2\varphi'^2\varphi'' \cos^2\theta_2}{(1+\varphi'^2)^{3/2}}.$$

We can write the Gauss map (12) , clearly, as follow:

$$\mathbf{e_R} = \frac{1}{\sqrt{1+\varphi'^2}} \begin{pmatrix} \varphi' \cos\theta_1 \cos\theta_2 \\ \varphi' \sin\theta_1 \cos\theta_2 \\ \varphi' \sin\theta_2 \\ -1 \end{pmatrix}. \tag{14}$$

Using the differentials of the Gauss map (14) of the rotational hypersurface (10) with respect to r, θ_1, θ_2, we get

$$\frac{\partial}{\partial r}\mathbf{e_R} = \frac{\varphi''}{(1+\varphi'^2)^{3/2}} \begin{pmatrix} \cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 \\ \sin\theta_2 \\ -1 \end{pmatrix},$$

$$\frac{\partial}{\partial \theta_1}\mathbf{e_R} = \frac{\varphi'}{\sqrt{1+\varphi'^2}} \begin{pmatrix} -\sin\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2 \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial \theta_2}\mathbf{e_R} = \frac{\varphi'}{\sqrt{1+\varphi'^2}} \begin{pmatrix} -\cos\theta_1\sin\theta_2 \\ -\sin\theta_1\sin\theta_2 \\ \cos\theta_2 \\ 0 \end{pmatrix}.$$

After some computations, we have the third quantities as follow:

$$\mathbf{III} = \begin{pmatrix} X & Y & U \\ Y & Z & J \\ U & J & O \end{pmatrix} = \begin{pmatrix} \frac{\varphi''^2}{(1+\varphi'^2)^2} & 0 & 0 \\ 0 & \frac{\varphi'^2 \cos^2\theta_2}{1+\varphi'^2} & 0 \\ 0 & 0 & \frac{\varphi'^2}{1+\varphi'^2} \end{pmatrix}.$$

So, we get

$$\det \mathbf{III} = \frac{\varphi'^4 \varphi''^2 \cos^2\theta_2}{(1+\varphi'^2)^4}.$$

We calculate the shape operator matrix of the rotational hypersurface (10) , using (5) , as follows:

$$\mathbf{S} = \begin{pmatrix} -\frac{\varphi''}{(1+\varphi'^2)^{3/2}} & 0 & 0 \\ 0 & -\frac{\varphi'}{r(1+\varphi'^2)^{1/2}} & 0 \\ 0 & 0 & -\frac{\varphi'}{r(1+\varphi'^2)^{1/2}} \end{pmatrix}.$$

Finally, using (6) and (7) , respectively, we calculate the Gaussian curvature and the mean curvature of the rotational hypersurface (10) as follow:

$$K = -\frac{\varphi'^2 \varphi''}{r^2 \left(1+\varphi'^2\right)^{5/2}},$$

and

$$H = -\frac{r\varphi'' + 2\varphi'^3 + 2\varphi'}{3r\left(1+\varphi'^2\right)^{3/2}}.$$

Corollary 1. *Let* $\mathbf{R} : M^3 \longrightarrow \mathbb{E}^4$ *be an isometric immersion given by* (10). *Then* M^3 *has constant Gaussian curvature if and only if*

$$\varphi'^4 \varphi''^2 - Cr^2\left(1+\varphi'^2\right)^5 = 0.$$

Corollary 2. *Let* $\mathbf{R} : M^3 \longrightarrow \mathbb{E}^4$ *be an isometric immersion given by* (10). *Then* M^3 *has constant mean curvature (CMC) if and only if*

$$\left(r\varphi'' + 2\varphi'^3 + 2\varphi'\right)^2 - 9Cr^2\left(1+\varphi'^2\right)^3 = 0.$$

Corollary 3. *Let* $\mathbf{R} : M^3 \longrightarrow \mathbb{E}^4$ *be an isometric immersion given by* (10). *Then* M^3 *has zero Gaussian curvature if and only if*

$$\varphi(r) = c_1 r + c_2.$$

Proof. Solving the 2nd order differential eq. $K = 0$, i.e. $\varphi'^2 \varphi'' = 0$, we get the solution.

