REPRESENTING SUMS OF FINITE PRODUCTS OF CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KINDS BY CHEBYSHEV POLYNOMIALS[†]

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Abstract. Here we consider sums of finite products of Chebyshev polynomials of the third and fourth kinds. Then we represent each of those sums of finite products as linear combinations of the four kinds of Chebyshev polynomials which involve the hypergeometric function ${}_3F_2$.

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1. Introduction and preliminaries

We first recall here that, for any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $< x >_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1), \quad (x)_0 = 1, \tag{1}$$

$$< x >_n = x(x+1)\cdots(x+n-1), \quad (n \geq 1), \quad < x >_0 = 1. \tag{2}$$

The two factorial polynomials are related by

$$(x)_n = (-1)^n < -x >_n, \quad < x >_n = (-1)^n(-x)_n. \tag{3}$$

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We will make use of the following.

$$\frac{(2n-2s)!}{(n-s)!} = \frac{2^{2n-2s}(-1)^s < \frac{1}{2} >_n}{< \frac{1}{2} - n >_s}, \tag{4}$$

for any integers n, s with $n \geq s \geq 0$.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (\mathrm{Re}(x), \mathrm{Re}(y) > 0), \tag{5}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\Gamma(\frac{1}{2})}{2^{2n}n!}, \quad (n \geq 0). \tag{6}$$

Here $B(x,y)$ and $\Gamma(x)$ are respectively the beta and gamma functions.

The hypergeometric function ${}_pF_q\left(\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}; x\right)$ is defined by

$$\begin{aligned} {}_pF_q\left(\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_p \end{matrix}; x\right) &= \sum_{n=0}^{\infty} \frac{< a_1 >_n \cdots < a_p >_n}{< b_1 >_n \cdots < b_q >_n} \frac{x^n}{n!} \\ & \qquad (p \leq q+1, \quad |x| \ < 1). \end{aligned} \tag{7}$$

In this paper, we will need only some basic knowledge about Chebyshev polynomials which we recall here in below. Interested reader may want to refer to [2,3,12] for full accounts of this fascinating area of orthogonal polynomials.

The Chebyshev polynomials of the first, second, third and fourth kinds are respectively defined by the following generating functions.

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{8}$$

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{9}$$

$$F(t,x) = \frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{10}$$

$$G(t,x) = \frac{1}{1-xt-t^2} = \sum_{n=0}^{\infty} W_n(x)t^n. \tag{11}$$

They are explicitly expressed as in the following.

$$\begin{aligned} T_n(x) &= {}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right) \\ &= \frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l \frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l}, \quad (n \geq 1), \end{aligned} \tag{12}$$

$$\begin{aligned} U_n(x) &= (n+1)\ {}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-x}{2}\right) \\ &= \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n-l}{l} (2x)^{n-2l}, \quad (n \geq 0), \end{aligned} \tag{13}$$

$$\begin{aligned} V_n(x) &= \ {}_2F_1\left(-n, n+1; \frac{1}{2}; \frac{1-x}{2}\right) \\ &= \sum_{l=0}^{n} \binom{2n-l}{l} 2^{n-l} (x-1)^{n-l}, \quad (n \geq 0), \end{aligned} \tag{14}$$

$$\begin{aligned} W_n(x) &= (2n+1)\ {}_2F_1\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right) \\ &= (2n+1) \sum_{l=0}^{n} \frac{2^{n-l}}{2n-2l+1} \binom{2n-l}{l} (x-1)^{n-l}, \quad (n \geq 0). \end{aligned} \tag{15}$$

The Chebyshev polynomials of the first, second, third and fourth kinds are also given by Rodrigues' formulas.

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} (1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n} (1-x^2)^{n-\frac{1}{2}}, \tag{16}$$

$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!} (1-x^2)^{-\frac{1}{2}} \frac{d^n}{dx^n} (1-x^2)^{n+\frac{1}{2}}, \tag{17}$$

$$(1-x)^{-\frac{1}{2}} (1+x)^{\frac{1}{2}} V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n} (1-x)^{n-\frac{1}{2}} (1+x)^{n+\frac{1}{2}}, \tag{18}$$

$$(1-x)^{\frac{1}{2}} (1+x)^{-\frac{1}{2}} W_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n} (1-x)^{n+\frac{1}{2}} (1+x)^{n-\frac{1}{2}}. \tag{19}$$

They have the following orthogonalities with respect to various weight functions.

