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Article

Umbral Methods and Harmonic Numbers

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Abstract: The theory of harmonic based function is discussed here within the framework of umbral operational methods. We derive a number of results based on elementary notions relying on the properties of Gaussian integrals.

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1. Introduction

Methods employing the concepts and the formalism of umbral calculus have been exploited in [1] to guess the existence of generating functions involving Harmonic Numbers [2]. The conjectures put forward in [1] have been proven in [3]-[4], further elaborated in subsequent papers [5] and generalized to Hyper-Harmonic Numbers in [6].

In this note we use the same point of view of [1], by discussing the possibility of exploiting the formalism developed therein in a wider context.

2. Harmonic Numbers and Generating Functions

We remind that harmonic numbers are defined as [2]

$$h_n := \sum_{r=1}^{n} \frac{1}{r}, \quad \forall n \in \mathbb{N} \tag{1}$$

It is furthermore evident that the integral representation for this family of numbers can be derived using a standard procedure, reported below.

Proposition 1. *By applying the Laplace transform, we obtain*

$$h_n = \sum_{r=1}^{n} \int_0^\infty e^{-s\,r} ds \tag{2}$$

thereby getting [7] the Euler's integral

$$h_n = \int_0^1 \frac{1-x^n}{1-x} dx \tag{3}$$

Proof.

$$
\begin{aligned}
h_n &= \sum_{r=1}^{n} \int_0^{\infty} e^{-sr} ds = \int_0^{\infty} \left[\left(\sum_{r=0}^{n} e^{-sr} \right) - 1 \right] ds = \\
&= \int_0^{\infty} \frac{1-(e^{-s})^{n+1}}{1-e^{-s}} - 1\, ds = \int_{-\infty}^{0} \frac{1-(e^{s})^{n+1}}{1-e^{s}} - 1\, ds = \\
&= \int_{-\infty}^{0} \frac{e^{(n+1)s}-e^{s}}{e^{s}-1} ds
\end{aligned}
$$

and by applying the change of variables $e^s \to x$ we obtain

$$
h_n = \int_0^1 \frac{1-x^n}{1-x} dx
$$

The definition in eq. (3) can be extended to non-integer values of n and, therefore, it can be exploited as an alternative definition holding for n (not-necessarily) a positive real.

Definition 1. *The function*

$$
\varphi_h(z) := \int_0^1 \frac{1-x^z}{1-x} dx, \quad \forall z \in \mathbb{R} \tag{4}
$$

is called harmonic number umbral vacuum, or simply the vacuum.

Definition 2. *The operator*

$$
\hat{h} := e^{\partial_z} \tag{5}
$$

is the vacuum shift operator.

Theorem 1. *The umbral operator, $\hat{h}^n$, defines the harmonic numbers, h_n, as the action of the operator (5) on the vacuum (4):*

$$
\hat{h}^n \varphi_h(z)\Big|_{z=0} = h_n \tag{6}
$$

or simply

$$
\begin{aligned}
\hat{h}^n &= h_n, \\
h_0 &= 0
\end{aligned} \tag{7}
$$

Proof.

$$
\begin{aligned}
\hat{h}^n \varphi_h(z)\Big|_{z=0} &= e^{n\partial_z} \varphi_h(z)\Big|_{z=0} = \varphi_h(z+n)|_{z=0} = \int_0^1 \frac{1-x^{z+n}}{1-x} dx \Big|_{z=0} = \\
&= \int_0^1 \frac{1-x^n}{1-x} dx = h_n
\end{aligned}
$$

Properties 1.

$$
\hat{h}^n \hat{h}^m = \hat{h}^{n+m} \tag{8}
$$

The proof follows from eq. (5). We just note that n, m are not necessarily positive integers.

Definition 3. *We call Harmonic Based Exponential Function (HBEF) the series*

$$
{}_h e(x) := e^{\hat{h}\, x} = 1 + \sum_{n=1}^{\infty} \frac{h_n}{n!} x^n \tag{9}
$$

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This function, as already discussed in [1], has quite remarkable properties.

