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Article

A generalized Fejér-Hadamard inequality for harmonically convex functions via generalized fractional integral operator and related results

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Abstract: In the present research, we will develop some integral inequalities of Hermite Hadamard type for differentiable η -convex function. Moreover, our results include several new and known results as special cases.

Keywords: Harmonically convex functions, Hadamard inequality, Generalized fractional integral operator, Mittag-Leffler function

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1. Introduction and preliminary results

Inequalities for convex functions, for example the celebrated one is the Hadamard inequality provide a new horizon in the field of mathematical analysis. Many authors have been working on it continuously and several Hadamard like integral inequalities have been established for many kinds of functions related to convex functions. Recently a lot of integral inequalities of the Hadamard type for harmonically convex functions via fractional integrals have been published (see, [3,6–9] and references there in). The Hadamard inequality for convex functions is stated in the following theorem.

Theorem 1. *Let I be an interval of real numbers and $f : I \to \mathbb{R}$ be a convex function on I. Then for all $a, b \in I$ the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Fejér gave a weighted version of the Hadamard inequality stated as follows.

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Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $g : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

It is well known as the Fejér-Hadamard inequality. In the following we give the definition of harmonically convex functions.

Definition 3. [7] Let I be an interval of non-zero real numbers. Then a function $f : I \to \mathbb{R}$ is said to be harmonically convex function if the inequality

$$f\left(\frac{ab}{ta+(1-t)b}\right) \le tf(b) + (1-t)f(a) \tag{1}$$

holds for $a,b \in I$ and $t \in [0,1]$. If inequality in (1) is reversed, then f is said to be harmonically concave.

Definition 4. [6] A function $h : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically symmetric about $\frac{2ab}{a+b}$ if

$$h(x) = h\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for $x \in [a,b]$.

In the following we give the Hadamard inequality for harmonically convex functions.

Theorem 5. *[7] Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequality holds*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}. \tag{2}$$

A Fejér-Hadamard inequality for harmonically convex functions is stated as follows.

Theorem 6. *[3] Let $f : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $a,b \in I$ with $a < b$. If $f \in L[a,b]$ and $g : [a,b] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is a non negative integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then the following inequality holds*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \le \int_a^b \frac{f(x)g(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}. \tag{3}$$

The following definition of the Riemann-Liouville fractional integral is the asset of fractional calculus.

Definition 7. [16] Let $f \in L[a,b]$. Then two sided Riemann-Liouville fractional integral of f of order $\nu > 0$ is defined as

$$I_{a^+}^{\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}f(t)dt, \ x > a$$

and

$$I_{b_-}^{\nu}f(x) = \frac{1}{\Gamma(\nu)}\int_x^b (t-x)^{\nu-1}f(t)dt, \ x < b.$$

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A version of the Fejér-Hadamard inequality for harmonically convex functions via Riemann-Liouville fractional integrals is stated as follows.

Theorem 8. *[9] Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a function such that $a, b \in I$ with $a < b$. If $f \in L[a, b]$ and f is harmonically convex function, then the following inequality for Riemann-Liouville fractional integral holds*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\nu+1)}{2}\left(\frac{ab}{b-a}\right)^{\nu}\left[I^{\nu}_{\frac{1}{b}+}(f\circ h)(\frac{1}{a}) + I^{\nu}_{\frac{1}{a}-}(f\circ h)(\frac{1}{b})\right]$$
$$\leq \frac{f(a)+f(b)}{2}. \tag{4}$$

In the following we give the definition of a generalized fractional integral operator which will help us to give a generalized Fejér-Hadamard inequality for harmonically convex functions and related results.

Definition 9. [14] Let μ, ν, k, l, γ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing generalized Mittag-Leffler function for a real valued continuous function f are defined as follows:

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+} f\right)(x) = \int_a^x (x-t)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(x-t)^{\mu}) f(t)dt,$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,b_-} f\right)(x) = \int_x^b (t-x)^{\nu-1} E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(t-x)^{\mu}) f(t)dt,$$

where the function $E^{\gamma,\delta,k}_{\mu,\nu,l}$ is a generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k}_{\mu,\nu,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}. \tag{5}$$

For $\delta = l = 1$ in (9), the integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to an integral operator $\epsilon^{\gamma,1,k}_{\mu,\nu,1,\omega,a^+}$ containing generalized Mittag-Leffler function $E^{\gamma,1,k}_{\mu,\nu,1}$ introduced by Srivastava and Tomovski in [15]. Along with $\delta = l = 1$ in addition if $k = 1$ then (9) reduces to an integral operator defined by Prabhaker in [12] containing Mittag- Leffler function $E^{\gamma}_{\mu,\nu}$. For $\omega = 0$ in (9), integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ reduces to the Riemann-Liouville fractional integral operator [14].

