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Theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Sets

Khodabocus m. i. and Sookia n. u. h.

Abstract. Several specific types of generalized sets of a generalized topological space have been defined and investigated for various purposes from time to time in the literature of topological spaces. Our recent research in the field of a new class of generalized sets of a generalized topological space is reported herein as a starting point for more generalized classes.

Key words and phrases. *Generalized topological space, generalized operations, generalized sets*

1. Introduction

Just as the notion of $\mathcal{T}$-set[1] (open or closed set relative to ordinary topology) is fundamental and indispensable in the study of $\mathfrak{T}$-sets in $\mathcal{T}$-spaces (arbitrary sets in ordinary topological spaces) and in the formulation of the concept of $\mathfrak{g}$-$\mathcal{T}$-set (generalized $\mathcal{T}$-open or $\mathcal{T}$-closed set relative to ordinary topology) in the study of $\mathfrak{g}$-$\mathfrak{T}$-sets in $\mathcal{T}$-spaces (generalized sets in ordinary topological spaces) [18, 19, 23, 37, 39, 41], so is the notion of $\mathcal{T}_{\mathfrak{g}}$-set (open or closed set relative to generalized topology) in the study of $\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces (arbitrary sets in generalized topological spaces) and in the formulation of the concept of $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-set (generalized $\mathcal{T}_{\mathfrak{g}}$-open or $\mathcal{T}_{\mathfrak{g}}$-closed set relative to generalized topology) in the study of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces (generalized sets in generalized topological spaces) [14]. Thus, the $\mathfrak{g}$-topology maps $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ from the power set $\mathcal{P}(\Omega)$ of Ω into itself, thereby inducing $\mathfrak{g}$-topologies on the underlying set Ω, are classes of distinguished open subsets of a $\mathcal{T}$-space which are not $\mathcal{T}$-open sets but are $\mathcal{T}_{\mathfrak{g}}$-open sets which are related to the families of $\mathfrak{g}$-$\mathcal{T}$-open sets [25, 35]. Examples of $\mathfrak{g}$-$\mathfrak{T}$-sets in $\mathcal{T}$-spaces are α-open and α-closed sets, introduced by [33]; β-open sets, introduced by [1] and γ-open sets, introduced by [34]. Examples of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces are Δ_{μ}-sets and ∇_{μ}-sets, introduced by [24]; ω-open sets, introduced by [19] and θ-sets, introduced by [10]. From these α, β, γ-sets and, Δ_{μ}, ∇_{μ}, ω, θ-sets, the theories of $\mathfrak{g}$-$\mathfrak{T}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets then appear to be subjects of primary interest.

To the best of our knowledge, the theory of $\mathfrak{g}$-$\mathfrak{T}$-sets is well-known and that of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets less-known. The earliest works on the theory of $\mathfrak{g}$-$\mathfrak{T}$-sets are those of [27] [28, 27], [33], and [13, 12, 11, 10, 9], and the latest works on the theory of

[1]Notes to the reader: The structures $\mathfrak{T} = (\Omega, \mathcal{T})$ and $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, respectively, are called ordinary and generalized topological spaces (briefly, $\mathcal{T}$-space and $\mathcal{T}_{\mathfrak{g}}$-space). The symbols $\mathcal{T}$ and $\mathcal{T}_{\mathfrak{g}}$, respectively, are called ordinary topology and generalized topology (briefly, topology and $\mathfrak{g}$-topology). Subsets of $\mathfrak{T}$ and $\mathfrak{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{T}$-sets and $\mathfrak{T}_{\mathfrak{g}}$-sets; subsets of $\mathcal{T}$ and $\mathcal{T}_{\mathfrak{g}}$, respectively, are called $\mathcal{T}$-open and $\mathcal{T}_{\mathfrak{g}}$-open sets, and their complements are called $\mathcal{T}$-closed and $\mathcal{T}_{\mathfrak{g}}$-closed sets. Generalizations of $\mathfrak{T}$-sets, $\mathcal{T}$-open and $\mathcal{T}$-closed sets in $\mathcal{T}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}$-sets, $\mathfrak{g}$-$\mathcal{T}$-open and $\mathfrak{g}$-$\mathcal{T}$-closed sets; generalizations of $\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets in $\mathcal{T}_{\mathfrak{g}}$, respectively, are called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets.

$\mathfrak{g}$-$\mathfrak{T}$-sets are those of [36], [24, 23], [19], and [41], among others. [27] introduced and investigated the weaker forms of open sets, [33] introduced and investigated the structures of some classes of more or less nearly open sets, and [9] introduced the notion of $\mathfrak{g}$-topologies; [36] introduced the weaker forms of closed sets and studied some of their characterizations, [23] gave a unified framework for the study of several types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, [19] extended the notion of a type of $\mathfrak{g}$-$\mathfrak{T}$-sets in a $\mathcal{T}$-space to its analogue in a $\mathcal{T}_{\mathfrak{g}}$-space, and [41] introduced and investigated several types of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathcal{T}_{\mathfrak{g}}$-spaces.

Several other specific classes of $\mathfrak{g}$-$\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets have been defined and investigated by other authors for various purposes from time to time in the literature of $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$-spaces [2, 3, 4, 5, 8, 17, 15, 20, 21, 22, 26, 29, 31, 30, 32, 35, 38, 40]. The fruitfulness of all these references have made significant contributions to the theory of $\mathcal{T}$, $\mathcal{T}_{\mathfrak{g}}$-spaces, among others. In this paper, we will show how further contributions can be added to the field in a unified way.

2. Theory

2.1. Preliminaries. Our discussion starts by recalling a carefully chosen set of terms used in this study. Throughout this chapter, $\mathfrak{U}$ stands for the universe of discourse, fixed within the framework of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets and containing as elements all sets (Ω, Γ-sets; $\mathcal{T}$, $\mathfrak{g}$-$\mathcal{T}$, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-sets; $\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$, $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets) considered in this theory, and $I_n^0 \stackrel{\text{def}}{=} \{\nu \in \mathbb{N}^0 : \ \nu \leq n\}$; index sets I_∞^0, I_n^*, I_∞^* are defined similarly. A set $\Gamma \subset \mathfrak{U}$ is a subset of the set $\Omega \subset \mathfrak{U}$ and, for some $\mathcal{T}_{\mathfrak{g}}$-open set $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T} \cup \mathfrak{g}\text{-}\mathcal{T} \cup \mathcal{T}_{\mathfrak{g}} \cup \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}$, these implications hold:

$$(2.1)\ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathcal{T} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}} \ \Rightarrow \ \mathcal{O}_{\mathfrak{g}} \subset \Omega \subset \mathfrak{U}.$$

In a natural way, a monotonic map $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ from the power set $\mathcal{P}(\Omega)$ of Ω into itself can be associated to a given mapping $\pi_{\mathfrak{g}} : \Omega \to \Omega$, thereby inducing a $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} \subset \mathcal{P}(\Omega)$ on the underlying set Ω [35]. Therefore, the definition of a $\mathcal{T}_{\mathfrak{g}}$-space can be presented in a nice way. Thus, retaining the axioms to be satisfied by its $\mathfrak{g}$-topology [29], and assuming no separation axioms, unless otherwise stated, the following definition is suggestive:

Definition 2.1 ($\mathcal{T}_{\mathfrak{g}}$-Space). Let $\Omega \subset \mathfrak{U}$ be a given set and let $\mathcal{P}(\Omega) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\nu} \subseteq \Omega : \nu \in I_\infty^*\}$ be the family of all subsets $\mathcal{O}_{\mathfrak{g},1}$, $\mathcal{O}_{\mathfrak{g},2}$, ..., of Ω. Then every one-valued map of the type $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ satisfying the following axioms:

- Ax. i. $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$,
- Ax. ii. $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathcal{O}_{\mathfrak{g}}$,
- Ax. iii. $\mathcal{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_\infty^*} \mathcal{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$,

is called a "$\mathfrak{g}$-topology on Ω," and the structure $\mathfrak{T}_{\mathfrak{g}} \stackrel{\text{def}}{=} (\Omega, \mathcal{T}_{\mathfrak{g}})$ is called a "$\mathcal{T}_{\mathfrak{g}}$-space."

In Def. 2.1, by Ax. i., Ax. ii. and Ax. iii., respectively, are meant that the unary operation $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ preserves nullary union, is contracting and preserves binary union. Any element $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Omega)$ of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ is called a $\mathcal{T}_{\mathfrak{g}}$-open set and its complement element $\complement(\mathcal{O}_{\mathfrak{g}}) = \mathcal{K}_{\mathfrak{g}} \notin \mathcal{T}_{\mathfrak{g}}(\Omega)$ is called a $\mathcal{T}_{\mathfrak{g}}$-closed set. If there exists a $\nu \in I_\infty^*$ such that $\mathcal{O}_{\mathfrak{g},\nu} = \Omega$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a strong $\mathcal{T}_{\mathfrak{g}}$-space [11, 35]. Moreover, if $\mathcal{T}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_n^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ holds for any index set $I_n^* \subset I_\infty^*$ such that $n < \infty$, then $\mathfrak{T}_{\mathfrak{g}}$ is called a quasi $\mathcal{T}_{\mathfrak{g}}$-space [13].

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DEFINITION 2.2 ($\mathfrak{g}$-Closure, $\mathfrak{g}$-Interior Operators). Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space on the set $\Omega \subset \mathfrak{U}$ with a $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$. Then:

- I. The operator $\mathrm{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its closure $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{g}$-closure operator."
- II. The operator $\mathrm{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ carrying each $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ into its interior $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} - \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ is called a "$\mathfrak{g}$-interior operator."

By convention, we let $\mathcal{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathcal{T}_{\mathfrak{g}}(\Omega)$, respectively, stand for the classes of all $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$. Their proper definitions are contained below.

DEFINITION 2.3. Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space, let $\complement : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ denotes the absolute complement with respect to the underlying set $\Omega \subset \mathfrak{U}$, and let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set. The classes

$$\mathcal{T}_{\mathfrak{g}}(\Omega) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}\}, \quad \neg\mathcal{T}_{\mathfrak{g}}(\Omega) \stackrel{\text{def}}{=} \{\mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \complement(\mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}}\}, \tag{2.2}$$

respectively, denote the classes of all $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$, and the classes

$$\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}} :\ \mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}\}, \quad \mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}} :\ \mathcal{K}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}}\}, \tag{2.3}$$

respectively, denote the classes of $\mathcal{T}_{\mathfrak{g}}$-open subsets and $\mathcal{T}_{\mathfrak{g}}$-closed supersets (complements of the $\mathcal{T}_{\mathfrak{g}}$-open subsets) of the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$.

