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Article

Hermite Hadamard type inequalities for functions whose derivatives are η- convex via fractional integrals

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Abstract: In the present research, we will develop some integral inequalities of Hermite Hadamard type for differentiable η -convex function. Moreover, our results include several new and known results as special cases.

Keywords: Convex function, η -convex function, Hermite Hadamard type inequality, fractional integral.

1. Introduction

Through this paper let I be an interval in the real line $\mathbb{R}$. Also consider $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as the Hadamard's inquality for convex function.
In [4], Fejer gave a generalization of (1) as follows.
If $f : [a, b] \to \mathbb{R}$ is a convex function and $g : [a, b] \to \mathbb{R}$ is non-neagative, integrable and symmetric about $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{2}$$

Since Hermite Hadamard's and Fractional integrals have a wide range of applications, many researchers extend their studies to Hermite Hadamard type inequalities involving fractional integrals. For recent generalizations, one can see [8], [9], [10] and [11].
In [5], [7], [8] Bo-Yan Xi, M. E. Ozdemir, M. Z. Sarikaya established the following Hermite Hadamard type inequalities for convex function.

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Theorem 1. *[8] Let $f, g : [a, b] \to [0, \infty)$ be convex functions on $[a, b] \subset \mathbb{R}$, $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) \tag{3}$$

and

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b), \tag{4}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \ \text{ and } \ N(a,b) = f(a)g(b) + f(b)g(a).$$

Theorem 2. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$ and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2}f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}\Big\{(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}} \\
&\times \left[\left(4-9\lambda+12\lambda^2-2\lambda^3\right)|f'(a)|^q + \left(2-3\lambda+2\lambda^3\right)|f'(b)|^q\right]^{\frac{1}{q}} \\
&+ \left(1-2\mu+2\mu^2\right)^{1-\frac{1}{q}}\Big[\left(2-3\mu+2\mu^3\right)|f'(a)|^q \\
&+ \left(4-9\mu+12\mu^2-2\mu^3\right)|f'(b)|^q)\Big]^{\frac{1}{q}}\Big\}.
\end{aligned} \tag{5}$$

Corollary 3. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $0 \leq \lambda \leq 1$, and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda}{2}[f(a)+f(b)] + (1-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}}\Big\{\Big[\left(4-9\lambda+12\lambda^2-2\lambda^3\right) \\
&\times |f'(a)|^q + \left(2-3\lambda+2\lambda^3\right)|f'(b)|^q\Big]^{\frac{1}{q}} + \Big[\left(2-3\lambda+2\lambda^3\right) \\
&\times |f'(a)|^q + \left(4-9\lambda+12\lambda^2-2\lambda^3\right)|f'(b)|^q)\Big]^{\frac{1}{q}}\Big\}.
\end{aligned} \tag{6}$$

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Corollary 4. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f^{'} \in L[a, b]$. If $|f^{'}(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{16}\left(\frac{1}{12}\right)^{\frac{1}{q}}\left\{[9|f^{'}(a)|^q+3|f^{'}(b)|^q]^{\frac{1}{q}}+[3|f^{'}(a)|^q+9|f^{'}(b)|^q]^{\frac{1}{q}}\right\}, \qquad (7)\\
&\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{5(b-a)}{72}\left(\frac{1}{90}\right)^{\frac{1}{q}}\left\{[74|f^{'}(a)|^q+16|f^{'}(b)|^q]^{\frac{1}{q}}+[16|f^{'}(a)|^q+74|f^{'}(b)|^q]^{\frac{1}{q}}\right\},\\
&\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{5(b-a)}{72}\left(\frac{1}{90}\right)^{\frac{1}{q}}\left\{[61|f^{'}(a)|^q+29|f^{'}(b)|^q]^{\frac{1}{q}}+[29|f^{'}(a)|^q+61|f^{'}(b)|^q]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 5. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f^{'} \in L[a, b]$. If $|f^{'}(x)|^q$ is convex on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{16}\left[|f^{'}(a)|+|f^{'}(b)|\right],\\
&\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{144}\left[|f^{'}(a)|+|f^{'}(b)|\right],\\
&\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{72}\left[|f^{'}(a)|+|f^{'}(b)|\right]. \qquad (8)
\end{aligned}$$