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_n(x) T_m(x) dx = \frac{\pi}{\epsilon_n} \delta_{n,m}, \tag{20}$$

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} U_n(x) U_m(x) dx = \frac{\pi}{2} \delta_{n,m}, \tag{21}$$

$$\int_{-1}^{1} (\frac{1+x}{1-x})^{\frac{1}{2}} V_n(x) V_m(x) dx = \pi \delta_{n,m}, \tag{22}$$

$$\int_{-1}^{1} (\frac{1-x}{1+x})^{\frac{1}{2}} W_n(x) W_m(x) dx = \pi \delta_{n,m}, \tag{23}$$

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where

$$\epsilon_n = \begin{cases} 1, & \text{if } n = 0, \\ 2, & \text{if } n \geq 1, \end{cases} \quad \delta_n = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases} \tag{24}$$

To proceed further, we let

$$\alpha_{n,r}(x) = \sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x)V_{i_2}(x)\cdots V_{i_{r+1}}(x), \quad (n \geq 0, r \geq 1), \tag{25}$$

$$\beta_{n,r}(x) = \sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x)W_{i_2}(x)\cdots W_{i_{r+1}}(x), \quad (n \geq 0, r \geq 1). \tag{26}$$

We note here that both $\alpha_{n,r}(x)$ and $\beta_{n,r}(x)$ are polynomials of degree n.

In this paper, we will consider the sums of finite products of Chebyshev polynomials of the third and fourth kinds in (25) and (26). Then we are going to express each of them as linear combinations of the four kinds of Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$. We obtain them by explicit computations and using Propositions 2.1 and Lemma 2.2. The general formulas in Proposition 2.1 can be derived by using orthogonalities and Rodrigues' formulas for Chebyshev polynomials and integration by parts.

The next two theorems are our main results in which the terminating hypergeometric functions ${}_3F_2\left(\begin{smallmatrix} -n, & a, & b \\ d, & e \end{smallmatrix}; 1\right)$ appear.

Theorem 1.1. *Let n, r be integers with $n \geq 0$, $r \geq 1$. Then we have following.*

$$\begin{aligned} &\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x)V_{i_2}(x)\cdots V_{i_{r+1}}(x) \\ &= \frac{(-1)^n(2n+2r)!}{r!2^{2r}(n+r-\frac{1}{2})_r} \\ &\times \sum_{k=0}^{n} \frac{(-1)^k \epsilon_k}{(n-k)!(n+k)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n, & \frac{1}{2}-n-r \\ \frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) T_k(x) \end{aligned} \tag{27}$$

$$\begin{aligned} &= \frac{(-1)^n(2n+2r)!}{r!2^{2r-2}(n+r-\frac{1}{2})_{r-1}} \\ &\times \sum_{k=0}^{n} \frac{(-1)^k(k+1)}{(n-k)!(n+k+2)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-2, & \frac{1}{2}-n-r \\ -\frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) U_k(x) \end{aligned} \tag{28}$$

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$$= \frac{(-1)^n(2n+2r)!}{r!2^{2r}(n+r-\frac{1}{2})_r} \times \sum_{k=0}^{n} \frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-1, & \frac{1}{2}-n-r \\ & \frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) V_k(x) \quad (29)$$

$$= \frac{(-1)^n(2n+2r)!}{r!2^{2r-1}(n+r-\frac{1}{2})_{r-1}} \times \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(n+k+1)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-1, & \frac{1}{2}-n-r \\ & -\frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) W_k(x). \quad (30)$$

Theorem 1.2. *Let n, r be integers with $n \geq 0$, $r \geq 1$. Then we have following.*

$$\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x) W_{i_2}(x) \cdots W_{i_{r+1}}(x)$$

$$= \frac{(-1)^n(2n+2r)!}{r!2^{2r}(n+r+\frac{1}{2})_r} \times \sum_{k=0}^{n} \frac{(-1)^k \epsilon_k}{(n-k)!(n+k)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n, & -\frac{1}{2}-n-r \\ & \frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) T_k(x) \quad (31)$$

$$= \frac{(-1)^n(2n+1)(2n+2r)!}{r!2^{2r-1}(n+r+\frac{1}{2})_r} \times \sum_{k=0}^{n} \frac{(-1)^k(k+1)}{(n-k)!(n+k+2)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-2, & -\frac{1}{2}-n-r \\ & -\frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) U_k(x) \quad (32)$$

$$= \frac{(-1)^n(2n+2r)!}{r!2^{2r}(n+r+\frac{1}{2})_r} \times \sum_{k=0}^{n} \frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-1, & -\frac{1}{2}-n-r \\ & \frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) V_k(x) \quad (33)$$

$$= \frac{(-1)^n(2n+1)(2n+2r)!}{r!2^{2r}(n+r+\frac{1}{2})_r} \times \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(n+k+1)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-1, & -\frac{1}{2}-n-r \\ & -\frac{1}{2}-n, & -2n-2r \end{matrix}; 1\right) W_k(x). \quad (34)$$