That $\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}] \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega)$ and $\neg\mathcal{T}_{\mathfrak{g}}(\Omega) \supseteq \mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]$ are true for the $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in question are clear from the context. To this end, the $\mathfrak{g}$-closure and the $\mathfrak{g}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space define themselves as

$$\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathcal{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}}, \quad \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathcal{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathcal{T}_{\mathfrak{g}}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{K}_{\mathfrak{g}}. \tag{2.4}$$

We note in passing that, $\mathrm{cl}_{\mathfrak{g}}(\cdot) \neq \mathrm{cl}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot) \neq \mathrm{int}(\cdot)$, because the resulting sets obtained from the intersection of all $\mathcal{T}_{\mathfrak{g}}$-closed supersets and the union of all $\mathcal{T}_{\mathfrak{g}}$-open subsets, respectively, relative to the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}$ are not necessarily equal to those which would be obtained from the intersection of all $\mathcal{T}$-closed supersets and the union of all $\mathcal{T}$-open subsets relative to the topology $\mathcal{T}$ [3]. Throughout this work, by $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot)$, $\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)$, and $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)$, respectively, are meant $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\cdot))$, $\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot))$, and $\mathrm{cl}_{\mathfrak{g}}(\mathrm{int}_{\mathfrak{g}}(\mathrm{cl}_{\mathfrak{g}}(\cdot)))$; other composition operators are defined in a similar way. Also, the backslash $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}$ refers to the set-theoretic difference $\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}$.

DEFINITION 2.4 ($\mathfrak{g}$-Operation). Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space on the set $\Omega \subset \mathfrak{U}$ with a $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$. The mapping $\mathrm{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ is called a "$\mathfrak{g}$-operation" on $\mathcal{P}(\Omega)$ if the following statements hold:

$$\begin{aligned} &\forall \mathcal{S}_{\mathfrak{g}} \in \mathcal{P}(\Omega) \setminus \{\emptyset\},\ \exists (\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}} \setminus \{\emptyset\} \times \neg\mathcal{T}_{\mathfrak{g}} \setminus \{\emptyset\} : \\ &\left(\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}(\emptyset) = \emptyset\right),\ \left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right), \end{aligned} \tag{2.5}$$

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where $\neg\,\mathrm{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is called the "complementary $\mathfrak{g}$-operation" on $\mathcal{P}(\Omega)$ and, for all $\mathfrak{T}_{\mathfrak{g}}$-sets $\mathcal{S}_{\mathfrak{g}}$, $\mathcal{S}_{\mathfrak{g},\nu}$, $\mathcal{S}_{\mathfrak{g},\mu} \in \mathcal{P}(\Omega) \setminus \{\emptyset\}$, the following axioms are satisfied:

- Ax. I. $(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))$,
- Ax. II. $(\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))$,
- Ax. III. $(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{S}_{\mathfrak{g},\mu} \;\rightarrow\; \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})) \vee (\mathcal{S}_{\mathfrak{g},\mu} \subseteq \mathcal{S}_{\mathfrak{g},\nu} \;\leftarrow\; \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))$,
- Ax. IV. $(\mathrm{op}_{\mathfrak{g}}(\bigcup_{\sigma=\nu,\mu} \mathcal{S}_{\mathfrak{g},\sigma}) \subseteq \bigcup_{\sigma=\nu,\mu} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma})) \vee (\neg\,\mathrm{op}_{\mathfrak{g}}(\bigcup_{\sigma=\nu,\mu} \mathcal{S}_{\mathfrak{g},\sigma}) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\sigma}))$,

for some $\mathcal{T}_{\mathfrak{g}}$-open sets $\mathcal{O}_{\mathfrak{g}}$, $\mathcal{O}_{\mathfrak{g},\nu}$, $\mathcal{O}_{\mathfrak{g},\mu} \in \mathcal{T}_{\mathfrak{g}} \setminus \{\emptyset\}$ and $\mathcal{T}_{\mathfrak{g}}$-closed sets $\mathcal{K}_{\mathfrak{g}}$, $\mathcal{K}_{\mathfrak{g},\nu}$, $\mathcal{K}_{\mathfrak{g},\mu} \in \neg\mathcal{T}_{\mathfrak{g}}$.

The formulation of Def. 2.5 is based on the axioms of the Čech closure operator [5] and the various axioms used by many mathematicians to define closure operators [32]. The class $\mathcal{L}_{\mathfrak{g}}[\Omega]$ stands for the class of all possible $\mathfrak{g}$-operators and their complementary $\mathfrak{g}$-operators in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

Definition 2.5 ($\mathbf{op}_{\mathfrak{g}}(\cdot)$-Elements). Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. The elements of the class $\mathcal{L}_{\mathfrak{g}}[\Omega] = \mathcal{L}_{\mathfrak{g}}^{\omega}[\Omega] \times \mathcal{L}_{\mathfrak{g}}^{\kappa}[\Omega]$, where

$$\mathcal{L}_{\mathfrak{g}}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\mathfrak{g},\nu\mu}(\cdot) = (\mathrm{op}_{\mathfrak{g},\nu}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g},\mu}(\cdot)) : \; (\nu,\mu) \in I_3^0 \times I_3^0\}, \tag{2.6}$$

in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are defined as:

$$\begin{aligned}
\mathrm{op}_{\mathfrak{g}}(\cdot) \;&\in\; \mathcal{L}_{\mathfrak{g}}^{\omega}[\Omega] \stackrel{\text{def}}{=} \{\mathrm{op}_{\mathfrak{g},0}(\cdot),\; \mathrm{op}_{\mathfrak{g},1}(\cdot),\; \mathrm{op}_{\mathfrak{g},2}(\cdot),\; \mathrm{op}_{\mathfrak{g},3}(\cdot)\} \\
&= \{\mathrm{int}_{\mathfrak{g}}(\cdot),\; \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot),\; \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot),\; \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot)\}; \\
\neg\,\mathrm{op}_{\mathfrak{g}}(\cdot) \;&\in\; \mathcal{L}_{\mathfrak{g}}^{\kappa}[\Omega] \stackrel{\text{def}}{=} \{\neg\,\mathrm{op}_{\mathfrak{g},0}(\cdot),\; \neg\,\mathrm{op}_{\mathfrak{g},1}(\cdot),\; \neg\,\mathrm{op}_{\mathfrak{g},2}(\cdot),\; \neg\,\mathrm{op}_{\mathfrak{g},3}(\cdot)\} \\
&= \{\mathrm{cl}_{\mathfrak{g}}(\cdot),\; \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\cdot),\; \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot),\; \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\cdot)\}.
\end{aligned} \tag{2.7}$$

We remark in passing that, $\mathbf{op}_{\mathfrak{g},11}(\cdot) = \neg\,\mathbf{op}_{\mathfrak{g},22}(\cdot)$, and the use of $\mathbf{op}_{\mathfrak{g}}(\cdot) = (\mathrm{op}_{\mathfrak{g}}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g}}(\cdot)) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$ on a class of $\mathfrak{T}_{\mathfrak{g}}$-sets will construct a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, just as the use of $\mathcal{L}[\Omega] \stackrel{\text{def}}{=} \{\mathbf{op}_{\nu}(\cdot) = (\mathrm{op}_{\nu}(\cdot), \neg\,\mathrm{op}_{\nu}(\cdot)) : \; \nu \in I_3^0\}$ on the class of $\mathfrak{T}$-sets have constructed the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets. But since $\mathrm{cl}_{\mathfrak{g}}(\cdot) \neq \mathrm{cl}(\cdot)$ and $\mathrm{int}_{\mathfrak{g}}(\cdot) \neq \mathrm{int}(\cdot)$, in general, it follows that $\mathbf{op}_{\mathfrak{g}}(\cdot) \neq \mathbf{op}(\cdot)$ and, therefore, the new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets that will be obtained from the first construction will, in general, differ from the new class of $\mathfrak{g}$-$\mathfrak{T}$-sets that had been obtained from the second construction.

Definition 2.6 ($\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-Set). A $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space is called a "$\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set" if and only if there exist a pair $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets, and a $\mathfrak{g}$-operator $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$ such that the following statement holds:

$$(\exists \xi)\,[(\xi \in \mathcal{S}_{\mathfrak{g}}) \wedge ((\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})))]\,. \tag{2.8}$$

The $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ is said to be of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-sets:

$$\begin{aligned}
\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \; &(\exists \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\
&[(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}}))]\}.
\end{aligned} \tag{2.9}$$

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It is called a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open set if it satisfies the first property in $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ and a $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed set if it satisfies the second property in $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. The classes of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ (\exists \mathcal{O}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\,[\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})]\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ (\exists \mathcal{K}_{\mathfrak{g}}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot))\,[\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})]\}. \end{aligned} \tag{2.10}$$

From the class $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$, consisting of the classes $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν, where $\nu \in I_3^0$, there results in the following definition.

Definition 2.7. Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. If, for each $\nu \in I_3^0$, $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, denote the classes of $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of category ν, then

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] &= \textstyle\bigcup_{\nu\in I_3^0}(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]) \\ &= (\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]) \cup (\textstyle\bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]) \\ &= \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]. \end{aligned} \tag{2.11}$$

In the sequel, it is interesting to view the concepts of open, semi-open, preopen, semi-preopen sets as $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2, and 3; likewise, to view the concepts of closed, semi-closed, preclosed, semi-preclosed sets as $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2, and 3. These can be realised by omitting the subscript "$\mathfrak{g}$" in all symbols of the above definitions.

Definition 2.8 ($\mathfrak{g}$-ν-$\mathfrak{T}$-Set). A $\mathfrak{T}$-set $\mathcal{S} \subset \mathfrak{T}$ in a $\mathcal{T}$-space is called a "$\mathfrak{g}$-$\mathfrak{T}$-set" if and only if there exists a pair $(\mathcal{O}, \mathcal{K}) \in \mathcal{T} \times \neg\mathcal{T}$ of $\mathcal{T}$-open and $\mathcal{T}$-closed sets, and an operator $\mathbf{op}(\cdot) \in \mathcal{L}[\Omega]$ such that the following statement holds:

$$(\exists \xi)\,[(\xi \in \mathcal{S}) \wedge ((\mathcal{S} \subseteq \mathrm{op}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}(\mathcal{K})))]\,. \tag{2.12}$$

The $\mathfrak{g}$-$\mathfrak{T}$-set $\mathcal{S} \subset \mathfrak{T}$ is said to be of category ν if and only if it belongs to the following class of $\mathfrak{g}$-ν-$\mathcal{T}$-sets:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}] \stackrel{\text{def}}{=} \{\mathcal{S} \subset \mathfrak{T} :\ &(\exists \mathcal{O}, \mathcal{K}, \mathbf{op}_{\nu}(\cdot)) \\ &[(\mathcal{S} \subseteq \mathrm{op}_{\nu}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathcal{K}))]\}. \end{aligned} \tag{2.13}$$

It is called a $\mathfrak{g}$-ν-$\mathfrak{T}$-open set if it satisfies the first property in $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}]$ and a $\mathfrak{g}$-ν-$\mathfrak{T}$-closed set if it satisfies the second property in $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}]$. The classes of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets, respectively, are defined by

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathcal{S} \subset \mathfrak{T} :\ (\exists \mathcal{O}, \mathbf{op}_{\nu}(\cdot))\,[\mathcal{S} \subseteq \mathrm{op}_{\nu}(\mathcal{O})]\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}] &\stackrel{\text{def}}{=} \{\mathcal{S} \subset \mathfrak{T} :\ (\exists \mathcal{K}, \mathbf{op}_{\nu}(\cdot))\,[\mathcal{S} \supseteq \neg\,\mathrm{op}_{\nu}(\mathcal{K})]\}. \end{aligned} \tag{2.14}$$

As in the previous definitions, from the class $\mathfrak{g}$-ν-$\mathrm{S}[\mathfrak{T}]$, consisting of the classes $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}]$, respectively, of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets of category ν, where $\nu \in I_3^0$, there results in the following definition.

