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Normal Calculus on Moving Surfaces

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Contents

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Abstract

This paper presents an extension for principles of Differential Geometry on Surfaces (re-hashed through the budding field of CMS, the Calculus of Moving Surfaces). It analyzes mostly 2D Hypersurfaces with Riemannian Geometry and proposes the construction of a 3D Static Frame combining the Surface Basis Vectors with the Orthogonal Normal Field as a 3D Orthogonal Vector Frame. The paper introduces conventions for manipulating Tensors defined using this 3D Orthogonal Vector Frame as well as Curvature Connections associated with this Vector Frame. It then finally introduces Symbols and Tensors to describe Inner Products and Variance within the 3D Vector Frame and then extends all the above concepts to a surface which is Dynamic utilizing principles from CMS. This formulation has potential to extend identities and concepts from CMS and from Differential Geometry in a compact Tensorial Framework, which agrees with the new Framework proposed by CMS.

1 Introduction

1.1 Need For a new Calculus

All too often, when Cellular Mathematical Analysis is performed, it can be seen that there is an implicit connection between events which happen on the cytosolic side of the phospholipid membrane, Biochemical and Physical environment on the extracellular side of the phospholipid membrane, and most importantly an aspect lying in between the two faces of the Membrane: the Cellular Membrane Morphology itself. Most of interactions between cells, especially paracellularly, involve events occuring on the surface of cells. [1, 19]

As expected from existing Physical Laws and Biochemical analyses of the membrane, the motions of the Cell Membrane are not exaggerated, but rather lethargic and appear to be periodic or at least bounded in some spatiotemporal nature [20, 21]. Methods to model this motion have yielded some, but still quite few analytical results — the bulk of which have been obtained over the past four decades [1, 2]. Most advances in this field of modelling have arose by modelling investigations which approximated Constitutive Equations of the Membrane and various Cytoskeletal components, and then proceeded apply various facets of Continuum Mechanics to the surface in an attempt to understand the effects of Microtubule Dynamics on the membrane [2, 3] and the effects of local Actin Concentrations to the surface [4].

Though these approaches are effective to an extent, much is left unanswered that appear to be un-accountable for at the present. In some cases, unanswered questions arise due to the lack of extensive research in the field; in other important cases however, unanswered queries are due to the inherent limitations of the theoretical framework as it pertains to accounting for various interactions.

- The effects of shear stresses on the Membrane can be well approximated by the field of Continuum Mechanics, but accounting for any shear stresses by other cells with their own receptors cannot be well accounted for by the discipline

- It is already well understood that there exist cytoslic and extracellular interaction methods such as chemoattractants and various signalling pathyways that may change the Cell Membrane [1, 5, 6, 7], but the exact way to incorporate these into Continuum Mechanics is unclear.

- To this day, there, are little-to-none methods, to provide a direct Analytic link between genetic regulation factors, and any aspects of the Central Dogma of Biology to their influence on Mechanical changes to the Membrane, though there is a very clear link [8, 9].

- In addition, there is no method which prominently exists to relate the interaction of a physical field theory (such as Electrodynamic Theory, QFT) to their influence on Membrane shapes although there is much evidence to suggest there is [10, 11]

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- Ultimately, there exists no theory which encompasses all the above interactions, and it is unclear what type of Mathematical theory would unite all the above, though it can be expected that such a theory would be intuitive, compact, easy to understand, geometric in nature, not be restricted to an ambient coordinate description, and capable of supporting theories sich as Field Theories, Biochemical interactions, and movements of Manifolds.

1.2 Existing Disciplines

The Need for Calculus

Right now, both the movements of Manifolds and Field Theories are united because they are described using the same language: Tensor Calculus.

Field Theories are described using Lagrangian Fields, Calculus of Variations, and Classic Tensor Calculus, which are all summarized into Gauge Theories. These are important since they outline a compact way of describing all the already existing physical laws such as Electrodynamics and Quantum Field Theory in a compact and intuitive way [12].

The Movement of Manifolds over the years has had a few methods of description. Most recently, Movement of Manifolds was described by the new discipline of the Calculus of Moving Surfaces (CMS) which essentially describes a Geometric and Tensorial method of analyzing differential objects intrinsic and extrinsic to a manifold's embedding [13], and introduces an operator which preserves the tensorial nature of Tensors under time differentiation [13, 18].

In reality, Lagrangian Mechanics, Hamiltonian, Newtonian, or Continuum Mechanics all provide ample methods of analyzing manifolds in their own respects, but can become "unwieldy" when analyzing surfaces wholistically. CMS, though effective, lacks the application to solid 3D processes occurring within cells. Thus CMS could benefit from an investigation in attempting to alow it to acknowledge the 3D space it is embedded within before adressing 3D processes ocurring within cells.

Therefore, the paper will proceed with attempting to introduce new concepts to allow CMS to better capture the essential 3D Nature of the Surfaces upon which CMS is defined in addition to introducing a new convention for denoting components on surfaces in the Normal direction, and will con-

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struct a 3D Orthogonal Basis on a Surface unifying the Normal with the Surface Basis Vectors in an Orthogonal and Tensorial Fashion as the Surface deforms in time all in accordance with CMS.

2 An Intrinsic Surface Volumetric Dynamic Coordinate Frame

2.1 Already Established Conventions

As a foreword, there are several conventions which will be used:

- CMS uses Einstein Summation Convention and draws on many of the conventions from Classical Tensor Calculus [13, 14, 15, 16]
- Different classes of indices imply different properties of tensors:
  - Indices which refer to ambient space are denoted using Latin Characters (i,j,k,m,n,p,q); ie. in ambient space, the N-dimensional coordinate system is given by $Z^i = (Z^1, Z^2, ..., Z^N)$ and the basis vectors with which the N-dimensional coordinate space that embeds a surface is given by is denoted by $\mathbf{Z}_i$ and the ambient space's Metric Tensor which denotes distances in standard N-dimensional space is denoted by Z_{ij}.
  - Indices which refer to the hypersurface's $(N-1)$ dimensional coordinate space are typically denoted using the first half of the Greek Alphabet. Here, the characters (α,β,γ,δ,ϵ,η,κ)will be used to denote surface components of tensors; ie. on the surface, the two-coordinate system required to specify its parameters are given by $S^\alpha = (S^1, S^2, ..., S^{N-1})$ and the surface's tangent vector basis will be indicated by $\mathbf{S}_\alpha$, and the Surface Metric Tensor which indicates distances across the surface are given by $S_{\alpha\beta}$)
  - Indices which will be used to refer to the surface's N-dimensional constructed coordinate space will be denoted using the second half of the Greek Alphabet. Here, we will use (μ,ν,λ,σ,ω,ζ,ϕ) to denote the components of a Dynamic Frame Tensor. ie. in this reconstructed space, it will posess the coordinates $\xi^\mu = (\xi^1, \xi^2, ..., \xi^N)$ and the basis in this coordinate frame will be given by $\boldsymbol{\xi}_\mu$)
- At times when there are several arguments that can be abbreviated by an index, only the letter of the argument will be included for brevity ie. $\psi(t, S^1, S^2, ...S^\alpha) = \psi(t, S)$ [13].

 doi:10.20944/preprints201806.0148.v1

- Apostrophes on an index imply a different coordinate change, and are related to the un-apostrophed coordinate systems by Jacobians. In transforming from one coordinate system to another, several different jacobians will be thrown into usage identifiable by the class of index they are using:
  - In transforming from one ambient coordinate system to another, we will follow the convention that:

$$\frac{\partial Z^i(Z')}{\partial Z^{i'}} = J^i_{i'} \tag{1}$$

  - In transforming from one surface coordinate system to another, we will require two jacobians. This is because the transformation does not just depend on the new coordinate system, but also depends on time because the surface is in motion.

$$\frac{\partial S^\alpha(t, S')}{\partial S^{\alpha'}} = J^\alpha_{\alpha'} \quad , \quad \frac{\partial S^\alpha(t, S')}{\partial t} = J^\alpha_t \tag{2}$$

    These all have inverses associated for them and upon each's multiplication with its respective inverse, you produce the Kronecker Delta of the systems:

$$J^i_{i'} J^{i'}_j = \delta^i_j \quad , \quad J^\alpha_{\alpha'} J^{\alpha'}_\beta = \delta^\alpha_\beta \tag{3}$$

- At times, tensors can have relevance to more than one coordinate system, and they do not necessarily require to be explicitly dependent on both of the coordinates. For example, $Z^i{}_\alpha(t, S)$
- There are several tensors which already have already been reserved for describing surfaces [13, 14, 15, 16]:
  - The Position Tensor, $\mathbf{R}(t, S) = Z^i(t, S)\mathbf{Z}_i$: This tensor describes the position of every point of the surface.
  - The Ambient Velocity, $\mathbf{V} = \partial_t \mathbf{R} = V^i \mathbf{Z}_i$: This object can be proven to not transform like a tensor, but is still of crucial importance to describing the movement of the surface and much like the Christoffel Symbols, combine with other objects to form a tensor.
  - The Shift Tensor, $Z^i{}_\alpha = \nabla_\alpha Z^i$: This tensor is an explicit relation between the surface and the ambient space. It has the property that multiplying it by either the Surface Basis Vectors or the Ambient Vectors can produce the opposite basis ($Z^i{}_\alpha \mathbf{Z}_i = \mathbf{S}_\alpha, Z^i{}_\alpha \mathbf{S}^\alpha = \mathbf{Z}^i$), and is an accurate representation of the Tangent Space of the Surface