As we know, the Bernoulli polynomials are not orthogonal polynomials but Appell polynomials. In [7], the sums of finite products of Chebyshev polynomials in (25) and (26) were expressed as linear combinations of Bernoulli polynomials. Also, the same has been done for the sums of finite products of Bernoulli, Euler and Genocchi polynomials in [1,8,9]. All of these were found by deriving Fourier series expansions for the functions closely connected with those various sums of finite products. For some other applications of Chebyshev polynomials, we let the reader refer to [4,10,11].

2. Proof of Theorem 1.1

Here we will prove Theorem 1.1. For this purpose, we first state Proposition 2.1 and Lemma 2.2 that will be used in Sections 2 and 3.

The results in Proposition 2.1 can be derived by using the orthogonalities in (20)-(23) and the Rodrigues formulas in (16)-(19). The statements (a) and (b) in Proposition 2.1 are respectively from the equations (23) and (35) of [6], while (c) and (d) are respectively from the equations (22) and (37) of [5].

Proposition 2.1. *Let $q(x) \in \mathbb{R}[x]$ be a polynomials of degree n. Then we have the following.*

$$(a) \quad q(x) = \sum_{k=0}^{n} c_{k,1} T_k(x),$$

$$where \quad c_{k,1} = \frac{(-1)^k 2^k k! \epsilon_k}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx,$$

$$(b) \quad q(x) = \sum_{k=0}^{n} c_{k,2} U_k(x),$$

$$where \quad c_{k,2} = \frac{(-1)^k 2^{k+1} (k+1)!}{(2k+1)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\frac{1}{2}} dx,$$

$$(c) \quad q(x) = \sum_{k=0}^{n} c_{k,3} V_k(x),$$

$$where \quad c_{k,3} = \frac{(-1)^k 2^k k!}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx,$$

$$(d) \quad q(x) = \sum_{k=0}^{n} c_{k,4} W_k(x),$$

$$where \quad c_{k,4} = \frac{(-1)^k 2^k k!}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx.$$

Lemma 2.2. *Let l, m be nonnegative integers. Then we have the following.*

$$\begin{aligned} &\int_{-1}^{1} (1-x)^{m-\frac{1}{2}} (1+x)^{l-\frac{1}{2}} dx \\ &= \frac{2^{l+m}}{(l+m)!} \Gamma(l+\frac{1}{2}) \Gamma(m+\frac{1}{2}) \\ &= \frac{(2l)!\ (2m)!\ \pi}{2^{l+m}\ (l+m)!\ l!\ m!}. \end{aligned} \tag{35}$$

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Proof. By changing the variables $1+x=2y$, the integral in (35) becomes

$$\begin{aligned} 2^{l+m}\int_0^1 y^{l+\frac{1}{2}-1}(1-y)^{m+\frac{1}{2}-1}dy &= 2^{l+m}\frac{\Gamma(l+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(l+m+1)} \\ &= \frac{2^{l+m}\,(2l)!\,\Gamma(\frac{1}{2})\,(2m)!\,\Gamma(\frac{1}{2})}{(l+m)!\,2^{2l}\,l!\,2^{2m}\,m!}, \end{aligned}$$

where we used (5) and (6). □

As it was shown in [7], the following lemma can be obtained by differentiating the equation (10). It expresses the sums of finite products in (25) very neatly which plays an important role for the following discussion.

Lemma 2.3. *Let n, r be integers with $n \geq 0$, $r \geq 1$. Then we have the identity.*

$$\sum_{l=0}^{n}\sum_{i_1+i_2+\cdots+i_{r+1}=l}\binom{r-1+n-l}{r-1}V_{i_1}(x)\cdots V_{i_{r+1}}(x)=\frac{1}{2^r\,r!}V_{n+r}^{r}(x), \tag{36}$$

where the inner sum runs over all nonnegative integers i_1, i_2, $\cdots$, i_{r+1}, with $i_1+i_2+\cdots+i_{r+1}=l$.