Definition 2.9. Let $\mathfrak{T}$ be a $\mathcal{T}$-space. If, for each $\nu \in I_3^0$, $\mathfrak{g}$-ν-$\mathrm{O}[\mathfrak{T}]$ and $\mathfrak{g}$-ν-$\mathrm{K}[\mathfrak{T}]$, respectively, denote the classes of $\mathfrak{g}$-ν-$\mathfrak{T}$-open and $\mathfrak{g}$-ν-$\mathfrak{T}$-closed sets of category ν,

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then

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}] = \bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}] &= \bigcup_{\nu\in I_3^0}\big(\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}]\cup\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}]\big)\\ &= \big(\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}]\big)\cup\big(\bigcup_{\nu\in I_3^0}\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}]\big)\\ &= \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}]\cup\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}]. \qquad (2.15)\end{aligned}$$

The classes of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ as well as the classes of $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets in a $\mathcal{T}$-space $\mathfrak{T}$ are defined as thus:

Definition 2.10. Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ be a $\mathcal{T}_{\mathfrak{g}}$-space and let $\mathfrak{T} = (\Omega, \mathcal{T})$ be a $\mathcal{T}$-space.

- I. The classes $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ denote the families of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{g}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]\cup\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$.
- II. The classes $\mathrm{O}[\mathfrak{T}]$ and $\mathrm{K}[\mathfrak{T}]$ denote the families of $\mathfrak{T}$-open and $\mathfrak{T}$-closed sets, respectively, in $\mathfrak{T}$, with $\mathrm{S}[\mathfrak{T}] = \mathrm{O}[\mathfrak{T}]\cup\mathrm{K}[\mathfrak{T}]$.

In the following sections, the main results of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets are presented.

3. Main Results

Theorem 3.1. *Let* $\mathrm{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ *and* $\mathrm{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$*, respectively, be* $\mathfrak{g}$*-closure and* $\mathfrak{g}$*-interior operators in the* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*. Then:*

- I. $\mathrm{cl}_{\mathfrak{g}}(\cdot)$ *and* $\mathrm{int}_{\mathfrak{g}}(\cdot)$ *are enhancing and contracting, respectively.*
- II. $\mathrm{cl}_{\mathfrak{g}}(\cdot)$ *and* $\mathrm{int}_{\mathfrak{g}}(\cdot)$ *are idempotent.*
- III. $\mathrm{cl}_{\mathfrak{g}}(\cdot)$ *and* $\mathrm{int}_{\mathfrak{g}}(\cdot)$ *are monotone.*

Proof. I. Since the following logical statement

$$\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \ (\forall\xi)\left[(\xi\in\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \leftarrow \xi\in\mathcal{S}_{\mathfrak{g}}) \vee (\xi\in\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \to \xi\in\mathcal{S}_{\mathfrak{g}})\right],$$

holds, it follows that $\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ or $\mathcal{S}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, which prove I.

II. If $\mathcal{S}_{\mathfrak{g}}$ is open, then $\mathcal{S}_{\mathfrak{g}} = \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$; if it is closed, $\mathcal{S}_{\mathfrak{g}} = \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. Consequently, the substitutions $\mathcal{S}_{\mathfrak{g}} \mapsto \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ and $\mathcal{S}_{\mathfrak{g}} \mapsto \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, respectively, give $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathrm{int}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, which prove II.

III. Let $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that $\mathcal{R}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$. Then, $\mathcal{R}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}})$, $\mathcal{R}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}})$, $\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, and $\mathcal{S}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ by I. Consequently, $\mathrm{int}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}}) \subseteq \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$, which prove III. Q.E.D.

Lemma 3.2. *Let* $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set of a* $\mathcal{T}_{\mathfrak{g}}$*-space. Then:*

- I. $(\mathcal{S}_{\mathfrak{g}} = \emptyset)\wedge(\Omega\in\mathcal{T}_{\mathfrak{g}}) \ \Rightarrow\ (\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \emptyset)\wedge(\mathrm{cl}_{\mathfrak{g}}(\emptyset) = \emptyset)$;
- II. $(\mathcal{S}_{\mathfrak{g}} = \emptyset)\wedge(\Omega\notin\mathcal{T}_{\mathfrak{g}}) \ \Rightarrow\ (\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \emptyset)\wedge(\mathrm{cl}_{\mathfrak{g}}(\emptyset) \neq \emptyset)$.

Proof. If $\mathcal{S}_{\mathfrak{g}} = \emptyset$ and $\Omega\in\mathcal{T}_{\mathfrak{g}}$, then $\big(\emptyset\in\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\big)\wedge\big(\emptyset\in\mathrm{C}^{\mathrm{sup}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\big)$. Consequently, $\mathrm{int}_{\mathfrak{g}}(\emptyset), \mathrm{cl}_{\mathfrak{g}}(\emptyset) = \emptyset$

If $\mathcal{S}_{\mathfrak{g}} = \emptyset$ and $\Omega\notin\mathcal{T}_{\mathfrak{g}}$, then $\big(\emptyset\in\mathrm{C}^{\mathrm{sub}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\big)\wedge\big(\emptyset\notin\mathrm{C}^{\mathrm{sup}}_{\mathcal{T}_{\mathfrak{g}}}[\emptyset]\big)$. Consequently, $\mathrm{int}_{\mathfrak{g}}(\emptyset) = \emptyset$ and $\mathrm{int}_{\mathfrak{g}}(\emptyset) \neq \emptyset$. These prove the lemma. Q.E.D.

Theorem 3.3. *If* $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$ *are* $n\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathcal{T}_{\mathfrak{g}}$*-space, then:*

- I. $\mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$,
- II. $\mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$.

PROOF. Expressed in set-builder notation, the $\mathfrak{g}$-closure and the $\mathfrak{g}$-interior of a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space can also be defined as thus:

$$\begin{aligned} \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) &\stackrel{\text{def}}{=} \{\xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g}} \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\}, \\ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) &\stackrel{\text{def}}{=} \{\xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g}} \cap \mathrm{int}(\mathcal{O}_{\mathfrak{g}}) = \mathrm{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\}, \end{aligned}$$

respectively, from which it is easily seen that,

$$\begin{aligned} \mathrm{cl}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \bigcup_{\nu\in I_n^*}\{\xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g},\nu} \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\} \\ &= \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \big(\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset\big) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\big\} \\ &= \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \big(\textstyle\bigcup_{\nu\in I_n^*}(\mathcal{S}_{\mathfrak{g},\nu} \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}})) \neq \emptyset\big) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\big\} \\ &= \big\{\xi \in \mathfrak{T}_{\mathfrak{g}} : \textstyle\bigvee_{\nu\in I_n^*}\big((\mathcal{S}_{\mathfrak{g},\nu} \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})\big)\big\} \\ &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}). \end{aligned}$$

To prove that $\mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$, it suffices to substitute $\mathcal{S}_{\mathfrak{g},\nu} \cap \mathrm{int}(\mathcal{O}_{\mathfrak{g}}) = \mathrm{int}(\mathcal{O}_{\mathfrak{g}})$ for $\mathcal{S}_{\mathfrak{g},\nu} \cap \mathrm{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset$ in the above proof. This completes the proof. Q.E.D.

COROLLARY 3.4. *If $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$ are $n \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathcal{T}_{\mathfrak{g}}$-space, then:*

- I. $\mathrm{cl}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$,
- II. $\mathrm{int}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$.

PROPOSITION 3.5. *For any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the following statement holds:*

$$\mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) - \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}) = \emptyset. \tag{3.1}$$

PROOF. Let $\xi \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}})$. Then, $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}$ since, $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}})$. But, $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}})$ and, consequently, $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. Hence, there follows that, $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}) \subseteq \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. Conversely, let $\xi \in \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. Then, $\xi \in \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$, since $\mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$. But, since $\mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$, and, consequently, $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. Hence, $\mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}})$.

Since $\mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ is equivalent to

$$\big(\mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}) \subseteq \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big) \wedge \big(\mathrm{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}) \supseteq \mathfrak{T}_{\mathfrak{g}} - \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\big),$$

the proof of the proposition at once follows. Q.E.D.

PROPOSITION 3.6. *Let $\mathrm{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ and $\mathrm{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, respectively, be $\mathfrak{g}$-closure and $\mathfrak{g}$-interior operators in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. If $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$ are $n \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, then:*

- I. $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$,
- II. $\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$.

PROOF. Since the relations

$$\mathrm{cl}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}), \quad \mathrm{int}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$$

hold, it follows that

$$\begin{aligned}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \mathrm{cl}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})\big)\\ &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})\\ \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})\big),\\ &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}),\end{aligned}$$

which were to be proved. Q.E.D.

From the above proposition, it is obvious that

$$\begin{aligned}\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})\\ \mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big) &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{int}_{\mathfrak{g}}\circ\mathrm{cl}_{\mathfrak{g}}\circ\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}).\end{aligned}\tag{3.2}$$

On this basis, we have the following corollary:

Corollary 3.7. *Let* $\mathbf{op}_{\mathfrak{g}}(\cdot)\in\mathcal{L}_{\mathfrak{g}}[\Omega]$ *be a* $\mathfrak{g}$*-operator in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*. If* $\mathcal{S}_{\mathfrak{g},1}$, $\mathcal{S}_{\mathfrak{g},2}$, …, $\mathcal{S}_{\mathfrak{g},n}\subset\mathfrak{T}_{\mathfrak{g}}$ *are* $n\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of the* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*, then:*

$$\mathrm{op}_{\mathfrak{g}}\circ\neg\,\mathrm{op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\big)=\textstyle\bigcup_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g}}\circ\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}).\tag{3.3}$$

Theorem 3.8. *If* $\mathcal{S}_{\mathfrak{g},1}$, $\mathcal{S}_{\mathfrak{g},2}$, …, $\mathcal{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ *are* $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets of a class* $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ *in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*, then* $\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. The statement $\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\nu\in I_n^*$ is identical to the logical statement:

$$\exists(\mathcal{O}_{\mathfrak{g},\nu},\mathcal{K}_{\mathfrak{g},\nu})\in\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}:\ (\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}))\vee(\mathcal{S}_{\mathfrak{g},\nu}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})).$$

On the other hand, if $\mathbf{op}_{\mathfrak{g}}(\cdot)\in\mathcal{L}_{\mathfrak{g}}[\Omega]$ is a $\mathfrak{g}$-operator in the $\mathcal{T}_{\mathfrak{g}}$-space, then

$$\begin{aligned}\mathrm{op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}\big) &= \textstyle\bigcup_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}),\\ \neg\,\mathrm{op}_{\mathfrak{g}}\big(\textstyle\bigcup_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}\big) &= \textstyle\bigcup_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}).\end{aligned}$$

Consequently,

$$\begin{aligned}&\bigvee_{\nu\in I_n^*}\big((\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}))\vee(\mathcal{S}_{\mathfrak{g},\nu}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))\big)\\ \Rightarrow\ &\big((\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\subseteq\bigcup_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}))\vee(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\supseteq\bigcup_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))\big)\\ \Rightarrow\ &\big((\textstyle\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\subseteq\mathrm{op}_{\mathfrak{g}}(\bigcup_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}))\vee(\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\bigcup_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}))\big).\end{aligned}$$

But, $\bigcup_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}\in\mathcal{T}_{\mathfrak{g}}$ and $\bigcup_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}\in\neg\mathcal{T}_{\mathfrak{g}}$. Hence, $\bigcup_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. This proves the theorem. Q.E.D.

