Peer-reviewed version available at Entropy 2018, 20, 610; doi:10.3390/e20080610

Article

SU(2) decomposition for the quantum information dynamics in 2d-partite two-level quantum systems

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Abstract: Quantum Computation, in the gate array version, uses logical gates adopting convenient forms for computational algorithms based on those of classical computation. There, two-level quantum systems are the basic elements connecting the binary nature of classical computation with the settlement of quantum processing. Despite, their design depends on specific quantum systems and physical interactions involved, exacerbating the dynamics analysis. Predictable and controllable manipulation should be addressed to control the quantum states, but resources are restricted to limitations imposed by the physical settlement. This work presents a formalism to decompose the quantum information dynamics in $SU(2^{2d})$ for $2d$-partite two-level systems into 2^{2d-1} $SU(2)$ quantum subsystems. Decomposition lets to set control procedures, to generate large entangled states and to design specialized dedicated quantum gates. There, easy and traditional operations proposed by quantum computation are recovered for more complex and large systems. Alternating the parameters of local and non-local interactions, the procedure states a universal exchange semantics on the generalized Bell states basis. It could be understood as a momentary splitting of the $2d$ information channels into 2^{2d-1} pairs of 2 level quantum information subsystems and a settlement of the quantum information manipulation free of the imposed restrictions by the underlying physical system.

Keywords: Quantum information; Quantum dynamics; Entanglement

1. Introduction

Quantum Information is generating new applications and tentative future technologies as Quantum Computation [1–3] and Quantum Cryptography [4,5] based on disruptive phenomena. In this arena, understanding of quantum information dynamics and control of quantum systems is a compulsory development to manage the quantum resources involved. Applications require a tight control of resources and interactions, in particular those related with coherence and entanglement, which are fundamental in the most of applications. Thus, Quantum Control has developed fine management of physical variables to prepare, maintain and transform quantum states in order to exploit them for concrete purposes. The outstanding high-tech commercial appliances as D-Wave and IBM-Q use qubits in the form of two-level systems, whatever superconducting circuits or ions as well as several approaches to their interconnecting architecture.

For multipartite systems, research in control is oriented to achieve different goals involved in quantum applications. The most of them are numerical approaches more than analytical due to the inherited complexity in the quantum information dynamics when the number of parts grows. For a single system with a two-level spectrum, the control problem has been extensively studied in terms of exact optimal control for energy or time cost [6,7]. Recently, research for the anisotropic Heisenberg-Ising model for bipartite systems in $SU(4)$ [8] has shown how this model exhibits $SU(2)$ block decomposition when it is written in the non-local basis of Bell states instead of the traditional computational basis. It means that $\mathcal{H}^2$ becomes a direct sum of two subspaces, each one generated by a pair of Bell states, while U underlies in the semi-direct product $U(1) \times SU(2)^2$. Thus, control

can be reduced to two $SU(2)$ control problems in each block and exact solutions for some control procedures can be found [9,10]. There, controlled blocks can be selected by the direction of external driven interactions introduced. That scheme lets controlled transformations between Bell states on demand and therefore on general states. Thus, the procedure sets a method of control to manipulate quantum information on magnetic systems where computational grammar is based on Bell states instead of traditional computational basis, letting an easier programmed transformation among any pair of elements in that basis. This result become an inspiration to reproduce similar decomposition schemes for larger systems in terms of simpler problems based on quasi-isolated two-level subsystems, stating easier and universal (not necessarily optimal) controlled manipulation procedures for quantum information. Technology to set up the possible architecture of these generic systems is being currently achieved through trapped-ion qubits [11] and superconducting qubits [12].

Thus, the generalization of $SU(2)$ block decomposition is a formalism to express the dynamics in a convenient basis revealing certain quantum information states algebraically free of the most of complexity introduced by the entangling operations (which does few convenient the use of computational basis based on eigenstates from single systems before the entangling interactions) but still conserving their entangled properties. It reveals how the probability exchange happens together with the structure of entanglement behind of randomness introduced by the complexity for large quantum information systems. Still, as for their predecessor, those basis maintain certain degree of universality including several alternative local and non-local interactions, which as for their $SU(4)$ predecessor, when they are combined state a series of punctual operations able to set: a) fine control based on well known $SU(2)$ control procedures, b) construction of universal gates for the entire process based on two-channel like operations, and c) design of more complex dedicated multi-channel gates by factorization.

The general aim of this paper is demonstrate that such decomposition and reduction is achievable for large qubit systems more than those in [8,10]. The second section states the general Hamiltonian to be analyzed. The third section shows how the $SU(2)$ decomposition procedure can be generalized on a general n-partite two-level systems (not only for the driven Heisenberg-Ising interactions), reducing them to 2^{n-1} selectable transformations between pairs of specific quantum states. Then, these transformations could be based on known control schemes for $SU(2)$ systems as those in [6,7]. The selection of these 2^{n-1} pairs of states has room to be settled on convenience of the quantum process being considered and the resources involved. Thus, the basis on which the decomposition can be established works as a computational grammar for the quantum procedures being attained. These basis are not completely arbitrary, thus, the fourth section shows how a kind of Bell basis generalization is able to generate the $SU(2)$ decomposition for an even number of parts, $n = 2d$. Then, the fifth section is devoted to analyze the restrictions on the Hamiltonian to get the $SU(2)$ decomposition, the inherited states and the block properties. This analysis includes a classification of interactions able to generate the $SU(2)$ decomposition. Due to the procedure being presented could reproduce complex quantum gates, generate large entangled states and introduce control procedures in $SU(2^{2d})$ when the grammar is based on the proposed generalized Bell states, the sixth section analyzes potential applications in these trends. The final section concludes summarizing the findings and settling the related future work to be developed.

2. Generalized Hamiltonian

The problem can be established for a general Hamiltonian for n coupled two-level systems on $U(2^n)$ conforming a closed system. It can be written as a general combination of tensor products of Pauli matrices for each subsystem:

$$\tilde{H} = \sum_{\{i_k\}} h_{\{i_k\}} \bigotimes_{k=1}^{n} \sigma_{i_k} = \sum_{\mathcal{I}=0}^{4^n-1} h_{\mathcal{I}_4^n} \bigotimes_{k=1}^{n} \sigma_{\mathcal{I}_{4,k}^n} \tag{1}$$

where $\{i_k\} = \{i_1, i_2, \ldots, i_n\}$, $i_k = 0,1,2,3$ and $h_{\{i_k\}}$ is a set of time dependent real functions in general. Sometimes, as in the second expression in (1), $\{i_k\}$ will be represented as the number $\mathcal{I} \in \{0,1,\ldots,4^n-1\}$ as it is expressed in base-4 with n digits, $\mathcal{I}_4^n$. Then, $\mathcal{I}_{4,k}^n = i_k$ for $k = 1,2,\ldots,n$. σ_{i_k} for $i_k = 0,1,2,3$ are the traditional Pauli matrices in the computational basis $|0\rangle, |1\rangle \in \mathcal{H}^2$ for each part k. Note that due to $SU(2)$ algebra of Pauli matrices, this Hamiltonian comprises all analytical Hamiltonians based on two-level systems with n parts. The Hamiltonian obeys the Schrödinger equation for its associated evolution operator $\tilde{U}$:

$$\tilde{H}\tilde{U} = i\hbar\frac{\partial \tilde{U}}{\partial t} \tag{2}$$

Despite $h_{\{0,\ldots,0\}}$ is not necessarily zero, if $\{\tilde{E}_j \,|j = 1,\ldots,2^n\}$ are the eigenvalues of $\tilde{H}$ and $\mathcal{E} \equiv \sum_{j=1}^{2^n}\tilde{E}_j = 2^n h_{\{0,\ldots,0\}}$, then defining:

$$H \equiv \tilde{H} - h_{\{0,\ldots,0\}}\bigotimes_{k=1}^{n}\sigma_0, \quad U \equiv \tilde{U}e^{\frac{i}{\hbar}h_{\{0,\ldots,0\}}t} \tag{3}$$

these operators become the equivalent traceless Hamiltonian and its corresponding evolution operator with eigenvalues $E_j = \tilde{E}_j - h_{\{0,\ldots,0\}}$, both fulfilling (2) too. H and $\tilde{H}$ have the same set of eigenvectors $\{|b_j\rangle \in \mathcal{H}^{2^n} | j = 1,\ldots,2^n\}$. Thus, the Hamiltonian can be written alternatively as $H = \sum_{j=1}^{2^n} E_j |b_j\rangle\langle b_j|$. Thus, in the following, the Hamiltonian could be assumed traceless without lost of generality. Note while $\tilde{U} \in U(2^n)$, then $U \in SU(2^n)$. In the further development only H and U symbols will be used as equivalent to $\tilde{H}$ and $\tilde{U}$. H can be split in two parts, the local H_l and the non-local H_nl interactions:

$$H_\mathrm{l} = \sum_{k=1}^{n}\sum_{i=1}^{3} h_{(i4^{k-1})_4^n}\bigotimes_{s=1}^{n}\sigma_{(i4^{k-1})_{4,s}^n} \rightarrow \tilde{H} = \tilde{H}_\mathrm{nl} + H_\mathrm{l} \tag{4}$$

where $(i4^{k-1})_4^n$ is the number $i4^{k-1}$ represented in base-4 with n digits and $(i4^{k-1})_{4,s}^n$ is its s^{th} term in that base.

3. SU(2) decomposition generalities

The $SU(2)$ decomposition procedure, as it was found in [8], can be induced by considering a set of 2^n orthogonal states: $\{|\alpha_i\rangle\}$ and 2^{n-1} pairs $\{j(i),k(i)\}, i = 1,2,...,2^{n-1}$ with $k(i) = j(i)+1 \in \{2,4,...,2^n\}$ related with the eigenvalues through a mixing matrix, in such way that they fulfill:

$$\begin{aligned} |b_{2i-1}\rangle &= A_i\left|\alpha_{j(i)}\right\rangle + B_i\left|\alpha_{k(i)}\right\rangle &\rightarrow \quad \left|\alpha_{j(i)}\right\rangle &= A_i^*|b_{2i-1}\rangle - B_i e^{-i\phi}|b_{2i}\rangle \\ |b_{2i}\rangle &= -B_i^* e^{i\phi}\left|\alpha_{j(i)}\right\rangle + A_i^* e^{i\phi}\left|\alpha_{k(i)}\right\rangle &\rightarrow \quad \left|\alpha_{k(i)}\right\rangle &= B_i^*|b_{2i-1}\rangle + A_i e^{-i\phi}|b_{2i}\rangle \end{aligned} \tag{5}$$

with: $|A_i|^2 + |B_i|^2 = 1$, where last relations are clearly true because orthogonality (note that energies E_j become ordered as the states are paired). States $\{|\alpha_j\rangle\}$ are then defined by the A_i, B_i selection. Each pair sets one of the orthogonal subspaces:

$$\mathcal{H}_i^2 = \mathrm{span}(\{|b_{2i-1}\rangle, |b_{2i}\rangle\}) = \mathrm{span}(\{\left|\alpha_{j(i)}\right\rangle, \left|\alpha_{k(i)}\right\rangle\}) \rightarrow \mathcal{H}^{2^n} = \bigoplus_{i=1}^{2^{n-1}}\mathcal{H}_i^2 \tag{6}$$