Theorem 6. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and $f^{'} \in L[a, b]$. If $|f^{'}(x)|^q$ for $q \geq 1$ is convex on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\leq \frac{b-a}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}}\\
&\times\left\{\left(\left[(q+3-\lambda)(1-\lambda)^{q+1}+(2q+4-\lambda)\lambda^{q+1}\right]|f^{'}(a)|^q\right.\right.\\
&\left.+\left[(q+1+\lambda)(1-\lambda)^{q+1}+\lambda^{q+2}\right]|f^{'}(b)|^q\right)^{\frac{1}{q}}\\
&+\left(\left[(q+1+\mu)(1-\mu)^{q+1}+\mu^{q+2}\right]|f^{'}(a)|^q\right.\\
&\left.\left.+\left[(q+3-\mu)(1-\mu)^{q+1}+(2q+4-\mu)\mu^{q+1}\right]|f^{'}(b)|^q\right)^{\frac{1}{q}}\right\}. \qquad (9)
\end{aligned}$$

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Corollary 7. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda \leq 1$ and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda}{2}\Big[f(a)+f(b)\Big]+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}}\Bigg\{\Bigg(\Big[(q+3-\lambda)(1-\lambda)^{q+1} \\
&+(2q+4-\lambda)\lambda^{q+1}\Big]|f'(a)|^q+\Big[(q+1+\lambda)(1-\lambda)^{q+1}+\lambda^{q+2}\Big]|f'(b)|^q\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\Big[(q+1+\lambda)(1-\lambda)^{q+1}+\lambda^{q+2}\Big]|f'(a)|^q \\
&+\Big[(q+3-\lambda)(1-\lambda)^{q+1}+(2q+4-\lambda)\lambda^{q+1}\Big]|f'(b)|^q\Bigg)^{\frac{1}{q}}\Bigg\}. \qquad (10)
\end{aligned}$$

Corollary 8. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{8}\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}}\Bigg\{\Big[(3q+6)|f'(a)|^q+(q+2)|f'(b)|^q\Big]^{\frac{1}{q}} \\
&+\Big[(q+2)|f'(a)|^q+(3q+6)|f'(a)|^q\Big]^{\frac{1}{q}}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{12}\left[\frac{1}{18(q+1)(q+2)}\right]^{\frac{1}{q}}\Bigg\{\Bigg[\left(11+6q+(3q+8)2^{q+1}\right)|f'(a)|^q \\
&+\left(1+(3q+4)2^{q+1}\right)|f'(b)|^q\Bigg]^{\frac{1}{q}}+\Bigg[\left(1+(3q+4)2^{q+1}\right)|f'(a)|^q \\
&+\left(11+6q+(3q+8)2^{q+1}\right)|f'(b)|^q\Bigg]^{\frac{1}{q}}\Bigg\}. \qquad (11)
\end{aligned}$$

Theorem 9. *[7] Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a,b]$, where a, $b \in I$ with $a < b$. If $|f|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{192}\left\{|f''(a)|+6\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right\}. \qquad (12)
\end{aligned}$$

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Theorem 10. *[7] Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f'' \in L[a,b]$, where a, $b \in I$ with $a < b$. If $|f''|^q$ for $q \geq 1$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left\{ \left(\frac{|f''(a)|^q}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \right. \\ &\left. + \left(\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{|f''(b)|^q}{3}\right)^{\frac{1}{q}} \right\}. \end{aligned} \tag{13}$$

Lemma 11. *[5] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ and $\lambda, \mu \in \mathbb{R}$, then*

$$\begin{aligned}&\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2-\lambda-\mu}{2} f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{b-a}{4}\int_0^1 \left[(1-\lambda-t)f'\left(ta + (1-t)\frac{a+b}{2}\right)\right. \\ &\left. + (\mu - t)f'\left(t\frac{a+b}{2} + (1-t)b\right)\right]dt. \end{aligned} \tag{14}$$

Lemma 12. *[5] For $s > 0$ and $0 \leq \epsilon \leq 1$, one has*

$$\begin{aligned}\int_0^1 |\epsilon - t|^s dt &= \frac{\epsilon^{s+1} + (1-\epsilon)^{s+1}}{s+1} \\ \int_0^1 t|\epsilon - t|^s dt &= \frac{\epsilon^{s+2} + (s+1+\epsilon)(1-\epsilon)^{s+1}}{(s+1)(s+2)}. \end{aligned} \tag{15}$$

In [1], S.S. Dragomir introduces the important generalization of convexity known as η- convexity.

Definition 13. [1] A function $f : I \to \mathbb{R}$ is called η- convex if

$$f(\alpha x + (1-\alpha)y) \leq f(y) + \alpha\eta(f(x), f(y)) \tag{16}$$

for all $x, y \in I$ and $\alpha \in [0,1]$.