- The Surfacial Levi Civita Symbols, $\epsilon_{\alpha\beta}$: This completely anti-symmetric tensor is a representation of the permutation algebra which is inherent in the cross product and in the definition of antisymmetric tensorial components.
- The Normal, $\mathbf{N} = N_i \mathbf{Z}^i$. The Normal is an essential tensor which describes the direction that is projected 'outwards' infinitesimally from each tangent space. It is of crucial importance in defining objects such as curvature and also is explicitly constructable from the shift tensor according to the following formula:$N_i = \frac{1}{2}\epsilon_{ijk}\epsilon^{\alpha\beta} Z^j_\alpha Z^k_\beta$.
- The Surface Metric Tensor, $S_{\alpha\beta} = \mathbf{S}_\alpha \cdot \mathbf{S}_\beta$: This symmetric tensor follows from analogue from the standard metric tensor of space and describes the cooefficients required to measure distances along the surface. Much like the Ambient metric tensor, it can be used to define Christoffel Symbols of its own and is crucial when specifying Laplacians on the surface
- The Surface Christoffel Symbols, $\Gamma^\gamma_{\alpha\beta}$: Like the Ambient Christoffel Symbols, this object is not a tensor, but can be used to create the definition of the Covariant Derivative, ∇_α which operates on various ambient and surface tensors and can also be constructed explicitly from the Surface Metric Tensor in a similar fashion as the Ambient Christoffel Symbols can be constructed from the Ambient Metric Tensor.
- The Normal Velocity, $C = \mathbf{V} \cdot \mathbf{N}$: This is an invariant tensor quantity constructed from a tensor and non-tensor, and intuitively describes the surface's speed in the normal direction. In addition, it can also be seen in a Vector form which is defined as $\mathbf{C} = C\mathbf{N} = (\mathbf{V} \cdot \mathbf{N})\mathbf{N}$.
- The Surface Velocity, $V^\alpha = \mathbf{V} \cdot \mathbf{S}^\alpha$: This object is not a tensor, but like the Christoffel Symbols, is of essential importance to defining the tensorial operator which captures time differentiation in an invariant way.
- The Curvature Tensor, $B_{\alpha\beta} = \mathbf{N} \cdot \nabla_\alpha \mathbf{S}_\beta = -\mathbf{S}_\beta \cdot \nabla_\alpha \mathbf{N}$: This symmetric tensor is one of the most important tensors in describing Static & Dynamic Surfaces. This tensor can be used to obtain the Mean and Gaussian Curvature of the surface. As a general rule, its trace is twice the surfaces mean curvature, and its determinant is the surface's Gaussian Curvature. It also satisfies many interesting equivalencies such as the Codazzi Equation ($\nabla_{[\alpha} B_{\beta]\gamma} = 0$) and can be used to create several other tensors

- The Surface Riemann Curvature Tensor, $R^{\gamma}_{\delta\alpha\beta} = 2(\Gamma^{\gamma}_{\delta[\beta,\alpha]} + \Gamma^{\gamma}_{\epsilon[\alpha}\Gamma^{\epsilon}_{\beta]\delta}) = 2B^{\gamma}_{[\alpha}B_{\beta]\delta}$: This Tensor is an Surfacial analogue of the Riemann Curvature Tensor which describes the curvature of space and is determined from the antisymmetric components of the Christoffel or Curvature Tensors. This is crucial for defining Ricci Curvature on the Surface.
- The Tensorial Time Derivative, $\dot{\nabla}$: Much like the Covariant Derivative, this Derivative's form changes depending on the Tensor it is acting on. This operator is an effective method to capture infinitesimal change of a Dynamic Field defined on a surface in the normal direction, and therefore is invariant under changes to the surface coordinates. This makes it a critical operator. For invariant fields, the derivative assumes the form: $\dot{\nabla}\psi = \dot{\psi} - V^{\alpha}\nabla_{\alpha}\psi$.
- The Temporal Symbols, $\dot{\Gamma}^{\alpha}_{\beta} = \nabla_{\beta}V^{\alpha} - CB^{\alpha}_{\beta}$: These symbols are not Tensors by no means, but effectively assume the role in the Tensorial Time Derivative of Surface Tensors that the Christoffel Symbols play in the Covariant Derivative
- The Temporal Curvature Tensor, $\dot{R}^{\gamma}_{\alpha\beta} = 2S^{\gamma\delta}B_{\alpha[\beta}\nabla_{\delta]}C$: This Tensor Describes the commutation between the Covariant Derivative and the Tensorial Time Derivative acting on a Surface Vector [18].

2.2 Creating the Static Frame

Letting Σ_S be the set of all possible deformed surfaces from an original surface, S. For all surfaces, and more specifically - for the subset of surfaces that can be classified as isosurfaces for all cells, $B_S \subset \Sigma_S$ - the surfaces possess intrinsically two Surface Basis Tensors, $\mathbf{S}_{\alpha}$. For these two surface tensors a third tensor is defined which is orthogonal to the other two surface tensors. This tensor is commonly known as the Normal (of unit length) and is explicitly given by [14]:

$$\mathbf{N} = \mathbf{S}_1 \times \mathbf{S}_2 \tag{4}$$

This may be formulated into a component by component form [13]:

$$\mathbf{N} = \frac{1}{2}\mathbf{Z}^i \epsilon_{ijk}\epsilon^{\alpha\beta} Z^j_{\alpha} Z^k_{\beta} \tag{5}$$

This is critical for the definition of the coordinate frame. The collection of the Vectors $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{N}\}$ is referred to as the **Static** Frame and is extremely useful. This frame is useful since it has no extrinsic preferred form, but rather has an intrinsically geometric significance. It is exactly the local euclidean space which is rotated to be tangent with the surface locally. To highlight this, considering the flat Surface described parametrically by $z(x, y) = 0$. It can be intuitively seen that its two Surface Basis Vectors are in fact $\hat{\imath}$ & $\hat{\jmath}$. In addition, the normal can be explicitly derived to reveal the intuitive notion that its normal is given by $\mathbf{N} = \hat{k}$. Thus the collection of the Static Frame Vectors can be identified as $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ which highlights the relationship of the static frame in relation to the Local Euclidean Basis.

For every surface, it will have its own intrinsic static frame. This frame is assumed to be composed of a curvilinear varying basis, denoted by $\boldsymbol{\xi}_\mu$. As a general formality, the Shift Kronecker Tensor is introduced $\tilde{\delta}^{\alpha}_{\mu}$. to relate the surface indices with the Static Frame's indices. If the local coordinate space is considered within the Static Frame, along with the geometric nature of the surface that the Static Frame is embedded upon, the three basis vectors which define this frame can be explicitly stated:

$$\boldsymbol{\xi}_\mu = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{N}\} \tag{6}$$

Using this basis, any tensor from a surface's apex may be defined. Suppose $\mathbf{T}$ is a tensor. In terms of the local coordinate basis, this may be expressed as:

$$\mathbf{T} = T^\mu \boldsymbol{\xi}_\mu = T^\alpha \mathbf{S}_\alpha + T^3 \mathbf{N} \tag{7}$$

The interesting aspect of this decomposition is the reduction for a covariant tensor:

$$\mathbf{T} = T_\mu \boldsymbol{\xi}^\mu = T^\alpha \mathbf{S}_\alpha + T_3 \mathbf{N} \tag{8}$$

Most interestingly, it seems that the decomposition implies the following universal equality: $T^3 = T_3$. This is due to the fact that for basis vectors, their dual is defined from the geometric condition that regarding the aspect of magnitude, they must be unit vectors (ie. $\mathbf{S}_1 \cdot \mathbf{S}^1 = 1, \mathbf{S}_2 \cdot \mathbf{S}^2 = 1$). This holds water with the first two basis vectors and even has the Kronecker Delta defined from their multplication: $\mathbf{S}_\alpha \cdot \mathbf{S}^\beta = \delta^\beta_\alpha$. However, the Normal cannot be continued from the definition; this is because although it is orthogonal to the other basis vectors, it is not defined independently of the Surface Basis Vectors as highlighted by Eq.(4). The definition of the Normal is exactly from the algebraic product of the Surface Basis Vectors, and thus cannot have the index exteded to to being the 'Third' Surface Basis Vector. In fact, it is completely orthogonal to the other basis vectors:

$$\mathbf{S}_\alpha \cdot \mathbf{N} = 0 \rightarrow (\boldsymbol{\xi}_\mu \cdot \boldsymbol{\xi}^3 = 0 \;\;,\;\; \boldsymbol{\xi}^\mu \cdot \boldsymbol{\xi}_3 = 0) \;\;,\;\; \mu = 1, 2 \tag{9}$$

Therefore, in order to rectify its independence from the surface basis vectors, it actually is its **own** dual. Without specifying any indices (making it a Tensor of Rank 0, with Rank-1-components), it is its own dual:

$$\mathbf{N} \cdot \mathbf{N} = 1 \rightarrow \boldsymbol{\xi}_3 \cdot \boldsymbol{\xi}^3 = 1 \tag{10}$$

Therefore, any tensor components associated to the Normal Basis vectors will have no distinction between being 'Contravariant or Covariant. They simply are independent Scalar Fields who have an arbitrarily Raised/Lowered index for the purposes of a placeholder. This is illustrated when the Metric Tensor is obtained later on.