There are lots of possibilities for this selection, but not necessarily all practical basis fit in this construction. In particular, separability or entanglement properties are not necessarily assured for $\{|\alpha_i\rangle\}$ as in [8]. Clearly, because these states are assumed unitary, then $A_i = \left\langle b_{2i}|\alpha_{k(i)}\right\rangle e^{i\phi} = \left\langle b_{2i-1}|\alpha_{j(i)}\right\rangle^*, B_i = \left\langle b_{2i-1}|\alpha_{k(i)}\right\rangle^* = -\left\langle b_{2i}|\alpha_{j(i)}\right\rangle e^{i\phi}$. By applying H on (5) and considering that $|b_i\rangle$ has the eigenvalue E_i, it is possible arrive to the following expressions:

$$\begin{aligned} H\left|\alpha_{j(i)}\right\rangle &= (|A_i|^2E_{2i-1} + |B_i|^2E_{2i})\left|\alpha_{j(i)}\right\rangle + A_i^*B_i(E_{2i-1} - E_{2i})\left|\alpha_{k(i)}\right\rangle \\ H\left|\alpha_{k(i)}\right\rangle &= A_iB_i^*(E_{2i-1} - E_{2i})\left|\alpha_{j(i)}\right\rangle + (|A_i|^2E_{2i} + |B_i|^2E_{2i-1})\left|\alpha_{k(i)}\right\rangle \end{aligned} \tag{7}$$

which gives the Hamiltonian components in this basis:

$$\begin{aligned} \left\langle\alpha_{j(i)}|H|\alpha_{j(i)}\right\rangle &= |A_i|^2E_{2i-1} + |B_i|^2E_{2i} \\ \left\langle\alpha_{k(i)}|H|\alpha_{k(i)}\right\rangle &= |A_i|^2E_{2i} + |B_i|^2E_{2i-1} \\ \left\langle\alpha_{j(i)}|H|\alpha_{k(i)}\right\rangle &= A_iB_i^*(E_{2i-1} - E_{2i}) \end{aligned} \tag{8}$$

which can be alternatively obtained from (5). Note that phase ϕ is non-physical. This basis transformation changes the diagonal structure for the basis $\{|b_i\rangle\}$ into a 2×2 diagonal block structure for the basis $\{|\alpha_j\rangle\}$. For simplicity, we define the following quantities:

$$\begin{aligned} A_i = r_{A_i}e^{i\gamma_{A_i}}, B_i = r_{B_i}e^{i\gamma_{B_i}} \\ \Delta_i^{\pm} = \tfrac{1}{2\hbar}(E_{2i} \pm E_{2i-1}) \\ \Gamma_i = \gamma_{A_i} - \gamma_{B_i} \end{aligned} \tag{9}$$

then, each 2×2 block in H, labeled as $\mathbb{S}_{Hi}$, can be written in matrix form as:

$$\begin{aligned} \mathbb{S}_{Hi} &= \begin{pmatrix} \Delta_i^+ - ({r_{A_i}}^2 - {r_{B_i}}^2)\Delta_i^- & -2r_{A_i}r_{B_i}\Delta_i^-e^{i\Gamma_i} \\ -2r_{A_i}r_{B_i}\Delta_i^-e^{-i\Gamma_i} & \Delta_i^+ + ({r_{A_i}}^2 - {r_{B_i}}^2)\Delta_i^- \end{pmatrix} \\ &= \Delta_i^+\mathbb{I}_i - 2r_{A_i}r_{B_i}\Delta_i^-\cos\Gamma_i\mathbb{X}_i + 2r_{A_i}r_{B_i}\Delta_i^-\sin\Gamma_i\mathbb{Y}_i - ({r_{A_i}}^2 - {r_{B_i}}^2)\Delta_i^-\mathbb{Z}_i \qquad (10) \\ &\equiv \Delta_i^+\mathbb{I}_i + \mathbb{S}_{Hi}^0 \qquad (11) \end{aligned}$$

where despite ${r_{A_i}}^2 + {r_{B_i}}^2 = 1$, we use both terms r_{A_i} and r_{B_i} for the symmetry. In addition, $\mathbb{I}_i, \mathbb{X}_i, \mathbb{Y}_i$ and $\mathbb{Z}_i$ are respectively the 2×2 unitary matrix and the Pauli matrices settled for the block $\mathbb{S}_{Hi}$. Thus, H can be written as a sum of operators acting on the different subspaces $\mathcal{H}_i^2$ or as the following direct sum structure of 2^{n-1} 2×2 block-diagonal matrices:

$$H = \bigotimes_{i=1}^{2^{n-1}}\mathbb{S}_{Hi} = \left(\begin{array}{c|c|c|c} \mathbb{S}_{H1} & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{S}_{H2} & \dots & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \dots & \mathbb{S}_{H2^{n-1}} \end{array}\right) \tag{12}$$

with $\mathbf{0}$, the 2×2 zero matrix. Because this structure is preserved under matrix products, it is inherited by the evolution matrix U. In particular, if Hamiltonian (1) is not time dependent, then $U = \sum_{j=1}^{2^n} e^{-\frac{i}{\hbar}E_jt}\left|b_j\right\rangle\left\langle b_j\right|$. Thus, when the basis is changed:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 June 2018 doi:10.20944/preprints201806.0034.v1

Peer-reviewed version available at Entropy 2018, 20, 610; doi:10.3390/e20080610

$$\begin{aligned}\mathbb{S}_{Ui} &= e^{-\frac{i}{\hbar}E_{2i-1}t}\left|b_{2i-1}\right\rangle\left\langle b_{2i-1}\right| + e^{-\frac{i}{\hbar}E_{2i}t}\left|b_{2i}\right\rangle\left\langle b_{2i}\right| = \\ &= e^{-i\Delta_i^+ t}\Big((e^{i\Delta_i^- t} - 2i{r_{B_i}}^2\sin\Delta_i^- t)\left|\alpha_j(i)\right\rangle\left\langle\alpha_j(i)\right| + \\ &\quad 2ir_{A_i}r_{B_i}\sin\Delta_i^- t(e^{i\Gamma_i}\left|\alpha_j(i)\right\rangle\left\langle\alpha_k(i)\right| + e^{-i\Gamma_i}\left|\alpha_k(i)\right\rangle\left\langle\alpha_j(i)\right|) + \\ &\quad (e^{-i\Delta_i^- t} + 2i{r_{B_i}}^2\sin\Delta_i^- t)\left|\alpha_k(i)\right\rangle\left\langle\alpha_k(i)\right|\Big) \end{aligned} \tag{13}$$

similarly, in matrix form or in terms of $\mathbb{I}_i, \mathbb{X}_i, \mathbb{Y}_i$ and $\mathbb{Z}_i$:

$$\begin{aligned}\mathbb{S}_{Ui} &= e^{-i\Delta_i^+ t}\begin{pmatrix}\cos\Delta_i^- t + i({r_{A_i}}^2 - {r_{B_i}}^2)\sin\Delta_i^- t & 2ir_{A_i}r_{B_i}e^{i\Gamma_i}\sin\Delta_i^- t \\ 2ir_{A_i}r_{B_i}e^{-i\Gamma_i}\sin\Delta_i^- t & \cos\Delta_i^- t - i({r_{A_i}}^2 - {r_{B_i}}^2)\sin\Delta_i^- t\end{pmatrix} \\ &= e^{-i\Delta_i^+ t}\Big(\cos\Delta_i^- t\mathbb{I}_i + 2ir_{A_i}r_{B_i}\cos\Gamma_i\sin\Delta_i^- t\mathbb{X}_i - \\ &\quad 2ir_{A_i}r_{B_i}\sin\Gamma_i\sin\Delta_i^- t\mathbb{Y}_i + i({r_{A_i}}^2 - {r_{B_i}}^2)\sin\Delta_i^- t\mathbb{Z}_i\Big)\end{aligned} \tag{14}$$

note that the election of Γ_i lets simplify last expression to contain only one operator between $\mathbb{X}_i$ and $\mathbb{Y}_i$ (as in [8,9]). This property is useful for the optimal control in each block based on the prescriptions in [6]. Then, similarly as H:

$$U = \bigotimes_{i=1}^{2^{n-1}}\mathbb{S}_{Ui} = \left(\begin{array}{c|c|c|c} \mathbb{S}_{U1} & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathbf{0} & \mathbb{S}_{U2} & \dots & \mathbf{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \dots & \mathbb{S}_{U2^{n-1}} \end{array}\right) \tag{15}$$

where in general for the time-dependent Hamiltonian:

$$\mathbb{S}_{Ui} = \tau\{e^{-\frac{i}{\hbar}\int_0^t \mathbb{S}_{Hi}dt'}\} = e^{-i\Delta_i^+ t}\tau\{e^{-\frac{i}{\hbar}\int_0^t \mathbb{S}_{Hi}^0 dt'}\} \tag{16}$$

being τ, the time ordered integral. It implies that U is an element in the semi-direct product $U(1)^{2^{n-1}-1} \times SU(2)^{2^{n-1}} \subset SU(2^n)$ (due to any factor phase $e^{-i\Delta_i^+ t}$ depends on the remaining phase factors through $\mathcal{E}$). In the following, informally, we will call to this factorization as the $SU(2)$ decomposition (in reality, each block has the form $U(1) \times SU(2)$), due to the block structure. As a consequence, Hilbert space $\mathcal{H}^n$ becomes the direct sum of 2^{n-1} subspaces generated by each pair $\{\left|\alpha_{j(i)}\right\rangle, \left|\alpha_{k(i)}\right\rangle\}$, $i = 1, 2, \dots, 2^{n-1}$. In each subspace, the dynamics mixes the probabilities, but probabilities among subspaces remain unmixed while there are not an rearrangement in the pairing between $\{|b_i\rangle\}$ and $\{|\alpha_j\rangle\}$ (clearly, in this rearrangement the basis $\{|\alpha_j\rangle\}$ could change). Thus, if $|\psi_0\rangle = \sum_{i=1}^{2^{n-1}}|\psi_{0_i}\rangle$ is the initial state with $|\psi_{0_i}\rangle = \psi_{0_{i,j(i)}}\left|\alpha_{j(i)}\right\rangle + \psi_{0_{i,k(i)}}\left|\alpha_{k(i)}\right\rangle$, then its evolved component in the subspace $i = 1, 2, ..., 2^{n-1}$ fulfills $\| |\psi_{t_i}\rangle \| = \|\mathbb{S}_{U_i}^0 |\psi_{0_i}\rangle \| = \| |\psi_{0_i}\rangle \|$.

Finally, note that $SU(2)$ decomposition is not the only available, despite it is the most valuable for the binary inheritance from the classical computation. In fact, other decompositions involving bigger subgroups are possible whether using bigger systems than two-level ones and/or simply involving more than two eigenvectors in (5). Inclusively, a mixed sized blocks structure can be realized.

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Peer-reviewed version available at Entropy 2018, 20, 610; doi:10.3390/e20080610

4. GBS: a non-local basis fitting in $\{|\alpha_j\rangle\}$

Non-local basis are been used as theoretical resource to show explicitly how evolution [13] and measurement [14] could generate entangled states. In [8] was shown that Heisenberg-Ising model including driven magnetic fields in a fix direction lets to generate the block structure depicted there for the traditional Bell basis. Thus, the Bell basis for two-level bipartite systems has shown fit in the $U(1) \times SU(2)^2$ decomposition of $SU(4)$. Despite the added complexity to manage non-local states, recent work goes in that direction, the control of entangled states [15]. This basis works as a universal basis for the Heisenberg-Ising interaction including an external magnetic field in any specific direction on a couple of qubits [8–10]. This model includes other interactions models as XXX [16] , XY [17] and XXZ [18]. In the current development, the most obvious guess is the Generalized Bell States (GBS) basis for $n = 2d$ presented in [19] as tensor products of Bell states. In the next sections some further useful formulas are obtained to show then how GBS basis fits in $SU(2)$ decomposition for larger systems than bipartite ones.

4.1. GBS basis and Hamiltonian components

For $n = 2d$, GBS basis [19] forms an orthogonal basis of partial entangled states for $2d$ particles. Each element in such basis can be written in a brief way as:

$$\left|\Psi_{\mathcal{I}_4^d}\right\rangle = \bigotimes_{s=1}^{d} \frac{1}{\sqrt{2}} \sum_{\epsilon_s,\delta_s=0}^{1} (\tilde{\sigma}_{i_s})_{\epsilon_s,\delta_s} |\epsilon_s \delta_s\rangle \tag{17}$$

$$= \frac{1}{\sqrt{2^d}} \sum_{\{\epsilon_j\},\{\delta_k\}} (\tilde{\sigma}_{i_1} \otimes \ldots \otimes \tilde{\sigma}_{i_d})_{\epsilon_1 \ldots \epsilon_d, \delta_1 \ldots \delta_d} |\epsilon_1 \ldots \epsilon_d\rangle \otimes |\delta_1 \ldots \delta_d\rangle \tag{18}$$

$$= \frac{1}{\sqrt{2^d}} \sum_{\mathcal{E},\mathcal{D}=0}^{2^d-1} (\tilde{\sigma}_{i_1} \otimes \ldots \otimes \tilde{\sigma}_{i_d})_{\mathcal{E}_2^d,\mathcal{D}_2^d} \left|\mathcal{E}_2^d\right\rangle \otimes \left|\mathcal{D}_2^d\right\rangle \tag{19}$$

where $\{\epsilon_j\} = \{\epsilon_1, \ldots, \epsilon_d\}, \{\delta_k\} = \{\delta_1, \ldots, \delta_d\}; \epsilon_j, \delta_k = 0, 1$. At this point, $\tilde{\sigma}_i$ can be considered as proportional unitaries to the traditional Pauli matrices [19]. In addition, $\mathcal{I}_4^d$ is a brief expression of $\{i_1, i_2, \ldots, i_d\}$ as the digits set of $\mathcal{I} \in \{0, 1, \ldots, 4^d - 1\}$ when it is written in base-4 with d digits. In a similar way, $\mathcal{E}_2^d, \mathcal{D}_2^d$ are numbers written in base-2 with d digits ($\mathcal{E}, \mathcal{D} \in \{0, 1, \ldots, 2^d - 1\}$) representing $\{\epsilon_1, \ldots, \epsilon_d\}, \{\delta_1, \ldots, \delta_d\}$ respectively (be aware about the fact that digits are reversed as commonly they appear in $\mathcal{E}_2^d$ or $\mathcal{I}_4^d$ expressions). In the following, for simplicity, we use $\mathcal{I}_b^d$ and $\mathcal{I}$ indistinctly, the base b could be normally inferred from the context. Each element in this basis is not maximally entangled, instead they have maximally entangled bipartite subsystems, which are separable from the remaining system. Separable pairs contain the parts $[s, s + d]$, $s = 1, 2, ..., d$ (in the following, square brackets will be used to point out a subsystem of parts in the whole system).