Theorem 3.9. *If* $\mathcal{S}_{\mathfrak{g},1}$, $\mathcal{S}_{\mathfrak{g},2}$, …, $\mathcal{S}_{\mathfrak{g},n}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ *are* $n\geq 1$ $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets of a class* $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ *in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$*, then*

$$\big(\textstyle\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\big)\vee\big(\bigcap_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}\notin\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\big).\tag{3.4}$$

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Proof. Because, $\mathcal{S}_{\mathfrak{g},1}$, $\mathcal{S}_{\mathfrak{g},2}$, …, $\mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ by hypothesis, the trueness of $\bigcap_{\nu\in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ and $\bigcap_{\nu\in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ evidently depend on the following property:

$$\bigwedge_{\nu\in I_n^*} \left(\left(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}\left(\mathcal{K}_{\mathfrak{g},\nu}\right)\right)\right),$$

where $(\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ for every $\nu \in I_n^*$. Furthermore, because the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-theoretic operations concern finite intersections, it suffices to prove the theorem for $n = 2$. Set the first property preceding $\vee$ to $\mathrm{P}(\nu)$ and that following $\vee$ to $\mathrm{Q}(\nu)$. Then, its decomposition gives

$$\begin{aligned} \textstyle\bigwedge_{\nu\in I_2^*} \left(\mathrm{P}(\nu) \vee \mathrm{Q}(\nu)\right) &= \left(\textstyle\bigwedge_{\nu\in I_2^*} \mathrm{P}(\nu)\right) \vee \left(\textstyle\bigwedge_{\nu\in I_2^*} \mathrm{Q}(\nu)\right) \\ &= \left(\mathrm{P}(1) \wedge \mathrm{Q}(2)\right) \vee \left(\mathrm{P}(2) \wedge \mathrm{Q}(1)\right). \end{aligned}$$

If $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ are both $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets then $\bigwedge_{\nu\in I_2^*} \mathrm{P}(\nu)$ is true, and if they are both $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed sets then $\bigwedge_{\nu\in I_2^*} \mathrm{Q}(\nu)$ is true. In these two cases, $\bigcap_{\nu\in I_2^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Because, in general, there does not necessarily exists $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set which is simultaneously $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathcal{T}_{\mathfrak{g}}$-closed, both $\mathrm{P}(1) \wedge \mathrm{Q}(2)$ and $\mathrm{P}(2) \wedge \mathrm{Q}(1)$ are untrue; thus, $\bigcap_{\nu\in I_2^*} \mathcal{S}_{\mathfrak{g},\nu} \notin \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. These prove the theorem. Q.E.D.

Theorem 3.10. *Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and let $\mathbf{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, then*

$$\left(\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\right) \vee \left(\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\right). \tag{3.5}$$

Proof. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Then, $\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right)$ for some pair $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets relative to $\mathcal{T}_{\mathfrak{g}}$. Consequently, $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. But, $\mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ and $\neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. Thus, there follows that $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$, and, hence, $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, which proves the theorem. Q.E.D.

Proposition 3.11. *Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ and suppose the logical statement*

$$(\exists \mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}) \left[\left(\mathcal{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\right) \vee \left(\mathcal{R}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\right)\right] \tag{3.6}$$

holds, then $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$.

Proof. Let there exists a $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ such that $\mathcal{R}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ or $\mathcal{R}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$. But $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ implies $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ or $\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Thus, $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. This completes the proof. Q.E.D.

Corollary 3.12. *Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ denotes a class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, and $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ denotes a class of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets, then*

$$\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]. \tag{3.7}$$

An important remark should be pointed out at this stage.

REMARK 3.13. The converse of the statement "if $\mathcal{S}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$" is obviously untrue. Because, the negation of this statement gives

$$\left(\mathcal{S}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\right) \wedge \left(\neg\left(\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]\right)\right),$$

which is an untrue statements.

THEOREM 3.14. *Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. If $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, then*

$$\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \Leftrightarrow \left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}} \circ \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\right). \tag{3.8}$$

PROOF. *Sufficiency.* Let

$$\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}} \circ \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})\right).$$

Then, the substitution of $\neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathcal{O}_{\mathfrak{g}}$ in the logical statement preceding $\vee$ and $\mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathcal{K}_{\mathfrak{g}}$ in that following $\vee$ gives $\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right)$.

Necessity. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. Then, $\left(\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})\right) \vee \left(\mathcal{S}_{\mathfrak{g}} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})\right)$. Consequently, substituting $\mathcal{O}_{\mathfrak{g}} = \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ in the logical statement preceding $\vee$ and $\mathcal{K}_{\mathfrak{g}} = \mathrm{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ in that following $\vee$, the required logical statement at once follows, which proves the theorem. Q.E.D. Q.E.D.

The class $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ forms a $\mathfrak{g}$-topology on Ω, which will be denoted by $\mathcal{T}_{\mathfrak{g}\text{-}\mathrm{S}}$.

THEOREM 3.15. *Let $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ be a given $\mathfrak{g}$-class in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. Then the one-valued map $\mathcal{T}_{\mathfrak{g}\text{-}S} : \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \to \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ forms a $\mathfrak{g}$-topology on Ω in the $\mathcal{T}_{\mathfrak{g}}$-space.*

PROOF. By definition, $\left(\emptyset = \mathrm{op}_{\mathfrak{g}}(\emptyset)\right) \vee \left(\emptyset = \neg\, \mathrm{op}_{\mathfrak{g}}(\emptyset)\right)$. Since, either $\mathrm{op}_{\mathfrak{g}}(\emptyset) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$ or $\neg\, \mathrm{op}_{\mathfrak{g}}(\emptyset) \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$ holds, where $\mathcal{O}_{\mathfrak{g}}$, $\mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, respectively, are some $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets in $\mathfrak{T}_{\mathfrak{g}}$, it follows that $\emptyset \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ and, hence, $\mathcal{T}_{\mathfrak{g}\text{-}\mathrm{S}}(\emptyset) = \emptyset$.

Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$. Then, since $\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that $\mathcal{S}_{\mathfrak{g}}$ is a superset of $\mathcal{T}_{\mathfrak{g}\text{-}\mathrm{S}}(\mathcal{S}_{\mathfrak{g}})$. Hence, $\mathcal{T}_{\mathfrak{g}\text{-}\mathrm{S}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathcal{S}_{\mathfrak{g}}$.

Let $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \ldots$ be $\mathfrak{T}_{\mathfrak{g}}$-sets satisfying, for every $\nu \in I_{\infty}^{*}$, $\mathcal{S}_{\mathfrak{g},\nu}$. Then, there exist classes $\{\mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}} : \nu \in I_{\infty}^{*}\}$ and $\{\mathcal{K}_{\mathfrak{g},\nu} \in \neg\mathcal{T}_{\mathfrak{g}} : \nu \in I_{\infty}^{*}\}$, respectively, of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets such that

$$\left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu}\right)\right) \vee \left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\, \mathrm{op}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{K}_{\mathfrak{g},\nu}\right)\right),$$

a relation established on the following expressions:

$$\begin{aligned} \bigcup_{\nu \in I_{\infty}^{*}} \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) &= \mathrm{op}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu}\right), \\ \bigcup_{\nu \in I_{\infty}^{*}} \neg\, \mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}) &= \neg\, \mathrm{op}_{\mathfrak{g}}\left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{K}_{\mathfrak{g},\nu}\right). \end{aligned}$$

Consequently, $\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$, since $\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}}$ is a $\mathcal{T}_{\mathfrak{g}}$-open set and $\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{O}_{\mathfrak{g},\nu} \in \neg\mathcal{T}_{\mathfrak{g}}$ is a $\mathcal{T}_{\mathfrak{g}}$-closed set. Hence,

$$\mathcal{T}_{\mathfrak{g}\text{-}\mathrm{S}}\left(\bigcup_{\nu \in I_{\infty}^{*}} \mathcal{S}_{\mathfrak{g},\nu}\right) = \bigcup_{\nu \in I_{\infty}^{*}} \mathcal{T}_{\mathfrak{g}\text{-}\mathrm{S}}(\mathcal{S}_{\mathfrak{g},\nu}).$$

These prove the theorem. Q.E.D. Q.E.D.

An immediate consequence of the above theorem is the following corollary.

COROLLARY 3.16. *Let a $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. Then the structure $(\Omega, \mathcal{T}_{\mathfrak{g}\text{-}S})$, where $\mathcal{T}_{\mathfrak{g}\text{-}S} : \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \to \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$, is a $\mathcal{T}_{\mathfrak{g}}$-space.*

To condense the set-builder notation describing the classes $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ and then classify it into sub-classes, predicates must be introduced, and the choice made is to consider the so-called *Boolean-valued functions* on $\mathfrak{T}_{\mathfrak{g}}\times\mathcal{T}_{\mathfrak{g}}\cup\neg\mathcal{T}_{\mathfrak{g}}\times\mathcal{L}_{\mathfrak{g}}[\Omega]\times\{\subseteq,\supseteq\}$, the definition of which are given below.

Definition 3.17. Let $(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}})\in\mathfrak{T}_{\mathfrak{g}}\times\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}$ and let $\mathbf{op}_{\mathfrak{g}}(\cdot)\in\mathcal{L}_{\mathfrak{g}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$. The first two predicates

$$\begin{aligned}
\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq) &\stackrel{\text{def}}{=} (\exists\mathcal{O}_{\mathfrak{g}},\mathrm{op}_{\mathfrak{g}}(\cdot))\,(\mathcal{S}_{\mathfrak{g}}\subseteq\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})),\\
\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq) &\stackrel{\text{def}}{=} (\exists\mathcal{K}_{\mathfrak{g}},\neg\,\mathrm{op}_{\mathfrak{g}}(\cdot))\,(\mathcal{S}_{\mathfrak{g}}\supseteq\neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})),\\
\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq,\supseteq) &\stackrel{\text{def}}{=} \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq)\\
&\qquad\vee\,\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq)
\end{aligned}\tag{3.9}$$

are called a Boolean-valued functions on $\mathfrak{T}_{\mathfrak{g}}\times\mathcal{T}_{\mathfrak{g}}\cup\neg\mathcal{T}_{\mathfrak{g}}\times\mathcal{L}_{\mathfrak{g}}[\Omega]\times\{\subseteq,\supseteq\}$.

In this respect, $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]\stackrel{\text{def}}{=}\{\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\ \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq,\supseteq)\}$. Moreover, employing the set-builder notations, the class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, denoted by $\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, respectively, may then be defined as thus:

Definition 3.18. Let $\mathfrak{T}_{\mathfrak{g}}$ be a $\mathcal{T}_{\mathfrak{g}}$-space. The classes

$$\begin{aligned}
\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\ \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq)\},\\
\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}:\ \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq)\},
\end{aligned}\tag{3.10}$$

respectively, such that $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]=\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\bigcup\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, denote the families of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in $\mathfrak{T}_{\mathfrak{g}}$.

It is interesting to demonstrate their usefulness. In this direction, let us prove in a different way that $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-theoretic operations is closed under arbitrary unions.