The relevant derivatives can accordingly be expressed as (see the concluding part of the paper for further comments)

$$\begin{aligned} &\left(\frac{d}{dx}\right)^m {}_h e(x) := {}_h e(x,m) = \hat{h}^m e^{\hat{h}\,x} = h_m + \sum_{n=1}^{\infty} \frac{h_{n+m}}{n!} x^n, \\ &\left(\frac{d}{dx}\right)^m {}_h e(x,k) = {}_h e(x,k+m) \end{aligned} \tag{10}$$

and, according to eq. (9) we also find that

$$\int_0^{\infty} {}_h e(-\alpha\, x)\, e^{-x} dx = \int_0^{\infty} e^{-(\alpha\,\hat{h}+1)\,x} dx = \frac{1}{\alpha\,\hat{h}+1} \tag{11}$$

Corollary 1. *By expanding the umbral function on the r.h.s. of eq. (11), we obtain*

$$\frac{1}{\alpha\,\hat{h}+1} = 1 + \sum_{n=1}^{\infty} (-1)^n \alpha^n h_n \tag{12}$$

Proof. Using the Taylor expansion and the eq. (7), we have

$$\frac{1}{\alpha\,\hat{h}+1} = \sum_{n=0}^{\infty} (-\alpha\hat{h})^n = 1 + \sum_{n=1}^{\infty} (-1)^n \alpha^n \hat{h}^n = 1 + \sum_{n=1}^{\infty} (-1)^n \alpha^n h_n$$

which is an expected conclusion, achievable by direct integration, underscored here to stress the consistency of the procedure.

A further interesting example comes from the following "Gaussian" integral:

Example 1.

$$\int_{-\infty}^{\infty} {}_h e(-\alpha\, x)\, e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(\alpha\,\hat{h}\,x+x^2)}\, dx = \sqrt{\pi} e^{\frac{\alpha^2\,\hat{h}^2}{4}} \tag{13}$$

The last term in eq. (13) has been obtained by treating $\hat{h}$ as an ordinary algebraic quantity and then by applying the standard rules of the Gaussian integration.

We notice that, using eq. (9), we obtain

$${}_{h^2} e\left(\frac{\alpha^2}{4}\right) := e^{\frac{\hat{h}^2 \alpha^2}{4}} = 1 + \sum_{r=1}^{\infty} \frac{h_{2r}}{r!} \left(\frac{\alpha}{2}\right)^{2\,r} \tag{14}$$

Let us now consider the following slightly more elaborated example, involving the integration of two "Gaussians", namely the ordinary case and its HBEF analogous.

Example 2.

$$\int_{-\infty}^{\infty} {}_h e(-\,\alpha\, x^2)\, e^{-\,x^2} dx = \int_{-\infty}^{\infty} e^{-(\hat{h}\,\alpha+1)x^2}\, dx = \sqrt{\frac{\pi}{1+\alpha\,\hat{h}}} \tag{15}$$

This last result, obtained after applying elementary rules, can be worded as it follows: the integral in eq. (15) depends on the operator function on its r.h.s., for which we should provide a computational meaning. The use of the Newton binomial yields

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$$\sqrt{\frac{\pi}{1+\alpha\hat{h}}} = \sqrt{\pi}\sum_{r=0}^{\infty}\binom{-\frac{1}{2}}{r}\left(\alpha\,\hat{h}\right)^r = \sqrt{\pi}\left(1+\sqrt{\pi}\sum_{r=1}^{\infty}\frac{\alpha^r h_r}{\Gamma\left(\frac{1}{2}-r\right)\,r!}\right), \tag{16}$$
$$|\alpha|<1$$

The correctness of this conclusion has been confirmed by the numerical check.

It is evident that the examples we have provided show that the use of concepts borrowed from umbral theory offers a fairly powerful tool to deal with the "harmonic based" functions.

3. Harmonic Based Functions and Differential Equations

In the following we will further push the formalism to stress the associated flexibility.

We note indeed that the function

$$_{\sqrt{h}}e(x) := e^{\hat{h}^{\frac{1}{2}}\,x} = 1+\sum_{n=1}^{\infty}\frac{\left(\sqrt{\hat{h}}\,x\right)^n}{n!} = 1+\sum_{n=1}^{\infty}\frac{h_{n/2}}{n!}\,x^n \tag{17}$$

defines a *HBEF* through the following Gauss transform

$$\int_{-\infty}^{+\infty} {}_{\sqrt{h}}e(\alpha\,x)\,e^{-x^2}dx = \int_{-\infty}^{+\infty} e^{\hat{h}^{\frac{1}{2}}\,\alpha\,x-x^2}dx = \sqrt{\pi}e^{\hat{h}\,\left(\frac{\alpha}{2}\right)^2} = \sqrt{\pi}\;{}_he\left(\left(\frac{\alpha}{2}\right)^2\right) \tag{18}$$

On the other side, the function (17) can be expressed in terms of the HBEF, ${}_he(x)$, using appropriate integral transform methods [8].