In order $\{\left|\Psi_{\mathcal{I}_4^d}\right\rangle\}$ ($\mathcal{I} \in \{0, 1, \ldots, 4^d - 1\}$) reaches the kind of sets $\{|\alpha_j\rangle\}$ stated in the previous section where H and U achieve the $SU(2)$ block structure, then H should fulfill some restrictions. In the current subsections we are interested to set them. Combining expressions (1) and (17) we can express the components of H in the GBS basis. First, we note [20]:

$$\begin{aligned}\left\langle \Psi_{\mathcal{I}_4^d} | \sigma_{j_1} \otimes \ldots \otimes \sigma_{j_{2d}} | \Psi_{\mathcal{K}_4^d} \right\rangle &= \\ &= \prod_{s=1}^{d} \frac{1}{\sqrt{2}} \sum_{\epsilon_s,\delta_s=0}^{1} (\tilde{\sigma}_{i_s}^*)_{\epsilon_s,\delta_s} \frac{1}{\sqrt{2}} \sum_{\gamma_s,\phi_s=0}^{1} (\tilde{\sigma}_{k_s})_{\gamma_s,\phi_s} \left\langle \epsilon_s | \sigma_{j_s} | \gamma_s \right\rangle \left\langle \delta_s | \sigma_{j_{s+d}} | \phi_s \right\rangle \\ &= \frac{1}{2^d} \sum_{\substack{\mathcal{E},\mathcal{D} \\ \mathcal{F},\mathcal{G}}} (\tilde{\sigma}_{i_1}^* \otimes \ldots \otimes \tilde{\sigma}_{i_d}^*)_{\mathcal{E}_2^d,\mathcal{D}_2^d} (\sigma_{j_1} \otimes \ldots \otimes \sigma_{j_{2d}})_{\mathcal{E}_2^d\mathcal{D}_2^d,\mathcal{F}_2^d\mathcal{G}_2^d} (\tilde{\sigma}_{k_1} \otimes \ldots \otimes \tilde{\sigma}_{k_d})_{\mathcal{F}_2^d,\mathcal{G}_2^d} \\ &= \frac{1}{2^d} \prod_{s=1}^{d} \mathrm{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T) \end{aligned} \tag{20}$$

where combined subscripts as $\mathcal{E}_2^d\mathcal{D}_2^d$ represent the set of subscripts obtained by merging $\{\epsilon_1 \ldots \epsilon_d\}$ and $\{\delta_1 \ldots \delta_d\}$. Therefore, the final expression for the Hamiltonian components becomes [20]:

$$\left\langle \Psi_{\mathcal{I}_4^d} | H | \Psi_{\mathcal{K}_4^d} \right\rangle = \frac{1}{2^d} \sum_{\mathcal{J}=0}^{4^{2d}-1} h_{\mathcal{J}_4^{2d}} \prod_{s=1}^{d} \mathrm{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T) \tag{21}$$

where $\mathcal{J} \in \{0, 1, \ldots, 4^{2d}-1\}$ (here, $\mathcal{J} = 0$ can be removed in spite of the discussion in the first section). In the last expressions, the product $\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T$ has some properties inherited from Pauli matrices. Because $\sigma_1, \sigma_2, \sigma_3$ are traceless and $\sigma_i^T = \pm\sigma_i$ (negative sign only if $i = 2$), then $\mathrm{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T)$ is non-zero only if i_s, j_{d+s}, k_s, j_s are: a) completely different between them, or b) equal by pairs.

A remark is convenient at this point. In some works, as in [19], GBS are preferred be defined using $\tilde{\sigma}_i = \sigma_i$ for $i = 0, 1, 3$ and $\tilde{\sigma}_2 = i\sigma_2$ because it lets to have real coefficients when they are expressed in the computational basis $|0\rangle, |1\rangle$ (other alternative definitions introduce specific phase factors in $\tilde{\sigma}_i$). We will adopt last definition in the following, which does not produce changes in the previous discussion, and $\tilde{\sigma}_i{}^* = \sigma_i^T = \sigma_i$. Last expression should be fitted to (12), in particular with the non-diagonal block entries. In the following sections we will show that GBS basis naturally generates the $SU(2)$ decomposition if Hamiltonian fulfills certain restrictions. The use of GBS basis lets manage this analysis because it is based on Pauli matrices.

4.2. Case $d = 1$

For $d = 1$ there are three possibilities to arrange the pairs in the corresponding GBS basis (reduced in this case to the traditional Bell states: $\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$), a direct but large analysis shows by fitting (21) to (12), that Hamiltonian should be reduced to the forms shown in Table 1 (assuming always $h_{0_4^{2d}} = 0$ and $H_0 = \sum_{j=1}^{3} h_{jj} \sigma_j \otimes \sigma_j$). The first column shows the pairs arrangement to construct the blocks. These results generalize those found in [8,9] for the anisotropic Heisenberg-Ising model reached if the crossed interaction terms like $h_{ij}\sigma_i \otimes \sigma_j$ with $i, j = 1, 2, 3; i \neq j$ are not present. These terms are alike to the Dzyaloshinskii-Moriya model [21,22], opening additional possibilities for control in the pair exchange. It will be seen that case $d = 1$ is special in the current context due to for $d > 1$ these kind of crossed terms can be present only for a unique pair in order to keep the $SU(2)$ decomposition.

Table 1. Basis pairs and Hamiltonian required to get the $SU(2)$ block decomposition for case $d = 1$.

Basis arrangement	Hamiltonian
$\{\{\|\beta_{00}\rangle, \|\beta_{01}\rangle\}, \{\|\beta_{11}\rangle, \|\beta_{10}\rangle\}\}$	$H = H_0 + h_{01}\sigma_0 \otimes \sigma_1 + h_{10}\sigma_1 \otimes \sigma_0 + h_{23}\sigma_2 \otimes \sigma_3 + h_{32}\sigma_3 \otimes \sigma_2$
$\{\{\|\beta_{00}\rangle, \|\beta_{11}\rangle\}, \{\|\beta_{01}\rangle, \|\beta_{10}\rangle\}\}$	$H = H_0 + h_{02}\sigma_0 \otimes \sigma_2 + h_{20}\sigma_2 \otimes \sigma_0 + h_{13}\sigma_1 \otimes \sigma_3 + h_{31}\sigma_3 \otimes \sigma_1$
$\{\{\|\beta_{00}\rangle, \|\beta_{10}\rangle\}, \{\|\beta_{01}\rangle, \|\beta_{11}\rangle\}\}$	$H = H_0 + h_{03}\sigma_0 \otimes \sigma_3 + h_{30}\sigma_3 \otimes \sigma_0 + h_{12}\sigma_1 \otimes \sigma_2 + h_{21}\sigma_2 \otimes \sigma_1$

In spite of eigenvalues $\{E_j\}$ do not follow a specific order, expressions in (21) could be arranged in several orders as function of pairs selected $\{\left|\alpha_{j(i)}\right\rangle, \left|\alpha_{k(i)}\right\rangle\}$, being related with the decomposition process. In general, there are $\frac{(2^{2d})!}{(2^{2d-1})!2^{2^{2d-1}}}$ combinations for these pairs, which grow very fast with d (3 for $d=1$; 2,027,025 for $d=2$ and so on!), doing unmanageable the cases for $d>1$ under an analogous direct analysis.

4.3. Case $d>1$

Exponential growing of the problem with d does impossible an exhaustive analysis for $d>1$ based on a large algebraic equation system as in the previous case. The previous case and the results in [8,9] suggest some possible Hamiltonians for more complex cases. Thus, some of the following forms could let the $SU(2)$ decomposition for the basis (17):

$$H_0 = \sum_{j=1}^{3} H_0^{(j)}, \qquad H_0^{(j)} = h_{(j\frac{4^{2d}-1}{3})_4^{2d}} \sigma_j^{\otimes 2d} \tag{22}$$

$$H_{\mathrm{nl}_i} = \sum_{k'>k=1}^{2d} H_{\mathrm{nl}_i}^{(k,k')}, \qquad H_{\mathrm{nl}_i}^{(k,k')} = h_{(i(4^{k-1}+4^{k'-1}))_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(i(4^{k-1}+4^{k'-1}))_{4,s}^{2d}} \tag{23}$$

$$H_{\mathrm{cnl}_i} = \sum_{k'>k=1}^{2d} H_{\mathrm{cnl}_i}^{(k,k')}, \qquad H_{\mathrm{cnl}_i}^{(k,k')} = \sum_{p=0}^{1} h_{(j_p 4^{k-1}+k_p 4^{k'-1})_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(j_p 4^{k-1}+k_p 4^{k'-1})_{4,s}^{2d}} \tag{24}$$

$$H_{\mathrm{l}_i} = \sum_{k=1}^{2d} H_{\mathrm{l}_i}^{(k)}, \qquad H_{\mathrm{l}_i}^{(k)} = h_{(i4^{k-1})_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(i4^{k-1})_{4,s}^{2d}} \tag{25}$$

where, $(i4^{k-1})_4^{2d}$ is the base-4 representation with $2d$ digits of $i4^{k-1}$, a number with only one i in the position k and zero in other; $(i4^{k-1})_{4,s}^{2d}$ is its element s; and $(j\frac{4^{2d}-1}{3})_4^{2d}$ is the base-4 representation with $2d$ digits of $j\frac{4^{2d}-1}{3}$, the number with j in each digit position. Note that $i \in \{1,2,3\}$ is fix in all expressions. Physically, H_0 represents a full simultaneous interaction between all particles (as in the bipartite Heisenberg-Ising anisotropic interaction), despite this kind of interaction is non-physical for $d>1$, it is included here as reference. H_{nl_i} represents the component i of the spin interaction between pairs of particles as in the Heisenberg-Ising model. H_{cnl_i} is the crossed non-local interactions by pairs in the direction i (as those for $d=1$ in Table 1), a label used to characterize these interactions. Note there i, j_p, k_p is a permutation of $1,2,3$ with parity $p=0,1$, even and odd respectively. Finally, H_{l_i} is the component i of the local interactions with $h_{(i4^{k-1})_4^{2d}}$ as strengths (as instance, magnetic fields in the direction i for magnetic systems). These cases generalize the bipartite models presented in [8,9] and those found for $d=1$.

Some observations are useful at this point: a) $\tilde{\sigma}_i = \alpha_i \sigma_i$, $\alpha_i \in \{1, i\}$; b) $\sigma_i^T = \beta_i \sigma_i$, $\beta_i \in \{-1, 1\}$; c) $\sigma_i \sigma_j = \gamma_{i,j} \sigma_j \sigma_i$, $\gamma_{i,j} \in \{-1, 1\}$. Thus, $2c_{j_s, j_{d+s}}^{i_s, k_s} \equiv \mathrm{Tr}(\tilde{\sigma}_{i_s} \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T) = \alpha_{i_s} \alpha_{k_s} \beta_{j_s} \beta_{k_s} \gamma_{k_s j_s} \gamma_{k_s i_s} \mathrm{Tr}(\sigma_{i_s} \sigma_{k_s} \sigma_{j_{d+s}} \sigma_{j_s}) \in \{0, \pm 2, \pm 2i\}$. We do not provide extensive formulas for coefficients $\alpha_i, \beta_i, \gamma_{i,j}, c_{j_s, j_{d+s}}^{i_s, k_s}$ but they are trivially constructed departing from the Pauli matrices properties.

At this point, a convenient definition is introduced for the following cases. We will say that two particles or parts, i, j, are *correspondents* if $j = i + d$, with $i, j - d \in \{1, 2, ..., d\}$. It means simply that one is in the same position of the first group of the Hamiltonian subscripts $1, 2, ..., d$ as the other is in the second group $d+1, d+2, ..., 2d$. Then, the analysis of $\left\langle \Psi_{\mathcal{I}_4^d} | H_0 | \Psi_{\mathcal{K}_4^d} \right\rangle$, $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$, $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{cnl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ and $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ conducts to the following results.

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Peer-reviewed version available at Entropy 2018, 20, 610; doi:10.3390/e20080610

4.3.1. Analysis of $\left\langle \Psi_{\mathcal{I}_4^d} | H_0 | \Psi_{\mathcal{K}_4^d} \right\rangle$

Because $\mathcal{J} = j\frac{4^{2d}-1}{3}$ in (21), then $j_{d+s} = j_s = j \quad \forall s = 1,2,...,d$ implying $c^{i_s,k_s}_{j_s,j_{d+s}} \neq 0$ only if $i_s = k_s \quad \forall s = 1,2,\ldots,d$, thus H_0 is diagonal in the GBS basis representation and each entry will contain the three same terms $h_{(j\frac{4^{2d}-1}{3})_4^{2d}}$ for $j = 1,2,3$, but with diverse signs each one. Nevertheless the similitude of H_0 with the bipartite case ($d = 1$), for multipartite cases this interaction is non-physical but it lets to introduce and to understand the main idea in the remaining analysis.