A Notation for the Static Frame

The Static Frame uses Tensors with the indices (μ,ν,λ,σ,ω,ζ,ϕ). Whenever a static frame index is noted, it will run from, $1..N$. The problem with that is that since it is a N- Vector Space constructed from a Manifold of (N-1) Dimension, the N index values can actually be classified into two sets. The first (N-1) index values needed for the surface parametrization, and the Nth index value which actually represents the Normal Direction. Therefore, when representing a vector for example, the first (N-1) entries are all presided over by CMS, and the Nth Entry corresponds to the last entry directly linked to the static frame. For this reason, suppose a (N=28) dimension Static Frame is being observed. The first 27 index values are all abbreviated as the surface indices.

Therefore for example; a Vector may be represented within a 28 dimension Static Frame as a **2-entry** Vector. One entry representing the 27 surface entries and the second entry for the Normal Representation. For example if the Vector on the surface was given in $\mathbb{R}^3$, on a hypersurface of dimension of 2:

$$\mathbf{V} = V^1\mathbf{S}_1 + V^2\mathbf{S}_2 + V^3\mathbf{N}$$

Using the conventions of CMS, this can be abbreviated into a surface component and a Normal Component:

$$\mathbf{V} = V^\mu \boldsymbol{\xi}_\mu = V^\alpha \mathbf{S}_\alpha + V^3\mathbf{N}$$

Therefore, in abrreviating vectors in this space with respect to the bases, it can be represented in the following manner:

$$V^\mu = \begin{pmatrix} V^\alpha \\ V^3 \end{pmatrix} \tag{11}$$

A Matrix may be constructed on the surface in the following manner: If it has components given by

$$\bar{\mathbf{V}} = V^{\mu\nu}(\boldsymbol{\xi}_\mu \otimes \boldsymbol{\xi}_\nu) = V^{\alpha\beta}(\mathbf{S}_\alpha \otimes \mathbf{S}_\beta) + V^{\alpha 3}(\mathbf{S}_\alpha \otimes \mathbf{N}) + V^{3\beta}(\mathbf{N} \otimes \mathbf{S}_\beta) + V^{33}(\mathbf{N} \otimes \mathbf{N})$$

Then it can be represented as:

$$V^{\mu\nu} = \begin{pmatrix} V^{\alpha\beta} & V^{\alpha 3} \\ V^{3\beta} & V^{33} \end{pmatrix} \tag{12}$$

This convention will be largely for notating Vectors And Matrices in the Static Frame.

2.3 The Frame Coordinates & the Normal Coordinate

Unlike other analyses of space, here there is use for the coordinates to be obtained **after** the basis vectors have been defined. In following with the general convention, the two coordinates, S^α which are parallel to the surface basis vectors are identified in the following manner [13]:

$$\frac{\partial \mathbf{R}}{\partial S^\alpha} = \mathbf{S}_\alpha \tag{13}$$

Issues arise when attempting to identify the same analogy with the Normal. With the Normal, there is no logical coordinate that one might use to define the 'direction' of the Normal since the manifold is of $(N-1)$ dimensions. By analogy, the Normal Derivative might be defined as the partial derivative which goes infinitesimally in the general direction of this new informal 'Normal Coordinate' denoted as n; the identity of this derivative as a directional derivative will become more obvious momentarily. As expected, the action of this partial derivative on the Position Vector yields the Normal [15]:

$$\frac{\partial \mathbf{R}}{\partial n} = \mathbf{N} \tag{14}$$

However, we already know that the normal can also be expressed in its coordinate form as $\mathbf{N} = N^i \mathbf{Z}_i$, and the ambient basis is just the partial derivative of the position vector in each of the ambient coordinates, (Z^1, Z^2, Z^3). Since the position vector is an invariant tensor of rank zero, we can also express its partial derivative as its covariant derivative. Thus we can also express the above as the following equivalency:

$$\frac{\partial \mathbf{R}}{\partial n} = N^i \mathbf{Z}_i \rightarrow \frac{\partial \mathbf{R}}{\partial n} = N^i \nabla_i \mathbf{R} \rightarrow \left(\frac{\partial}{\partial n} - N^i \nabla_i\right) \mathbf{R} = \mathbf{0} \tag{15}$$

In this equation, the definition of the normal coordinate is given in a differential form. This is the theoretical properties of the coordinate's derivative. We also directly obtain the normal derivative. This is given as:

$$\frac{\partial}{\partial n} = (\mathbf{N} \cdot \nabla) \tag{16}$$

Searching for a rigourous definition of the coordinate, we see that the following equality should hold:

$$\frac{\partial}{\partial n} n = 1 = (\mathbf{N} \cdot \nabla) n \tag{17}$$

Since we know that the Normal is a unit vector, then we see that the normal coordinate is the ambient scalar field which satisfies the following equation:

$$\nabla n = \mathbf{N} \tag{18}$$

Therefore, we see that there is no exact definition of the normal coordinate yet, but it still satisfies very real differential properties, and thus has an existence.

2.4 Change of Coordinates

If we allow the frame to change (such as a change of surface coordinates, we see that to maintain an invariant tensor, we have the following difference

$$\mathbf{T}' = T^{\mu'} \boldsymbol{\xi}_{\mu'} = T^{\mu'} \tilde{J}^{\mu}_{\mu'} \boldsymbol{\xi}_{\mu} = T^{\alpha'} \tilde{J}^{\alpha}_{\alpha'} \mathbf{S}_{\alpha} + T^{3'} \tilde{J}^{\alpha}_{3'} \mathbf{S}_{\alpha} + T^{\alpha'} \tilde{J}^{3}_{\alpha'} \mathbf{N} + T^{3'} \tilde{J}^{3}_{3'} \mathbf{N} \tag{19}$$

However from the linear nature of the tensor, we know that the form of the tensor under transformation must be as following:

$$\mathbf{T}' = T^{\mu'} \boldsymbol{\xi}_{\mu'} = T^{\alpha'} \mathbf{S}_{\alpha'} + T^{3'} \mathbf{N} = T^{\alpha'} J^{\alpha}_{\alpha'} \mathbf{S}_{\alpha} + T^{3'} \mathbf{N} \tag{20}$$

Therefore, we see that by equating the two equations, the form of a jacobian must be of the following:

$$J^{\mu}_{\mu'} = \begin{bmatrix} J^{\alpha}_{\alpha'} & 0 \\ 0 & 1 \end{bmatrix} \tag{21}$$

This has a form intricately connected to the Surface Jacobian, and also demonstrates the invariant of the normal coordinate.

3 Differential Connections in the Static Frame

3.1 Distances Measured in the Static Frame

Much like the ambient space, the Static Frame's 3d space has a Metric Which denoted distance along its coordinates. We allow this metric to be denoted by $\xi_{\mu\nu}$. Much as in the Surface Metric Tensor and as in the Ambient Metric Tensor, we define the Metric in the Static Frame to be:

$$\xi_{\mu\nu} = \boldsymbol{\xi}_\mu \cdot \boldsymbol{\xi}_\nu \tag{22}$$

Since this has a particular form in the case that either of the indices are 1 & 2, or 3, there will be 4 distinct components for the Metric Tensor which should be symmetric. We find that the 4 cases are well summarized as the following table:

$$\begin{matrix} \tilde{\delta}^\mu_\alpha \tilde{\delta}^\nu_\beta \xi_{\mu\nu} = S_{\alpha\beta} & \tilde{\delta}^\mu_\alpha \xi_{\mu 3} = 0 \\ \tilde{\delta}^\mu_\beta \xi_{3\mu} = 0 & \xi_{33} = 1 \end{matrix} \tag{23}$$

This Metric can be well summarized in the following Matrix Form:

$$\xi_{\mu\nu} = \begin{bmatrix} S_{\alpha\beta} & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

For such a metric, we can say that the length of any infinitesimal distances in this space are given by:

$$ds^2 = S_{\alpha\beta} dS^\alpha dS^\beta + dn^2 \tag{25}$$

And the dot product of any two vectors, **A** and **B**, in this space is given by the following:

$$\mathbf{A} \cdot \mathbf{B} = A^\mu B_\mu = \xi_{\mu\nu} A^\mu B^\nu = S_{\alpha\beta} A^\alpha B^\beta + A^3 B^3 \tag{26}$$