The paper is organized as follows:
In the second section we will establish Hermite Hadamard and Fejer type inequalities for η- convex function. In the last section, we will derive Fractional integral inequalities for η- convex function.

2. Hermite Hadamard and Fejer type inequalities

Theorem 14. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be η- convex function, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. Then*

$$\begin{aligned}&f(\frac{a+b}{2}) - \frac{1}{2(b-a)}\int_a^b \eta(f(a+b-x), f(x))dx \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(b) + \frac{1}{2}\eta(f(a), f(b)). \end{aligned} \tag{17}$$

Proof. According to (16), with $x = ta + (1-t)b$ and $y = (1-t)a + tb$ and $\alpha = \frac{1}{2}$, we find that

$$f(\frac{a+b}{2}) \leq f((1-t)a + tb) + \frac{1}{2}\eta(f(ta + (1-t)b), f((1-t)a + tb).$$

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Thus, by integrating, we obtain

$$
\begin{aligned}
f(\frac{a+b}{2}) &\leq \int_0^1 f((1-t)a+tb) + \frac{1}{2}\int_0^1 \eta(f(ta+(1-t)b), f((1-t)a+tb)dt \\
&\leq \frac{1}{b-a}\int_a^b f(x)dx + \frac{1}{2(b-a)}\int_a^b \eta(f(a+b-x), f(x))dx
\end{aligned}
$$

so,

$$
f(\frac{a+b}{2}) - \frac{1}{2(b-a)}\int_a^b \eta(f(a+b-x), f(x))dx \leq \frac{1}{b-a}\int_a^b f(x)dx \tag{18}
$$

and the first inequality is proved. The proof of second inequality follows by using (16) with $x = a$ and $y = b$ and integrating with respect to α over $[0\,,1]$. That is,

$$
\frac{1}{b-a}\int_a^b f(x)dx \leq f(b) + \frac{1}{2}\eta(f(a), f(b)) \tag{19}
$$

20 Ofcourse, (18) and (19) yields (17). □

21 **Remark 1.** If we take $\eta(x,y) = x - y$, (17) reduces to the inequality (2).

Theorem 15. *Let f and g be non-negative η- convex functions $a, b \in I$, $a < b$ such that $fg \in L_1[a,b]$, then*

$$
\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq M'(a,b), \tag{20}
$$

where

$$
\begin{aligned}
M'(a,b) = f(b)g(b) &+ \frac{1}{2}f(b)\eta(g(a), g(b)) + \frac{1}{2}g(b)\eta(f(a), f(b)) \\
&+ \frac{1}{3}\eta(f(a), f(b))\eta(g(a), g(b))
\end{aligned}
$$

Proof. Since, f and g are η- convex functions, we have

$$
\begin{aligned}
f(ta+(1-t)b) &\leq f(b) + t\eta(f(a), f(b)) \\
g(ta+(1-t)b) &\leq g(b) + t\eta(f(a), f(b))
\end{aligned}
$$

for all $t \in [a,b]$. Since f and g are non-negative, so

$$
\begin{aligned}
&f(ta+(1-t)b)g(ta+(1-t)b) \\
&\qquad \leq f(b)g(b) + tf(b)\eta(g(a), g(b)) \\
&\qquad + tg(b)\eta(f(a), f(b)) + t^2\eta(f(a), f(b))\eta(g(a), g(b)).
\end{aligned}
$$

Integrating both sides of above inequality over $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt \\
&\qquad \leq f(b)g(b) + \frac{1}{2}f(b)\eta(g(a), g(b)) + \frac{1}{2}g(b)\eta(f(a), f(b)) \\
&\qquad + \frac{1}{3}\eta(f(a), f(b))\eta(g(a), g(b)).
\end{aligned}
$$

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Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le M^{'}(a,b).$$

□

Remark 2. If we take $\eta(x,y) = x - y$, (20) reduces to inequality (4).