In [14,15] properties of the generalized fractional integral operator $\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,a^+}$ and the generalized Mittag-Leffler function $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ are studied in brief. In [14] it is proved that $E^{\gamma,\delta,k}_{\mu,\nu,l}(t)$ is absolutely convergent for $k < l + \mu$ and $t \in \mathbb{R}$.
Since

$$|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \leq \sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}\right|.$$

If we say that $\sum_{n=0}^{\infty}\left|\frac{(\gamma)_{kn} t^n}{\Gamma(\mu n+\nu)(\delta)_{ln}}\right| = S$, then

$$|E^{\gamma,\delta,k}_{\mu,\nu,l}(t)| \leq S.$$

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We use this property of generalized Mittag-Leffler function in sequal in our results.

Also we use in sequal the following definitions of special functions known as beta and hypergeometric functions, (see, [10])

$$\beta(\mu,\nu)=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}=\int_0^1 t^{\mu-1}(1-t)^{\nu-1}dt,\quad x,y>0,$$

$${}_2F_1(a,b;c;w)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-wt)^{-a}dt,$$

where $0<b<c$ and $|w|<1$.

In this paper we give a generalized version of the Fejér-Hadamard inequality for harmonically convex functions via generalized fractional integral operator. We also obtain bounds of the absolute differences of this generalized Fejér-Hadamard inequality for harmonically convex functions. Being generalizations, we reproduce the results proved in [8].

2. Main Results

To obtain our main results we need the following lemmas.

Lemma 10. *[13] For $0\le a<b$ and $0<\mu\le 1$, we have*

$$|a^\mu-b^\mu|\le(b-a)^\mu.$$

Lemma 11. *Let $g:[a,b]\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}, a<b$, be integrable and harmonically symmetric function with respect to $\frac{2ab}{a+b}$. Then for generalized fractional integrals we have*

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{b}+}g\circ h\right)\left(\frac{1}{a}\right)=\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{a}-}g\circ h\right)\left(\frac{1}{b}\right)$$
$$=\frac{\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{b}+}g\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,\frac{1}{a}-}g\circ h\right)\left(\frac{1}{b}\right)}{2}$$

where $h(t)=\frac{1}{t}$ for all $t\in[\frac{1}{b},\frac{1}{a}]$.

Proof. Since f is harmonically symmetric about $\frac{2ab}{a+b}$, we have $f(\frac{1}{x})=f(\frac{1}{\frac{1}{a}+\frac{1}{b}-x})$. By definition of generalized fractional integral operator

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}f\circ g\right)\left(\frac{1}{a}\right)=\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-t\right)^{\mu}\right)f\left(\frac{1}{t}\right)dt \tag{6}$$

replace t by $\frac{1}{a}+\frac{1}{b}-x$ in equation (6), we have

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}f\circ g\right)\left(\frac{1}{a}\right)=\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)dx$$
$$=\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(x-\frac{1}{b}\right)^{\mu}\right)f(x)dx.$$

This implies

$$\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}f\circ g\right)\left(\frac{1}{a}\right)=\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}f\circ g\right)\left(\frac{1}{b}\right). \tag{7}$$

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By adding $\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+} f\circ g\right)\left(\frac{1}{a}\right)$ in both sides of (7), we have

$$2\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+} f\circ g\right)\left(\frac{1}{a}\right)=\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+} f\circ g\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-} f\circ g\right)\left(\frac{1}{b}\right) \tag{8}$$

(7) and (8) give the required result. □

Theorem 12. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a harmonically convex function. Let for $a, b \in I, a < b, f \in L[a,b]$ and also let $g : [a,b] \to \mathbb{R}$ be a non-negative, integrable and harmonically symmetric function about $\frac{2ab}{a+b}$. Then the following inequalities for generalized fractional integrals hold*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+} g\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-} g\circ h\right)\left(\frac{1}{b}\right)\right] \\ & \le \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+} fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-} fg\circ h\right)\left(\frac{1}{b}\right) \\ & \le \frac{f(a)+f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{b}+} g\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega',\frac{1}{a}-} g\circ h\right)\left(\frac{1}{b}\right)\right], \end{aligned} \tag{9}$$

where $\omega' = \omega\left(\frac{ab}{b-a}\right)^{\mu}$ and $h(t) = \frac{1}{t}$ for all $t \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

Proof. Since f is harmonically convex function, therefore for $t \in [0,1]$, we have

$$2f\left(\frac{2ab}{a+b}\right) \le f\left(\frac{ab}{ta+(1-t)b}\right)+f\left(\frac{ab}{tb+(1-t)a}\right). \tag{10}$$

Multiplying both sides of (10) by $t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)$ and then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} & 2f\left(\frac{2ab}{a+b}\right)\int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)dt \\ & \le \int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt \\ & + \int_0^1 t^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega t^{\mu})f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt. \end{aligned} \tag{11}$$

By choosing $x = \frac{tb+(1-t)a}{ab}$ that is $\frac{ab}{ta+(1-t)b} = \frac{1}{\frac{1}{a}+\frac{1}{b}-x}$ in (11), we have

$$\begin{aligned} & 2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu}\right)g\left(\frac{1}{x}\right)dx \\ & \le \int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-x}\right)g\left(\frac{1}{x}\right)dx \\ & + \int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{ab}{b-a}\right)^{\nu}\left(x-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{ab}{b-a}\right)^{\mu}\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx. \end{aligned} \tag{12}$$