Again, as expected, the Normal Components are completely separate from the Surface Components, but they both contribute to the result. In here, we see that in deriving a volume for the surrounding area, we must have a volume element which is determined by the determinant of the Static Frame's Metric. As it can be seen, from the matrix form, the Determinant of the Metric Tensor is identical to the Surface Metric Tensor's determinant. However this can be verified algebraically by the following:

$$
\begin{aligned}
|\xi_{\mu\nu}| &= \frac{1}{3!}e^{\mu\nu\lambda}e^{\sigma\omega\upsilon}\xi_{\mu\sigma}\xi_{\nu\omega}\xi_{\lambda\upsilon} \\
|\xi_{\mu\nu}| &= \frac{1}{3!}e^{\mu\nu\alpha}e^{\sigma\omega\beta}\xi_{\mu\sigma}\xi_{\nu\omega}S_{\alpha\beta} + \frac{1}{3!}e^{\mu\nu3}e^{\sigma\omega3}\xi_{\mu\sigma}\xi_{\nu\omega} \\
|\xi_{\mu\nu}| &= \frac{1}{3!}e^{3\gamma\alpha}e^{3\delta\beta}S_{\gamma\delta}S_{\alpha\beta} + \frac{1}{3!}e^{\gamma3\alpha}e^{\delta3\beta}S_{\gamma\delta}S_{\alpha\beta} + \frac{1}{3!}e^{\alpha\beta3}e^{\gamma\delta3}S_{\alpha\gamma}S_{\beta\delta} \\
|\xi_{\mu\nu}| &= \frac{1}{3!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} + \frac{1}{3!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} + \frac{1}{3!}e^{\alpha\beta}e^{\gamma\delta}S_{\alpha\gamma}S_{\beta\delta} \\
|\xi_{\mu\nu}| &= \frac{1}{3!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} + \frac{1}{3!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} + \frac{1}{3!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} \\
|\xi_{\mu\nu}| &= \frac{3}{3!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} \\
|\xi_{\mu\nu}| &= \frac{1}{2!}e^{\gamma\alpha}e^{\delta\beta}S_{\gamma\delta}S_{\alpha\beta} \\
|\xi_{\mu\nu}| &= |S_{\alpha\beta}|
\end{aligned}
$$

We also define an interaction between the Static Frame and the Ambient Coordinate Frame; we define this as the Twist Tensor. This is given by:

$$\zeta^{i}{}_{\mu} = \mathbf{Z}^{i} \cdot \boldsymbol{\xi}_{\mu} \tag{27}$$

The Twist Tensor is realized as the following matrix:

$$\zeta^{i}{}_{\mu}\mathbf{Z}_{i} \otimes \boldsymbol{\xi}^{\mu} = \begin{bmatrix} Z^{1}_{\ 1} & Z^{1}_{\ 2} & N^{1} \\ Z^{2}_{\ 1} & Z^{2}_{\ 2} & N^{2} \\ Z^{3}_{\ 1} & Z^{3}_{\ 2} & N^{3} \end{bmatrix} \tag{28}$$

Or the equivalent:

$$\zeta^{i}{}_{\mu} = \begin{bmatrix} Z^{i}{}_{\alpha} & N^{i} \end{bmatrix} \tag{29}$$

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We can semi-use the definition of the determinant [13] to derive the property of the Twist Tensor:

$$|\zeta^i{}_\mu| = \frac{1}{3!}e_{ijk}e^{\mu\nu\lambda}\zeta^i{}_\mu\zeta^j{}_\nu\zeta^k{}_\lambda$$

$$|\zeta^i{}_\mu| = \frac{1}{3!}e_{ijk}e^{\mu\nu\alpha}\zeta^i{}_\mu\zeta^j{}_\nu Z^k{}_\alpha + \frac{1}{3!}e_{ijk}e^{\mu\nu 3}\zeta^i{}_\mu\zeta^j{}_\nu N^k$$

$$|\zeta^i{}_\mu| = \frac{1}{3!}e_{ijk}e^{\mu\beta\alpha}\zeta^i{}_\mu Z^j{}_\beta Z^k{}_\alpha + \frac{1}{3!}e_{ijk}e^{\mu 3\alpha}\zeta^i{}_\mu N^j Z^k{}_\alpha + \frac{1}{3!}e_{ijk}e^{\mu\nu 3}\zeta^i{}_\mu\zeta^j{}_\nu N^k$$

$$|\zeta^i{}_\mu| = \frac{1}{3!}e_{ijk}e^{3\beta\alpha}N^i Z^j{}_\beta Z^k{}_\alpha + \frac{1}{3!}e_{ijk}e^{\beta 3\alpha}Z^i{}_\beta N^j Z^k{}_\alpha + \frac{1}{3!}e_{ijk}e^{\alpha\beta 3}Z^i{}_\alpha Z^j{}_\beta N^k$$

$$|\zeta^i{}_\mu| = \frac{1}{3!}e_{ijk}e^{\beta\alpha 3}Z^j{}_\beta Z^k{}_\alpha N^i + \frac{1}{3!}e_{jik}e^{\beta\alpha 3}Z^i{}_\beta Z^k{}_\alpha N^j + \frac{1}{3!}e_{kij}e^{\alpha\beta 3}Z^i{}_\alpha Z^j{}_\beta N^k$$

$$|\zeta^i{}_\mu| = \frac{1}{3}\left(\frac{1}{2!}e_{ijk}e^{\beta\alpha}Z^j{}_\beta Z^k{}_\alpha\right)N^i + \frac{1}{3}\left(\frac{1}{2!}e_{jik}e^{\beta\alpha}Z^i{}_\beta Z^k{}_\alpha\right)N^j + \frac{1}{3}\left(\frac{1}{2!}e_{kij}e^{\alpha\beta}Z^i{}_\alpha Z^j{}_\beta\right)N^k$$

$$|\zeta^i{}_\mu| = \frac{1}{3}\frac{\sqrt{|S_{\alpha\beta}|}}{\sqrt{|Z_{ij}|}}\left(\left(\frac{1}{2!}\epsilon_{ijk}\epsilon^{\beta\alpha}Z^j{}_\beta Z^k{}_\alpha\right)N^i + \left(\frac{1}{2!}\epsilon_{jik}\epsilon^{\beta\alpha}Z^i{}_\beta Z^k{}_\alpha\right)N^j + \left(\frac{1}{2!}\epsilon_{kij}\epsilon^{\alpha\beta}Z^i{}_\alpha Z^j{}_\beta\right)N^k\right)$$

$$|\zeta^i{}_\mu| = \frac{1}{3}\frac{\sqrt{|S_{\alpha\beta}|}}{\sqrt{|Z_{ij}|}}(N_iN^i + N_jN^j + N_kN^k) = \frac{\sqrt{|S_{\alpha\beta}|}}{\sqrt{|Z_{ij}|}} = \sqrt{\frac{S}{Z}}$$

3.2 Surface Variance in the Surface Basis Vectors

At this point it is appropiate to analyze how the Basis Varies from point to point. We may begin by first analyzing the definition of the Christoffel Symbols in terms of the Surface Vector's partial derivatives:

$$\Gamma^\gamma_{\alpha\beta} = \mathbf{S}^\gamma \cdot \frac{\partial \mathbf{S}_\alpha}{\partial S^\beta} \tag{30}$$

Any variance of the surface vectors in the direction of the Normal is referred to the Curvature Tensor, this is delineated by:

$$B_{\alpha\beta} = \mathbf{N} \cdot \frac{\partial \mathbf{S}_\alpha}{\partial S^\beta} \tag{31}$$

As we would expect, applying the definition of the Covariant Derivative to the Surface Vector's partial derivative, and also using the known definition of the Surface Vector's covariant derivative, $\nabla_\beta \mathbf{S}_\alpha = \mathbf{N}B_{\beta\alpha}$ we see the following expected identity arise:

$$\frac{\partial \mathbf{S}_\alpha}{\partial S^\beta} = \nabla_\beta \mathbf{S}_\alpha + \Gamma^\gamma_{\alpha\beta}\mathbf{S}_\gamma = \mathbf{N}B_{\alpha\beta} + \Gamma^\gamma_{\alpha\beta}\mathbf{S}_\gamma \tag{32}$$

This identity agrees with the two observations in the above two equations. We can also abbreviate this using the Static Frame's basis:

$$\frac{\partial \mathbf{S}_\alpha}{\partial S^\beta} = \Gamma^\mu_{\alpha\beta} \boldsymbol{\xi}_\mu \tag{33}$$

Where the Christoffel Symbols of the Static Frame will be discussed in more detail.