Theorem 16. *Let f be an η- convex function, a, $b \in I$ with $a < b$, $f \in L_1[a,b]$ and $g : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric about $\frac{(a+b)}{2}$. Then*

$$\int_a^b f(x)g(x)dx \le [f(b) + \frac{1}{2}\eta(f(a), f(b))] \int_a^b g(x)dx. \tag{21}$$

Proof. since, f is an η- convex function and g is non-negative, integrable and symmetric about $\frac{(a+b)}{2}$, we find that

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= \frac{1}{2}\left[\int_a^b f(x)g(x)dx + \int_a^b f(a+b-x)g(a+b-x)dx\right] \\
&= \frac{1}{2}\int_a^b [(f(x) + f(a+b-x))\, g(x)dx] \\
&= \frac{1}{2}\int_a^b \left[f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right)\right] g(x)dx
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2}\int_a^b \left[\left(f(b) + \frac{b-x}{b-a}\eta(f(a), f(b))\right)\right. \\
&+ \left.\left(f(b) + \frac{x-a}{b-a}\eta(f(a), f(b))\right)\right] g(x)dx \\
&\le [f(b) + \frac{1}{2}\eta(f(a), f(b))] \int_a^b g(x)dx.
\end{aligned}$$

□

Remark 3. If we choose $\eta(x,y) = x - y$ and $g(x) = 1$,then (21) reduces to second inequality in (2), and if we take $\eta(x,y) = x - y$, then (21) reduces to second inequality in (3).

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3. Fractional integral inequalities

Theorem 17. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \geq 1$ is η- convex on $[a,b]$, then*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left\{(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}}\left[\left(6-12\lambda+12\lambda^2\right)|f'(b)|^q\right.\right.$$
$$\left.+\left(4-9\lambda+12\lambda^2-2\lambda^3\right)\eta(|f'(a)|^q,|f'(b)|^q)\right]^{\frac{1}{q}}$$
$$+\left(1-2\mu+2\mu^2\right)^{1-\frac{1}{q}}\left[\left(6-12\mu+12\mu^2\right)|f'(b)|^q\right.$$
$$\left.\left.+\left(2-3\mu+2\mu^3\right)\eta(|f'(a)|^q,|f'(b)|^q)\right]^{\frac{1}{q}}\right\}. \tag{22}$$

Proof. For $q > 1$, by lemma (11), the η- convexity of $|f'(x)|^q$ on $[a,b]$ and the noted Holder's integral inequality, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left[\int_0^1|1-\lambda-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt\right.$$
$$\left.+\int_0^1|\mu-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\right]$$

$$\leq \frac{b-a}{4}\left\{\left(\int_0^1|1-\lambda-t|dt\right)^{1-\frac{1}{q}}\left[\int_0^1|1-\lambda-t|\left(|f'(b)|^q\right.\right.\right.$$
$$\left.+(\frac{1+t}{2})\eta(|f'(a)|^q,|f'(b)|^q)\ right)dt\right]^{\frac{1}{q}}+\left(\int_0^1|\mu-t|dt\right)^{1-\frac{1}{q}}$$
$$\left.\times\left[\int_0^1|\mu-t|\left(|f'(b)|^q+(\frac{t}{2})\eta(|f'(a)|^q,|f'(b))\right)dt\right]^{\frac{1}{q}}\right\}. \tag{23}$$

By using Lemma (12), a direct calculation yields

$$\int_0^1|1-\lambda-t|\left(|f'(b)|^q+(\frac{1+t}{2})\eta(|f'(a)|^q,|f'(b)|^q)\right)dt$$
$$=\left(|f'(b)|^q+\frac{1}{2}\eta(|f'(a)|^q,|f'(b)|^q)\right)\int_0^1|1-\lambda-t|dt$$
$$+\frac{1}{2}\eta(|f'(a)|^q,|f'(b)|^q)\int_0^1 t|1-\lambda-t|dt$$
$$=\left(|f'(b)|^q+\frac{1}{2}\eta(|f'(a)|^q,|f'(b)|^q)\right)\left(\frac{1}{2}-\lambda+\lambda^2\right)$$
$$+\frac{1}{12}\eta(|f'(a)|^q,|f'(b)|^q)[(1-\lambda)^3+\lambda^2(3-\lambda)]$$
$$=\frac{1}{2}(1-2\lambda+2\lambda^2)|f'(b)|^q+\frac{1}{12}(4-9\lambda+12\lambda^2-2\lambda^3)$$
$$\times\eta(|f'(a)|^q,|f'(b)|^q)$$

and

$$\begin{aligned}
&\int_0^1 |\mu-t|\Big(|f'(b)|^q+(\frac{t}{2})\eta(|f'(a)|^q,|f'(b)|^q)\Big)dt\\
&\quad=|f'(b)|^q\int_0^1|\mu-t|dt+\frac{1}{2}\eta(|f'(a)|^q,|f'(b)|^q)\int_0^1 t|\mu-t|dt\\
&\quad=|f'(b)|^q\left(\frac{1}{2}-\mu-\mu^2\right)+\frac{1}{12}\eta(|f'(a)|^q,|f'(b)|^q)(\mu^3+(2+\mu)(1-\mu)^2)\\
&\quad=\frac{1}{2}(1-2\mu+2\mu^2)|f'(b)|^q+\frac{1}{12}(2-3\mu+2\mu^3)\eta(|f'(a)|^q,|f'(b)|^q).
\end{aligned}$$