Definition 4. *Let*

$$g_{\frac{1}{2}}(\eta) = \frac{1}{2\sqrt{\pi\eta^3}}e^{-\frac{1}{4\eta}} \tag{19}$$

the Levy distribution of order $\frac{1}{2}$ *then [8]*

$$e^{-p^{\frac{1}{2}}\,x} = \int_0^{\infty} e^{-p\,\eta\,x^2}\,g_{\frac{1}{2}}(\eta)\,d\eta \tag{20}$$

is the associated Levy integral transform.

The use of eq. (17) allows to write the identity

Corollary 2.

$$\begin{aligned} {}_{\sqrt{h}}e(-x) &= \int_0^{\infty} {}_he(-\eta\,x^2)\,g_{\frac{1}{2}}(\eta)\,d\eta, \\ g_{\frac{1}{2}}(\eta) &= \frac{1}{2\sqrt{\pi\eta^3}}e^{-\frac{1}{4\eta}} \end{aligned} \tag{21}$$

Proof.

$$_{\sqrt{h}}e(-x) = e^{-\hat{h}^{\frac{1}{2}}\,x} = \int_0^{\infty} e^{-\hat{h}\eta x^2}g_{\frac{1}{2}}(\eta)\,d\eta = \int_0^{\infty} {}_he(-\eta\,x^2)\,g_{\frac{1}{2}}(\eta)\,d\eta$$

The possibility of defining ${}_{\sqrt[k]{h}}e(x)$ will be discussed elsewhere.

Theorem 2. *The function* ${}_he(x)$ *satisfies the first order non homogeneous differential equation*

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$$\begin{cases} {}_he'(x) = \frac{d}{dx}{}_he(x) = {}_he(x) + \frac{e^x - x - 1}{x}, & \forall x \in \mathbb{R} \\ {}_he(0) = 1 \end{cases} \tag{22}$$

Proof. Eq. (10) for $m = 1$ yields

$$_he'(x) =_h e(x,1) = 1 + \sum_{n=1}^{\infty} \frac{h_{n+1}}{n!} x^n \tag{23}$$

Being $h_{n+1} = h_n + \frac{1}{n+1}$ we find

$$1 + \sum_{n=1}^{\infty} \frac{h_{n+1}}{n!} x^n = {}_he(x) + \frac{1}{x}(e^x - x - 1) \tag{24}$$

hence eq. (22) follows.

Corollary 3. *The solution of eq. (22) yields for the HBEF the explicit expression in terms of ordinary special functions.*

$$\begin{aligned} &{}_he(x) = 1 + e^x\left(\ln(x) + E_1(x) + \gamma\right), \\ &E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt, \\ &\left(\ln(x) + E_1(x) + \gamma\right) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n\, n!}, \\ &\gamma \equiv Euler - Mascheroni - constant \end{aligned} \tag{25}$$

The previous expression is the generating function of harmonic numbers originally derived by Gosper (see [2]).

By iterating the previous procedure we find the following general recurrence

Corollary 4.

$$_he(x,m) = {}_he(x) + \sum_{r=0}^{m-1} \left(\frac{d}{dx}\right)^r \frac{e^x - 1 - x}{x} \tag{26}$$

Definition 5. *The binomial expansion*

$$h_n(x) = (x + \hat{h})^n = x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, h_s \tag{27}$$

specifies the Harmonic Polynomials.

They are easily shown to be linked to the HBEF by means of the generating function:

Corollary 5.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = e^{x\,t}\, {}_he(t) \tag{28}$$

Proof.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (x + \hat{h})^n = e^{t(x+\hat{h})} = e^{x\,t}\, {}_he(t)$$

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69 They belong to the family of Appél polynomials and satisfy the recurrences:

Properties 2.

$$i) \ \frac{d}{dx} h_n(x) = n\, h_{n-1}(x) \tag{29}$$

$$\begin{aligned} ii) \ & h_{n+1}(x) = (x+1)\, h_n(x) + f_n(x), \\ & f_n(x) := \sum_{s=1}^{n} \frac{n!}{(n-s)!} \frac{x^{n-s}}{(s+1)!} = \int_0^1 (x+y)^n dy - x^n \end{aligned} \tag{30}$$