3.3 Normal Variance in the Basis Vectors

We see that in general, it is very difficult to define the Normal Derivative on Surface Differential Objects. Here we use the $\phi_{,i}$ notation to indicate the partial derivative of ϕ. We know that for a 3d ambient coordinate system, for a spatiotemporal ambient field, $\phi(t, Z)$ the following identity holds [17]:

$$\phi_{,[ij]} = 0 \tag{34}$$

Normally for a surface coordinate system, we also see that for a spatiotemporal field, $\psi(t, S)$ the following identity holds [16]:

$$\psi_{,[\alpha\beta]} = 0 \tag{35}$$

Motivated by the following two examples, we state a requirement of our coordinate system; for a field defined in the region of a surface in the Static Frame, $\Omega(\xi)$, we state that the coordinate's partial derivatives must commute:

$$\Omega_{,[\mu\nu]} = 0 \tag{36}$$

As we can see by the above examples, this holds for the Surface Coordinate Partial Derivatives, but is left ambiguous by the Normal Derivative. Following from the above requirement, we state the resultant commutation condition that:

$$\left[\frac{\partial}{\partial n}, \frac{\partial}{\partial S^\alpha}\right]\Omega = 0 \tag{37}$$

3.4 A more General Normal Derivative

Thus, we then re-define a normal derivative which manifests more generally than the other definition. We first state that this derivative must satisfy the following requirements:

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial \xi^3} \quad , \quad \frac{\partial \mathbf{R}}{\partial n} = \mathbf{N} \quad , \quad \phi_{[\mu\nu]} = 0 \tag{38}$$

We can utilize this derivative antisymmetric relation to obtain all sorts of relations within CMS. If we apply the last condition to the position vector:

$$\left[\frac{\partial}{\partial n}, \frac{\partial}{\partial S^\alpha}\right]\mathbf{R} = 0 \rightarrow \frac{\partial \mathbf{S}_\alpha}{\partial n} = \nabla_\alpha \mathbf{N} \tag{39}$$

The following fundamental identity forms:

$$\frac{\partial \mathbf{S}_\alpha}{\partial n} = -B^\beta_\alpha \mathbf{S}_\beta \tag{40}$$

As expected, the normal derivative of the tangent vectors, too, lies in the tangent plane [13]. This also extends to the Surface Metric Tensor, and using the relation $S_{\alpha\beta}S^{\beta\gamma} = \delta^\gamma_\alpha$ can also be extended to the inverse Metric Tensor:

$$\frac{\partial S_{\alpha\beta}}{\partial n} = -2B_{\alpha\beta} \ , \ \frac{\partial S^{\alpha\beta}}{\partial n} = 2B^{\alpha\beta} \tag{41}$$

Therefore, based on the symmetry of the Inverse Metric Tensor, there exists a Normal Derivative on the contravariant Metric Tensor:

$$\frac{\partial \mathbf{S}^\alpha}{\partial n} = B^\alpha_\beta \mathbf{S}^\beta \tag{42}$$

It can be automatically seen that the new Normal Derivative preserves the Tensorial Identity of its Tensors that it operates on by assuming the Commutation of the Frame's partial derivatives. Allowing the Normal Derivative of the Normal to be assumed as the following form:

$$\frac{\partial \mathbf{N}}{\partial n} = K^\mu \boldsymbol{\xi}_\mu$$

By applying the Normal Derivative to two Normal Vector identities, we can specify the exact form of the Derivative. First by applying the Normal Derivative to both sides of the equation of its unit length condition $\mathbf{N}\cdot\mathbf{N} = 1$, and assuming the theNormal Derivative satisfies the Liebniz Product Rule, then we see that:

$$\mathbf{N} \cdot \frac{\partial \mathbf{N}}{\partial n} = 0$$

Also by considering its orthogonality with the Tangent Vectors and Normal Differentiating both sides of the equation $\mathbf{N} \cdot \mathbf{S}_\alpha = 0$, the following identity also arises:

$$\frac{\partial \mathbf{N}}{\partial n} \cdot \mathbf{S}_\alpha = 0$$

Therefore, we see that the only way to rectify these two conditions is if we establish the fundamental result:

$$\frac{\partial \mathbf{N}}{\partial n} = \mathbf{0} \tag{43}$$

This fundamental result arises as expected: this is because since the normal is in the direction of the normal derivative, it should have no variation in that direction. Both of the fundamental identities arise because of the assumption that this new Normal Derivative must commute with the Surface Partial Derivatives.

3.5 Extending the Surface-Normal Commutator

If we apply the commutation to the invariant field that is the Normal, we see that:

$$\frac{\partial}{\partial n}(\nabla_\alpha \mathbf{N}) = \mathbf{0} \rightarrow \frac{\partial}{\partial n}(B^\beta_\alpha \mathbf{S}_\beta) = \mathbf{0} \tag{44}$$

This can be utilized to obtain the following relation:

$$\frac{\partial B^\beta_\alpha}{\partial n} = \tau^\beta_\alpha \tag{45}$$

This can be used to obtain the similar relation to the other forms of the Curvature Tensor and We finally see that:

$$\frac{\partial B^{\alpha\beta}}{\partial n} = 3\tau^{\alpha\beta} \quad , \quad \frac{\partial B^\beta_\alpha}{\partial n} = \tau^\beta_\alpha \quad , \quad \frac{\partial B_{\alpha\beta}}{\partial n} = -\tau_{\alpha\beta} \tag{46}$$

In addition, we can apply the commutator to the Tangent Vectors and we obtain the following relation:

$$\left[\frac{\partial}{\partial n}, \frac{\partial}{\partial S^\alpha}\right] \mathbf{S}_\beta = \frac{\partial}{\partial n}\left(\Gamma^\gamma_{\alpha\beta}\mathbf{S}_\gamma + B_{\alpha\beta}\mathbf{N}\right) + \frac{\partial}{\partial S^\alpha}\left(B^\gamma_\beta \mathbf{S}_\gamma\right)$$

$$\left[\frac{\partial}{\partial n}, \frac{\partial}{\partial S^\alpha}\right] \mathbf{S}_\beta = \mathbf{S}_\gamma \frac{\partial}{\partial n}\Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} B^\delta_\gamma \mathbf{S}_\delta + \frac{\partial B_{\alpha\beta}}{\partial n}\mathbf{N} + \frac{\partial B^\gamma_\beta}{\partial S^\alpha}\mathbf{S}_\gamma + B^\gamma_\beta\left(\Gamma^\delta_{\alpha\gamma}\mathbf{S}_\delta + B_{\alpha\gamma}\mathbf{N}\right)$$

$$\left[\frac{\partial}{\partial n}, \frac{\partial}{\partial S^\alpha}\right] \mathbf{S}_\beta = \mathbf{S}_\gamma \frac{\partial}{\partial n}\Gamma^\gamma_{\alpha\beta} + \mathbf{S}_\gamma \nabla_\alpha B^\gamma_\beta$$

Assuming that the Commutation Vanishes, we see that:

$$\frac{\partial}{\partial n}\Gamma^\gamma_{\alpha\beta} = -\nabla_\alpha B^\gamma_\beta \tag{47}$$

We can use this to find how the Riemann Curvature Tensor of the Surface Varies in the Normal Direction. For example, if we define the Riemann Curvature Tensor on the Surface as [13]:

$$R^\gamma{}_{\delta\alpha\beta} = 2\Gamma^\gamma_{\delta[\beta,\alpha]} + 2\Gamma^\epsilon_{\delta[\beta}\Gamma^\gamma_{\alpha]\epsilon} = \Gamma^\gamma_{\delta\beta,\alpha} - \Gamma^\gamma_{\delta\alpha,\beta} + \Gamma^\epsilon_{\delta\beta}\Gamma^\gamma_{\alpha\epsilon} - \Gamma^\epsilon_{\delta\alpha}\Gamma^\gamma_{\beta\epsilon}$$

And seeing that we know the normal derivative of the Christoffel Symbols, we can definitively define the Normal Derivative of both sides of the Equation

$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = \frac{\partial}{\partial n}\left(2\Gamma^{\gamma}_{\delta[\beta,\alpha]} + 2\Gamma^{\epsilon}_{\delta[\beta}\Gamma^{\gamma}_{\alpha]\epsilon}\right)$$
$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = 2\frac{\partial}{\partial n}\Gamma^{\gamma}_{\delta[\beta,\alpha]} + 2\Gamma^{\epsilon}_{\delta[\beta}\frac{\partial}{\partial n}\Gamma^{\gamma}_{\alpha]\epsilon} + 2\Gamma^{\gamma}_{\epsilon[\alpha}\frac{\partial}{\partial n}\Gamma^{\epsilon}_{\beta]\delta}$$
$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = -2\left(\nabla_{\delta}B^{\gamma}_{[\beta}\right)_{,\alpha]} - 2\Gamma^{\epsilon}_{\delta[\beta}\nabla_{\alpha]}B^{\gamma}_{\epsilon} - 2\Gamma^{\gamma}_{\epsilon[\alpha}\nabla_{\beta]}B^{\epsilon}_{\delta}$$

Here we see that we can turn the first term into a Covariant Derivative form and Insert the Chrisoffel Symbols obtained by this conversion into Covariant Derivative format. Here we know that: $\left(\nabla_{\delta}B^{\gamma}_{\beta}\right)_{,\alpha} = \nabla_{\alpha}\nabla_{\delta}B^{\gamma}_{\beta} + \Gamma^{\epsilon}_{\alpha\beta}\nabla_{\delta}B^{\gamma}_{\epsilon} + \Gamma^{\epsilon}_{\alpha\delta}\nabla_{\epsilon}B^{\gamma}_{\beta} - \Gamma^{\gamma}_{\alpha\epsilon}\nabla_{\delta}B^{\epsilon}_{\beta}$. Since there is a commutation on all the terms, and by the symmetry of the Christoffel Symbols, we know that the second term will vanish and using the Codazzi Equation [14, 16], we are left with the following result:

$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = -2\left(\nabla_{[\alpha}\nabla_{\beta]}B^{\gamma}_{\delta} - \Gamma^{\gamma}_{\epsilon[\alpha}\nabla_{\beta]}B^{\epsilon}_{\delta} + \Gamma^{\epsilon}_{\delta[\alpha}\nabla_{\beta]}B^{\gamma}_{\epsilon}\right) - 2\Gamma^{\epsilon}_{\delta[\beta}\nabla_{\alpha]}B^{\gamma}_{\epsilon} - 2\Gamma^{\gamma}_{\epsilon[\alpha}\nabla_{\beta]}B^{\epsilon}_{\delta}$$

From here we see that the second and last terms will cancel out, and because the third terms and fourth terms are symmetric, they will vanish under the commutation. Therefore we are left with the following equation:

$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = -2\nabla_{[\alpha}\nabla_{\beta]}B^{\gamma}_{\delta}$$

And we can utilize the definition of the Riemann Curvature Tensor to finally obtain the following:

$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = R^{\epsilon}{}_{\delta\alpha\beta}B^{\gamma}_{\epsilon} - R^{\gamma}{}_{\epsilon\alpha\beta}B^{\epsilon}_{\delta} \tag{48}$$

This formula can also be put into an Eigen-operator form:

$$\frac{\partial}{\partial n}R^{\gamma}{}_{\delta\alpha\beta} = \left(\delta^{\eta}_{\delta}B^{\gamma}_{\epsilon} - \delta^{\gamma}_{\epsilon}B^{\eta}_{\delta}\right)R^{\epsilon}{}_{\eta\alpha\beta} \tag{49}$$

While this operator may seem not useful to use at first, it actually can be used to obtain serveral relationships. For example, using this we can obtain the Normal Derivative of the Ricci Tensor by contracting the Two Indices required using the Kronecker Delta:

$$\frac{\partial}{\partial n}R_{\delta\beta} = \left(\delta^{\eta}_{\delta}B^{\alpha}_{\epsilon} - \delta^{\alpha}_{\epsilon}B^{\eta}_{\delta}\right)R^{\epsilon}{}_{\eta\alpha\beta} = B^{\alpha}_{\epsilon}R^{\epsilon}{}_{\eta\alpha\beta} - R_{\eta\beta}B^{\eta}_{\delta} \tag{50}$$

We can also obtain the Normal Derivative of the Ricci Curvature by contracting the equation with the Metric Tensor:

$$\frac{\partial R^{\alpha}_{\alpha}}{\partial n} = 2R^{\alpha}_{\beta}B^{\beta}_{\alpha}$$

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3.6 Table of Normal Derivatives of Static Objects

Here we have obtained several relationships on the objects defined on a surface and will concisely state them here:

$$
\begin{aligned}
&\mathbf{R}_{,\mu} = \boldsymbol{\xi}_\mu \quad , \quad \mathcal{T}_{[\mu\nu]} = 0 \\
&\frac{\partial \mathbf{S}_\alpha}{\partial n} = -B^\beta_\alpha \mathbf{S}_\alpha \quad , \quad \frac{\partial \mathbf{S}^\alpha}{\partial n} = B^\alpha_\beta \mathbf{S}^\beta \quad , \quad \frac{\partial \mathbf{N}}{\partial n} = \mathbf{0} \\
&\frac{\partial S_{\alpha\beta}}{\partial n} = -2B_{\alpha\beta} \quad , \quad \frac{\partial S^{\alpha\beta}}{\partial n} = 2B^{\alpha\beta} \\
&\frac{\partial B^{\alpha\beta}}{\partial n} = 3\tau^{\alpha\beta} \quad , \quad \frac{\partial B^\beta_\alpha}{\partial n} = \tau^\beta_\alpha \quad , \quad \frac{\partial B_{\alpha\beta}}{\partial n} = -\tau_{\alpha\beta} \\
&\frac{\partial}{\partial n}\Gamma^\gamma_{\alpha\beta} = -\nabla_\alpha B^\gamma_\beta \quad , \quad \frac{\partial}{\partial n} R^\gamma{}_{\delta\alpha\beta} = \left(\delta^\eta_\delta B^\gamma_\epsilon - \delta^\gamma_\epsilon B^\eta_\delta\right) R^\epsilon{}_{\eta\alpha\beta} \\
&\frac{\partial}{\partial n} R_{\delta\beta} = B^\alpha_\epsilon R^\epsilon{}_{\eta\alpha\beta} - R_{\eta\beta} B^\eta_\delta \quad , \quad \frac{\partial R^\alpha_\alpha}{\partial n} = 2R^\alpha_\beta B^\beta_\alpha
\end{aligned}
$$

4 Covariant Derivatives in the Static Frame

4.1 Formulating a Covariant Derivative

It is at this point that we consider the covariant derivatives of the coordinate frame. We begin by establishing that for a invariant tensor, $\mathbf{T}$, its Static Frame Covariant Derivative, $\tilde{\nabla}_\mu$ will be defined by:

$$\tilde{\nabla}_\mu \mathbf{T} = \frac{\partial \mathbf{T}}{\partial \xi^\mu} \tag{51}$$

If we assume that $\mathbf{T}$ is of the form $\mathbf{T} = T^\mu \boldsymbol{\xi}_\mu$ (ie. it is a contravariant tensor), then we can simplify the expression further:

$$\tilde{\nabla}_\mu \mathbf{T} = \frac{\partial T^\nu}{\partial \xi^\mu}\boldsymbol{\xi}_\nu + T^\nu \frac{\partial \boldsymbol{\xi}_\nu}{\partial \xi^\mu}$$

The second term of the equation has benefit if it is separated into its Normal and Surface Components due to the partial derivative of the basis vectors becoming more clear:

$$\tilde{\nabla}_\mu \mathbf{T} = \frac{\partial T^\nu}{\partial \xi^\mu}\boldsymbol{\xi}_\nu + T^\alpha \frac{\partial \mathbf{S}_\alpha}{\partial \xi^\mu} + T^3 \frac{\partial \mathbf{N}}{\partial \xi^\mu}$$

Since we know how the Basis Frame behaves under the Normal Derivative, then in the special case that $\mu = 3$ the Covariant Derivative reduces to:

$$\tilde{\nabla}_3 \mathbf{T} = \frac{\partial T^\nu}{\partial n}\boldsymbol{\xi}_\nu + T^\nu \frac{\partial \boldsymbol{\xi}_\nu}{\partial n} = \frac{\partial T^\nu}{\partial n}\boldsymbol{\xi}_\nu - \mathbf{S}_\alpha B^\alpha_\beta T^\beta \tag{52}$$

The other case where the indices are in fact Surface Indices are more complex. We see that we can expand the partial derivatives of the Basis Vectors to obtain:

$$\tilde{\nabla}_\beta \mathbf{T} = \frac{\partial T^\nu}{\partial S^\beta}\boldsymbol{\xi}_\nu + T^\alpha(\mathbf{N}B_{\alpha\beta} + \Gamma^\gamma_{\alpha\beta}\mathbf{S}_\gamma) - T^3\mathbf{S}_\gamma B^\gamma_\beta \tag{53}$$

4.2 The Frame Christoffel Symbols

All the terms can be grouped together if an object denoted as the **Frame Christoffel Symbols** are defined by:

$$\tilde{\Gamma}^\nu_{\beta\mu} = \begin{bmatrix} \Gamma^\alpha_{\beta\gamma} & -B^\alpha_\beta \\ B_{\beta\gamma} & 0 \end{bmatrix} \quad , \quad \tilde{\Gamma}^\nu_{3\mu} = \begin{bmatrix} -B^\alpha_\gamma & 0 \\ 0 & 0 \end{bmatrix} \tag{54}$$

Much like the other Christoffel Symbols, these are Symmetrical in Nature:

$$\tilde{\Gamma}^\nu_{\mu\sigma} = \tilde{\Gamma}^\nu_{\sigma\mu} \tag{55}$$

Using the definition of the Frame Christoffel Symbols, the Covariant Derivative can be summarized in the form as:

$$\tilde{\nabla}_\beta \mathbf{T} = \left(\frac{\partial T^\nu}{\partial S^\beta} + \tilde{\Gamma}^\nu_{\beta\sigma}T^\sigma\right)\boldsymbol{\xi}_\nu \tag{56}$$

In fact, if we use the definition of $\tilde{\Gamma}^\mu_{3\sigma}$, then

$$\tilde{\nabla}_\mu \mathbf{T} = \left(\frac{\partial T^\nu}{\partial \xi^\mu} + \tilde{\Gamma}^\nu_{\mu\sigma}T^\sigma\right)\boldsymbol{\xi}_\nu \tag{57}$$