Substituting the above two inequalities into the inequality (23) and utilizing lemma (12) results in the inequality (22) for $q>1$.
For $q=1$, from Lemmas (11) and (12), it follows that

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\quad\le\frac{b-a}{4}\Big\{\int_0^1|1-\lambda-t|\left(|f'(b)|+(\frac{1+t}{2})\eta(|f'(a)|,|f'(b)|)\right)dt\\
&\quad+\int_0^1|\mu-t|\left(|f'(b)|+\frac{t}{2}\eta(|f'(a)|,|f'(b)|)\right)dt\Big\}\\
&\quad=\frac{b-a}{48}\Big\{\left(6-12\lambda+12\lambda^2\right)|f'(b)|+\left(4-9\lambda+12\lambda^2-2\lambda^3\right)\\
&\quad\times\eta(|f'(a)|,|f'(b)|)+\left(6-12\mu+12\mu^2\right)|f'(b)|\\
&\quad+\left(2-3\mu+2\mu^3\right)\eta(|f'(a)|,|f'(b)|)\Big\}.
\end{aligned}\tag{24}$$

28 □

29 **Remark 4.** If we take $\eta(x,y)=x-y$, then inequality (22) reduces to inequality (5).

30 If taking $\lambda=\mu$ in Theorem (22), we derive the following corollary.

Corollary 18. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $0\le\lambda\le1$, and $f'\in L[a,b]$. If $|f'(x)|^q$ for $q\ge1$ is η- convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda}{2}[f(a)+f(b)]+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\quad\le\frac{b-a}{8}\left(\frac{1}{6}\right)^{\frac{1}{q}}(1-2\lambda+2\lambda^2)^{1-\frac{1}{q}}\Big[(6-12\lambda+12\lambda^2)|f'(b)|^q\\
&\quad+(4-9\lambda+12\lambda^2-2\lambda^3)\eta(|f'(a)|^q,|f'(b)|^q)\Big]^{\frac{1}{q}}\\
&\quad+\Big[(6-12\lambda+12\lambda^2)|f'(b)|^q+(2-3\lambda+2\lambda^3)\eta(|f'(a)|^q,|f'(b)|^q)\Big]^{\frac{1}{q}}.
\end{aligned}\tag{25}$$

31 **Remark 5.** If we take $\eta(x,y)=x-y$, then inequality (25) reduces to inequality (6).

32 If letting $\lambda=\mu=\frac{1}{2},\frac{2}{3},\frac{1}{3}$, respectively in Theorem (22), we can deduce the inequalities below.

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Corollary 19. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \le \lambda, \mu \le 1$, and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \ge 1$ is η- convex on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{16}\left(\frac{1}{12}\right)^{\frac{1}{q}}\left\{[12|f'(b)|^q+9\eta(|f'(a)|^q,|f'(b)|^q)]^{\frac{1}{q}}\right.$$
$$\left.+[12|f'(b)|^q+3\eta(|f'(a)|^q,|f'(b)|^q)]^{\frac{1}{q}}\right\}$$
$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{5(b-a)}{72}\left(\frac{1}{90}\right)^{\frac{1}{q}}\left\{[90|f'(b)|^q+74\eta(|f'(a)|^q,|f'(b)|^q)]^{\frac{1}{q}}\right.$$
$$\left.+[90|f'(b)|^q+16\eta(|f'(a)|^q,|f'(b)|^q)]^{\frac{1}{q}}\right\}$$
$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{5(b-a)}{72}\left(\frac{1}{90}\right)^{\frac{1}{q}}\left\{[90|f'(b)|^q+61\eta(|f'(a)|^q,|f'(b)|^q)]^{\frac{1}{q}}\right.$$
$$\left.+[90|f'(b)|^q+29\eta(|f'(a)|^q,|f'(b)|^q)]^{\frac{1}{q}}\right\}. \tag{26}$$

If setting $q = 1$ in Corollary (26), then one has the following.