4.3.2. Analysis of $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$

The treatment for the remaining cases is compressed in the explanation of the current case. Considering first only an isolated term $H^{(k)}_{\mathrm{l}_i}$, in this case $\mathcal{J} = i4^{k-1}$ for some $i \in \{1,2,3\}$ and $k = 1,2,\ldots,2d$ in (21), then $\mathcal{J}$ in the base-4 representation contains only one i (in the position k) while other digits are zero. Thus, there are only two meaningful possibilities for each correspondent parts: 1) $j_{d+s} = j_s = 0$ in the most cases, so $i_s = k_s$ is the only case with $c^{i_s,k_s}_{j_s,j_{d+s}} \neq 0$; or 2) one and only one position $s = k$ or $d + s = k$ in $\mathcal{J}_4^d$ has $j_k = i$ either for j_s or j_{d+s} while other is zero. For this last case, it implies only two possibilities for i_s, k_s: Case A) one of i_s, k_s is i and other is zero (both possibilities are possible); or Case B) i, i_s, k_s are different among them and from zero, thus they are a permutation i, i', i'' of $1,2,3$. In this case, there are two possibilities, $i_s = i', k_s = i''$ or $i_s = i'', k_s = i'$.

Case A is depicted in the Figure 1 for indexes $\mathcal{I}, \mathcal{K}$ being considered in $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$. There are a pair of entries whose labels for rows and columns have 0 or i in the position $s = k$: $((i_1, i_2, \ldots, i, \ldots, i_d), (i_1, i_2, \ldots, 0, \ldots, i_d))$ and $((i_1, i_2, \ldots, 0, \ldots, i_d), (i_1, i_2, \ldots, i, \ldots, i_d))$. This will be named the $0 \leftrightarrow i$ association rule.

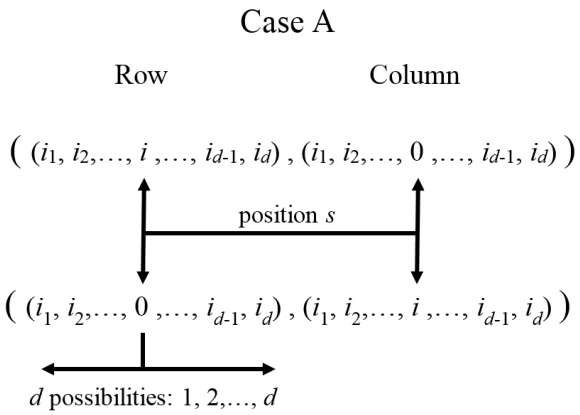

Figure 1. First case for a pair of entries in which $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ is non-zero. In them, for a fixed position $s = k$ in the row and the column labels appears i or 0 alternatively, while other corresponding positions in the row and in the column have the same values.

Case B is depicted in the Figure 2. Here, there are a pair of entries whose labels for rows and columns have i' or i'' in the position $s = k$: $((i_1, i_2, \ldots, i', \ldots, i_d), (i_1, i_2, \ldots, i'', \ldots, i_d))$ and $((i_1, i_2, \ldots, i'', \ldots, i_d), (i_1, i_2, \ldots, i', \ldots, i_d))$, being i, i', i'' a permutation of $1,2,3$. This will be named the $i' \leftrightarrow i''$ association rule.

At the end, clearly in each case, A or B, for each pair of correspondent interaction terms with i and k fix ($k \leq d$ and $k + d$ positions), there are only two pairs on non-zero entries in rows $(i_1, i_2, \ldots, i, \ldots, i_d)$, $(i_1, i_2, \ldots, 0, \ldots, i_d)$ for the case A and in rows $(i_1, i_2, \ldots, i', \ldots, i_d)$, $((i_1, i_2, \ldots, i'', \ldots, i_d))$ for the case B (with the corresponding column labels exchanged in both cases). They, together with diagonal entries generated by other adequate Hamiltonians (by example H_0 or H_{nl_i} as it will be seen) will form 2×2 blocks. In fact, each non-zero entry for H_{l_i} will have only two $h_{\mathcal{J}}$ terms corresponding with

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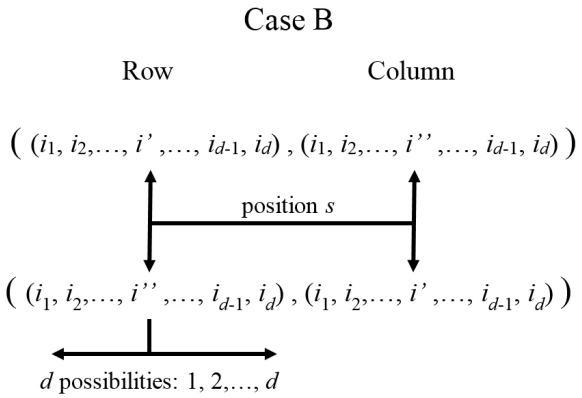

Figure 2. Second case for a pair of entries in which $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ is non-zero. In them, for a fixed position $s = k$ in the row and the column labels appears i' or i'' alternatively (being i, i', i'' a permutation of $1, 2, 3$), while other corresponding positions in the row and in the column have the same values.

$h_{0,0\ldots0,i,0,\ldots,0}$ with i in positions s or $d+s$ (meaning local interaction with each element of the pair of correspondents parts in position k). Noting that labels in the position $s = k$ in $\mathcal{I}$ (row) and $\mathcal{K}$ (column) for the non-zero entries are $0, i; i, 0; i', i''$ or i'', i', then they cover all possibilities $i_k = 0, 1, 2, 3$. Thus, for a fixed column and defined i, k values in $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$, there is exactly one non-zero row; still, if two correspondent k elements are considered (local interactions on each element of a correspondent pair), they still generate only one non-zero row (each one with the two terms explained before).

Despite there are $2d$ possibilities to select the position $s = k$ in (25), they do not account as separate blocks because they appear in other entries in (21). Instead, for each term in non-correspondent terms, it will appear in a different non-zero row, giving d non-zero rows as total; for each i direction of interaction being included, additional non-zero rows will appear; last implies that $3d$ rows could appear when all parts have local interactions in the three spatial directions at time, destroying in this case the 2×2 block structure. Thus, maintaining local interactions in only one direction and on only one correspondent pair elements utmost, together, cases A and B form $\frac{1}{2}4^d$ 2×2 blocks as it was required in the previous section. In any case, each non-zero entry will have the same 2 terms $h_{(i4^{k-1})_4^{2d}}$ with different signs depending on $c_{j_s,j_{d+s}}^{i_s,k_s}$ involved in each factor of $H_{\mathrm{l}_i}^{(k)}$. Clearly, blocks can be rearranged to order adequately the GBS basis elements getting the form (12). A brief analysis shows that there are not more diagonal-off elements in addition to last cases being generated by local terms. Additional diagonal-off elements are becoming from the non-local terms as those in Table 1.

4.3.3. Analysis of $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$

With the correspondent parts definition and the analysis for H_{l_i}, we can identify two cases for the different terms $H_{\mathrm{nl}_i}^{(k,k')}$: a) non-local interactions between correspondent parts, and b) non-local interactions between non-correspondents parts. Discussion is similar to the previous subsection.

Correspondent terms $H_{\mathrm{nl}_i}^{(k,k+d)}$. This term in the Hamiltonian H_{nl_i} contains $\sigma_0 \otimes \ldots \otimes \sigma_i \otimes \ldots \otimes \sigma_i \otimes \ldots \otimes \sigma_0$ with σ_i in positions k and $k+d$, and σ_0 in any other. When this term is allocated in $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ in agreement with (21), it does not cancel if each factor in the product become different from zero, implying $i_s = k_s \quad \forall s = 1, 2, \ldots, d$. Thus, this term gives non-zero entries only in the diagonal elements. Thus, each non-zero entry of H_{nl_i} will have d different terms in each diagonal element (one for each pair of correspondent particles interacting). Those terms will appear with different signs in each diagonal element in spite of $c_{j_s,j_{d+s}}^{i_s,k_s}$. At this point, note that results for $H_{\mathrm{nl}_i}^{(k,k+d)}$ and $H_{\mathrm{l}_i}^{(k)}$ were expected due to results in [8,23] and the separability of GBS basis in their constitutive entangled pairs.

Non-correspondent terms $H_{\mathrm{nl}_i}^{(k,k'\neq k+d)}$. These terms have a different behavior. Each term contains $\sigma_0 \otimes ... \otimes \sigma_i \otimes ... \otimes \sigma_i \otimes ... \otimes \sigma_0$ with σ_i in positions k and k', and σ_0 in any other. It defines two pairs of correspondent parts involving σ_i: $[k, k+d, k', k'+d]$ if $k < k' \leq d$ or $k, k+d, k'-d, k'$ if $k \leq d < k' \leq 2d$. Then, each factor in (21) related with those two pairs ($s = k, k'$ or $s = k, k'-d$) will include now $\mathrm{Tr}(\sigma_{i_s}\sigma_i\sigma_{k_s})$ (until unitary factors), which is non-zero only if: a) i_s or k_s are one of the pairs 0 and i or i and 0; b) i, i_s, k_s are a permutation i, i', i'' of 1, 2, 3 (having two cases depending of the parity). Last situation is similar to the local terms in the previous subsection, but in two parts simultaneously. While, remaining factors for $s \neq k, k'$ or $s \neq k, k'-d$ will require $i_s = k_s$ in order to become non-zero. Last scenario gives 16 possibilities for each term $h_{(i(4^{k-1}+4^{k'-1}))_4^{2d}}$, which will appear in diagonal-off positions obtained departing from diagonal position $(i_1, ..., i_d; i_1, ..., i_d)$ in $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$, by changing each one of indexes in the pair (i_k, i'_k) in the row, following the rules depicted in the cases A and B. Thus, for each column and i, k, k' fixed, only one row becomes non-zero in agreement with the previous rule. Each entry of this kind involves four terms including the four combinations of each two non-correspondent parts selected from the set $[k, k', k+d, k'+d]$. Instead, when all values i and k, k' are considered, a total of $3 \cdot \frac{1}{2}d(d-1)$ non-zero rows appear in each column (clearly, by considering all these terms, $SU(2)$ decomposition is not achieved).

4.3.4. Analysis of $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{cnl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$

Correspondent terms $H_{\mathrm{cnl}_i}^{(k,k+d)}$. For each term $H_{\mathrm{cnl}_i}^{(k,k+d)}$, the behavior is similar as for $H_{\mathrm{l}_i}^{(k')}$. Due to only one correspondent pair has $j_p = j_s \neq 0 \neq j_{d+s} = k_p$ in (21), then i_s, k_s for $s = k'$ should be $0, i$ or j_p, k_p. For $s \neq k'$, $i_s = k_s$. As before, it means that each term is diagonal-off by combining the values of index k' in $\mathcal{I}$ and $\mathcal{K}$ as before: $0, i; i, 0; j_p, k_p$ and k_p, j_p. For a fixed column and i, k it will give four possibilities and two $SU(2)$ blocks. Each entry will have two terms corresponding to the different parities p. Note that only one i and k' can be considered to achieve the $SU(2)$ decomposition. Otherwise, for each column, could appear $3d$ rows different from zero, breaking the $SU(2)$ decomposition as for the local interaction case.

Non-correspondent terms $H_{\mathrm{cnl}_i}^{(k,k'\neq k+d)}$. As for $H_{\mathrm{nl}_i}^{(k,k'\neq k+d)}$, in this case the only non-zero terms have $i_s = k_s$ for $s \neq k, k', k-d, k'-d$. While, for the two remaining cases $s \in \{k, k', k-d, k'-d\} \cap \{1, 2, \ldots, d\}$, each i_s, k_s should be selected from the set $0, j_p;\ j_p, 0;\ i, k_p;\ k_p, i$ or $0, k_p;\ k_p, 0;\ i, j_p; j_p, i$. In a specific column and fixing i, it will give 16 possibilities and eight blocks in $SU(2)$ as for the $H_{\mathrm{nl}_i}^{(k,k'\neq k+d)}$ case. Note that parity p should be fixed in this case because each one give a different decomposition. Each entry will contain four terms for each parity p combining the four possible interaction terms. Again, if all options for i and k, k', p are considered, then will appear $3 \cdot d(d-1)$ non-zero rows for each column, breaking the $SU(2)$ decomposition. These kind of terms are non commonly introduced in models as Heisenberg-Ising and those related, instead for magnetic systems they are the first order approximation in the spin-orbit coupling introducing antisymmetric exchange as the Dzyaloshinskii-Moriya model: $H_{DM} = \overrightarrow{D} \cdot (\overrightarrow{\sigma_1} \times \overrightarrow{\sigma_2})$. There, $\overrightarrow{D}$ is the Dzyaloshinskii-Moriya vector defining the orientation of coupling. Here, as only one term could be included in order to preserve the $SU(2)$ reduction property, this coupling should be strictly oriented.