4.3 The Frame Covariant Derivative & Metrilinic Properties

Now, noticing that the Covariant Derivative must obey the Liebniz Product Rule on an invariant, we obtain the following identity:

$$\tilde{\nabla}_\mu T^\nu = \frac{\partial T^\nu}{\partial \xi^\mu} + \tilde{\Gamma}^\nu_{\mu\sigma}T^\sigma \tag{58}$$

This Covariant Derivative does not seem different from the ones given in standard literature. However, this also implies a very crucial identity about the Covariant Derivative we have formed. We also notice that the following identity is going to hold:

$$\tilde{\nabla}_\mu \boldsymbol{\xi}_\nu = \mathbf{0} \tag{59}$$

The beautiy of this equation is that as expected, it follows all the conventions as you would expect in a covariant derivative. For any three dimensional ambient space, we know that:$\nabla_i \mathbf{Z}_j = \mathbf{0}$; this reaffirms the central concept that the Static Frame is as good of a basis vector frame as any other ambient space coordinate system. When applied to a covariant tensor, it has the following form:

$$\tilde{\nabla}_\mu T_\nu = \frac{\partial T_\nu}{\partial \xi^\mu} - \tilde{\Gamma}^\sigma_{\mu\nu} T_\sigma \tag{60}$$

The Frame Christoffel Symbols play a very interesting role in defining a new Calculus on the Basis Vectors. There is a very crucial difference between the Surface Covariant Derivative and the Surface Projection of the Static Frame Covariant Derivative; this can be concisely summarized as: $\tilde{\nabla}_\alpha \mathbf{S}_\beta \neq \nabla_\alpha \mathbf{S}_\beta$. We see expanding the covariant form of the Static Frame Covariant Derivative:

$$\tilde{\nabla}_\alpha \mathbf{S}_\beta = \frac{\partial \mathbf{S}_\beta}{\partial S^\alpha} - \tilde{\Gamma}^\sigma_{\alpha\beta} \boldsymbol{\xi}_\sigma = \frac{\partial \mathbf{S}_\beta}{\partial S^\alpha} - \tilde{\Gamma}^\gamma_{\alpha\beta} \mathbf{S}_\gamma - \tilde{\Gamma}^3_{\alpha\beta} \mathbf{N}$$

After expanding the definition of the partial derivative of the Surface Vectors, then we see that the covariant derivatives vanish; this of course is different than the Surface Covariant Derivative of the Surface Vectors, because as it is already known: $\nabla_\alpha \mathbf{S}_\beta = \mathbf{N} B_{\alpha\beta}$. This is accomplished entierely by the unique new Frame Christoffel Symbols.As a general rule we see that in general when acting on Tensors of Rank greater than (0,0), then the following identity holds:

$$\tilde{\nabla}_\alpha \neq \nabla_\alpha \tag{61}$$

For a Contravariant Tensor of rank (1,0) embedded in the Frame Space $\mathbf{T} = T^\mu \boldsymbol{\xi}_\mu$, we see that by expanding the Frame Christoffel Symbols, we can obtain the following for the Tensor's Surface Components:

$$\tilde{\nabla}_\beta T^\alpha = \nabla_\beta T^\alpha - B^\alpha_\beta T^3 \tag{62}$$

For a Covariant Tensor, then we see that for the Surface Components:

$$\tilde{\nabla}_\beta T_\alpha = \nabla_\beta T_\alpha - B_{\alpha\beta} T_3 \tag{63}$$

We see that we can also use the equation to find the Divergence in the Static Frame of a Frame Tensor. This is given on an Invariant Tensor in the Frame by:

$$\tilde{\nabla} \cdot \mathbf{T} = \tilde{\nabla}_\mu T^\mu \tag{64}$$

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4.4 Local Curvature of the Static Frame Space

We notice that when analyzing the Commutation of the Static Frame's Covariant Derivatives, we should have the following identity:

$$2\nabla_{[\mu}\nabla_{\nu]}\psi = 0 \tag{65}$$

However, what does occur us that when the commutation is applied to a Static Frame Tensor, the folliwing identity forms itself:

$$2\nabla_{[\mu}\nabla_{\nu]}\psi^{\sigma} = \tilde{R}^{\sigma}{}_{\lambda\mu\nu}\psi^{\lambda} \tag{66}$$

Where $\tilde{R}^{\sigma}_{\lambda\mu\nu}$ has the same definition as the regular Curvature Tensor:

$$\tilde{R}^{\sigma}{}_{\lambda\mu\nu} = 2(\tilde{\Gamma}^{\sigma}_{\lambda[\nu}),_{\nu]} + 2\tilde{\Gamma}^{\sigma}_{\omega[\mu}\tilde{\Gamma}^{\omega}_{\nu]\lambda} \tag{67}$$

If we calculate the curvature's 81 components, using the following identity:

$$\frac{\partial}{\partial n}B_{\alpha\beta} = -\tau_{\alpha\beta} \tag{68}$$

Then the stunning result follows:

$$\tilde{R}^{\sigma}{}_{\lambda\mu\nu} = 0 \tag{69}$$

This is interestingly strong result; It is seen that the Christoffel Symbols are definitely non-zero, and depend on the structure of the Surface, but however based on the organization of the Coordinate System, the whole Riemann Curvature Tensor Vanishes. This means that the Static Frame Covariant Derivatives Vanish commute with a tensor. Therefore we also can state that for the 3 dimensional space in the neighbourhood of a Surface:

$$\tilde{R}_{\mu\nu} = \tilde{R}^{\sigma}{}_{\mu\sigma\nu} = 0, \tilde{R} = \xi^{\mu\nu}\tilde{R}_{\mu\nu} = 0 \tag{70}$$

5 The Dynamic Frame

We now move on an essential and grand generalization; Normally, our surface will depend on time. In the case that the surface is moving, the calculus which is formed upon tensors changes. In fact, we see that now, the frame which was assumed to be static is now in fact dynamic. Therefore we proceed to state the Dynamic Frame's Coordinates:

$$\boldsymbol{\xi}_{\mu} = \boldsymbol{\xi}_{\mu}(t, \xi) \tag{71}$$

5.1 Invariant Theta Symbols

Using certain elements from the Extension of CMS [13], we see that we can express the Invariant Time Derivatives of all the Basis Vectors. If we recall the following Elements from the Extension of CMS:

$$\dot{\nabla}\mathbf{S}_\alpha = (\nabla_\alpha C)\mathbf{N} \ \ , \ \ \dot{\nabla}\mathbf{N} = (-\nabla^\alpha C)\mathbf{S}_\alpha$$

We can therefore abbreviate this in a symbol:

$$\dot{\nabla}\boldsymbol{\xi}_\mu = \Theta^\nu{}_\mu \boldsymbol{\xi}_\nu \tag{72}$$

Where we define the **Invariant Theta Symbols** as the following Frame Matrix:

$$\Theta^\nu{}_\mu = \begin{bmatrix} 0 & -\nabla^\alpha C \\ \nabla_\beta C & 0 \end{bmatrix} \tag{73}$$

In addition, we identify that the Invariant Time Derivative of the Contravariant Basis can be obtained by inverting the equation to obtain:

$$\dot{\nabla}\boldsymbol{\xi}^\mu = -\Theta^\mu{}_\nu \boldsymbol{\xi}^\nu \tag{74}$$

This can also be used to talk about the Invariant Time Derivatives of the Metric Tensors:

$$\dot{\nabla}\xi_{\mu\nu} = 2\Theta^{\bullet}{}_{(\mu\nu)} \ \ , \ \ \dot{\nabla}\xi^{\mu\nu} = -2\Theta^{(\mu\nu)}{}_{\bullet} \tag{75}$$

Interestingly when these are calculated, the Invariant Theta Symbols are in fact Antisymmetric. Thus their symmetry vanishes and we obtain the following equivalency:

$$\dot{\nabla}\xi_{\mu\nu} = \dot{\nabla}\xi^{\mu\nu} = 0 \tag{76}$$

5.2 Omega Time Symbols

We will also introduce another useful symbol which also characterizes the motion of basis vectors. First we identify the Partial Time Derivatives of the Basis Vectors. In order to do this, we need to recall two elements from the Extension of CMS [13]

$$\dot{\Gamma}^\beta_\alpha = \nabla_\alpha V^\beta - CB^\beta_\alpha \ \ , \ \ \hat{\eta}^\alpha = \nabla^\alpha C + V^\beta B^\beta_\alpha$$

Then we can state the Partial Time Derivatives of the Frame Basis Vectors:

$$\frac{\partial \mathbf{S}_\alpha}{\partial t} = \dot{\Gamma}^\beta_\alpha \mathbf{S}_\beta + \hat{\eta}_\alpha \mathbf{N} \ \ , \ \ \frac{\partial \mathbf{N}}{\partial t} = -\hat{\eta}^\alpha \mathbf{S}_\alpha \tag{77}$$