Corollary 20. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'(x)|$ is η- convex on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{16}\left[2|f'(b)|+\eta(|f'(a)|,|f'(b)|)\right]$$
$$\left|\frac{1}{3}\left[f(a)+f(b)+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{5(b-a)}{72}\left[2|f'(b)|+\eta(|f'(a)|,|f'(b)|)\right]$$
$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{5(b-a)}{72}\left[2|f'(b)|+\eta(|f'(a)|,|f'(b)|)\right]. \tag{27}$$

Remark 6. If we take $\eta(x,y) = x - y$, then inequalities (26) and (27) reduces to inequalities (7) and (8).

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Theorem 21. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \le \lambda, \mu \le 1$, and $f' \in L[a,b]$. If $|f'(x)|^q$ for $q \ge 1$ is η- convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{b-a}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}} \\
&\times\Bigg\{\Bigg[\Big([2(q+2)(1-\lambda)^{q+1}+2(q+2)\lambda^{q+1}]\Big)|f'(b)|^q \\
&+[(q+3-\lambda)(1-\lambda)^{q+1}+(2q+4-\lambda)\lambda^{q+1}]\eta(|f'(a)|^q,|f'(b)|^q)\Big)\Bigg]^{\frac{1}{q}} \\
&+\Bigg[(2(q+2)(1-\mu)^{q+1}+2(q+2)\mu^{q+1})|f'(b)|^q \qquad (28)\\
&+(\mu^{q+2}+(q+1+\mu)(1-\mu)^{q+1})\eta(|f'(a)|^q,|f'(b)|^q)\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Proof. For $q > 1$, by η- convexity of $|f'(x)|^q$ on $[a,b]$, Lemma (11), and Holder's integral inequality, it follows that

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{b-a}{4}\left[\int_0^1|1-\lambda-t|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt\right. \\
&\left.+\int_0^1|\mu-t|\left|f'\left(t\frac{a+b}{2}+(1-t)b\right)\right|dt\right] \\
&\le \frac{b-a}{4}\Bigg\{\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\Bigg[\int_0^1|1-\lambda-t|^q\Big(|f'(b)|^q+(\frac{1+t}{2}) \\
&\times\eta(|f'(a)|^q,|f'(b)|^q)\Big)dt\Bigg]^{\frac{1}{q}}+\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\Bigg[\int_0^1|\mu-t|^q \\
&\times\left(|f'(b)|^q+(\frac{t}{2})\eta(|f'(a)|^q,|f'(b))\right)dt\Bigg]^{\frac{1}{q}}\Bigg\} \\
&\le \frac{b-a}{4}\Bigg\{\left[\int_0^1|1-\lambda-t|^q\left(|f'(b)|^q+(\frac{1+t}{2})\eta(|f'(a)|^q,|f'(b)|^q)\right)dt\right]^{\frac{1}{q}} \\
&+\left[\int_0^1|\mu-t|^q\left(|f'(b)|^q+(\frac{t}{2})\eta(|f'(a)|^q,|f'(b))\right)dt\right]^{\frac{1}{q}}\Bigg\}. \qquad (29)
\end{aligned}$$

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By Lemma (12) we have

$$
\begin{aligned}
&\int_0^1 |1-\lambda-t|^q \left(|f^{'}(b)|^q + (\frac{1+t}{2})\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \right) dt \\
&= \left(|f^{'}(b)|^q + \frac{1}{2}\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \right) \int_0^1 |1-\lambda-t|^q dt \\
&+ \frac{1}{2}\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \int_0^1 t|1-\lambda-t|^q dt \\
&= \left(|f^{'}(b)|^q + \frac{1}{2}\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \right) \left(\frac{(1-\lambda)^{q+1}+\lambda^{q+1}}{q+1} \right) \\
&+ \frac{1}{2}\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \left(\frac{(1-\lambda)^{q+2}+(q+2-\lambda)\lambda^{q+1}}{(q+1)(q+2)} \right) \\
&= \frac{1}{2(q+1)(q+2)} \left[2(q+2)(1-\lambda)^{q+1} + 2(q+2)\lambda^{q+1} \right] |f^{'}(b)|^q \\
&+ \left[2(q+2)(1-\lambda)^{q+1} + (q+2)\lambda^{q+1} + (1-\lambda)^{q+2} + (q+2-\lambda)\lambda^{q+1} \right] \\
&\times \eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \\
&= \frac{1}{2(q+1)(q+2)} \left[2(q+2)(1-\lambda)^{q+1} + 2(q+2)\lambda^{q+1} \right] |f^{'}(b)|^q \\
&+ \left[(q+3-\lambda)(1-\lambda)^{q+1} + (2q+4-\lambda)\lambda^{q+1} \right] \eta(|f^{'}(a)|^q, |f^{'}(b)|^q).
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 |\mu-t|^q \left(|f^{'}(b)|^q + (\frac{t}{2})\eta(|f^{'}(a)|^q, |f^{'}(b)|^q \right) dt \\
&= |f^{'}(b)|^q \int_0^1 |\mu-t|^q dt + \frac{1}{2}\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \int_0^1 t|\mu-t|^q dt \\
&= |f^{'}(b)|^q \left(\frac{\mu^{q+1}+(1-\mu)^{q+1}}{q+1} \right) + \frac{1}{2}\eta(|f^{'}(a)|^q, |f^{'}(b)|^q) \\
&\times \left(\frac{\mu^{q+2}+(q+1+\mu)(1-\mu)^{q+1}}{(q+1)(q+2)} \right) \\
&= \frac{1}{2(q+1)(q+2)} \left\{ \left[2(q+2)(1-\mu)^{q+1} + 2(q+2)\mu^{q+1} \right] |f^{'}(b)|^q \right. \\
&+ \left. \left[\mu^{q+2} + (q+1+\mu)(1-\mu)^{q+1} \right] \eta(|f^{'}(a)|^q, |f^{'}(b)|^q \right\}.
\end{aligned}
$$