4.4. Explicit analytical formulas for Hamiltonians components

After last analysis, it is clear that other candidates to generate $SU(2)$ decomposition are possible but they involve more than two parts at time (as the H_0 case), which are non-physical for common point-like interactions, nevertheless these terms could appear for the quantum mechanical extended objects in which (1) is a merely expansion of the interactions. Therefore, we will restrict our remaining discussion to local or pairwise interactions. In this section, analytical formulas for $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$, $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ and $\left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{cnl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle$ are provided to summarize the previous findings and due to

their utility for computer simulation optimality purposes for larger systems. In order to simplify the expressions, we introduce the definition of the following generalized Kronecker delta:

$$\delta^{\mathrm{S}}_{\mathcal{IK}} \equiv \prod_{\substack{s=1 \\ s\notin \mathrm{S}}}^{d} \delta_{i_s k_s} \tag{26}$$

where S is a set of scripts of excluded parts in the product. Thus, for H_{l_i}:

$$\begin{aligned} \left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{l}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle &= \sum_{k'=1}^{d} \delta^{\{k'\}}_{\mathcal{IK}} \mathcal{H}^{k'}_{\mathrm{l}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} \\ \text{with}: \quad \mathcal{H}^{k'}_{\mathrm{l}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} &= \sum_{t'=0}^{1} h_{(i4^{k'+dt'-1})^{2d}_4} \mathcal{F}^{i\delta_{0,t'},i\delta_{1,t'}}_{i,k'} \end{aligned} \tag{27}$$

by noting $c^{i_s,i_s}_{0,0} = 1$. In $\mathcal{H}^{k'}_{\mathrm{l}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d}$, $[k', k'+d]$ is the correspondent pair where each local interaction is being applied. There, the exchange factor generating the diagonal-off entries in the $SU(2)$ blocks is:

$$\mathcal{F}^{j,k}_{i,k'} = \delta_{i_{k'}0}\delta_{k_{k'}i} c^{0,i}_{j,k} + \delta_{i_{k'}i}\delta_{k_{k'}0} c^{i,0}_{j,k} + \sum_{i',i''=1}^{3} \epsilon^2_{ii'i''} \delta_{i_{k'}i'} \delta_{k_{k'}i''} c^{i',i''}_{j,k} \tag{28}$$

For H_{nl_i}:

$$\begin{aligned} \left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{nl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle &= \sum_{k'=1}^{d} \delta^{\{k'\}}_{\mathcal{IK}} \mathcal{H}^{c,k'}_{\mathrm{nl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} + \sum_{k''>k'=1}^{d} \delta^{\{k',k''\}}_{\mathcal{IK}} \mathcal{H}^{nc,k'k''}_{\mathrm{nl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} \\ \text{with}: \quad \mathcal{H}^{c,k'}_{\mathrm{nl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} &= h_{(i(4^{k'-1}+4^{k'+d-1}))^{2d}_4} \delta_{i_{k'}k_{k'}} c^{i_{k'},i_{k'}}_{i,i} \\ \mathcal{H}^{nc,k'k''}_{\mathrm{nl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} &= \sum_{t',t''=0}^{1} h_{(i(4^{k'+dt'-1}+4^{k''+dt''-1}))^{2d}_4} \mathcal{F}^{i\delta_{0,t'},i\delta_{1,t'}}_{i,k'} \mathcal{F}^{i\delta_{0,t''},i\delta_{1,t''}}_{i,k''} \end{aligned} \tag{29}$$

each term belongs to correspondent and non-correspondent interactions respectively. In $\mathcal{H}^{c,k'}_{\mathrm{nl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d}$ and $\mathcal{H}^{nc,k'k''}_{\mathrm{nl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d}$, $[k', k'']$ are the parts with non-local interactions between them. Similarly, for $\mathcal{H}_{\mathrm{cnl}_i}$:

$$\begin{aligned} \left\langle \Psi_{\mathcal{I}_4^d} | H_{\mathrm{cnl}_i} | \Psi_{\mathcal{K}_4^d} \right\rangle &= \sum_{k'=1}^{d} \delta^{\{k'\}}_{\mathcal{IK}} \mathcal{H}^{c,k'}_{\mathrm{cnl}_i \mathcal{I}_4^d,\mathcal{K}_4^d} + \sum_{p=0}^{1} \sum_{k''>k'=1}^{d} \delta^{\{k',k''\}}_{\mathcal{IK}} \mathcal{H}^{nc,k'k''p}_{\mathrm{cnl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} \\ \text{with}: \quad \mathcal{H}^{c,k'}_{\mathrm{cnl}_i \mathcal{I}_4^d,\mathcal{K}_4^d} &= \sum_{p=0}^{1} h_{(j_p 4^{k'-1}+k_p 4^{k'+d-1})^{2d}_4} \mathcal{F}^{j_p,k_p}_{i,k'} \\ \mathcal{H}^{nc,k'k''p}_{\mathrm{cnl}_i\ \mathcal{I}_4^d,\mathcal{K}_4^d} &= \sum_{t',t''=0}^{1} h_{(j_p 4^{k'+dt'-1}+k_p 4^{k''+dt''-1})^{2d}_4} \mathcal{F}^{j_p\delta_{0,t'},j_p\delta_{1,t'}}_{j_p,k'} \mathcal{F}^{k_p\delta_{0,t''},k_p\delta_{1,t''}}_{k_p,k''} \end{aligned} \tag{30}$$

again, $\mathcal{H}^{c,k'}_{\mathrm{cnl}_i}$ and $\mathcal{H}^{nc,k'k''p}_{\mathrm{cnl}_i}$ are the correspondent and non-correspondent interactions in the Hamiltonian, being $[k', k'']$ the parts where there are non-local interactions. It shows explicitly the existence of four (for $\mathcal{H}^{k'}_{\mathrm{l}_i}$ and $\mathcal{H}^{c,k'}_{\mathrm{cnl}_i}$) and sixteen (for $\mathcal{H}^{nc,k'k''}_{\mathrm{nl}_i}$ and $\mathcal{H}^{nc,k'k''p}_{\mathrm{cnl}_i}$) diagonal-off entries respectively in agreement with cases A and B depicted by Figures 1 and 2 (if only single specific values of $i, k', k''$are considered instead of the whole sum), generating $2 \times 4^{d-1} = \frac{1}{2} \times 4^d$ and $8 \times 4^{d-2} = \frac{1}{2} \times 4^d$ blocks respectively.

Note then, the $SU(2)$ decomposition could be achieved only by: a) including any desired non-local terms $\mathcal{H}_{\mathrm{nl}_i}^{c,k'}$ (to generate the diagonal elements); and b) including only one type of interaction among $\mathcal{H}_{\mathrm{l}_i}^{k'}$, $\mathcal{H}_{\mathrm{nl}_i}^{nc,k'k''}$, $\mathcal{H}_{\mathrm{cnl}_i}^{c,k'}$ or $\mathcal{H}_{\mathrm{cnl}_i}^{nc,k'k''p}$ for concrete values for i, k', k'' and p.

An important property used later for $\mathcal{F}_{i,k'}^{j_s,j_{d+s}}$ is that only one term utmost in (28) remains with the election of $i_{k'}$ and $k_{k'}$. Due to each $c_{j,k}^{j_s,j_{d+s}}$ is real or imaginary, and more concretely, as a brief analysis shows that if it is not zero, then it becomes imaginary only if j_s or j_{d+s} is equal to 2, therefore this property is transferred to $\mathcal{F}_{i,k'}^{j_s,j_{d+s}}$.

5. Specific interactions generating $SU(2)$ decomposition

In this section, we summarize and organize the global findings to reach the $SU(2)$ block structure on the GBS basis. Finally, we will conclude that there are three great types of interactions able to generate the block structure as it was depicted in section 3.

5.1. General depiction of interactions having $SU(2)$ decomposition for the GBS basis

Based on the previous discussion, there are three groups of interactions able to generate the $SU(2)$ decomposition on the GBS basis. The first one (Type I) comprehend all kind of non-local and non-crossed interactions between any two correspondent parts in any direction. These terms generate the diagonal terms depicted before in the Hamiltonian. Together, only two local interactions in only one specific direction and on only one pair of correspondent parts, k_l, should be included to generate the diagonal-off entries. Thus, this group of interactions conform the $SU(2)$ blocks. Note that local interaction terms could be intended as external driven fields as in [8,23]. The second interaction (Type II) is obtained by substituting the previous local interactions with non-local interactions among only those non-correspondent elements included in two pairs of correspondent parts. It means, if $k, k', k+d, k'+d$ with $k < k' \leq d$ are these elements in the two correspondent parts, then only the interactions between the following non-correspondent elements are allowed: $[k, k']$, $[k, k'+d]$, $[k', k+d]$ and $[k+d, k'+d]$. This group of four interactions generates the diagonal-off terms to conform the $SU(2)$ blocks. Nevertheless, Type II interaction normally should be understood as a non driven process of control. Note that Type II interaction could be classified in two other subclasses: a) Type IIa for non-crossed interactions $\mathcal{H}_{\mathrm{nl}_i}^{nc,k'k''}$, and b) Type IIb for crossed interactions $\mathcal{H}_{\mathrm{cnl}_i}^{nc,k'k''p}$. Finally, the third interaction (Type III) comprehend together: the non-local and non-crossed interactions with the inclusion of crossed interactions between one specific correspondent pair.

Figure 3 resumes the three types of interactions depicted before. In particular, note this description is in agreement with the results in the Table 1 for $d = 1$, despite it is an special case because diagonal-off entries for Type I, II and III coincide in the same diagonal-off entries, so both interactions could be combined at same time preserving the $SU(2)$ decomposition. This case has a richer structure for control in terms of the number of free parameters involved with respect to the number of parts to be controlled. Note that while Type I and III only are able to modify the inner entanglement of the correspondent pairs, Type II interaction lets to generate of modify the global entanglement between different correspondent pairs, thus letting spread it on the entire system by switching the pairs involving interactions generating diagonal-off entries.

5.2. General structure of $SU(2)$ blocks

In this subsection a complementary analysis of $SU(2)$ blocks obtained for the last interactions will be given. Their form is particularly useful as a connection with optimal control schemes as those presented in [6]. In any case (Type I, II or III), each block $\mathbb{S}_{H\mathcal{I},\mathcal{I}'}$ (with $\mathcal{I}, \mathcal{I}'$ the rows in which is situated) has the form:

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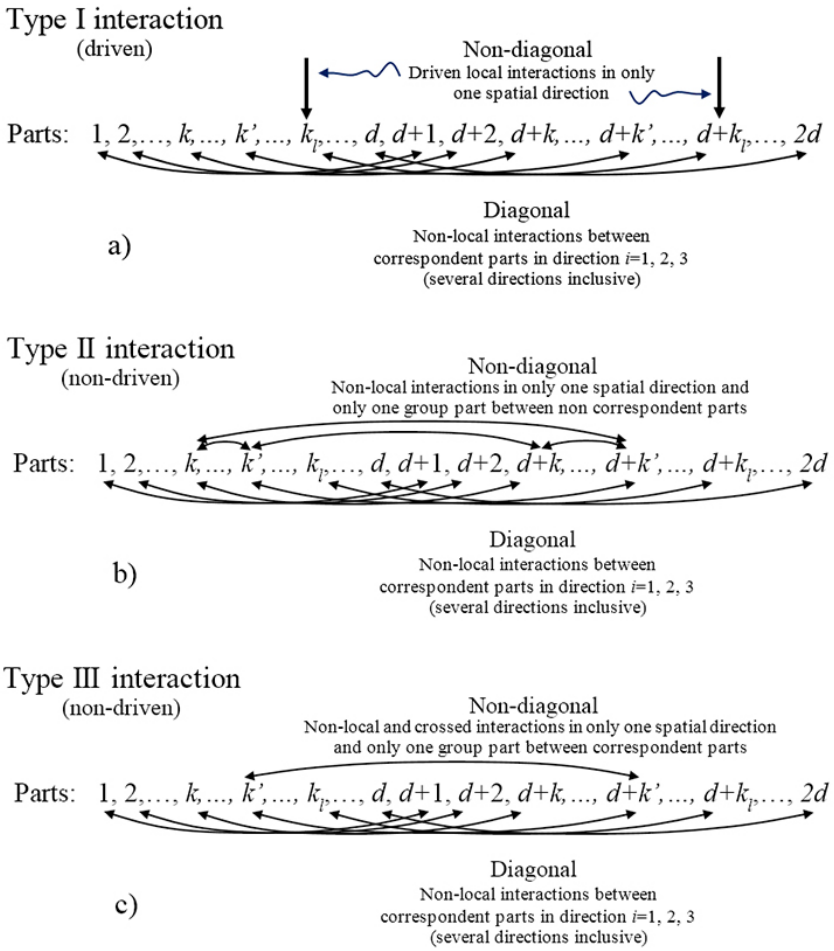

Figure 3. Three types of physical interactions able to generate the block decomposition. Non-local and non-crossed interactions among any correspondent parts combined with: a) local interactions on only two correspondent parts $(k_l, k_l + d)$, b) any two non-correspondent parts in only two specific pairs of correspondent parts of only one subtype, non-crossed or crossed, and c) crossed interactions between a specific pair of correspondent parts.