Corollary 4. *Let* $\mathbf{R} : M^3 \longrightarrow \mathbb{E}^4$ *be an isometric immersion given by* (10). *Then* M^3 *has zero mean curvature if and only if*

$$\begin{aligned} \varphi(r) &= \pm\sqrt{c_1} \int \frac{dr}{\sqrt{r^4 - c_1}} + c_2 \\ &= EllipticF(Ir, I), \end{aligned}$$

where $EllipticF(Ir, I)$ *is incomplete elliptic integral of the first kind.*

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Proof. When we solve the 2nd order differential eq. $H = 0$, i.e.

$$r\varphi'' + 2\varphi'^3 + 2\varphi' = 0,$$

73 we get the solution.

74 4. The third Laplace-Beltrami operator

The inverse of the matrix

$$(e_{ij}) = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

is as follows:

$$\frac{1}{e}\begin{pmatrix} e_{22}e_{33} - e_{23}e_{32} & -(e_{12}e_{33} - e_{13}e_{32}) & e_{12}e_{23} - e_{13}e_{22} \\ -(e_{21}e_{33} - e_{31}e_{23}) & e_{11}e_{33} - e_{13}e_{31} & -(e_{11}e_{23} - e_{21}e_{13}) \\ e_{21}e_{32} - e_{22}e_{31} & -(e_{11}e_{32} - e_{12}e_{31}) & e_{11}e_{22} - e_{12}e_{21} \end{pmatrix},$$

75 where

$$\begin{aligned} e &= \det(e_{ij}) \\ &= e_{11}e_{22}e_{33} - e_{11}e_{23}e_{32} + e_{12}e_{31}e_{23} \\ &\quad - e_{12}e_{21}e_{33} + e_{21}e_{13}e_{32} - e_{13}e_{22}e_{31}. \end{aligned}$$

Definition 7. *The third Laplace-Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3) \mid_{\mathbf{D}} (\mathbf{D} \subset \mathbb{R}^3)$ of class C^3 with respect to the third fundamental form of hypersurface $\mathbf{M}$ is the operator Δ which is defined by as follows:*

$$\Delta^{\mathbf{III}}\phi = \frac{1}{\sqrt{e}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{e}e^{ij}\frac{\partial\phi}{\partial x^j}\right). \tag{15}$$

76 *where $(e^{ij}) = (e_{kl})^{-1}$ and $e = \det(e_{ij})$.*

Clearly, we write $\Delta^{\mathbf{III}}\phi$ as follows:

$$\frac{1}{\sqrt{e}}\left[\begin{array}{c} \frac{\partial}{\partial x^1}\left(\sqrt{e}e^{11}\frac{\partial\phi}{\partial x^1}\right) - \frac{\partial}{\partial x^1}\left(\sqrt{e}e^{12}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(\sqrt{e}e^{13}\frac{\partial\phi}{\partial x^3}\right) \\ -\frac{\partial}{\partial x^2}\left(\sqrt{e}e^{21}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\sqrt{e}e^{22}\frac{\partial\phi}{\partial x^2}\right) - \frac{\partial}{\partial x^2}\left(\sqrt{e}e^{23}\frac{\partial\phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(\sqrt{e}e^{31}\frac{\partial\phi}{\partial x^1}\right) - \frac{\partial}{\partial x^3}\left(\sqrt{e}e^{32}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\sqrt{e}e^{33}\frac{\partial\phi}{\partial x^3}\right) \end{array}\right]. \tag{16}$$

So, using more clear notation, we get

$$\mathbf{III}^{-1} = \frac{1}{\det\mathbf{III}}\begin{pmatrix} OZ - J^2 & JU - OY & JY - UZ \\ JU - OY & OX - U^2 & UY - JX \\ JY - UZ & UY - JX & XZ - Y^2 \end{pmatrix},$$

where

$$\det\mathbf{III} = \left(XZ - Y^2\right)O - J^2X + 2JUY - U^2Z.$$

77 Hence, using more transparent notation we get the third Laplace-Beltrami operator of a smooth
78 function $\phi = \phi(r, \theta_1, \theta_2)$ as follow:

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$$\Delta^{\mathrm{III}}\phi = \frac{1}{\sqrt{|\det \mathbf{III}|}}\left[\begin{array}{c} \frac{\partial}{\partial r}\left(\frac{(OZ-J^2)\phi_r-(JU-OY)\phi_{\theta_1}+(JY-UZ)\phi_{\theta_2}}{\sqrt{|\det \mathbf{III}|}}\right) \\ -\frac{\partial}{\partial \theta_1}\left(\frac{(JU-OY)\phi_r-(OX-U^2)\phi_{\theta_1}+(UY-JX)\phi_{\theta_2}}{\sqrt{|\det \mathbf{III}|}}\right) \\ +\frac{\partial}{\partial \theta_2}\left(\frac{(JY-UZ)\phi_r-(UY-JX)\phi_{\theta_1}+(XZ-Y^2)\phi_{\theta_2}}{\sqrt{|\det \mathbf{III}|}}\right) \end{array}\right]. \tag{17}$$

We continue our calculations to find the third Laplace-Beltrami operator $\Delta^{\mathrm{III}}\mathbf{R}$ of the rotational hypersurface $\mathbf{R}$ using (17) to the (10).