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Since f is harmonically symmetric about $\frac{2ab}{a+b}$, therefore after simplification, (12) becomes

$$\begin{aligned} &2f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu}\right)g\left(\frac{1}{x}\right)dx \\ &\leq \int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{a}-x\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'\left(\frac{1}{a}-x\right)^{\mu}\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx \\ &+\int_{\frac{1}{b}}^{\frac{1}{a}}\left(x-\frac{1}{b}\right)^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega'\left(x-\frac{1}{b}\right)^{\mu}\right)f\left(\frac{1}{x}\right)g\left(\frac{1}{x}\right)dx. \end{aligned} \tag{13}$$

This implies

$$\begin{aligned} &2f\left(\frac{2ab}{a+b}\right)\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}g\circ h\right)\left(\frac{1}{b}\right) \\ &\leq \left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{b}\right). \end{aligned}$$

Using Lemma 11 in above inequality, we have

$$\begin{aligned} &f\left(\frac{2ab}{a+b}\right)\left[\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}g\circ h\right)\left(\frac{1}{b}\right)+\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}g\circ h\right)\left(\frac{1}{a}\right)\right] \\ &\leq \left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{b}\right). \end{aligned} \tag{14}$$

To prove the second half of inequality, again from harmonically convexity of f on $[a,b]$ and for $t \in [0,1]$ we have

$$f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)\leq f(a)+f(b). \tag{15}$$

Multiplying both sides of (15) by $t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)$, then integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f\left(\frac{ab}{tb+(1-t)a}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt \\ &+\int_0^1 t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})f\left(\frac{ab}{ta+(1-t)b}\right)g\left(\frac{ab}{tb+(1-t)a}\right)dt \\ &\leq [f(a)+f(b)]\int_0^1 t^{\nu-1}E_{\mu,\nu,l}^{\gamma,\delta,k}(\omega t^{\mu})g\left(\frac{ab}{tb+(1-t)a}\right)dt. \end{aligned} \tag{16}$$

Setting $x=\frac{tb+(1-t)a}{ab}$ and by using harmonically symmetry of f with respect to $\frac{2ab}{a+b}$ in (16), after simplification we have

$$\begin{aligned} &\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{b}\right) \\ &\leq [f(a)+f(b)]\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}g\circ h\right)\left(\frac{1}{a}\right). \end{aligned} \tag{17}$$

Using Lemma 11 in (17), we have

$$\begin{aligned} &\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{a}\right)+\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}fg\circ h\right)\left(\frac{1}{b}\right) \\ &\leq \frac{[f(a)+f(b)]}{2}\left[\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{a}-}^{\gamma,\delta,k}g\circ h\right)\left(\frac{1}{b}\right)+\left(\epsilon_{\mu,\nu,l,\omega',\frac{1}{b}+}^{\gamma,\delta,k}g\circ h\right)\left(\frac{1}{a}\right)\right]. \end{aligned} \tag{18}$$

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By joining (14) and (18) we get (9). □

Remark 1. In Theorem 12,
(i) if we put $\omega' = 0$ along with $g(x) = 1$ and $\nu = 1$, then we get inequality (2) of Theorem 5.
(ii) if we put $\omega' = 0$ along with $g(x) = 1$, then we get inequality (4) of Theorem 8.
(iii) if we put $\omega' = 0$ along with $\nu = 1$, then we get inequality (3) of Theorem 6.

Lemma 13. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior of I and $f' \in L[a,b]$ where $a, b \in I$ and $a < b$. Also let $g : I \subset (0,\infty) \to \mathbb{R}$ be an integrable and harmonically symmetric function about $\frac{2ab}{a+b}$. Then the following equality holds for generalized fractional integrals*

$$\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)$$
$$-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)$$
$$=\left[\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\right.$$
$$\left.-\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt\right]$$

where $h(t) = \frac{1}{t}$ for $t \in [\frac{1}{b}, \frac{1}{a}]$.

Proof. To prove this lemma, we have

$$\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt$$
$$=\left|\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)(t)\right|_{\frac{1}{b}}^{\frac{1}{a}}$$
$$-\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-t\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-t\right)^{\mu}\right)(g\circ h)(t)(f\circ h)(t)dt$$
$$=\left(\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)f(a)$$
$$-\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fg\circ h)\left(\frac{1}{a}\right).$$

This implies

$$\int_{\frac{1}{b}}^{\frac{1}{a}}\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt \tag{19}$$
$$=f(a)\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(g\circ h)\left(\frac{1}{a}\right)-\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}+}(fg\circ h)\left(\frac{1}{a}\right).$$