$$\begin{aligned} \mathbb{S}_{H\mathcal{I},\mathcal{I}'} &= \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^* & h_{22} \end{pmatrix} \\ &= \frac{h_{11}+h_{22}}{2}\mathbb{I}_{\mathcal{I},\mathcal{I}'} + \mathrm{Re}(h_{12})\mathbb{X}_{\mathcal{I},\mathcal{I}'} - \mathrm{Im}(h_{12})\mathbb{Y}_{\mathcal{I},\mathcal{I}'} + \frac{h_{11}-h_{22}}{2}\mathbb{Z}_{\mathcal{I},\mathcal{I}'} \end{aligned} \tag{31}$$

where $\{\mathbb{I}_{\mathcal{I},\mathcal{I}'}, \mathbb{X}_{\mathcal{I},\mathcal{I}'}, \mathbb{Y}_{\mathcal{I},\mathcal{I}'}, \mathbb{Z}_{\mathcal{I},\mathcal{I}'}\}$ is the Pauli basis for the $SU(2)$ block. If the Hamiltonian coefficients involved in the block are time independent, then the corresponding $\mathbb{S}_{U\mathcal{I},\mathcal{I}'}$ block in the evolution matrix becomes:

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$$\begin{aligned}
\mathbb{S}_{U\mathcal{I},\mathcal{I}'} &= e^{i\mathbb{S}_{H\mathcal{I},\mathcal{I}'}\frac{t}{\hbar}} = e^{i\frac{h_{11}+h_{22}}{2\hbar}t}e^{i\omega\mathbf{n}\cdot\mathbf{s}_{\mathcal{I},\mathcal{I}'}t} = e^{i\frac{h_{11}+h_{22}}{2\hbar}t}(\cos\omega t + i\sin\omega t\mathbf{n}\cdot\mathbf{s}_{\mathcal{I},\mathcal{I}'}) \\
&= e^{i\frac{h_{11}+h_{22}}{2\hbar}t}\begin{pmatrix} \cos\omega t + i\frac{h_{11}-h_{22}}{2\hbar\omega}\sin\omega t & i\frac{h_{12}}{\hbar\omega}\sin\omega t \\ i\frac{h_{12}^*}{\hbar\omega}\sin\omega t & \cos\omega t - i\frac{h_{11}-h_{22}}{2\hbar\omega}\sin\omega t \end{pmatrix} \\
\text{with :} \quad & \mathbf{n} = \frac{1}{\hbar\omega}(\mathrm{Re}(h_{12}), -\mathrm{Im}(h_{12}), \frac{h_{11}-h_{22}}{2}) \\
& \mathbf{s}_{\mathcal{I},\mathcal{I}'} = (\mathbb{X}_{\mathcal{I},\mathcal{I}'}, \mathbb{Y}_{\mathcal{I},\mathcal{I}'}, \mathbb{Z}_{\mathcal{I},\mathcal{I}'}) \\
& \hbar\omega = \sqrt{|h_{12}|^2 + \frac{1}{4}|h_{11}-h_{22}|^2}
\end{aligned} \tag{32}$$

clearly belonging to $U(1)\times SU(2)$. As was stated before, $\mathcal{F}_{j,k'}^{j_s,j_{d+s}}$ is imaginary only if j_s or j_{d+s} is 2. Thus, only one component from n_1 or n_2 is different from zero because non-diagonal entries of block in (27,29,30) are always real or imaginary. It reduces the optimal control to the second case reported by [6]. An additional analysis will show that $h_{11}\pm h_{22}\neq 0$ in general (without impose restrictions to the non-local strengths $h_{\mathcal{J}}$), this aspect will be relevant after.

5.3. Structure of $SU(2)$ blocks for each interaction

Several classical interactions fitting in the current procedure have been analyzed. All them generate blocks (no necessarily $SU(2)$ blocks) when they are expressed in the GBS basis, denoting a kind of universality for this basis due to its ability to gather up similar interactions through simplified representations. In the sake of the search of $SU(2)$ decomposition, we discuss finally closed forms for the specific Hamiltonians able to achieve the $SU(2)$ decomposition.

5.3.1. Blocks in Type I interaction

This interaction includes non-crossed spin interactions between correspondent particles in all spatial directions and external local interactions on the pair $[k', k'+d]$ of correspondent particles in direction j. From (27-29), it can be written as:

$$\begin{aligned}
H_I = \quad & H_D + H_{ND_I}^{(j,k')} \\
\text{with :} \quad & H_D \equiv \sum_{i'=1}^{3}\sum_{k=1}^{d} h_{(i'(4^{k-1}+4^{k+d-1}))_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(i'(4^{k-1}+4^{k+d-1}))_{4,s}^{2d}} \\
& H_{ND_I}^{(j,k')} = \sum_{t'=0}^{1} h_{(j4^{k'+dt'-1})_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(j4^{k'+dt'-1})_{4,s}^{2d}}
\end{aligned} \tag{33}$$

generating $SU(2)$ blocks with the diagonal terms from non-local interactions between correspondent parts and the non-diagonal terms from local interactions. Departing from (27-29), we get for the Hamiltonian components:

$$\begin{aligned}
\left\langle \Psi_{\mathcal{I}_4^d} | H_I | \Psi_{\mathcal{K}_4^d} \right\rangle = \quad & \delta_{\mathcal{I}\mathcal{K}} \sum_{i'=1}^{3}\sum_{k''=1}^{d} \left((-1)^{\delta_{i',2}+(1-\delta_{i',i_{k''}})(1-\delta_{0,i_{k''}})} h_{(i'(4^{k''-1}+4^{k''+d-1}))_4^{2d}} \right) + \\
& \sum_{t'=0}^{1} h_{(j4^{k'+dt'-1})_4^{2d}} \delta_{\mathcal{I}\mathcal{K}}^{\{k'\}} \mathcal{F}_{j,k'}^{j\delta_{0,t'},j\delta_{1,t'}} \equiv H_{D\mathcal{I}\mathcal{K}} + H_{ND_I\,\mathcal{I}\mathcal{K}}^{(j,k')}
\end{aligned} \tag{34}$$

last formula is obtained noting that $c_{i,i}^{i_{k''},i_{k''}} = (-1)^{\delta_{i,2}+(1-\delta_{i,i_{k''}})(1-\delta_{0,i_{k''}})}$. The first term of last expressions denotes the diagonal terms of interaction. This formula shows that in general the pair of entries in

the diagonal of each $SU(2)$ block are different. Due to the block is formed by switching an index $i_{k''}$ in the rows labels (or two as in the following cases) in agreement with the association $0 \leftrightarrow j$ or $i \leftrightarrow k$ (j is the direction associated to the interaction and i, j, k a permutation of $1, 2, 3$), then for $i' \neq j$ the terms in $H_{D\mathcal{I}\mathcal{K}}$ have a sign change. It implies that in general $h_{11} \neq h_{22}$ in (31), generating non-diagonal $\mathbb{S}_{H\mathcal{I},\mathcal{I}'}$-blocks. The second term contains the four diagonal-off elements generating two blocks with two terms each one. Note that Hamiltonian terms ($h_{\mathcal{I}}$) are real together with $c_{i,i}^{i_{k''},i_{k''}}$, so diagonal terms are real as is expected. Diagonal-off terms will be real or imaginary depending from $\mathcal{F}_{j,k'}^{j,0}, \mathcal{F}_{j,k'}^{0,j}$. In any case, concretely, they are imaginary only if $j = 2$.

Clearly, note that this interaction applied on a combination of correspondent pairs with bipartite entangled states generate only non-local operations on each correspondent pair as those presented in [8,10]. Still switching the direction j and the correspondent pair k' on which the local interaction is applied, this kind of Hamiltonian cannot generate extended entanglement between correspondent pairs more than the included in the initial state. It means that if the initial state is separable by correspondent pairs, it will remain separable at this level (but should be able to entangle or untangle the parts of each pair). Conversely, it cannot untangling each correspondent pair from the remaining state in more complex cases. We deserve a later section to analyze these topics.

5.3.2. Blocks in Type II interaction

Type IIa : In this case, interaction is completely non-local between correspondent pairs to generate the diagonal entries and in only one direction between non-correspondent parts in two correspondent pairs to generate the diagonal-off entries. Hamiltonian becomes:

$$\begin{aligned} H_{IIa} = &\quad H_D + H_{ND_{IIa}}^{(j,k'k'')} \\ \text{with :} &\quad H_{ND_{IIa}}^{(j,k'k'')} = \sum_{t',t''=0}^{1} h_{(j(4^{k'+dt'-1}+4^{k''+dt''-1}))_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(j(4^{k'+dt'-1}+4^{k''+dt''-1}))_{4,s}^{2d}} \end{aligned} \tag{35}$$

with a non-local and non-crossed interaction in the direction j for the group of non-correspondent terms defined by $k' < k'' \leq d$. The Hamiltonian entries are similar to those in (27-29) but with the last restriction for the non-correspondent terms of interaction. Due to discussion in the previous subsection, diagonal-off entries in the Hamiltonian are now always real. Components become:

$$\begin{aligned} \left\langle \Psi_{\mathcal{I}_4^d} | H_{IIa} | \Psi_{\mathcal{K}_4^d} \right\rangle &= H_{D\mathcal{I}\mathcal{K}} + H_{ND_{IIa}\ \mathcal{I}\mathcal{K}}^{(j,k'k'')} \\ H_{ND_{IIa}\ \mathcal{I}\mathcal{K}}^{(j,k'k'')} &\equiv \sum_{t',t''=0}^{1} h_{(j(4^{k'+dt'-1}+4^{k''+dt''-1}))_4^{2d}} \delta_{\mathcal{I}\mathcal{K}}^{\{k',k''\}} \mathcal{F}_{j,k'}^{j\delta_{0,t'},j\delta_{1,t'}} \mathcal{F}_{j,k''}^{j\delta_{0,t''},j\delta_{1,t''}} \end{aligned} \tag{36}$$

Type IIb : For this interaction, the non-diagonal part generated by the non-local interaction between non-correspondent parts is supplied by a non-local and crossed interaction among non-correspondent parts of two pairs of correspondent pairs:

$$\begin{aligned} H_{IIb} = &\quad H_D + H_{ND_{IIb}}^{(i,k'k''p)} \\ \text{with :} &\quad H_{ND_{IIb}}^{(i,k'k''p)} \equiv \sum_{t',t''=0}^{1} h_{(j_p 4^{k'+dt'-1}+k_p 4^{k''+dt''-1})_4^{2d}} \bigotimes_{s=1}^{2d} \sigma_{(j_p 4^{k'+dt'-1}+k_p 4^{k''+dt''-1})_{4,s}^{2d}} \end{aligned} \tag{37}$$

as before, i, j_p, k_p is a permutation of $1, 2, 3$ with parity $p = 0, 1$ (even and odd respectively). Thus, the components become:

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$$\begin{aligned}\left\langle \Psi_{\mathcal{I}_4^d}|H_{IIb}|\Psi_{\mathcal{K}_4^d}\right\rangle &= H_{D\mathcal{I}\mathcal{K}}+H^{(i,k'k''p)}_{ND_{IIb}\ \mathcal{I}\mathcal{K}} \\ H^{(i,k'k''p)}_{ND_{IIb}\ \mathcal{I}\mathcal{K}} &= \sum_{t',t''=0}^{1} h_{(j_p4^{k'+dt'-1}+k_p4^{k''+dt''-1})_4^{2d}}\delta^{\{k',k''\}}_{\mathcal{I}\mathcal{K}}\mathcal{F}^{j_p\delta_{0,t'},j_p\delta_{1,t'}}_{j_p,k'}\mathcal{F}^{k_p\delta_{0,t''},k_p\delta_{1,t''}}_{k_p,k''}\end{aligned} \tag{38}$$

the non-diagonal entries are now imaginary, except for $i=2$.

5.3.3. Blocks in Type III interaction

Finally, for Type III interaction, the non-diagonal part is generated by the non-local and crossed interaction between a pair of correspondent parts k':

$$\begin{aligned}H_{III} = &\ H_D+H^{(i,k')}_{ND_{III}} \\ \text{with}: &\ H^{(i,k')}_{ND_{III}}=\sum_{p=0}^{1} h_{(j_p4^{k'-1}+k_p4^{k'+d-1})_4^{2d}}\bigotimes_{s=1}^{2d}\sigma_{(j_p4^{k'-1}+k_p4^{k'+d-1})^{2d}_{4,s}}\end{aligned} \tag{39}$$

with the Hamiltonian components:

$$\begin{aligned}\left\langle \Psi_{\mathcal{I}_4^d}|H_{III}|\Psi_{\mathcal{K}_4^d}\right\rangle &= H_{D\mathcal{I}\mathcal{K}}+H^{(i,k')}_{ND_{III}\mathcal{I}\mathcal{K}} \\ H^{(i,k')}_{ND_{III}\mathcal{I}\mathcal{K}} &= \sum_{p=0}^{1} h_{(j_p4^{k'-1}+k_p4^{k'+d-1})_4^{2d}}\delta^{\{k'\}}_{\mathcal{I}\mathcal{K}}\mathcal{F}^{j_p,k_p}_{i,k'}\end{aligned} \tag{40}$$

where non-diagonal entries are imaginary only if $i=2$.