Theorem 3.19. *If* $\{\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]:\ \nu\in I_n^*\}$ *and* $\{\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]:\ \nu\in I_n^*\}$, *respectively, are finite collections of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-open and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-closed sets in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$, *then*

$$\begin{aligned}
&\bigcup_{\mu\in I_n^*}\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ (\exists\nu\in I_\mu^*)\,(\xi\in\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}])\}\subseteq\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}],\\
&\bigcap_{\mu\in I_n^*}\{\xi\in\mathfrak{T}_{\mathfrak{g}}:\ (\forall\nu\in I_\mu^*)\,(\xi\in\mathcal{S}_{\mathfrak{g},\nu}\in\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}])\}\subseteq\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}].
\end{aligned}\tag{3.11}$$

Proof. For every $\nu\in I_\mu^*$, there exist $(\mathcal{O}_{\mathfrak{g},\nu},\mathcal{K}_{\mathfrak{g},\nu})\in\mathcal{T}_{\mathfrak{g}}\times\neg\mathcal{T}_{\mathfrak{g}}$ such that properties $\mathrm{P}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\nu},\mathcal{O}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq)$ and $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu},\mathcal{K}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq)$ hold for some pair $(\mathcal{R}_{\mathfrak{g},\nu},\mathcal{S}_{\mathfrak{g},\nu})\in\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\times\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently,

$$\begin{aligned}
\mathrm{P}_{\mathfrak{g}}(\textstyle\bigcup_{\nu\in I_\mu^*}\mathcal{R}_{\mathfrak{g},\nu},\bigcup_{\nu\in I_\mu^*}\mathcal{O}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq) &= \bigvee_{\nu\in I_\mu^*}\mathrm{P}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\nu},\mathcal{O}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq),\\
\mathrm{P}_{\mathfrak{g}}(\textstyle\bigcup_{\nu\in I_\mu^*}\mathcal{S}_{\mathfrak{g},\nu},\bigcup_{\nu\in I_\mu^*}\mathcal{K}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq) &= \bigwedge_{\nu\in I_\mu^*}\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu},\mathcal{K}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq).
\end{aligned}$$

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Hence, it suffices to set

$$\mathrm{P}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq) = \bigvee_{\nu\in I_{\mu}^{*}} \mathrm{P}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g},\nu},\mathcal{O}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq),$$

$$\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq) = \bigvee_{\nu\in I_{\mu}^{*}} \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu},\mathcal{K}_{\mathfrak{g},\nu};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq),$$

and the theorem is proved. Q.E.D. Q.E.D.

If in $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq,\supseteq)$ it be assumed that $(\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, we have the following theorem:

THEOREM 3.20. *Let* $(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}}) \in \mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}}$ *in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$. *If* $(\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, *then*

$$\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq,\supseteq)\} \subseteq \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]. \tag{3.12}$$

PROOF. It is clear that

$$\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq,\supseteq) = \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq) \vee \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq),$$

and the Boolean-valued functions surrounding $\vee$ hold on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq,\supseteq\}$. Consequently, the following two cases must be considered in proving the theorem:

CASE I. Let $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq)$ hold on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq,\supseteq\}$. Then, $\mathcal{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$. But, $\mathcal{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and, consequently, $\mathcal{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ and $\mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}} \circ \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ for some $\mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}}$, by the properties of the $\mathfrak{g}$-operator. Hence, $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g},\nu};\mathrm{op}_{\mathfrak{g}}(\cdot);\subseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq,\supseteq\}$.

CASE II. Let $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq)$ hold on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq,\supseteq\}$. Then, $\mathcal{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})$. But, $\mathcal{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and, consequently, $\mathcal{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})$ and $\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}} \circ \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu})$ for some $\mathcal{K}_{\mathfrak{g},\nu} \in \neg\mathcal{T}_{\mathfrak{g}}$, by the properties of the $\mathfrak{g}$-operator. Hence, $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g},\nu};\mathrm{op}_{\mathfrak{g}}(\cdot);\supseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathcal{T}_{\mathfrak{g}} \cup \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq,\supseteq\}$.

From CASE I. and CASE II., it follows that

$$\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq)\} \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}],$$

$$\{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} :\ \mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\supseteq)\} \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}].$$

But, since $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \bigcup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$, the proof of the theorem at once follows. Q.E.D. Q.E.D.

The following theorem shows that the class $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$, upon satisfaction of two conditions, is the smallest class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

THEOREM 3.21. *Let* $\mathfrak{g}\text{-S}_{o}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}_{o}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}_{o}[\mathfrak{T}_{\mathfrak{g}}]$ *be a class of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-sets in a* $\mathcal{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}$ *such that the following two conditions are satisfied:*

- I. *If* $(\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}_{o}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}_{o}[\mathfrak{T}_{\mathfrak{g}}]$ *and* $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}},\mathcal{O}_{\mathfrak{g}},\mathcal{K}_{\mathfrak{g}};\mathbf{op}_{\mathfrak{g}}(\cdot);\subseteq,\supseteq)$ *holds on* $\mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}_{o}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}_{o}[\mathfrak{T}_{\mathfrak{g}}] \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq,\supseteq\}$, *then* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}_{o}[\mathfrak{T}_{\mathfrak{g}}]$.
- II. *The relation* $\mathcal{S}_{\mathfrak{g}} \in \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ *implies* $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}_{o}[\mathfrak{T}_{\mathfrak{g}}]$.

Then, $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-S}_{o}[\mathfrak{T}_{\mathfrak{g}}]$.

PROOF. Let $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. Then $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}} \times \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$ for some pair $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$. But, $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ implies $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}_0[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}_0[\mathfrak{T}_{\mathfrak{g}}]$ by I., and the latter together with the trueness of $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$ on $\mathfrak{T}_{\mathfrak{g}} \times \mathfrak{g}\text{-O}_0[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}_0[\mathfrak{T}_{\mathfrak{g}}] \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$ implies $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}_0[\mathfrak{T}_{\mathfrak{g}}]$ by II. Thus, $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-S}_0[\mathfrak{T}_{\mathfrak{g}}]$, which completes the proof. Q.E.D. Q.E.D.

In the earlier discussion, the set $\Omega \subset \mathfrak{U}$ carried the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}(\Omega)$. A $\mathfrak{g}$-topology of this kind will be termed an *absolute* $\mathfrak{g}$*-topology.* To this end, if $\Gamma \subseteq \Omega$ is any subset of Ω then, obviously, we would expect Γ to carry the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}(\Gamma)$. But, since $\mathcal{T}_{\mathfrak{g}}(\Gamma) \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega)$, as a consequence of the fact that $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Gamma) \to \mathcal{P}(\Gamma)$ is the one-valued restriction map of $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, which follows from the statement, $\Gamma \subseteq \Omega$ implies $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Omega)$, it does make sense to term $\mathcal{T}_{\mathfrak{g}}(\Gamma)$ a *relative* $\mathfrak{g}$*-topology.* In order to determine what any $\mathfrak{g}$-set-theoretic concepts for the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$ becomes when discussion is restricted to $\Gamma \subseteq \Omega$, it merely suffices to regard Γ as the set which carries the relative $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}}(\Gamma)$ and carry over the discussion verbatim.

DEFINITION 3.22 ($\mathcal{T}_{\mathfrak{g}}$-Subspace). Let $\mathfrak{T}_{\mathfrak{g}}(\Omega) \stackrel{\text{def}}{=} (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$ be a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, where $\Omega \subset \mathfrak{U}$ carries the absolute $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, and let $\mathcal{P}(\Gamma) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\nu} \subset \Gamma : \nu \in I_{\infty}^{*}\}$ be the family of all subsets $\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \ldots$, of any subset $\Gamma \subseteq \Omega$ of Ω, then every one-valued restriction map of the type

$$\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Gamma) \longmapsto \mathcal{T}_{\mathfrak{g}}(\Gamma) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Omega)\}, \tag{3.13}$$

defines a "relative $\mathfrak{g}$-topology on Γ," and the structure $\mathfrak{T}_{\mathfrak{g}}(\Gamma) \stackrel{\text{def}}{=} (\Gamma, \mathcal{T}_{\mathfrak{g}}(\Gamma))$ is called a "$\mathcal{T}_{\mathfrak{g}}$-subspace."

THEOREM 3.23. *Let $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}(\Gamma) \subseteq \mathfrak{T}_{\mathfrak{g}}(\Omega)$, where $\mathfrak{T}_{\mathfrak{g}}(\Gamma) = (\Gamma, \mathcal{T}_{\mathfrak{g}}(\Gamma))$ is the $\mathcal{T}_{\mathfrak{g}}$-subspace of a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}(\Omega) = (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$. If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$, then $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$.*

PROOF. If $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$, then $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}}(\Omega) \times \mathcal{T}_{\mathfrak{g}}(\Omega) \cup \neg\mathcal{T}_{\mathfrak{g}}(\Omega) \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$. Therefore, if $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$, then $\mathrm{P}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}} \cap \Gamma, \mathcal{O}_{\mathfrak{g}} \cap \Gamma, \mathcal{K}_{\mathfrak{g}} \cap \Gamma; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$ holds on $\mathfrak{T}_{\mathfrak{g}}(\Gamma) \times \mathcal{T}_{\mathfrak{g}}(\Gamma) \cup \neg\mathcal{T}_{\mathfrak{g}}(\Gamma) \times \mathcal{L}_{\mathfrak{g}}[\Gamma] \times \{\subseteq, \supseteq\}$. But, since $\mathcal{S}_{\mathfrak{g}} \cap \Gamma = \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$, $\mathcal{O}_{\mathfrak{g}} \cap \Gamma = \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Gamma)$, and $\mathcal{K}_{\mathfrak{g}} \cap \Gamma = \mathcal{K}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Gamma)$, it follows that $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$ whenever $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$, and the theorem is proved. Q.E.D. Q.E.D.

DEFINITION 3.24 (Cartesian Product). The Cartesian product of an arbitrary family $\{\Omega_{\nu} \subset \mathfrak{U} : \nu \in I_n^{*}\}$ of sets is the set of functions $\phi : I_n^{*} \to \bigcup_{\nu \in I_n^{*}} \Omega_{\nu}$ such that $\phi : \nu \mapsto \Omega_{\nu}$ for every $\nu \in I_n^{*}$. It is denoted by $\bigtimes_{\nu \in I_n^{*}} \Omega_{\nu}$ and satisfies the following properties:

- I. $\bigtimes_{\nu=\mu} \Omega_{\nu} = \Omega_{\mu} \quad \forall \mu \in I_n^{*}$,
- II. $\bigtimes_{\nu \in I_{\mu+1}^{*}} \Omega_{\nu} = \left(\bigtimes_{\nu \in I_{\mu}^{*}} \Omega_{\nu}\right) \times \Omega_{\mu+1} \quad \forall \mu \in I_{n-1}^{*}$.

The projection map which gives the projection of the Cartesian product set $\bigtimes_{\nu \in I_n^{*}} \Omega_{\nu}$ onto the μ^{th} factor of $\bigtimes_{\nu \in I_n^{*}} \Omega_{\nu}$ is defined as thus.