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Similarly

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{t}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt$$
$$= \left| \left(\int_{t}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)(t)\right|_{\frac{1}{b}}^{\frac{1}{a}}$$
$$+ \int_{\frac{1}{b}}^{\frac{1}{a}} \left(t-\frac{1}{b}\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(t-\frac{1}{a}\right)^{\mu}\right)(g\circ h)(f\circ h)(t)dt$$
$$= -\left(\int_{\frac{1}{b}}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right) f(b)$$
$$+ \epsilon_{\mu,\nu,l,\omega,\frac{1}{a}-}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{b}\right).$$

This implies

$$\int_{\frac{1}{b}}^{\frac{1}{a}} \left(\int_{t}^{\frac{1}{a}} \left(s-\frac{1}{b}\right)^{\nu-1} E_{\mu,\nu,l}^{\gamma,\delta,k}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)(f\circ h)'(t)dt \tag{20}$$
$$= -f(b)\epsilon_{\mu,\nu,l,\omega,\frac{1}{a}^-}^{\gamma,\delta,k}(g\circ h)\left(\frac{1}{b}\right) + \epsilon_{\mu,\nu,l,\omega,\frac{1}{a}^-}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{b}\right)$$

on substracting (20) from (19) and using lemma 11, we get the result. □

Remark 2. In Lemma 13 if we take $g(x) = 1$ with $\omega = 0$, then it gives [9, Lemma 3].

Theorem 14. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior of I and $f' \in L[a,b]$ where $a,b \in I$ and $a < b$. If $|f'|$ is harmonically convex function on $[a,b]$, $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a continuous and harmonically symmetric function with respect to $\frac{2ab}{a+b}$, then the following inequality for generalized fractional integrals holds*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon_{\mu,\nu,l,\omega,\frac{1}{b}^+}^{\gamma,\delta,k}(g\circ h)\left(\frac{1}{a}\right) + \epsilon_{\mu,\nu,l,\omega,\frac{1}{a}-}^{\gamma,\delta,k}(g\circ h)\left(\frac{1}{b}\right)\right)\right.$$
$$\left. - \left(\epsilon_{\mu,\nu,l,\omega,\frac{1}{b}^+}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{a}\right) + \epsilon_{\mu,\nu,l,\omega,\frac{1}{a}-}^{\gamma,\delta,k}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|$$
$$\leq \frac{\| g \|_\infty S(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left(C_1(\nu)|f'(a)| + C_2(\nu)|f'(b)|\right)$$

where
$C_1(\nu) = \frac{b^{-2}}{(\nu+2)} {}_2F_1(2,1;\nu+3;1-\frac{a}{b}) - \frac{b^{-2}}{(\nu+1)(\nu+2)} {}_2F_1(2,\nu+1;\nu+3;1-\frac{a}{b}) + \frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)} {}_2F_1(2,\nu+1;\nu+3;\frac{b-a}{b+a})$
and
$C_2(\nu) = \frac{b^{-2}}{(\nu+1)(\nu+2)} {}_2F_1(2,2;\nu+3;1-\frac{a}{b}) - \frac{b^{-2}}{\nu+2} {}_2F_1(2,\nu+2;\nu+3;1-\frac{a}{b}) + \frac{4(a+b)^{-2}}{(\nu+1)} {}_2F_1(2,\nu+1;\nu+2;\frac{b-a}{b+a}) - \frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)} {}_2F_1(\nu,\nu+1;\nu+3;\frac{b-a}{b+a})$
with $0 < \nu \leq 1$, $h(t) = \frac{1}{t}$ for all $t \in [\frac{1}{b}, \frac{1}{a}]$.

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Proof. By Lemma 13, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\int_{\frac{1}{b}}^{\frac{1}{a}}\left|\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)\right.\\&\left.-\left(\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right||(f\circ h)'(t)|dt.\end{aligned}\tag{21}$$

Since g is Harmonically symmetric with respect to $\frac{2ab}{a+b}$ therefore $g(\frac{1}{t})=g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-t}\right)$ for all $t\in[\frac{1}{b},\frac{1}{a}]$, we have

$$\begin{aligned}&\left|\left(\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(\frac{1}{a}-s\right)^{\mu}\right)(g\circ h)(s)ds\right)\right.\\&\left.-\left(\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right|\\&=\left|\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right.\\&\left.+\left(\int_{\frac{1}{a}}^{t}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right|\\&=\left|\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)ds\right)\right|\end{aligned}$$

$$\leq\begin{cases}\int_t^{\frac{1}{a}+\frac{1}{b}-t}|(s-\frac{1}{b})^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-\frac{1}{b})^{\mu})g(s)|ds, & t\in[\frac{1}{a},\frac{a+b}{2ab}]\\ \int_{\frac{1}{a}+\frac{1}{b}-t}^{t}|(s-\frac{1}{b})^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}(\omega(s-\frac{1}{b})^{\mu})g(s)|ds, & t\in[\frac{a+b}{2ab},\frac{1}{a}].\end{cases}\tag{22}$$