Figure 4 depicts the three types of interactions associated with the $SU(2)$ decomposition. Rays represent non-local interactions (blue/dark between correspondent particles and yellow/bright between non-correspondent ones), while the yellow wedges mark particles subject to external local driving interactions. Distributed evolution on 2^{2d-1} Bloch spheres is shown for the states $|\psi_j\rangle=\alpha_{2j-2}|\Psi_{2j-2}\rangle+\alpha_{2j-1}|\Psi_{2j-1}\rangle$ which are part of the global state $|\psi\rangle=\sum_{j=1}^{2^{2d-1}}|\psi_j\rangle$, where each $|\Psi_k\rangle$ is an element of the GBS basis. Each state $|\psi_j\rangle$ evolves as a different curve on each Bloch sphere depending on parameters $h_{\mathcal{J}}$.

Finally, we should note that each one of the last interactions involves labels to be completely identified, namely: $H_I^{(j,k')}$, $H_{IIa}^{(j,k',k'')}$, $H_{IIb}^{(j,k',k'',p)}$ and $H_{III}^{(j,k',k'')}$. These labels will be omitted by simplicity at least their specification becomes needed. In any case, closed expressions (34, 36, 38, 40) are computationally efficient to generate matrix representations of Hamiltonians $H_I, H_{IIa,b}, H_{III}$, then for their respective U, inclusively in the time dependent case, despite a numerical approach to construct could be necessary.

5.4. Available parameters and structure of entries

The number of free parameters (coefficients $h_{\mathcal{I}}$ of Hamiltonian) and its availability are important to set control procedures. In this section, we count the entries and terms for each Hamiltonian, summarizing the previous findings. If $D\leq 3$ is the number of spatial dimensions involved in each interaction, then, the accounting of free parameters generating the $SU(2)$ decomposition, together with the maximum number of entries by column able to generate it (breaking the $SU(2)$ decomposition) is reported in the Table 2. Note the number of entries by column for all Hamiltonians -labeled with i, in some sense the direction of the interaction-, could be increased by a factor D if all directions are considered at time. In the table, each Hamiltonian analyzed is reported, arriving to the main Hamiltonians $H_I, H_{IIa,b}$ and H_{III}. Accounting shows few free parameters at time (compared with

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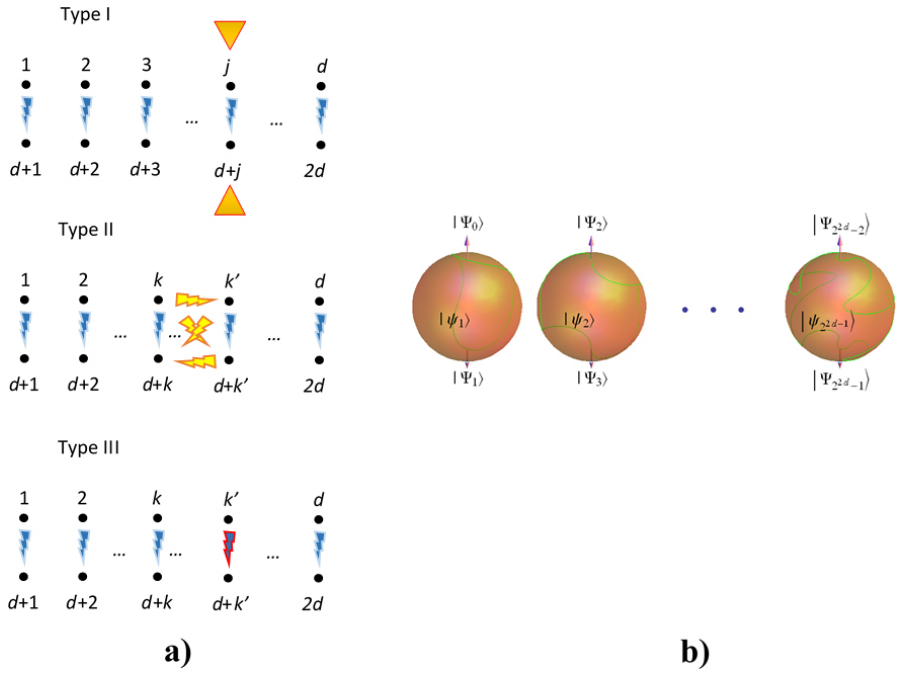

Figure 4. Representation of qubit interactions able to generate $SU(2)$ decomposition: a) Type I, II and III interactions among $2d$ qubits (Type III assumes the inclusion of crossed interactions in the pair k'), and b) Distributed evolution on 2^{2d-1} Bloch spheres, each one for the states $\left|\psi_j\right\rangle$.

the exponential grow of the matrix with the system size d) to set a whole control (over all blocks) in one period of constant driven parameters, suggesting the use of time dependent or at least a constant-piecewise parameters to increase the control.

Table 2. Rows generated and free parameters in each interaction considered in the text.

Hamiltonian	Entries type	Entries by column/row	Parameters by entry
H_0	Diagonal	1	$D \leq 3$
H_{l_i}	Non-diagonal	d	2
$H^{c}_{\mathrm{nl}_i}$	Diagonal	1	d
$H^{nc}_{\mathrm{nl}_i}$	Non-diagonal	$\frac{1}{2}d(d-1)$	4
$H^{c}_{\mathrm{cnl}_i}$	Non-diagonal	d	2
$H^{nc}_{\mathrm{cnl}_i}$	Non-diagonal	$d(d-1)$	4
H_I	2×2 block	2	$2+Dd \leq 2+3d$
$H_{IIa,b}$	2×2 block	2	$4+Dd \leq 4+3d$
H_{III}	2×2 block	2	$2+Dd \leq 2+3d$

5.4.1. Structure of diagonal entries belonging to a specific block

Other aspects should be remarked. The first one is related with terms in the diagonal entries generated by non-local interactions $\mathcal{H}^{c,s}_{\mathrm{nl}_j}$ among correspondent parts. Note that blocks are generated by other interactions to those, which are prescribed as a difference in one ($\mathcal{H}^{c,k'}_{\mathrm{nl}_i}$ or $\mathcal{H}^{c,k'}_{\mathrm{cnl}_i}$) or two ($\mathcal{H}^{nc,k'k''}_{\mathrm{nl}_i}$ or $\mathcal{H}^{nc,k'k''p}_{\mathrm{cnl}_i}$) terms in the scripts labels in agreement with the rules depicted in the Figures 1 and 2. It implies there will be two or eight blocks each one relating rows (and columns) differing in only one or two terms of their scripts respectively. We can note for the diagonal entries for $\mathcal{H}^{c,s}_{\mathrm{nl}_j}$ in (21) for the GBS defined as in [19]: $\mathrm{Tr}(\tilde{\sigma}^{*}_{i_s}\sigma_j\tilde{\sigma}^{T}_{i_s}\sigma^{T}_{j}) = 2(-1)^{\delta_{j,2}+(1-\delta_{j,i_s})(1-\delta_{0,i_s})}$. Then, for each three strengths for a

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fix correspondent pair, there will be only four signs combinations (none is the negative of another) depending on: a) the direction of the interaction involved (on the correspondent pair s) is $j = 2$ or $j \neq 2$, and b) i_s for the s^{th} script is in the set $\{0, j\}$ or in $\{i, k\}$ (with i, j, k a permutation of $1, 2, 3$). There, the factors corresponding to other correspondent pairs will be equal to one. Then, for the $3d$ terms included in all diagonal entries will be 4^d combinations for the whole terms, precisely the number of rows! It implies that all diagonal entries are different (but not linearly independent because there are only $3d$ parameters). Despite, for two rows differing in only one or two terms in their scripts, only the three or six terms corresponding with the strengths of $\mathcal{H}_{\text{nl}_j}^{c,s}$ for such correspondent pairs (associated with those terms in the scripts) will change their signs in the diagonal terms in their block. As a consequence, for such 4^{d-1} or 4^{d-2} groups of blocks having the same scripts exchange and generated by the whole combinations in the other $d-1$ or $d-2$ terms in their scripts, they will have the same $h_{11} - h_{22}$ parameters respectively. Thus, it will be only two or eight different $h_{11} - h_{22}$ parameters for the entire H. While, $h_{11} + h_{22}$ parameters could be different.

5.4.2. Structure of diagonal-off entries belonging to a specific block

The second aspect is related with the explicit calculation of $c_{j_s,j_{d+s}}^{i_s,k_s}$ for the basic interest cases in the diagonal-off entries. a) For H_I and $H_{IIa,b}$: $j_s = j, j_{d+s} = 0$ or $j_s = 0, j_{d+s} = j$ (being j the direction label involved in the local and non-local interactions between non-correspondent parts; b) and b) for H_{III}: $j_s = j_p, j_{d+s} = k_p$. Table 3 shows explicitly these values. Note the parallelism between the two halves of them (vertically and horizontally).

These cases generate the diagonal-off entries in each block in agreement with the exchange rules depicted before for the s^{th} scripts of such entries' rows: $(i_s, k_s) \in \{(0, j), (j, 0); (i, k), (k, i)\}$, with i, j, k a permutation from $1, 2, 3$ and j the associated direction for the corresponding interaction being used from H_I and $H_{II_{a,b}}$; and, $(i_s, k_s) \in \{(j_p, k_p), (k_p, j_p); (0, i), (i, 0)\}$, with i, j_p, k_p a permutation of parity p from $1, 2, 3$ and j_p, k_p are the associated directions for the interaction H_{III}.

We should note first that signs for each term in the diagonal-off entries do not depend on the entries' scripts in the other positions than the parts in which interaction is being applied, k', k'' in the expressions of the previous section (33,35,37,39). It is because $\text{Tr}(\tilde{\sigma}_{i_s}^* \sigma_{j_{d+s}} \tilde{\sigma}_{k_s}^T \sigma_{j_s}^T) = \text{Tr}(\tilde{\sigma}_{i_s}^* \sigma_0 \tilde{\sigma}_{k_s}^T \sigma_0^T) = 2$. Instead, signs only depend on the type of exchange indexes shown in the Table 3. Other fact already noted is that $c_{j_s,j_{d+s}}^{i_s,k_s}$ is imaginary only if $j_s = 2$ or $j_{d+s} = 2$. This property is then transferred to the corresponding $\mathcal{F}_{j,s}^{j_s,j_{d+s}}$ and then transformed to h_{12} as a function of the number of those factors in (34,36,38,40). Thus, by exchanging i_s, k_s (block transposing), only the cases with $h_{12} \in \mathbb{I}$ will change their sign.

The final fact is related with the different signs appearing in the terms of diagonal-off entries. This aspect will be important to analyze the number of independent blocks in the entire evolution matrix. For H_I and H_{III} the two different terms are obtained by the exchange of j_s, j_{d+s}. Thus, for H_I, only the $c_{j_s,j_{d+s}}^{i_s,k_s}$ with $(j_s, j_{s+d}) = (0, 2), (2, 0)$ and $(i_s, k_s) \in \{(0, 2), (2, 0)\}$ or $(j_s, j_{s+d}) = (0, j \neq 2), (j \neq 2, 0)$ and $(i_s, k_s) \in \{(i, k), (k, i)\}$ will change their sign (in the first four rows of Table 3). For H_{III}, if $(j_s, j_{s+d}) = (1, 3), (3, 1)$ and $(i_s, k_s) \in \{(0, 2), (2, 0)\}$ or $(j_s, j_{s+d}) = (0, 2), (2, 0)$ and $(i_s, k_s) \in \{(i, k), (k, i)\}$, then $c_{j_s,j_{d+s}}^{i_s,k_s}$ will change their sign (in the last four rows of Table 3). For $H_{IIa,b}$, two terms in the scripts are involved, so different aspects contribute: location of parts interacting, type of exchange and their order in the scripts.

This aspect exhibits the way in which each term in h_{12} will change or will not their sign. These aspects let to understand the diagonal-off structure of $H_I, H_{IIa,b}$ and H_{III} (considering that their diagonal components follow the properties discussed before). In the next subsections we analyze this structure for each interaction, particularly discussing the independence of blocks in terms of the free parameters, stating a distinction between the effective parameters (those appearing in the final expression of (31)) and the physical parameters (those appearing as coefficients $h_{\mathcal{I}}$ in the Hamiltonian). They are not the same because many physical parameters appear clustered in the same way in (31),

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due to the entries of $\mathbb{S}_U$ depend only on the parameters $h_{11} \pm h_{22}, h_{12}$. As a result, as we will seen, by grouping finally in the $U(1) \times SU(2)$ blocks, there will only two or eight different blocks $\mathbb{S}_U$ in U.