The third Laplace-Beltrami operator of the hypersurface (10) is given by

$$\Delta^{\mathrm{III}}\mathbf{R} = \frac{1}{\sqrt{|\det \mathbf{III}|}}\left(\frac{\partial}{\partial r}\mathbf{U} - \frac{\partial}{\partial \theta_1}\mathbf{V} + \frac{\partial}{\partial \theta_2}\mathbf{W}\right),$$

where

$$\begin{aligned} \mathbf{U} &= \frac{(OZ-J^2)\,\mathbf{R}_r-(JU-OY)\,\mathbf{R}_{\theta_1}+(JY-UZ)\,\mathbf{R}_{\theta_2}}{\sqrt{|\det \mathbf{III}|}}, \\ \mathbf{V} &= \frac{(JU-OY)\,\mathbf{R}_r-(OX-U^2)\,\mathbf{R}_{\theta_1}+(UY-JX)\,\mathbf{R}_{\theta_2}}{\sqrt{|\det \mathbf{III}|}}, \\ \mathbf{W} &= \frac{(JY-UZ)\,\mathbf{R}_r-(UY-JX)\,\mathbf{R}_{\theta_1}+(XZ-Y^2)\,\mathbf{R}_{\theta_2}}{\sqrt{|\det \mathbf{III}|}}. \end{aligned}$$

Here, using the hypersurface (10) , we get $Y = U = J = 0$. Therefore, we write $\mathbf{U}, \mathbf{V}, \mathbf{W}$ again, as follow:

$$\begin{aligned} \mathbf{U} &= \frac{OZ}{\sqrt{|\det \mathbf{III}|}}\mathbf{R}_r = \left(\frac{\varphi'^4\cos^2\theta_2}{(1+\varphi'^2)^2\sqrt{|\det \mathbf{III}|}}\right)\mathbf{R}_r, \\ \mathbf{V} &= -\frac{OX}{\sqrt{|\det \mathbf{III}|}}\mathbf{R}_{\theta_1} = \left(-\frac{\varphi'^2\varphi''^2}{(1+\varphi'^2)^3\sqrt{|\det \mathbf{III}|}}\right)\mathbf{R}_{\theta_1}, \\ \mathbf{W} &= \frac{XZ}{\sqrt{|\det \mathbf{III}|}}\mathbf{R}_{\theta_2} = \left(\frac{\varphi'^2\varphi''^2\cos^2\theta_2}{(1+\varphi'^2)^3\sqrt{|\det \mathbf{III}|}}\right)\mathbf{R}_{\theta_2}, \end{aligned}$$

where

$$\det \mathbf{III} = \frac{\varphi'^4\varphi''^2\cos^2\theta_2}{(1+\varphi'^2)^4}.$$

We obtain the matrices:

$$\mathbf{U} = \frac{\varphi'^2\cos\theta_2}{(1+\varphi'^2)^2\varphi''}\begin{pmatrix} \cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 \\ \sin\theta_2 \\ \varphi' \end{pmatrix},$$

$$\mathbf{V} = -\frac{\varphi''}{(1+\varphi'^2)\cos\theta_2}\begin{pmatrix} -r\sin\theta_1\cos\theta_2 \\ r\cos\theta_1\cos\theta_2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathbf{W} = \frac{\varphi''\cos\theta_2}{1+\varphi'^2}\begin{pmatrix} -r\cos\theta_1\sin\theta_2 \\ -r\sin\theta_1\sin\theta_2 \\ r\cos\theta_2 \\ 0 \end{pmatrix}.$$