Using (22) in (21), we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_{\frac{1}{b}}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)\right|ds\right)|(f\circ h)'(t)|dt\\&+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\nu-1}E^{\gamma,\delta,k}_{\mu,\nu,l}\left(\omega\left(s-\frac{1}{b}\right)^{\mu}\right)(g\circ h)(s)\right|ds\right)|(f\circ h)'(t)|dt.\end{aligned}\tag{23}$$

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using $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$ and absolute convergence of Mittag-Leffer function, above inequality becomes

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right. \\ &\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right| \\ &\leq \| g \|_\infty S\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\nu-1}ds\right)|(f\circ h)'(t)|dt\right. \\ &\left.+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(\frac{1}{a}-s\right)^{\nu-1}ds\right)|(f\circ h)'(t)|dt\right] \\ &= \| g \|_\infty S\left[\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\frac{(\frac{1}{a}-t)^\nu-(t-\frac{1}{b})^\nu}{\nu}\frac{1}{t^2}\right)|f'\left(\frac{1}{t}\right)|dt\right. \\ &\left.+\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\frac{\left(t-\frac{1}{b}\right)^\nu-(\frac{1}{a}-t)^\nu}{\nu}\frac{1}{t^2}\right)|f'\left(\frac{1}{t}\right)|dt\right]. \end{aligned} \tag{24}$$

Setting $t = \frac{ub+(1-u)a}{ab}$ in (24), we have

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right. \\ &\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right| \\ &\leq \frac{\| g \|_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}|f'\left(\frac{ab}{(ub+(1-u)a)}\right)|du\right. \\ &\left.+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}|f'\left(\frac{ab}{(ub+(1-u)a)}\right)|du\right]. \end{aligned} \tag{25}$$

Since $|f'|$ is harmonically convex on $[a,b]$, it can be written as

$$\left|f'\left(\frac{ab}{(ub+(1-u)a)^2}\right)\right| \leq u|f'(a)|+(1-u)|f'(b)|. \tag{26}$$

Using (26) in (25), we have

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(g\circ h)\left(\frac{1}{b}\right)\right)\right. \\ &\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_-}(fg\circ h)(\frac{1}{b})\right)\right| \\ &\leq \frac{\| g \|_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}\left(u|f'(a)|+(1-u)|f'(b)|\right)du\right. \\ &\left.+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}\left(u|f'(a)|+(1-u)|f'(b)|\right)du\right] \end{aligned}$$

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that is

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel g\parallel_{\infty}S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\right)|f'(a)|\right.\\&\left.+\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}(1-u)du+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}(1-u)du\right)|f'(b)|\right].\end{aligned}\tag{27}$$

One can has by using Lemma 10

$$\begin{aligned}&\int_0^{\frac{1}{2}}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\\&=\int_0^{1}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu-\int_{\frac{1}{2}}^{1}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu\\&+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\\&=\int_0^{1}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\\&\quad+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\\&=\int_0^{1}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu+2\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\end{aligned}\tag{28}$$

On simplification we get

$$\begin{aligned}&\int_0^{\frac{1}{2}}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}udu+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}udu\\&=\frac{b^{-2}}{(\nu+2)}{}_2F_1(2,1;\nu+3;1-\frac{a}{b})\\&-\frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;1-\frac{a}{b})\\&+\frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,\nu+1;\nu+3;\frac{b-a}{b+a})\\&=C_1(\nu).\end{aligned}\tag{29}$$

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Similarly

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}(1-u)du + \int_{\frac{1}{2}}^{1} \frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}(1-u)du \\
&= \frac{b^{-2}}{(\nu+1)(\nu+2)}{}_2F_1(2,2;\nu+3;1-\frac{a}{b}) \\
&- \frac{b^{-2}}{\nu+2}{}_2F_1(2,\nu+2;\nu+3;1-\frac{a}{b}) \\
&+ \frac{4(a+b)^{-2}}{(\nu+1)}{}_2F_1(2,\nu+1;\nu+2;\frac{b-a}{b+a}) \\
&- \frac{2(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1\left(\nu,\nu+1;\nu+3;\frac{b-a}{b+a}\right) \\
&= C_2(\nu).
\end{aligned} \tag{30}
$$

Using (29) and (30) in (27), we get the result. □

Remark 3. In Theorem 14,
(i) if we put $\omega = 0$, then we get [8, Theorem 6].
(ii) if we take $\nu = 1$ along with $\omega = 0$, then we get [8, Corollary 1(1)].
(iii) if we take $g(x) = 1$ along with $\omega = 0$, then we get [8, Corollary 1(2)].
(iv) if we take $\nu = 1$, $g(x) = 1$ along with $\omega = 0$, then we get [8, Corollary 1(3)].