Proof. The derivation of eq. (29) is trivial; regarding eq. (30) we have:

$$\begin{aligned} h_{n+1}(x) &= (x+\hat{h})(x+\hat{h})^n = (x+\hat{h}) \left(x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, \hat{h}^s \right) = \\ &= x\, h_n + 1 \cdot x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, \hat{h}^{s+1} = \\ &= x\, h_n(x) + \left(x^n + \sum_{s=1}^{n} \binom{n}{s} x^{n-s}\, \hat{h}^s \right) + \sum_{s=1}^{n} \frac{n!\, x^{n-s}}{(n-s)!(s+1)!} = \\ &= (x+1)\, h_n(x) + \sum_{s=1}^{n} \frac{n!\, x^{n-s}}{(n-s)!(s+1)!} \end{aligned}$$

and

$$\begin{aligned} & \sum_{s=1}^{n} \frac{n!}{(n-s)!} \frac{x^{n-s}}{(s+1)!} = \sum_{s=1}^{n} \frac{n!}{s!(n-s)!} \frac{x^{n-s}}{s+1} y^{s+1} \Big|_{y=1} = \\ & = \sum_{s=1}^{n} \binom{n}{s} x^{n-s} \int_0^1 y^s dy = \int_0^1 \sum_{s=0}^{n} \binom{n}{s} x^{n-s} y^s - x^n dy = \\ & = \int_0^1 (x+y)^n dy - x^n \end{aligned}$$

70 **Corollary 6.** *The identity*

$$h_n(-1) = (-1)^n \left(1 - \frac{1}{n}\right) \tag{31}$$

follows from the eq. (30) *after setting* $x = -1$.
The identity

$$h_n = 1 + \sum_{s=1}^{n} \binom{n}{s} h_s(-1) \tag{32}$$

71 *is a consequence of the fact that* $\hat{h}^n = ((\hat{h}-1)+1)^n$.

72 The harmonic Hermite polynomials (touched on in ref. [1]-[3]-[9]) can also be written as

Definition 6.

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_hH_n(x) = e^{x\,t}\, {}_he(t^2), \\ & {}_hH_n(x) := H_n(x, \hat{h}) = e^{\hat{h}\partial_x^2} x^n = x^n + n! \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2\,r} \hat{h}^r}{(n-2\,r)!\, r!} \end{aligned} \tag{33}$$

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Properties 3. *The recurrences identity of the umbral Hermite polynomials*

$$\begin{aligned}
&i)\ \frac{d}{dx}{}_hH_n(x) = n\,{}_hH_{n-1}(x)\\
&ii)\ {}_hH_{n+1}(x) = \left(x + 2\hat{h}\,\frac{d}{dx}\right){}_hH_n(x) = \left(x + 2\frac{d}{dx}\right){}_hH_n(x) + 2\,\alpha_n'(x)\\
&\alpha_n(x) = n!\sum_{s=1}^{\lfloor \frac{n}{2}\rfloor}\frac{x^{n-2s}}{(s+1)!\,(n-2s)!}\\
&\alpha_n'(x) = \frac{d}{dx}\alpha_n(x)
\end{aligned} \tag{34}$$

are a by-product of the previous identities and a consequence of the monomiality principle in ref. [10].

Corollary 7. *The umbral Hermite satisfy the second order non homogeneous ODE*

$$\left(x\frac{d}{dx} + 2\left(\frac{d}{dx}\right)^2\right){}_hH_n(x) = n\,{}_hH_n(x) - 2\,\alpha_n'(x) \tag{35}$$

4. Final Comments

Before closing the paper, we want to stress the possibility of extending the present procedure to the truncated exponential numbers, namely

$$e_n := \sum_{r=0}^{n}\frac{1}{r!} \tag{36}$$

The relevant integral representation writes [11]

$$e_\alpha := \frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-s}(1+s)^\alpha ds \tag{37}$$

which holds for non-integer real values of α too. For example we find

Example 3.

$$e_{-\frac{1}{2}} = \frac{e}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},1\right) \tag{38}$$

with $\Gamma\left(1,\frac{1}{2}\right)$ *being the truncated Gamma function.*

According to the previous discussion and to eq. (38), setting $\hat{e}^\alpha \leftrightarrow e_\alpha$, we also find that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\hat{e}\,x^2}dx = \sqrt{\pi}e_{-\frac{1}{2}},\\
&e^{-\hat{e}\,x^2} = \sum_{r=0}^{\infty}(-1)^r\frac{e_r}{r!}x^{2r}
\end{aligned} \tag{39}$$

This last identity is a further proof that the implications offered by the topics treated in this paper are fairly interesting and deserve further and more detailed investigation, which will be more accurately treated elsewhere.

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