Substituting the above two inequalities into the inequality (29) yields the inequality (28) for $q > 1$. For $q = 1$, the proof is the same as the deduction of (24) and the Theorem is proved. □

Remark 7. If we take $\eta(x,y) = x - y$, inequality (28) reduces to inequality (9).

As the derivation of corollaries of Theorem (22), we can obtain the following corollaries of Theorem (3.8).

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Corollary 22. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $0 \leq \lambda \leq 1$, and $f^{'} \in L[a, b]$. If $|f^{'}(x)|^q$ for $q \geq 1$ is η- convex on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{\lambda}{2}[f(a)+f(b)]+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\frac{1}{2(q+1)(q+2)}\right]^{\frac{1}{q}} \\
&\times\Bigg\{\Bigg[\left(2(q+2)(1-\lambda)^{q+1}+2(q+2)\lambda^{q+1}\right)|f^{'}(b)|^q \\
&+\left((q+3-\lambda)(1-\lambda)^{q+1}+(2q+4-\lambda)\lambda^{q+1})\eta(|f^{'}(a)|^q,|f^{'}(b)|^q)\right)\Bigg]^{\frac{1}{q}} \\
&+\Bigg[\left(2(q+2)(1-\lambda)^{q+1}+2(q+2)\lambda^{q+1}\right)|f^{'}(b)|^q \qquad (30)\\
&+\left(\lambda^{q+2}+(q+1+\lambda)(1-\lambda)^{q+1}\right)\eta(|f^{'}(a)|^q,|f^{'}(b)|^q)\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

41 **Remark 8.** If we take $\eta(x, y) = x - y$ and $h(t) = t$, inequality (30) reduces to inequality (10).

Corollary 23. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and $f^{'} \in L[a, b]$. If $|f^{'}(x)|^q$ for $q \geq 1$ is η- convex on $[a, b]$, then*

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{8}\left[\frac{1}{4(q+1)(q+2)}\right]^{\frac{1}{q}} \\
&\times\left\{\left[\left((4q+8)|f^{'}(b)|^q+(3q+6)\eta(|f^{'}(a)|^q,|f^{'}(b)|^q)\right)\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(4q+8)|f^{'}(b)|^q+(q+2)\eta(|f^{'}(a)|^q,|f^{'}(b)|^q)\right)\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{12}\left[\frac{1}{18(q+1)(q+2)}\right]^{\frac{1}{q}}\Bigg\{\Bigg[\left((3q+6)(2)^{q+2}+6(q+2)\right)|f^{'}(b)|^q \\
&+\left((3q+8)(2)^{q+1}+(6q+11)\eta(|f^{'}(a)|^q,|f^{'}(b)|^q)\right)\Bigg]^{\frac{1}{q}} \qquad (31)\\
&+\left[\left((3q+6)(2)^{q+2}+6(q+2)\right)|f^{'}(b)|^q+\left(1+(3q+4)(2)^{q+1}\right)\eta(|f^{'}(a)|^q,|f^{'}(b)|^q)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