Theorem 15. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior of I and $f' \in L[a,b]$ where $a, b \in I$ and $a < b$. If $|f'|^q$, $q > 1$ is harmonically convex function on $[a,b]$, $g : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a continuous and harmonically symmetric function about $\frac{2ab}{a+b}$, then the following inequality for generalized fractional integrals holds*

$$
\begin{aligned}
&\left| \left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right) + \epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}^-}(g\circ h)\left(\frac{1}{b}\right)\right)\right. \\
&\left. - \left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right) + \epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}^-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right| \\
&\le \frac{\| g \|_\infty \, S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[(C_3^{1-\frac{1}{q}}(\nu)\left(C_4(\nu)|f'(a)|^q + C_4(\nu)|f'(b)|^q\right)^{\frac{1}{q}}\right. \\
&\left. +(C_6^{1-\frac{1}{q}}(\nu)\left(C_7(\nu)|f'(a)|^q + C_8(\nu)|f'(b)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

where
$C_3(\nu) = \frac{2(a+b)^{-2}}{\nu+1}{}_2F_1\left(2;\nu+1;\nu+3;\frac{b-a}{b+a}\right)$
$C_4(\nu) = \frac{(a+b)^{-2}}{(\nu+1)(\nu+2)}{}_2F_1\left(2;\nu+1;\nu+3;\frac{b-a}{b+a}\right)$
$C_5(\nu) = C_3(\nu) - C_4(\nu)$
$C_6(\nu) = \frac{b^{-2}}{\nu+1}{}_2F_1(2;1;\nu+1;(1-\frac{a}{b})) - \frac{b^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+2;(1-\frac{a}{b})) + C_3(\nu)$
$C_7(\nu) = \frac{b^{-2}}{\nu+1}{}_2F_1(2;1;\nu+2;(1-\frac{a}{b})) - \frac{b^{-2}}{\nu+1}{}_2F_1(2;\nu+1;\nu+2;(1-\frac{a}{b})) + C_4(\nu)$
$C_8(\nu) = C_6(\nu) - C_7(\nu)$
with $0 < \nu \le 1$, $h(t) = \frac{1}{t}$ *for all* $t \in [\frac{1}{b}, \frac{1}{a}]$.

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Proof. By inequality (25) of Theorem 14, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\le\frac{\parallel g\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{(ub+(1-u)a)}\right)\right|du\right.\\&\left.+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{(ub+(1-u)a)}\right)\right|du\right].\end{aligned}\tag{31}$$

Using power means, inequality (31) becomes

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\le\frac{\parallel g\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\right.\\&\times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{(ub+(1-u)a)^2}\right)\right|^q du\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\&\left.\times\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}\left|f'\left(\frac{ab}{(ub+(1-u)a)^2}\right)\right|^q du\right)^{\frac{1}{q}}\right].\end{aligned}\tag{32}$$

By using the harmonically convexity of $|f'|^q$ in (32), we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\le\frac{\parallel g\parallel_\infty S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\right.\\&\times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}(u|f'(a)|^q+(1-u)|f'(b)|^q)\right)^{\frac{1}{q}}\\&+\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\\&\left.\times\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(u|f'(a)|^q+(1-u)|f'(b)|^q)\right)^{\frac{1}{q}}\right].\end{aligned}\tag{33}$$

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That is

$$\Bigg|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(g\circ h)\left(\frac{1}{b}\right)\right)$$
$$-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^+}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}-}(fg\circ h)\left(\frac{1}{b}\right)\right)\Bigg|$$
$$\leq\| g\|_\infty S\frac{(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu|f'(a)|^q+\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}(1-u)du|f'(b)|^q\right)^{\frac{1}{q}}$$
$$+\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}udu|f'(a)|^q+\int_{\frac{1}{2}}^1\frac{u^\nu-(1-u)^\nu}{(ub+(1-u)a)^2}(1-u)du|f'(b)|^q\right)^{\frac{1}{q}}\right]. \tag{34}$$

Now we evaluate the integrals of (34) by using Lemma 10

$$\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du$$
$$\leq\int_0^{\frac{1}{2}}\frac{(1-2u)^\nu}{(ub+(1-u)a)^2}du$$
$$=\frac{1}{2}\int_0^1\frac{(1-u)^\nu}{(\frac{ub}{2}+(1-\frac{u}{2})a)^2}du. \tag{35}$$

Substitute $u=1-w$ in (35), we have

$$\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}du$$
$$\leq 2(a+b)^{-2}\int_0^1 w^\nu\left(1-w(\frac{b-a}{b+a})\right)^{-2}dw$$
$$=2\frac{(a+b)^{-2}}{\nu+1}{}_2F_1\left(2;\nu+1;\nu+2;\frac{b-a}{b+a}\right)$$
$$=C_3(\nu).$$

Similarly

$$\int_0^{\frac{1}{2}}\frac{(1-u)^\nu-u^\nu}{(ub+(1-u)a)^2}udu$$
$$\leq\int_0^{\frac{1}{2}}\frac{(1-2u)^\nu}{(ub+(1-u)a)^2}udu$$
$$=\frac{1}{4}\int_0^1\frac{u(1-u)^\nu}{(\frac{ub}{2}+(1-\frac{u}{2})a)^2}du. \tag{36}$$