From (14), the r th derivative of $V_n(x)$ is given by

$$V_n^{(r)}(x)=\sum_{l=0}^{n-r}\binom{2n-l}{l}2^{n-l}(n-l)_r(x-1)^{n-l-r}. \tag{37}$$

In particular, we have

$$V_{n+r}^{(r+k)}(x)=\sum_{l=0}^{n-k}\binom{2n+2r-l}{l}2^{n+r-l}(n+r-l)_{r+k}(x-1)^{n-k-l}. \tag{38}$$

Here we will show only (28) of Theorem 1.1, since (27), (29) and (30) can be proved similarly to (28).

With $\alpha_{n,r}(x)$ as in (25), we let

$$\alpha_{n,r}(x)=\sum_{k=0}^{n}c_{k,2}U_k(x). \tag{39}$$

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Then, from (b) of Proposition 2.1, (36), (38), and integration by parts k times, we have

$$\begin{aligned} c_{k,2} &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi} \int_{-1}^{1} \alpha_{n,r}(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx \\ &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \int_{-1}^{1} V_{n+r}^{(r)}(x) \frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}} dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \int_{-1}^{1} V_{n+r}^{(r+k)}(x)(1-x^2)^{k+\frac{1}{2}} dx \\ &= \frac{2^{k+1}(k+1)!}{(2k+1)!\pi 2^r r!} \sum_{l=0}^{n-k} (-1)^{n-k-l} \binom{2n+2r-l}{l} 2^{n+r-l} \\ &\quad \times (n+r-l)_{r+k} \int_{-1}^{1} (1-x)^{n-l+1-\frac{1}{2}} (1+x)^{k+1-\frac{1}{2}} dx. \end{aligned} \tag{40}$$

From (40), (35) and after some simplifications, we get

$$\begin{aligned} c_{k,2} &= \frac{(-1)^{n-k}(k+1)}{r!} \\ &\times \sum_{l=0}^{n-k} \frac{(-1)^l (2n+2r-l)!(n+r-l)!(2n+2-2l)!}{l!(n-k-l)!(n+k-l+2)!(2n+2r-2l)!(n+1-l)!}. \end{aligned} \tag{41}$$

Using (3) and (4), (41) is equal to

$$\begin{aligned} c_{k,2} &= \frac{(-1)^{n-k}(k+1)(2n+2r)!}{r!(n-k)!(n+k+2)!} \\ &\times \sum_{l=0}^{n-k} \frac{(-1)^l (n-k)_l (n+k+2)_l <\frac{1}{2}-n-r>_l 2^{2n-2l+2}(-1)^l <\frac{1}{2}>_{n+1}}{l!(2n+2r)_l 2^{2n+2r-2l}(-1)^l <\frac{1}{2}>_{n+r}<\frac{1}{2}-n-1>_l}. \\ &= \frac{(-1)^n (2n+2r)!}{r! 2^{2r-2}(n+r-\frac{1}{2})_{r-1}} \\ &\times \quad \frac{(-1)^k(k+1)}{(n-k)!(n+k+2)!} \sum_{l=0}^{n-k} \frac{<k-n>_l<-k-n-2>_l<\frac{1}{2}-n-r>_l}{<-\frac{1}{2}-n>_l<-2n-2r>_l\, l!} \\ &= \frac{(-1)^n (2n+2r)!}{r! 2^{2r-2}(n+r-\frac{1}{2})_{r-1}} \\ &\times \quad \frac{(-1)^k(k+1)}{(n-k)!(n+k+2)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n-2, & \frac{1}{2}-n-r \\ -\frac{1}{2}-n, & -2n-2r & \end{matrix} ;1 \right). \end{aligned} \tag{42}$$

Now, the equation (28) in Theorem 1.1 follows from (39) and (42).

3. Proof of Theorem 1.2

In this section, we will show (31) of Theorem 1.2, as (32)-(34) can be treated analogously to (31). The following lemma can be obtained by differentiating (11) and is stated as Lemma 3 in [7].

Lemma 3.1. *Let n, r be integers with $n \geq 0$, $r \geq 1$. Then we have the following identity.*

$$\begin{aligned} &\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x) W_{i_2}(x) \cdots W_{i_{r+1}}(x) \\ &= \frac{1}{2^r r!} W_{n+r}^{(r)}(x), \end{aligned} \tag{43}$$

where the inner sum runs over all nonnegative integers i_1, i_2, $\cdots$, i_{r+1}, with $i_1 + i_2 + \cdots + i_{r+1} = l$.