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Taking differentials with respect to r, θ_1, θ_2 on $\mathbf{U}, \mathbf{V}, \mathbf{W}$, respectively, we get

$$\frac{\partial}{\partial r}(\mathbf{U}) = \frac{\partial}{\partial r}\left(\frac{\varphi'^2}{(1+\varphi'^2)^2\,\varphi''}\right)\begin{pmatrix} \cos\theta_1\cos^2\theta_2 \\ \sin\theta_1\cos^2\theta_2 \\ \sin\theta_2\cos\theta_2 \\ \varphi'\cos\theta_2 \end{pmatrix} + \frac{\varphi'^2}{(1+\varphi'^2)^2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos\theta_2 \end{pmatrix},$$

$$\frac{\partial}{\partial\theta_1}(\mathbf{V}) = \left(-\frac{\varphi''}{(1+\varphi'^2)}\right)\begin{pmatrix} -r\cos\theta_1 \\ -r\sin\theta_1 \\ 0 \\ 0 \end{pmatrix},$$

$$\frac{\partial}{\partial\theta_2}(\mathbf{W}) = \left(\frac{\varphi''}{1+\varphi'^2}\right)\begin{pmatrix} r\cos\theta_1\cos 2\theta_2 \\ r\sin\theta_1\cos 2\theta_2 \\ -r\sin 2\theta_2 \\ 0 \end{pmatrix}.$$

Hence, we have

$$\Delta^{\mathbf{III}}\mathbf{R} = \left(\Delta^{\mathbf{III}}\mathbf{R}_1, \Delta^{\mathbf{III}}\mathbf{R}_2, \Delta^{\mathbf{III}}\mathbf{R}_3, \Delta^{\mathbf{III}}\mathbf{R}_4\right),$$

where

$$\begin{aligned}\Delta^{\mathbf{III}}\mathbf{R}_1 &= \eta.\{[-4\varphi'\varphi'' + 2\varphi'\left(1+\varphi'^2\right)\varphi''^2 - \left(1+\varphi'^2\right)\varphi''']\cos\theta_1\cos^2\theta_2 \\ &\quad + \left(1+\varphi'^2\right)\varphi''^3 r\cos\theta_1 + \left(1+\varphi'^2\right)\varphi''^3 r\cos\theta_1\cos 2\theta_2\},\end{aligned}$$

$$\begin{aligned}\Delta^{\mathbf{III}}\mathbf{R}_2 &= \eta.\{[-4\varphi'\varphi'' + 2\varphi'\left(1+\varphi'^2\right)\varphi''^2 - \left(1+\varphi'^2\right)\varphi''']\sin\theta_1\cos^2\theta_2 \\ &\quad + \left(1+\varphi'^2\right)\varphi''^3 r\sin\theta_1 + \left(1+\varphi'^2\right)\varphi''^3 r\sin\theta_1\cos 2\theta_2\},\end{aligned}$$

$$\begin{aligned}\Delta^{\mathbf{III}}\mathbf{R}_3 &= \eta.\{[-4\varphi'\varphi'' + 2\varphi'\left(1+\varphi'^2\right)\varphi''^2 - \left(1+\varphi'^2\right)\varphi''']\sin\theta_2\cos\theta_2 \\ &\quad - \left(1+\varphi'^2\right)\varphi''^3 r\sin 2\theta_2\},\end{aligned}$$

$$\begin{aligned}\Delta^{\mathbf{III}}\mathbf{R}_4 &= \eta.\{[-4\varphi'\varphi'' + 2\varphi'\left(1+\varphi'^2\right)\varphi''^2 - \left(1+\varphi'^2\right)\varphi''']\varphi'\cos\theta_2 \\ &\quad + \varphi'^2\varphi''^2\cos\theta_2\},\end{aligned}$$

and

$$\eta = \left(\varphi'^2\varphi''^3\cos\theta_2\right)^{-1}.$$

Remark 1. When the rotational hypersurface $\mathbf{R}$ has the equation $\Delta^{\mathbf{III}}\mathbf{R} = \mathbf{0}$, i.e. the rotational hypersurface (10) is $\mathbf{III}-$minimal, then we have to solve the system of eq. as follow:

$$\Delta^{\mathbf{III}}\mathbf{R}_i = 0,$$

where $1 \le i \le 4$. Here, finding the function φ is a hard problem for us.

Corollary 5. Here $\varphi \neq c = const.$ or $\varphi \neq c_1 r + c_2$, and $\theta_2 \neq \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$, then we have $\Delta^{\mathbf{III}}\mathbf{R} \neq \mathbf{0}$. Hence, the rotational hypersurface (10) is not $\mathbf{III}-$minimal.

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