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Substitute $u = 1 - w$ in (36), we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{(1-u)^\nu - u^\nu}{(ub+(1-u)a)^2} u du \\
&\leq (a+b)^{-2} \int_0^1 (1-w) w^\nu \left(1 - w(\frac{b-a}{b+a})\right)^{-2} dw \\
&= \frac{(a+b)^{-2}}{(\nu+1)(\nu+2)} {}_2F_1\left(2;\nu+1;\nu+3;\frac{b-a}{b+a}\right) \\
&= C_4(\nu).
\end{aligned} \tag{37}$$

$$\int_0^{\frac{1}{2}} \frac{(1-u)^\nu - u^\nu(1-u)}{(ub+(1-u)a)^2}(1-u)du \leq C_3(\nu) - C_4(\nu) = C_5(\nu). \tag{38}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \frac{u^\nu - (1-u)^\nu}{(ub+(1-u)a)^2} du \\
&= \int_0^1 \frac{u^\nu - (1-u)^\nu}{(ub+(1-u)a)^2} du + \int_0^{\frac{1}{2}} \frac{(1-u)^\nu - u^\nu}{(ub+(1-u)a)^2} du \\
&\leq \frac{b^{-2}}{\nu+1} {}_2F_1\left(2;1;\nu+2;(1-\frac{a}{b})\right) \\
&- \frac{b^{-2}}{\nu+1} {}_2F_1\left(2;\nu+1;\nu+2;(1-\frac{a}{b})\right) + C_3(\nu) \\
&= C_6(\nu).
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \frac{u^\nu - (1-u)^\nu}{(ub+(1-u)a)^2} u du \\
&= \int_0^1 \frac{u^\nu - (1-u)^\nu}{(ub+(1-u)a)^2} u du + \int_0^{\frac{1}{2}} \frac{(1-u)^\nu - u^\nu}{(ub+(1-u)a)^2} u du \\
&\leq \frac{b^{-2}}{\nu+2} {}_2F_1\left(2;1;\nu+3;(1-\frac{a}{b})\right) \\
&- \frac{b^{-2}}{(\nu+1)(\nu+2)} {}_2F_1\left(2;\nu+1;\nu+3;(1-\frac{a}{b})\right) + C_4(\nu) \\
&= C_7(\nu)
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
\int_{\frac{1}{2}}^1 \frac{u^\nu - (1-u)^\nu}{(ub+(1-u)a)^2}(1-u)du &\leq C_6(\nu) - C_7(\nu) \\
&= C_8(\nu).
\end{aligned} \tag{41}$$

76 Using (36)-(41) in (34), we get the result. □

77 **Remark 4.** Following results can be obtained by giving particular values to parameter in Theorem
78 15.
79 (i) If we take $\omega = 0$, then we get [8, Theorem 7].
80 (ii) If we take $\nu = 1$ along with $\omega = 0$, then we get [8, Corollary 2(1)].
81 (iii) If we take $g(x) = 1$ along with $\omega = 0$, then we get [8, Corollary 2(2)].
82 (iv) If we take $\nu = 1$, $g(x) = 1$ along with $\omega = 0$, then we get [8, Corollary 2(3)].

Theorem 16. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on the interior of I such that $f' \in L[a,b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q > 1$ is harmonically convex function on $[a,b]$, $g : [a,b] \subset (0,\infty) \to \mathbb{R}$*

be a continuous and harmonically symmetric function about $\frac{2ab}{a+b}$*, then the following inequality for generalized fractional integrals hold*

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel g\parallel_{\infty}S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left(C_9^{\frac{1}{p}}(\nu)\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right.\\&\left.+C_{10}^{\frac{1}{p}}(\nu)\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right)\end{aligned}$$

where
$C_9(\nu)=\frac{(a+b)^{-2p}}{2^{-2p+1}(\nu p+1)}{}_2F_1(2p,\nu p+1;\nu p+2;\frac{b-a}{b+a})$
and
$C_{10}(\nu)=\frac{b^{-2p}}{2(\nu p+1)}{}_2F_1(2p,1;\nu p+2;\frac{1}{2}(1-\frac{a}{b}))$
with $0\leq\nu<1$, $h(t)=\frac{1}{t}$ *for all* $t\in[\frac{1}{b},\frac{1}{a}]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By inequality (25) of Theorem 14, we have

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel g\parallel_{\infty}S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\int_0^{\frac{1}{2}}\frac{(1-u)^{\nu}-u^{\nu}}{(ub+(1-u)a)^2}|f'\left(\frac{ab}{(ub+(1-u)a)}\right)|du\right.\\&\left.+\int_{\frac{1}{2}}^{1}\frac{u^{\nu}-(1-u)^{\nu}}{(ub+(1-u)a)^2}|f'\left(\frac{ab}{(ub+(1-u)a)}\right)|du\right].\end{aligned}\tag{42}$$