Definition 3.25 (Projection). Let $\{\Omega_\nu \subset \mathfrak{U} : \nu \in I_n^*\}$ be any class of sets and let $\times_{\nu \in I_n^*} \Omega_\nu$ denotes the Cartesian product of these sets. The map

$$\operatorname{proj}_\mu : \times_{\nu \in I_n^*} \Omega_\nu \to \Omega_\mu \quad (\operatorname{proj}_\mu(\times_{\nu \in I_n^*} \Omega_\nu) = \Omega_\mu) \tag{3.14}$$

is called the projection of the Cartesian product set $\times_{\nu \in I_n^*} \Omega_\nu$ onto the μ^{th} factor of $\times_{\nu \in I_n^*} \Omega_\nu$.

To generate all $\mathcal{T}_\mathfrak{g}$-open sets in a $\mathcal{T}_\mathfrak{g}$-space $\mathfrak{T}_\mathfrak{g}$, a basis $\mathcal{B}[\mathcal{T}_\mathfrak{g}]$ for $\mathfrak{T}_\mathfrak{g}$ must be supplied, and the following definition is worth considering.

Definition 3.26 ($\mathcal{T}_\mathfrak{g}$-Basis). A subclass $\mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)] \subseteq \mathcal{T}_\mathfrak{g}(\Omega_\mu)$ of $\mathcal{T}_\mathfrak{g}$-open sets in a $\mathcal{T}_\mathfrak{g}$-space $\mathfrak{T}_\mathfrak{g}(\Omega_\mu) \stackrel{\text{def}}{=} (\Omega_\mu, \mathcal{T}_\mathfrak{g}(\Omega_\mu))$, defined by

$$\mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)] \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} : (\nu, \mu, \sigma(\nu,\mu)) \in I_\infty^* \times \{\mu\} \times I_\infty^*\}, \tag{3.15}$$

is said to be a base for $\mathcal{T}_\mathfrak{g} : \mathcal{P}(\Omega_\mu) \to \mathcal{P}(\Omega_\mu)$ if and only if

$$\begin{aligned} &\forall (\mu, \sigma(\mu), \mathcal{O}_{\mathfrak{g},\sigma(\mu)}) \in \{\mu\} \times I_\infty^* \times \mathcal{T}_\mathfrak{g}(\Omega_\mu), \ \exists I_{\sigma(\mu)} \subseteq I_\infty^* : \\ &\qquad \mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup\nolimits_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}. \end{aligned} \tag{3.16}$$

With regards to the terminology employed, $\mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)]$ is called a $\mathcal{T}_\mathfrak{g}$-basis and its elements, $\mathcal{B}_{\mathcal{T}_\mathfrak{g}}$-open sets, because they are $\mathcal{T}_\mathfrak{g}$-open sets of $\mathcal{T}_\mathfrak{g} : \mathcal{P}(\Omega_\mu) \to \mathcal{P}(\Omega_\mu)$. With regards to the definition itself, an immediate consequence follows. By the relation $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}$, is meant, for every $(\nu, \mu, \sigma(\mu), \sigma(\nu,\mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_\infty^* \times I_\infty^*$, that $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)]$ and $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathcal{T}_\mathfrak{g}(\Omega_\mu)$ in the relation $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}$, where $\mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)]$ and $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathcal{T}_\mathfrak{g}(\Omega_\mu)$ are given by

$$\begin{aligned} &\operatorname{proj}_\alpha : \times_{\mu \in I_n^*} \mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)] \to \mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\alpha)], \\ &\operatorname{proj}_\alpha : \times_{\mu \in I_n^*} \mathcal{T}_\mathfrak{g}(\Omega_\mu) \to \mathcal{T}_\mathfrak{g}(\Omega_\alpha) \quad \forall \alpha \in I_n^*, \end{aligned} \tag{3.17}$$

respectively. To this end, a Cartesian product topology (Cartesian $\mathcal{T}_\mathfrak{g}$-product) is one that having for $\mathcal{T}_\mathfrak{g}$-basis all $\mathcal{B}_{\mathcal{T}_\mathfrak{g}}$-open sets of the form $\operatorname{proj}_\mu^{-1}(\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)})$, where $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)]$ for every $(\nu, \mu, \sigma(\nu,\mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_\infty^*$. Therefore, in order to define a Cartesian product $\mathcal{T}_\mathfrak{g}$-space, it suffices to take the above descriptions into account and postulate a proper definition on this ground. The following definition presents itself.

Definition 3.27. Let $\{\mathfrak{T}_\mathfrak{g}(\Omega_\mu) \stackrel{\text{def}}{=} (\Omega_\mu, \mathcal{T}_\mathfrak{g}(\Omega_\mu)) : \mu \in I_n^*\}$ be a class of $n \geq 1$ $\mathcal{T}_\mathfrak{g}$-spaces and, for every $\mu \in I_n^*$, let $\mathcal{T}_{\mathfrak{g},\Omega_\mu} : \mathcal{P}(\Omega_\mu) \to \mathcal{P}(\Omega_\mu)$ be the $\mathfrak{g}$-topology for $\mathfrak{T}_\mathfrak{g}(\Omega_\mu)$. The Cartesian $\mathcal{T}_\mathfrak{g}$-product $\stackrel{\text{def}}{=} \times_{\mu \in I_n^*} \mathcal{T}_\mathfrak{g}(\Omega_\mu)$ on the Cartesian product set $\Omega \stackrel{\text{def}}{=} \times_{\mu \in I_n^*} \Omega_\mu$ is that having for $\mathcal{T}_\mathfrak{g}$-basis all $\mathcal{B}_{\mathcal{T}_\mathfrak{g}}$-open sets belonging to the following class:

$$\begin{aligned} \mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega)] \stackrel{\text{def}}{=} \{&\operatorname{proj}_\mu^{-1}(\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}) : \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_\mathfrak{g}(\Omega_\mu)] \\ &\forall (\nu, \mu, \sigma(\nu,\mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_\infty^*\}. \end{aligned} \tag{3.18}$$

The structure $\mathfrak{T}_\mathfrak{g}(\Omega) \stackrel{\text{def}}{=} (\Omega, \mathcal{T}_\mathfrak{g}(\Omega))$ is called a "Cartesian product $\mathcal{T}_\mathfrak{g}$-space."

The fact that $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})]$ and $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$ hold for every $(\nu,\mu,\sigma(\mu),\sigma(\nu,\mu)) \in I^*_{\sigma(\mu)} \times I^*_n \times I^*_\infty \times I^*_\infty$ makes it reasonable to write

$$\begin{aligned} \times_{\mu\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\mu)} &\in \times_{\mu\in I^*_n}\mathcal{T}_{\mathfrak{g}}(\Omega_\mu), \\ \times_{\mu\in I^*_n}\big(\bigcup\nolimits_{\nu\in I^*_{\sigma(\mu)}}\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}\big) &= \bigcup_{\vec{\nu}\in\times_{\alpha\in I^*_n}I^*_{\sigma(\alpha)}}\big(\times_{\alpha\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\big) \\ &\in \times_{\mu\in I^*_n}\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_\mu)], \end{aligned} \tag{3.19}$$

where $\vec{\nu} \overset{\text{def}}{=} (\nu_1,\nu_2,\ldots,\nu_n)$ and, for every $\alpha \in I^*_n$, $\nu_\alpha \in I^*_{\sigma(\alpha)}$. An immediate consequence of such relation is contained in the following lemma.

LEMMA 3.28. *If $\mathcal{T}_{\mathfrak{g}} : \mathcal{Q}(\Omega) \to \mathcal{Q}(\Omega)$ is a one-valued map on the Cartesian product set $\Omega = \times_{\mu\in I^*_n}\Omega_\mu$, where*

$$\mathcal{Q}(\Omega) \overset{\text{def}}{=} \Big\{\mathcal{O}_{\mathfrak{g},\sigma} = \bigcup_{\vec{\nu}\in\times_{\alpha\in I^*_n}I^*_{\sigma(\alpha)}}\big(\times_{\alpha\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\big) : \mathcal{O}_{\mathfrak{g},\sigma} \in \times_{\mu\in I^*_n}\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_\mu)]\Big\}, \tag{3.20}$$

then $\mathcal{T}_{\mathfrak{g}} : \mathcal{Q}(\Omega) \to \mathcal{Q}(\Omega)$ is a $\mathfrak{g}$-topology on the Cartesian product set $\times_{\mu\in I^*_n}\Omega_\mu$.

PROOF. Let $\mathcal{O}_{\mathfrak{g},\sigma} = \bigcup_{\vec{\nu}\in\times_{\alpha\in I^*_n}I^*_{\sigma(\alpha)}}\big(\times_{\alpha\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\big)$. Since $\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)} \in \mathcal{T}_{\mathfrak{g}}(\Omega_\mu)$ for every $(\nu_\alpha,\alpha,\sigma(\nu_\alpha,\alpha)) \in I^*_{\sigma(\alpha)} \times I^*_n \times I^*_\infty$, it is evident that $\mathcal{O}_{\mathfrak{g},\sigma} = \emptyset$ only if, for every $(\nu_\alpha,\alpha,\sigma(\nu_\alpha,\alpha)) \in I^*_{\sigma(\alpha)} \times I^*_n \times I^*_\infty$, $\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)} = \emptyset$. Thus, $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$.

Let $\mathcal{O}_{\mathfrak{g},\sigma} = \bigcup_{\vec{\nu}\in\times_{\alpha\in I^*_n}I^*_{\sigma(\alpha)}}\big(\times_{\alpha\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\big)$. Then, since $\mathcal{Q}(\Omega) \subseteq \mathcal{Q}(\Omega)$, it follows that $\mathcal{O}_{\mathfrak{g},\sigma}$ is a superset of $\mathcal{T}_{\mathfrak{g}}(\Omega)$. Thus, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma}) \subseteq \mathcal{O}_{\mathfrak{g},\sigma}$.

Let $\vec{\nu} = (\nu_1,\ldots,\nu_n)$ and $\vec{\kappa} = (\kappa_1,\ldots,\kappa_n)$, and consider

$$\begin{aligned} \mathcal{O}_{\mathfrak{g},\sigma} &= \bigcup_{\vec{\nu}\in\times_{\alpha\in I^*_n}I^*_{\sigma(\alpha)}}\big(\times_{\alpha\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\big). \\ \mathcal{O}_{\mathfrak{g},\tau} &= \bigcup_{\vec{\kappa}\in\times_{\beta\in I^*_n}I^*_{\tau(\beta)}}\big(\times_{\beta\in I^*_n}\mathcal{O}_{\mathfrak{g},\tau(\kappa_\beta,\beta)}\big). \end{aligned}$$

Further, let us assume that $\vec{\vec{\eta}} = (\nu_1,\ldots,\nu_n,\kappa_1,\ldots,\kappa_n)$, $\mathbb{I}^*_{\sigma(\alpha)} \overset{\text{def}}{=} \times_{\alpha\in I^*_n}I^*_{\sigma(\alpha)}$, and $\mathbb{I}^*_{\sigma(\beta)} \overset{\text{def}}{=} \times_{\beta\in I^*_n}I^*_{\tau(\beta)}$. Then

$$\mathcal{O}_{\mathfrak{g},\sigma} \cup \mathcal{O}_{\mathfrak{g},\tau} = \bigcup_{\vec{\vec{\eta}}\in\mathbb{I}^*_{\sigma(\alpha)}\times\mathbb{I}^*_{\sigma(\beta)}}\left(\mathop{\times}_{\mu\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha,\alpha)}\right) \cup \left(\mathop{\times}_{\beta\in I^*_n}\mathcal{O}_{\mathfrak{g},\sigma(\kappa_\beta,\beta)}\right)$$

Thus, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma} \cup \mathcal{O}_{\mathfrak{g},\tau}) \subseteq \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma}) \cup \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\tau})$. This completes the proof of the theorem. Q.E.D. Q.E.D.