42 If setting $q = 1$ in Corollary (3.12), then one has the following.

Corollary 24. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, $0 \leq \lambda, \mu \leq 1$, and $f' \in L[a,b]$. If $|f'(x)|$ is η- convex on $[a,b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left\{\left[\left(\frac{1}{4}\right)|f'(b)|+\eta(|f'(a)|,|f'(b)|)\int_0^1 h\left(\frac{1+t}{2}\right)\left|\frac{1}{2}-t\right|dt\right]\right.$$
$$\left.+\left[\left(\frac{1}{4}\right)|f'(b)|+\eta(|f'(a)|,|f'(b)|)\int_0^1 h\left(\frac{t}{2}\right)\left|\frac{1}{2}-t\right|dt\right]\right\},$$
$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{b-a}{4}\left\{\left[\left(\frac{5}{18}\right)|f'(b)|^q+\eta(|f'(a)|,|f'(b)|)\int_0^1 h\left(\frac{1+t}{2}\right)\left|\frac{2}{3}-t\right|dt\right]\right.$$
$$\left.+\left[\left(\frac{5}{18}\right)|f'(b)|+\eta(|f'(a)|,|f'(b)|)\int_0^1 h\left(\frac{t}{2}\right)\left|\frac{1}{3}-t\right|dt\right]\right\}. \tag{32}$$

Remark 9. If we take $\eta(x,y) = x - y$ and $h(t) = t$, inequalities (31) and (32) reduces to inequalities (11) and (8) respectively.

To prove our next results, we consider the following Lemma proved in [7].

Lemma 25. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f'' \in L[a,b]$, then the following equality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)$$
$$= \frac{(b-a)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{a+b}{2}+(1-t)a\right)dt\right.$$
$$\left.+\int_0^1 (t-1)^2 f''\left(tb+(1-t)\frac{a+b}{2}\right)dt\right]. \tag{33}$$

Theorem 26. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on $I\circ$ such that $f'' \in L[a,b]$, where a, $b \in I$ with $a < b$. If $|f''|$ is η - convex on $[a,b]$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left[\frac{1}{3}\left(|f''(a)|+\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right.$$
$$\left.+\frac{1}{4}\left(\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)+\frac{1}{3}\eta\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right)\right]. \tag{34}$$

Proof. From Lemma(25), we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\
&\left.+\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left(|f''(a)|+t\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)\right)dt\right] \\
&+\frac{(b-a)^2}{16}\left[\int_0^1 (t-1)^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|+t\eta\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right)dt\right] \\
&=\frac{(b-a)^2}{16}\left[\frac{1}{3}|f''(a)|+\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|+\frac{1}{4}\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)\right. \\
&\left.+\frac{1}{12}\eta\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right] \\
&=\frac{(b-a)^2}{16}\left[\frac{1}{3}\left(|f''(a)|+\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right. \\
&\left.+\frac{1}{4}\left(\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(a)|\right)+\frac{1}{3}\eta\left(|f''(b)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right)\right].
\end{aligned}
$$

This proves the inequality (34). □

Remark 10. If we take $\eta(x,y)=x-y$, inequality (34) reduces to inequality (12).

Theorem 27. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on $I\circ$ such that $f''\in L[a,b]$, where a, $b\in I$ with $a<b$. If $|f''|^q$ for $q\geq 1$ is η- convex on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left[\left(\frac{1}{3}|f''(a)|^q+\frac{1}{4}\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{12}\eta\left(|f''(b)|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
\tag{35}
$$

Proof. Suppose that $p\geq 1$. From Lemma (25) and using the power mean inequality, we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{(b-a)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\
&\left.+\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\leq \frac{(b-a)^2}{16}\left(\int_0^1 t^2dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2\left|f''\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{(b-a)^2}{16}\left(\int_0^1 (t-1)^2dt\right)^{\frac{1}{p}}\left(\int_0^1 (t-1)^2\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Because $|f''|^q$ is η- convex, we have

$$\int_0^1 t^2 \left| f''\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt$$
$$\leq \frac{1}{3}|f''(a)|^q + \frac{1}{4}\left(\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right)\right)$$

and

$$\int_0^1 (t-1)^2 \left| f''\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt$$
$$\leq \frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{12}\eta\left(|f''(b)|^q, \left|f''\left(\frac{a+b}{2}\right)\right|^q\right).$$

Therefore, we have

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{3}|f''(a)|^q + \frac{1}{4}\eta\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right)\right)^{\frac{1}{q}}\right.$$
$$\left. + \frac{1}{3}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{12}\eta\left(|f''(b)|^q, \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}.$$

□

Remark 11. If we take $\eta(x,y) = x - y$, inequality (35) reduces to inequality (13).

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