By using Hölder inequality and harmonically convexity of $|f'|^q$, (42) follows

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(g\circ h)\left(\frac{1}{b}\right)\right)\right.\\&\left.-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}_{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\right|\\&\leq\frac{\parallel g\parallel_{\infty}S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\right.\\&\times\left(\int_0^{\frac{1}{2}}(u|f'(a)|^q+(1-u)|f'(b)|^q)du\right)^{\frac{1}{q}}\\&\left.+\left(\int_{\frac{1}{2}}^{1}\frac{(u^{\nu}-(1-u)^{\nu})^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}(u|f'(a)|^q+(1-u)|f'(b)|^q)du\right)^{\frac{1}{q}}\right].\end{aligned}$$

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After simplification, we have

$$
\begin{aligned}
&\Bigg|\left(\frac{f(a)+f(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(g\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}^{-}}(g\circ h)\left(\frac{1}{b}\right)\right) \\
&-\left(\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{b}^{+}}(fg\circ h)\left(\frac{1}{a}\right)+\epsilon^{\gamma,\delta,k}_{\mu,\nu,l,\omega,\frac{1}{a}^{-}}(fg\circ h)\left(\frac{1}{b}\right)\right)\Bigg| \\
&\leq \frac{\| g \|_{\infty} S\,(b-a)^{\nu+1}}{\nu(ab)^{\nu-1}}\left[\left(\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^{p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\right. \\
&\times\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{8}\right)^{\frac{1}{q}} \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\frac{(u^{\nu}-(1-u)^{\nu})^{p}}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^{q}+3|f'(b)|^{q}}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{43}
$$

We evaluate the integrals by using Lemma 10

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^{p}}{(ub+(1-u)a)^{2p}}du \\
&\leq \int_0^{\frac{1}{2}}\frac{(1-2u)^{\nu p}}{(ub+(1-u)a)^{2p}}du \\
&= \frac{1}{2}\int_0^{1}\frac{(1-u)^{\nu p}}{(\frac{ub}{2}+(1-\frac{u}{2})a)^{2p}}du.
\end{aligned}
\tag{44}
$$

put $u=1-w$ in (44), we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\frac{((1-u)^{\nu}-u^{\nu})^{p}}{(ub+(1-u)a)^{2p}}du \\
&\leq \frac{1}{2}\int_0^{1} w^{\nu p}\left(\frac{a+b}{2}\right)^{-2p}\left(1-w\left(\frac{b-a}{b+a}\right)\right)^{-2p}dw \\
&= \frac{(a+b)^{-2p}}{2^{-2p+1}(\nu p+1)}\,{}_2F_1\left(2p,\nu p+1;\nu p+2;\frac{b-a}{b+a}\right) \\
&= C_9(\nu).
\end{aligned}
\tag{45}
$$

Similarly

$$
\int_{\frac{1}{2}}^{1}\frac{(u^{\nu}-(1-u)^{\nu})^{p}}{(ub+(1-u)a)^{2p}}du \leq \int_{\frac{1}{2}}^{1}\frac{(2u-1)^{\nu p}}{(ub+(1-u)a)^{2p}}du
\tag{46}
$$

put $u=1-w$ in on right hand side of inequality (46), we have

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1}\frac{(u^{\nu}-(1-u)^{\nu})^{p}}{(ub+(1-u)a)^{2p}}du \\
&\leq \int_0^{\frac{1}{2}}\frac{(1-2w)^{\nu p}}{((1-w)b+wa)^{2p}}dw \\
&= \frac{1}{2}\int_0^{1}\frac{(1-w)^{\nu p}}{(\frac{wa}{2}+(1-\frac{w}{2})b)^{2p}}dw. \\
&= \frac{b^{-2p}}{2(\nu p+1)}\,{}_2F_1(2p,1;\nu p+2;\frac{1}{2}(1-\frac{a}{b})) \\
&= C_{10}(\nu).
\end{aligned}
\tag{47}
$$

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Using (45) and (47) in (43) we get the result. □

Remark 5. On giving particular values to parameter in Theorem 16 we have the following results.
(i) If we put $\omega = 0$, then we get [8, Theorem 8].
(ii) If we put $\nu = 1$ along with $\omega = 0$, then we get [8, Corollary 3(1)].
(iii) If we put $g(t) = 1$ along with $\omega = 0$, then we get [8, Corollary 3(2)].
(iv) If we put $\nu = 1$, $g(t) = 1$ along with $\omega = 0$, then we get [8, Corollary 3(3)].

Conclusion

We have obtained generalized Fejér-Hadamard inequality for harmonically convex functions via generalized fractional integral operator. This inequality includes several inclusions for example Fejér-Hadamard and Hadamarad inequalities for harmonically convex functions via Riemann-Liouville fractional integral. Taking different specific values of parameters in the generalized Mittag-Leffler function one can obtain results for some known fractional integral operators for example, for fractional integral operators defined in [12,15]. Also we have established some bounds of the difference of the generalized Fejér-Hadamard inequality, in particular several bounds for particular values of parameters involved in the generalized Mittag-Leffler function.

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