If we so wish, we can abbreviate this into a symbol which will come in handy later. Stating that we can abbreviate all this according for the Frame Basis, we generalize the result into the following:

$$\frac{\partial \boldsymbol{\xi}_\mu}{\partial t} = \omega^\nu{}_\mu \boldsymbol{\xi}_\nu \tag{78}$$

Where we define the **Omega Time Symbols** as the following Frame Matrix:

$$\omega^\nu{}_\mu = \begin{bmatrix} \dot{\Gamma}^\beta_\alpha & -\hat{\eta}^\beta \\ \hat{\eta}_\alpha & 0 \end{bmatrix} \tag{79}$$

These Symbols seem to be antisymmetric but are in fact not symmetric nor antisymmetric. Therefore, in addition, we identify that the Partial Time Derivative of the Contravariant Basis can be obtained by inverting the equation to obtain:

$$\frac{\partial \boldsymbol{\xi}^\mu}{\partial t} = -\omega^\mu{}_\nu \boldsymbol{\xi}^\nu \tag{80}$$

This can also be used to talk about the Partial Time Derivatives of the Metric Tensors:

$$\frac{\partial \xi_{\mu\nu}}{\partial t} = 2\omega_{(\mu\nu)}^{\bullet} \quad , \quad \frac{\partial \xi^{\mu\nu}}{\partial t} = 2\omega^{(\mu\nu)}_{\bullet} \tag{81}$$

5.3 Invariant Time Derivative on Frame Vectors

These Symbols can be used to discuss the Invariant Time Derivative's action on Basis Frame Vectors. Here we state the form of the Vector is: $\mathbf{T} = T^\mu \boldsymbol{\xi}_\mu$. For such a vector, applying the definition of the Invariant Time Derivative, we see that:

$$\dot{\nabla}\mathbf{T} = \frac{\partial \mathbf{T}}{\partial t} - V^\alpha \nabla_\alpha \mathbf{T} = \frac{\partial T^\mu}{\partial t}\boldsymbol{\xi}_\mu + T^\mu \omega^\nu{}_\mu \boldsymbol{\xi}_\nu - \boldsymbol{\xi}_\mu V^\alpha \tilde{\nabla}_\alpha T^\mu$$
$$\dot{\nabla}\mathbf{T} = \left(\frac{\partial T^\mu}{\partial t} - V^\alpha \tilde{\nabla}_\alpha T^\mu + T^\nu \omega^\mu{}_\nu\right)\boldsymbol{\xi}_\mu$$

Also, by the product rule, we see that acting on the Vector:

$$\dot{\nabla}\mathbf{T} = \boldsymbol{\xi}_\mu \dot{\nabla} T^\mu + T^\mu \dot{\nabla}\boldsymbol{\xi}_\mu = \boldsymbol{\xi}_\mu \dot{\nabla} T^\mu + T^\mu \Theta^\nu{}_\mu \boldsymbol{\xi}_\nu$$
$$\dot{\nabla}\mathbf{T} = \left(\dot{\nabla} T^\mu + T^\nu \Theta^\mu{}_\nu\right)\boldsymbol{\xi}_\mu$$

Thus by combining the two, we see that:

$$\dot{\nabla} T^\mu = \frac{\partial T^\mu}{\partial t} - V^\alpha \tilde{\nabla}_\alpha T^\mu + \left(\omega^\mu{}_\nu - \Theta^\mu{}_\nu\right) T^\nu$$

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We can abbreviate the two symbols in the last term into a new symbol which we will denote the **Kappa Time Derivative Symbols**. This symbol will be used often:

$$\kappa^{\mu}{}_{\nu} = \omega^{\mu}{}_{\nu} - \Theta^{\mu}{}_{\nu} = \begin{bmatrix} \dot{\Gamma}^{\alpha}_{\beta} & -V^{\gamma}B^{\alpha}_{\gamma} \\ V_{\gamma}B^{\gamma}_{\beta} & 0 \end{bmatrix} \tag{82}$$

Therefore, we have a final formula for the Invariant Time Derivative of a Contravariant Vector in the Basis Frame. We will denote this with a special symbol, $\dot{\tilde{\nabla}}$ and it is given by the following:

$$\dot{\tilde{\nabla}} T^{\mu} = \frac{\partial T^{\mu}}{\partial t} - V^{\alpha}\tilde{\nabla}_{\alpha}T^{\mu} + \kappa^{\mu}{}_{\nu}T^{\nu} \tag{83}$$

For a Covariant Frame Tensor, it can also be shown that you obtain the following Formula:

$$\dot{\tilde{\nabla}} T_{\mu} = \frac{\partial T_{\mu}}{\partial t} - V^{\alpha}\tilde{\nabla}_{\alpha}T_{\mu} - \kappa^{\nu}{}_{\mu}T_{\nu} \tag{84}$$

The Beauty of the Formula is that when applied to the Surface Components of a Frame Tensor, it reduces to the Expression of the Invariant Time Derivative of a Surface Tensor. This is encapsulated algebraically:

$$\begin{aligned}
\dot{\tilde{\nabla}} T^{\alpha} &= \frac{\partial T^{\alpha}}{\partial t} - V^{\beta}\tilde{\nabla}_{\beta}T^{\alpha} + \kappa^{\alpha}{}_{\nu}T^{\nu} \\
\dot{\tilde{\nabla}} T^{\alpha} &= \frac{\partial T^{\alpha}}{\partial t} - V^{\beta}\left(T^{\alpha}{}_{,\beta} + \tilde{\Gamma}^{\alpha}_{\beta\mu}T^{\mu}\right) + \kappa^{\alpha}{}_{\nu}T^{\nu} \\
\dot{\tilde{\nabla}} T^{\alpha} &= \frac{\partial T^{\alpha}}{\partial t} - V^{\beta}\left(T^{\alpha}{}_{,\beta} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}T^{\gamma} + \tilde{\Gamma}^{\alpha}_{\beta 3}T^{3}\right) + \kappa^{\alpha}{}_{\beta}T^{\beta} + \kappa^{\alpha}{}_{3}T^{3} \\
\dot{\tilde{\nabla}} T^{\alpha} &= \frac{\partial T^{\alpha}}{\partial t} - V^{\beta}\left(\nabla_{\beta}T^{\alpha} - B^{\alpha}_{\beta}T^{3}\right) + \dot{\Gamma}^{\alpha}_{\beta}T^{\beta} - V^{\beta}B^{\alpha}_{\beta}T^{3} \\
\dot{\tilde{\nabla}} T^{\alpha} &= \frac{\partial T^{\alpha}}{\partial t} - V^{\beta}\nabla_{\beta}T^{\alpha} + \dot{\Gamma}^{\alpha}_{\beta}T^{\beta} \\
\dot{\tilde{\nabla}} T^{\alpha} &= \dot{\nabla} T^{\alpha}
\end{aligned}$$

The same can also be proved for T^3 in that it behaves like an Invariant Field which transforming dual to the Normal, it should. Therefore, the equation reproduces all the required amenities of the original Invariant Time Derivatives and reproduces the familiar identities of CMS. Therefore, we can finally state the following Transformation of the Invariant Time Derivative acting on a Frame Tensor:

$$\dot{\tilde{\nabla}} T^{\mu'} = J^{\mu'}_{\mu}\dot{\tilde{\nabla}} T^{\mu} \tag{85}$$

Therefore, this TIme Derivative is an appropiate method by which to measure the change in time of a Frame Vector.

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6 Conclusion

CMS, is a subdiscipline that is just at the inception of its growth. It has introduced several invariant tensors of utmost importance at describing the Dynamicism of Surfaces and has given a deeper insight into the foundations of our reality and the manner which surfaces are embedded within that reality.

The sub-discipline has already shown great utility being used to solve problems ranging from modelling Soap Films [20], modelling basic Liposphere [19] as well as solving classical problems with ease such as determining the shape of falling Droplets [22]. Laws from Physics such as the Young-Laplace Equation and basic Electromagnetics [21], as well as established Theorems such as the Hadamard Principle & Minimal Surface Problem, have been shown to be easily obtainable using CMS [13, 18].

Thus, the field demonstrates exciting new opportunities and models for Theoretical Physics and in addition, great promise for eventually being used to identify 3D surfaces such as entire biological cells. However, as with any new discipline (especially in mathematics), time must be taken to investigate the foundations of the discipline and convert the new subdiscipline of CMS into a well developed field of its own before it is of much use.

This paper has attempted to unite CMS with Differential Geometry generalizing it to accomodate CMS, in its unique type of analysis, with the concept that the Surface Basis Vectors and Normal form an Orthogonal Frame on a Surface. Objects such as the Frame Christoffel Symbols, Invariant Theta Symbols, Time Omega Symbols, and Kappa Time Derivative Symbols are powerful objects which define the Space on, and surrounding a surface, and which fit withing the framework of CMS. Several problems can be abbreviated using these symbols to generate laws that can be used to model several complex phenomena.

The further CMS is developed, the more fundamental laws can be derived to understand the Calculus of Moving Surfaces better, and understand how to incorporate the study of Surfaces with physics to generate compact, tensorial, powerful models & theorems which explain complex physical phenomena.

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