From (15), the r th derivative of $W_n(x)$ is given by

$$W_n^{(r)}(x) = (2n+1) \sum_{l=0}^{n-r} \frac{2^{n-l}}{2n+1-2l} \binom{2n-l}{l} (n-l)_r (x-1)^{n-l-r}. \tag{44}$$

In particular,

$$\begin{aligned} &W_{n+r}^{(r+k)}(x) \\ &= (2n+1) \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n+2r+1-2l} \binom{2n+2r-l}{l} (n+r-l)_{r+k} (x-1)^{n-k-l}. \end{aligned} \tag{45}$$

Here we will show only (31) of Theorem 1.2, since (32)-(34) can be proved analogously to (31). With $\beta_{n,r}(x)$ as in (26), we put

$$\beta_{n,r}(x) = \sum_{k=0}^{n} c_{k,1} T_k(x). \tag{46}$$

Then, from (a) of Proposition 2.1, (43), (45) and integration by parts k times, we have

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$$\begin{aligned}
c_{k,1} &= \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi} \int_{-1}^{1} \beta_{n,r}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \int_{-1}^{1} W_{n+r}^{(r)}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \int_{-1}^{1} W_{n+r}^{(r+k)}(x)(1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{(2n+1)2^k k! \epsilon_k}{(2k)!\pi 2^r r!} \sum_{l=0}^{n-k}) \frac{(-1)^{n-k-l} 2^{n+r-l}}{2n+2r+1-2l} \binom{2n+2r-l}{l} \\
&\quad \times (n+r-l)_{r+k} \int_{-1}^{1} (1-x)^{n-l-\frac{1}{2}}(1+x)^{k-\frac{1}{2}} dx.
\end{aligned} \tag{47}$$

From (47), (35) and after some simplifications, we get

$$\begin{aligned}
c_{k,2} &= \frac{(2n+1)\epsilon_k(-1)^{n-k}}{r!} \\
&\times \sum_{l=0}^{n-k} \frac{(-1)^l (2n+2r-l)!(n+r-l)!}{l!(n-k-l)!(n+k-l)!(2n+2r-2l+1)!(n-l)!} \\
&= \frac{2(2n+1)\epsilon_k(-1)^{n-k}}{r!} \\
&\times \sum_{l=0}^{n-k} \frac{(-1)^l (2n+2r-l)!(n+r-l+1)!(2n-2l)!}{l!(n-k-l)!(n+k-l)!(2n+2r-2l+2)!(n-l)!}.
\end{aligned} \tag{48}$$

Using (3) and (4), (48) is equal to

$$\begin{aligned}
c_{k,1} &= \frac{2(2n+1)(2n+2r)!\epsilon_k(-1)^{n-k}}{r!(n-k)!(n+k)!} \\
&\times \sum_{l=0}^{n-k} \frac{(-1)^l (n-k)_l (n+k)_l < \frac{1}{2}-n-r-1 >_l 2^{2n-2l}(-1)^l < \frac{1}{2} >_n}{l!(2n+2r)_l 2^{2n+2r+2-2l}(-1)^l < \frac{1}{2} >_{n+r+1} < \frac{1}{2}-n >_l} \\
&= \frac{(2n+1)(-1)^n(2n+2r)!}{r! 2^{2r+1}(n+r+\frac{1}{2})_{r+1}} \\
&\times \quad \frac{(-1)^k \epsilon_k}{(n-k)!(n+k)!} \sum_{l=0}^{k} \frac{< k-n >_l < -k-n >_l < -\frac{1}{2}-n-r >_l}{< \frac{1}{2}-n >_l < -2n-2r >_l} \\
&= \frac{(-1)^n(2n+2r)!}{r! 2^{2r}(n+r+\frac{1}{2})_r} \\
&\times \quad \frac{(-1)^k \epsilon_k}{(n-k)!(n+k)!} {}_3F_2\left(\begin{matrix} k-n, & -k-n, & -\frac{1}{2}-n-r \\ \frac{1}{2}-n, & -2n-2r & \end{matrix} ; 1 \right).
\end{aligned} \tag{49}$$

Now, the equation (31) in Theorem 1.2 follows from (46) and (49).

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