THEOREM 3.29. *Let $\mathfrak{T}_{\mathfrak{g},1}(\Omega)$, $\mathfrak{T}_{\mathfrak{g},2}(\Omega)$, $\ldots$, $\mathfrak{T}_{\mathfrak{g},n}(\Omega)$ be $n \geq 1$ $\mathcal{T}_{\mathfrak{g}}$-spaces and let $\mathfrak{T}_{\mathfrak{g}}(\Omega) \overset{\text{def}}{=} \times_{\nu\in I^*_n}\mathfrak{T}_{\mathfrak{g},\nu}(\Omega)$ be the $\mathcal{T}_{\mathfrak{g}}$-space product. If the relation $(\mathcal{S}_{\mathfrak{g},1},\ldots,\mathcal{S}_{\mathfrak{g},n}) \in \times_{\nu\in I^*_n}\mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{g},\nu}]$ holds, then $\times_{\nu\in I^*_n}\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-}\mathrm{S}[\times_{\nu\in I^*_n}\mathfrak{T}_{\mathfrak{g},\nu}(\Omega)]$.*

PROOF. For every $\sigma \in I_n^*$, let

$$\mathbf{op}_{\mathfrak{g},12\cdots\sigma}(\cdot) = (\mathrm{op}_{\mathfrak{g},12\cdots\sigma}(\cdot), \neg\,\mathrm{op}_{\mathfrak{g},12\cdots\sigma}(\cdot)) \in \mathcal{L}_{\mathfrak{g},12\cdots\sigma}[\Omega]$$

denotes the $\mathfrak{g}$-operator in $\times_{\nu\in I_\sigma^*}\mathfrak{T}_{\mathfrak{g},\nu}(\Omega)$ and, for every $\nu \in I_n^*$, let $(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\nu}] \times \mathcal{T}_{\mathfrak{g},\nu} \times \neg\mathcal{T}_{\mathfrak{g},\nu}$. Then,

$$\begin{aligned} \mathrm{op}_{\mathfrak{g},12\cdots n}(\times_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}) &= \times_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu}), \\ \neg\,\mathrm{op}_{\mathfrak{g},12\cdots n}(\times_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}) &= \times_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu}). \end{aligned}$$

On the other hand, for every $\nu \in I_n^*$, the logical statement

$$(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu}))$$

holds in $\mathfrak{T}_{\mathfrak{g},\nu}$. Consequently,

$$\begin{aligned} &\times_{\nu\in I_n^*}((\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu}))) \\ \Rightarrow\ &((\times_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu} \subseteq \times_{\nu\in I_n^*}\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\times_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu} \supseteq \\ &\qquad \times_{\nu\in I_n^*}\neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu}))) \\ \Rightarrow\ &((\times_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathrm{op}_{\mathfrak{g},12\cdots n}(\times_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu})) \vee (\times_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu} \supseteq \\ &\qquad \neg\,\mathrm{op}_{\mathfrak{g},12\cdots n}(\times_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}))). \end{aligned}$$

Therefore, the Boolean-valued functions

$$\mathrm{P}_{\mathfrak{g}}(\times_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu}, \times_{\nu\in I_n^*}\mathcal{O}_{\mathfrak{g},\nu}, \times_{\nu\in I_n^*}\mathcal{K}_{\mathfrak{g},\nu}; \mathbf{op}_{\mathfrak{g},12\cdots n}(\cdot); \subseteq, \supseteq)$$

holds on $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathcal{T}_{\mathfrak{g}} \times \neg\mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g},12\cdots n}[\Omega] \times \{\subseteq, \supseteq\}$ and, hence, it follows that $\times_{\nu\in I_n^*}\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-G}[\times_{\nu\in I_n^*}\mathfrak{T}_{\mathfrak{g},\nu}(\Omega)]$, which completes the proof. Q.E.D. Q.E.D.

The categorical classifications of $\mathfrak{T}$-sets and $\mathfrak{g}$-$\mathfrak{T}$-sets in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ and, $\mathfrak{T}_{\mathfrak{g}}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$ are discussed and diagrammed on this ground in the next sections.

4. Discussion

4.1. Categorical Classifications. Having adopted a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, the twofold purposes here are to establish the various relationships between the classes of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets and the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and to illustrate them through diagrams.

We have seen that, $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. But, $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}], \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathrm{S}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$; $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$. In Fig. 1, we present the relationships between the class $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{T}_{\mathfrak{g}}$-closed sets and the class $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$.

It is plain that $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}]$ and $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\nu \in I_3^0$. Moreover, it is also clear that $\mathfrak{g}\text{-2-O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-3-O}[\mathfrak{T}]$ and $\mathfrak{g}\text{-0-O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-1-O}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-3-O}[\mathfrak{T}]$, and $\mathfrak{g}\text{-2-O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-3-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-0-O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq$

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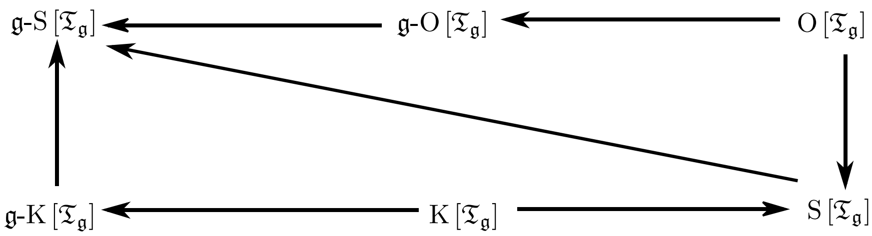

FIGURE 1. Relationships: classes of $\mathfrak{T}_{\mathfrak{g}}$-sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets.

$\mathfrak{g}\text{-1-O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-3-O}[\mathfrak{T}_{\mathfrak{g}}]$. In fact, for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relation $\mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ holds. Consequently,

$$\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \quad \forall \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}. \tag{4.1}$$

In FIG. 2, we present the relationships between the class $\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets of categories 0, 1, 2 and 3 in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and the class $\mathfrak{g}\text{-O}[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}]$ of $\mathfrak{g}$-$\mathfrak{T}$-open sets of categories 0, 1, 2 and 3 in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$.

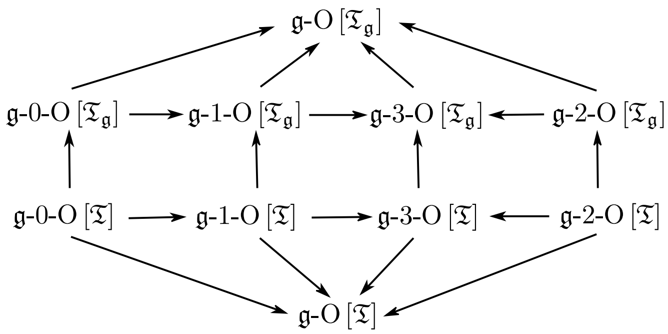

FIGURE 2. Relationships: classes of $\mathfrak{g}$-$\mathfrak{T}$-open sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets.

It is plain that, $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}]$ and $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ for every $\nu \in I_3^0$. Moreover, it is also clear that, $\mathfrak{g}\text{-2-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-3-K}[\mathfrak{T}]$ and $\mathfrak{g}\text{-0-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-1-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-3-K}[\mathfrak{T}]$, and $\mathfrak{g}\text{-2-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-3-K}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-0-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-1-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-3-K}[\mathfrak{T}_{\mathfrak{g}}]$. Because, for every $\mathfrak{T}_{\mathfrak{g}}$-set $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$, the relations $\mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ holds. Consequently,

$$\neg\,\mathrm{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \subseteq \neg\,\mathrm{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \;\; \forall \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}. \tag{4.2}$$

In FIG. 3, we present the relations between the class $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories 0, 1, 2 and 3 in the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and the class $\mathfrak{g}\text{-K}[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]$ of $\mathfrak{g}$-$\mathfrak{T}$-closed sets of categories 0, 1, 2 and 3 in the $\mathcal{T}$-space $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$.

As in the papers of [7], [16], [25], and [41], among others, the manner we have positioned the arrows is solely to stress that, in general, none of the implications in FIGS 1, 2 and 3 is reversible.

At this stage, a nice application is worth considering, and is presented in the following section.

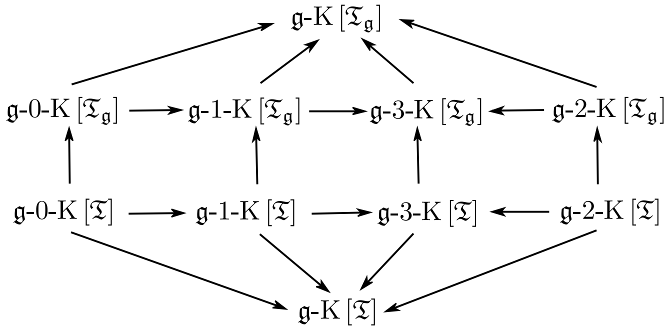

FIGURE 3. Relationships: classes of $\mathfrak{g}$-$\mathfrak{T}$-closed sets and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets.

4.2. A NICE APPLICATION. Concentrating on fundamental concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets, we shall now present a nice application. Let $\Omega = \{\xi_\nu : \nu \in I_5^*\}$ denotes the underlying set and consider the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$, where

$$\begin{aligned} \mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset, \{\xi_1\}, \{\xi_3, \xi_4\}, \{\xi_1, \xi_3, \xi_4\}\} \\ &= \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \mathcal{O}_{\mathfrak{g},3}, \mathcal{O}_{\mathfrak{g},4}\}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \neg\mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\Omega, \{\xi_2, \xi_3, \xi_4, \xi_5\}, \{\xi_1, \xi_2, \xi_5\}, \{\xi_2, \xi_5\}\} \\ &= \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},4}\}, \end{aligned} \tag{4.4}$$

respectively, stand for the classes of $\mathcal{T}_{\mathfrak{g}}$-open and $\mathcal{T}_{\mathfrak{g}}$-closed sets. Since conditions $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathcal{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, and $\mathcal{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_4^*} \mathcal{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_4^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$ are satisfied, it is clear that the one-valued map $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\{\xi_\nu : \nu \in I_5^*\})$ is a $\mathfrak{g}$-topology. Furthermore, it is easily checked that, $\mathcal{O}_{\mathfrak{g},\mu} \in \mathfrak{g}$-$\nu$-O$[\mathfrak{T}]$ for every $(\nu, \mu) \in I_3^0 \times I_4^*$. Hence, the $\mathcal{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ of the $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ are $\mathfrak{g}$-$\mathfrak{T}$-open sets relative to the $\mathcal{T}$-space $\mathfrak{T} = (\Omega, \mathcal{T})$.

After calculations, the classes $\mathfrak{g}$-ν-O$[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}$-ν-K$[\mathfrak{T}_{\mathfrak{g}}]$, respectively, of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets of categories $\nu \in \{0, 2\}$ then take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &= \mathcal{T}_{\mathfrak{g}} \cup \{\{\xi_3\}, \{\xi_4\}, \{\xi_1, \xi_3\}, \{\xi_1, \xi_4\}\}; \\ \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &= \neg\mathcal{T}_{\mathfrak{g}} \cup \{\{\xi_2, \xi_4, \xi_5\}, \{\xi_1, \xi_2, \xi_3, \xi_5\}, \\ &\qquad \{\xi_1, \xi_2, \xi_4, \xi_5\}, \{\xi_2, \xi_3, \xi_5\}\} \quad \forall \nu \in \{0, 2\}. \end{aligned} \tag{4.5}$$

On the other hand, those of categories $\nu \in \{1, 3\}$ take the following forms:

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] &= \mathcal{T}_{\mathfrak{g}} \cup \{\mathcal{O}_{\mathfrak{g}} : \mathcal{O}_{\mathfrak{g}} \in \mathcal{P}(\Omega) \setminus \mathcal{T}_{\mathfrak{g}}\}; \\ \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] &= \neg\mathcal{T}_{\mathfrak{g}} \cup \{\mathcal{K}_{\mathfrak{g}} : \mathcal{K}_{\mathfrak{g}} \in \mathcal{P}(\Omega) \setminus \neg\mathcal{T}_{\mathfrak{g}}\} \quad \forall \nu \in \{1, 3\}. \end{aligned} \tag{4.6}$$

The discussions carried out in the preceding sections can be easily verified from this nice application. The next section provides concluding remarks and future directions of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets discussed in the preceding sections.

4.3. Concluding Remarks. In this chapter, we developed a new theory, called *Theory of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-Sets*. In its own rights, the proposed theory has several advantages. The very first advantage is that the theory holds equally well when $(\Omega, \mathcal{T}_{\mathfrak{g}}) = (\Omega, \mathcal{T})$ and other features adapted on this basis, in which case it might be called *Theory of* $\mathfrak{g}$-$\mathfrak{T}$*-Sets*. Hence, in a $\mathcal{T}_{\mathfrak{g}}$-space the theoretical framework categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-closed sets, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-preclosed sets, and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-preopen and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-preclosed sets as $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a $\mathcal{T}$-space it categorises such pairs of concepts as $\mathfrak{g}$-$\mathfrak{T}$-open and $\mathfrak{g}$-$\mathfrak{T}$-closed sets, $\mathfrak{g}$-$\mathfrak{T}$-semi-open and $\mathfrak{g}$-$\mathfrak{T}$-semi-closed sets, $\mathfrak{g}$-$\mathfrak{T}$-preopen and $\mathfrak{g}$-$\mathfrak{T}$-preclosed sets, and $\mathfrak{g}$-$\mathfrak{T}$-semi-preopen and $\mathfrak{g}$-$\mathfrak{T}$-semi-preclosed sets as $\mathfrak{g}$-$\mathfrak{T}$-sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets of mixed categories. More precisely, for some pair $(\nu, \mu) \in I_3^0 \times I_3^0$ such that $\nu \neq \mu$, to develop the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open sets belonging to the class $\big\{\mathcal{O}_{\mathfrak{g}} = \mathcal{O}_{\mathfrak{g},\nu} \cup \mathcal{O}_{\mathfrak{g},\mu} : (\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ and the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed sets belonging to the class $\big\{\mathcal{K}_{\mathfrak{g}} = \mathcal{K}_{\mathfrak{g},\nu} \cup \mathcal{K}_{\mathfrak{g},\mu} :\ (\mathcal{K}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big] \times \mathfrak{g}\text{-}\mu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{g}}\big]\big\}$ in a $\mathcal{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, as [2] and [6] developed the theory of b-open and b-closed sets in a $\mathcal{T}$-space $\mathfrak{T}$. Such two theories are what we thought would certainly be worth considering, and the discussion of this chapter ends here.

References

1. D. Andrijević, Semi-Preopen Sets, *mat. vesnik*, vol. 38 (1), pp. 24–32, 1986.
2. ______, On b-Open Sets, *mat. vesnik*, vol. 48, pp. 59–64, 1996.
3. S. Bayhan, A. Kanibir, and I. L. Reilly, On Functions between Generalized Topological Spaces, *appl. gen. topol.*, vol. 14 (2), pp. 195–203, 2013.
4. P. Bhattacharyya and B.K. Lahiri, Semi-Generalized Closed Sets in Topology, *indian j. math.*, vol. 29, pp. 376–382, 1987.
5. C. Boonpok, On Generalized Continuous Maps in Čech Closure Spaces, *general mathematics*, vol. 19 (3), pp. 3–10, 2011.
6. M. Caldas and S. Jafari, On Some Applications of b-Open Sets in Topological Spaces, *kochi j. math.*, vol. 2, pp. 11–19, 2007.
7. M. Caldas, S. Jafari, and R. K. Saraf, Semi-θ-Open Sets and New Classes of Maps, *bulletin of the iranian mathematical society*, vol. 31 (2), pp. 37–52, 2005.
8. J. Cao, M. Ganster, and I. Reilly, On Generalized Closed sets, *topology and its applications*, vol. 123 (1), pp. 37–46, 2002.
9. Á. Császár, Generalized Open Sets, *acta math. hungar.*, vol. 75 (1-2), pp. 65–87, 1997.
10. ______, Generalized Topology, Generalized Continuity, *acta math. hungar.*, vol. 96 (4), pp. 351–357, 2002.
11. ______, Generalized Open Sets in Generalized Topologies, *acta math. hungar.*, vol. 106 (1-2), pp. 53–66, 2005.
12. ______, Further Remarks on the Formula for γ-Interior, *acta math. hungar.*, vol. 113 (4), pp. 325–332, 2006.
13. ______, Remarks on Quasi-Topologies, *acta math. hungar.*, vol. 119 (1-2), pp. 197–200, 2008.
14. A. Danabalan and C. Santhi, A Class of Separation Axioms in Generalized Topology, *mathematical journal of interdisciplinary sciences*, vol. 4 (2), pp. 151–159, 2016.
15. J. Dontchev, On Generalizing Semi-Preopen Sets, *mem. fac sci. kochi. univ. ser. a. math.*, vol. 16, pp. 35–48, 1995.
16. ______, On Some Separation Axioms Associated with the α-Topology, *mem. fac. sci. kochi univ. ser. a, math.*, vol. 18, pp. 31–35, 1997.
17. J. Dontchev and T. Noiri, Quasi-Normal Spaces and πg-Closed Sets, *acta math. hungar.*, vol. 89 (3), pp. 211–219, 2000.

18. S. Ersoy, M. Bilgin, and İ. İnce, Generalized Closed Set in Topological Spaces, *mathematica moravica*, vol. 19 (1), pp. 49–56, 2015.
19. S. Al Ghour and W. Zareer, Omega Open Sets in Generalized Topological Spaces, *j. nonlinear sci. appl.*, vol. 9, pp. 3010–3017, 2016.
20. Y. Gnanambal, On Generalized Preregular Closed Sets in Topological Spaces, *indian j. pure appl. math.*, vol. 28, pp. 351–360, 1997.
21. A. Gupta and R. D. Sarma, A Note on some Generalized Closure and Interior Operators in a Topological Space, *math. appl.*, vol. 6, pp. 11–20, 2017.
22. R. A. Hosny and D. Al-Kadi, Types of Generalized Sets with Ideal, *international journal of computer applications*, vol. 80 (4), pp. 11–14, 2013.
23. P. Jeyanthi, P. Nalayini, and M. Mocanu, g*λ_{μ}-Closed Sets and Generalized Topological Spaces, *bol. soc. paran. mat.*, vol. 34 (1), pp. 203–212, 2016.
24. P. Jeyanthi, P. Nalayini, and T. Noiri, Δ_{μ}-Sets and ∇_{μ}-Sets in Generalized Topological Spaces, *georgian mathematical journal*, vol. 24 (3), pp. 403–407, 2016.
25. Y. B. Jun, S. W. Jeong, H. J. Lee, and J. W. Lee, Applications of Pre-Open Sets, *applied general topology, universidad politécnica de valencia*, vol. 9 (2), pp. 213–228, 2008.
26. M. K. R. S. Veera Kumar, Between Closed Sets and g-Closed Sets, *mem. fac sci. kochi. univ. (math)*, vol. 21, pp. 1–19, 2000.
27. N. Levine, Semi-Open Sets and Semi-Continuity in Topological Spaces, *amer. math. monthly*, vol. 70, pp. 19–41, 1963.
28. ______, Generalized Closed Set in Topological Spaces, *rend. circ. mat. palermo*, vol. 19, pp. 89–96, 1970.
29. L. L. L. Lusanta and H. M. Rara, Generalized Star $\alpha-$b-Separation Axioms in Bigeneralized Topological Spaces, *applied mathematical sciences*, vol. 9 (75), pp. 3725–3737, 2015.
30. H. Maki, R. Devi, and K. Balachandran, Generalized α-Closed Sets in Topology, *bull. fukuoka univ. ed. part iii*, vol. 42, pp. 13–21, 1993.
31. ______, Associated Topologies of Generalized α-Closed Sets and α-Generalized Closed Sets, *mem. sci. kochi. univ. ser. a. math.*, vol. 15, pp. 51–63, 1994.
32. A. S. Mashhour, I. A. Hasanein, and S. N. E. Deeb, α-Continuous and α-Open Mappings, *acta. math. hungar.*, vol. 41 (3-4), pp. 213–218, 1983.
33. O. Njåstad, On Some Classes of Nearly Open Sets, *pacific j. of math.*, vol. 15 (3), pp. 961–970, 1965.
34. H. Ogata, Operations on Topological Spaces and Associated Topology, *math. japonica*, vol. 36, pp. 175–184, 1991.
35. V. Pavlović and A. S. Cvetković, On Generalized Topologies arising from Mappings, *vesnik*, vol. 38 (3), pp. 553–565, 2012.
36. K. Rajeshwari, T. D. Rayanagoudar, and S. M. Patil, On Semi Generalized ω-Closed Sets in Topological Spaces, *global journal of pure and applied mathematics*, vol. 13 (9), pp. 5491–5503, 2017.
37. I. Reilly, Generalized Closed Sets: A Survey of Recent Works, *general and geometric topology and its applications*, vol. 1248, pp. 1–11, 2002.
38. B. Roy, On a Type of Generalized Open Sets, *applied general topology*, vol. 12 (2), pp. 163–173, 2011.
39. D. Saravanakumar, N. Kalaivani, and G. S. S. Krishnan, On $\tilde{\mu}$-Open Sets in Generalized Topological Spaces, *malaya j. mat.*, vol. 3 (3), pp. 268–276, 2015.
40. P. Sundaram and M. Sheik John, On w-Closed Sets in Topology, *acta ciencia indica*, vol. 4, pp. 389–392, 2000.
41. B. K. Tyagi and Harsh V. S. Chauhan, On Generalized Closed Sets in a Generalized Topological Spaces, *cubo a mathematical journal*, vol. 18 (01), pp. 27–45, 2016.

Dr.Mohammad Irshad KHODABOCUS
Current address: Department of Mathematics, Faculty of Science, University of Mauritius
E-mail address: ikhodabo@gmail.com

Dr. Noor-Ul-Hacq SOOKIA
Current address: Department of Mathematics, Faculty of Science, University of Mauritius
E-mail address: sookian@uom.ac.mu