Table 3. Values of $c^{i_s,k_s}_{j_s,j_{d+s}}$ for all exchange scripts in $H_I, H_{IIa,b}, H_{III}$. i, j, k is an even permutation of $1, 2, 3$

(j_s, j_{s+d})	(i_s, k_s)	$c^{i_s,k_s}_{j_s,j_{d+s}}$	(i_s, k_s)	$c^{i_s,k_s}_{j_s,j_{d+s}}$	(i_s, k_s)	$c^{i_s,k_s}_{j_s,j_{d+s}}$	(i_s, k_s)	$c^{i_s,k_s}_{j_s,j_{d+s}}$
$(0,2)$	$(0,2)$	$-i$	$(2,0)$	i	$(1,3)$	i	$(3,1)$	$-i$
$(2,0)$	$(0,2)$	i	$(2,0)$	$-i$	$(1,3)$	i	$(3,1)$	$-i$
$(0, j \neq 2)$	$(0,j)$	1	$(j,0)$	1	(i,k)	$-(-1)^{\delta_{2k}}$	(k,i)	$-(-1)^{\delta_{2k}}$
$(j \neq 2, 0)$	$(0,j)$	1	$(j,0)$	1	(i,k)	$(-1)^{\delta_{2k}}$	(k,i)	$(-1)^{\delta_{2k}}$
$2 \in (j,k)$	(j,k)	$-i$	(k,j)	i	$(0,i)$	$-i(-1)^{\delta_{2k}}$	$(i,0)$	$i(-1)^{\delta_{2k}}$
$2 \in (k,j)$	(j,k)	i	(k,j)	$-i$	$(0,i)$	$-i(-1)^{\delta_{2k}}$	$(i,0)$	$i(-1)^{\delta_{2k}}$
$(1,3)$	$(1,3)$	1	$(3,1)$	1	$(0,2)$	-1	$(2,0)$	-1
$(3,1)$	$(1,3)$	1	$(3,1)$	1	$(0,2)$	1	$(2,0)$	1

5.4.3. Block entries of H_I

The diagonal-off entries have exactly the two terms $h_{(j4^{s+dt-1})^{2d}_4}$ for $t' = 0, 1$ and there are only two combinations: adding or subtracting terms. As was stated previously, they are imaginary only if local interactions are in direction $j = 2$. If in such case we separate the factor $\pm i$ for $j = 2$ cases in the diagonal-off entries, the remaining coefficients in the opposite corners in each block are equal as is expected in (31). Then, in general, there is one term with the same sign through all diagonal-off entries (depending if $k = 2$ or not, or otherwise if $j_s = 2$ in the first four rows in the Table 3), leaving only two possibilities for the remaining term. Thus, in each H_I matrix there are blocks with only two different diagonal-off entries, only depending on the indexes exchange type in the local interaction position but not on the remaining indexes. Thus, for a fixed set of indexes for the positions do not related with the part on which the interaction is applied, a pair of blocks exists, each one for the exchanges $(0,j), (j,0)$ and $(i,k), (k,i)$, with different relative signs in their diagonal-off terms. For the corresponding diagonal entries, in (32), only the difference $h_{11} - h_{22}$ is relevant. As was previously stated by analyzing the formula (34), it is also possible realize that in each diagonal entry there are only two terms from the $3d$ terms changing their sign with respect to other rows. Block scripts differ in only one index, those corresponding with $i' \neq j$ (the local interaction direction) and $k = k'$ (the correspondent pair on which local interaction is being applied), leaving only two terms and two different combinations for $h_{11} - h_{22}$. It implies there are only two different blocks for (32) through all U, operating each one with different exchange rules, $(0,j), (j,0)$ or $(i,k), (k,i)$. Each one is the same (until unitary factors which could be different) for all entries with different indexes in other positions than k'. This fact appoints, depending on the disposable number of parameters (five, including the time and excluding the parameters in the unitary factor of each block), to the independence between the two types of blocks in the evolution matrix (32).

5.4.4. Block entries of H_{IIa}

In this case, for the non-diagonal entries, due to the exchange factor $\mathcal{F}^{j_s,j_{d+s}}_{j,s}$ appears two times for each j, all them are real, so opposite corners of each block are always equal. Each entry has four terms alternating signs in agreement with the Table 3 as a function of the rows' subscripts involved. Signs only depend on both indexes exchanged, either they are the same type $(0,j), (j,0)$ or $(i,k), (k,i)$, or otherwise opposite with an exchange of each type. It will give only four sign combinations (a calculation does not developed explicitly here), except for $j = 2$ where the appearance of two factors i will change the overall factor, giving eight combinations, one half of them with opposite overall sign to the remaining. While, for the diagonal entries, based on the ideas in the previous case for H_I, there will four terms changing their relative signs with respect to other associated diagonal entries in the same block, but now differing in two part indexes (due to the related non-local interaction). As before,

one term has a sign fixed so there are only eight combinations for the three remaining terms from the 16 possible, it means eight different combinations for $h_{11} - h_{22}$ in (32), due to the values $k = k', k''$ for the non-correspondent parts with non-local interactions in H_D for this case. Thus, similarly for H_I, there will be in this case eight different blocks in U for (32), one for each one of the eight different combinations for the exchange rules. Each type applies in the same way for all entries with different indexes in other positions than k', k''. There are nine free parameters including the time and excluding the parameters in the unitary factor for the block, so independence among the eight types of blocks could be more elusive. Despite, located operations do not involving all GBS basis states appear as achievable.

5.4.5. Block entries of H_{IIb}

Despite the exchange factors $\mathcal{F}_{j,s}^{j_s,j_{d+s}}$ are crossed and j takes two different values in the subscripts, the discussion regards certain similitude to that for H_{IIa}. For the diagonal-off entries, in agreement with the Table 3, it implies now that only if $j = 2$ is not included in the crossed interaction ($i = 2$ in (38)), they will become real. Each entry will have four terms with alternating signs in agreement with the outcomes of products of exchange factors in the Table 3 as function of the rows' subscript involved. Here, there will be eight combinations (four and four with opposite overall signs) except for $j = 2$ with only four combinations. For the diagonal entries, $h_{11} - h_{22}$ in (32), the situation is identical to H_{IIa}. Then, there will be eight different block types in U for each combination of exchange rules on the indexes k', k''. Again, nine free parameters for the $SU(2)$ blocks are available.

5.4.6. Block entries of H_{III}

This is a special case excluded of the previous remarks due to j_s, j_{d+s} is not of the forms $0, j$ or $j, 0$. Nevertheless, $\mathcal{F}_{i,s}^{j_p,k_p}$ becomes in the same way on of $c_{j_p,k_p}^{0,i}$, $c_{j_p,k_p}^{i,0}$, $c_{j_p,k_p}^{j_p,k_p}$ or $c_{j_p,k_p}^{k_p,j_p}$. Despite, several aspects are identical to the H_I case. A brief analysis shows that entries become real only for $i = 2$ (see the last four rows of Table 3). Each diagonal-off entry has two terms with alternating signs as function of entries labels. For the diagonal entries, again only two types of terms change their sign in H_D from (34) for the rows forming the $SU(2)$ blocks with the exchange rules. It gives only two types of $h_{11} - h_{22}$ in (32), generating again only two different blocks in the whole U, each one for a kind of exchange rule involved here, and containing five free parameters.

To resume the findings, Figure 5 shows the relations exhibited in the exchange indexes for each interaction. This figure depicts each one of the exchanges relating pairs of GBS basis states under the interaction. Thus, Figures 5a and 5d, depict the two groups of exchange states for H_I and H_{III} generated by the two different blocks through the whole $SU(2^{2d})$ evolution matrix, both independent until five parameters and with h_{12} in (32). Figures 5b and 5c depict the double exchange indexes induced by the eight blocks generated by H_{IIa} and H_{IIb}. These eight blocks are independent until nine parameters. All representations in Figure 5 are for a single GBS basis state, but clearly one specific block is operating on any of them simultaneously. Note finally that there are for all cases a complementary number of free physical parameters in $h_{11} + h_{22}$: $3d + 2 - 4 = 3d - 2$ for H_I and H_{III} and $3d + 4 - 8 = 3d - 4$ for H_{IIa} and H_{IIb} (time t is not accounted due it was considered in the $SU(2)$ fitting). Then, there is a linear growing room to fit the blocks into a programmed operation in terms of the physical parameters, despite there is an exponential growing of those blocks.

6. Connectedness, Superposition, Entanglement and Separability

To understand how the dynamics is addressed under the interactions $H_I, H_{IIa,b}, H_{III}$ (used independently or in a combined way), some complementary analysis is convenient.

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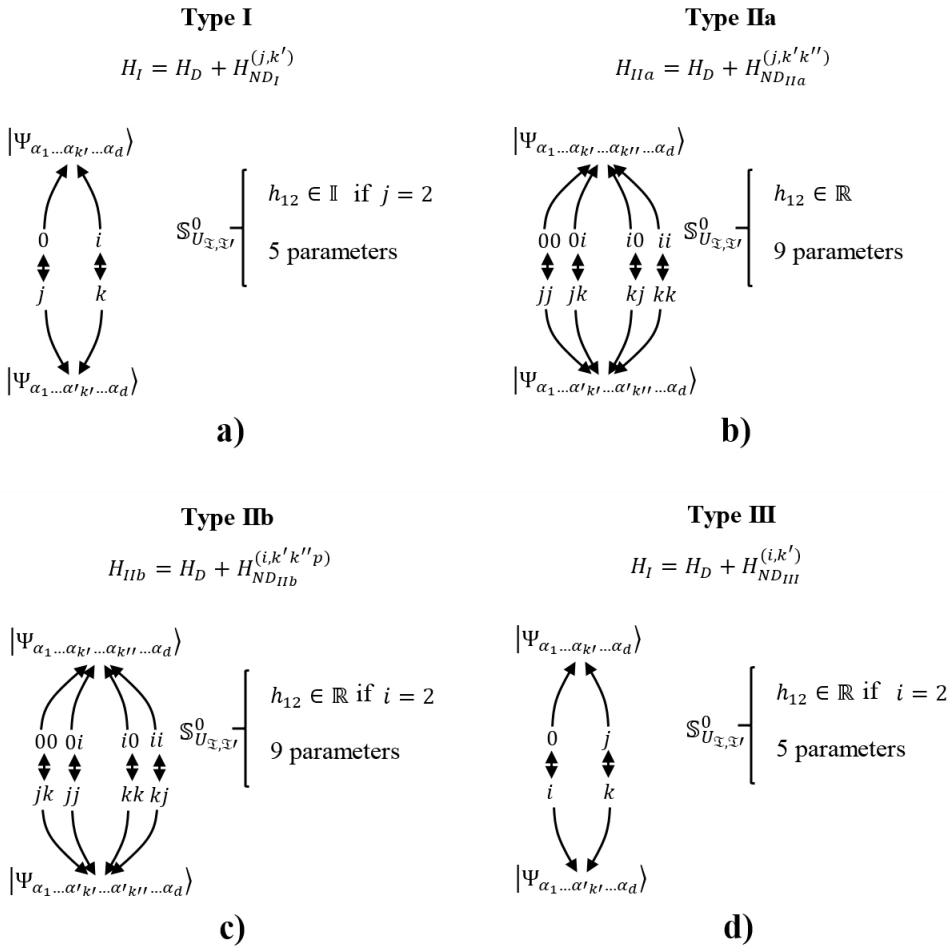

Figure 5. Exchange indexes relations involved for each interaction and highlighted properties for their correspondent $\mathbb{S}^0_{U_{\mathfrak{I},\mathfrak{I}'}}$: a) H_I, b) H_{IIa}, c) H_{IIb} and c) H_{III}. Exchange relations in b) and d) are double by considering the vertical switching in one of the indexes for each pair shown.

6.1. Exchange connectedness under interactions

Under the $SU(2)$ decomposition, pairs of states in GBS basis become related, showing a probability exchange between them. As was seen, each one of the $H_I, H_{IIa,b}, H_{III}$ interactions has rules for these exchange. In any case, it should be clear this exchange is achievable between any pair by combining all types of interactions obtained by switching the value of: a) interaction direction and correspondent pair j, k' in (34) for H_I, b) interaction direction and correspondent pairs j, k', k'' in (36) for H_{IIa}, c) interaction directions, correspondent pairs and parity i, k', k'', p in (38) for H_{IIb}, and d) interaction direction and correspondent pair i, k' in (40) for H_{III}. Several types of interactions could be combined in sequence. Combination of interactions is not precise for the basis elements connectedness but it is necessary to increase the entanglement, thus to connect two arbitrary quantum states. In this terms, there are only two types of states: 1) those exchanging one script (H_I and H_{III}), and b) those exchanging two scripts (H_{IIa} and H_{IIb}) in the GBS basis elements under the rules depicted in the Figure 5 (despite rules and connections are different). There, all basis states become connected under one or several interactions applied consecutively, depending on the number of necessary exchanges in their scripts. Figure 6 shows a graph with these relations for the cases $d = 1, 2, 3$. Green edges are for one script exchange and red lines are for two scripts exchange. Only in the first two cases, due to the low entanglement level, the connection could be achieved with a single interaction. Figure 6a corresponds to that presented in [8] for Bell states in $SU(4)$ systems.

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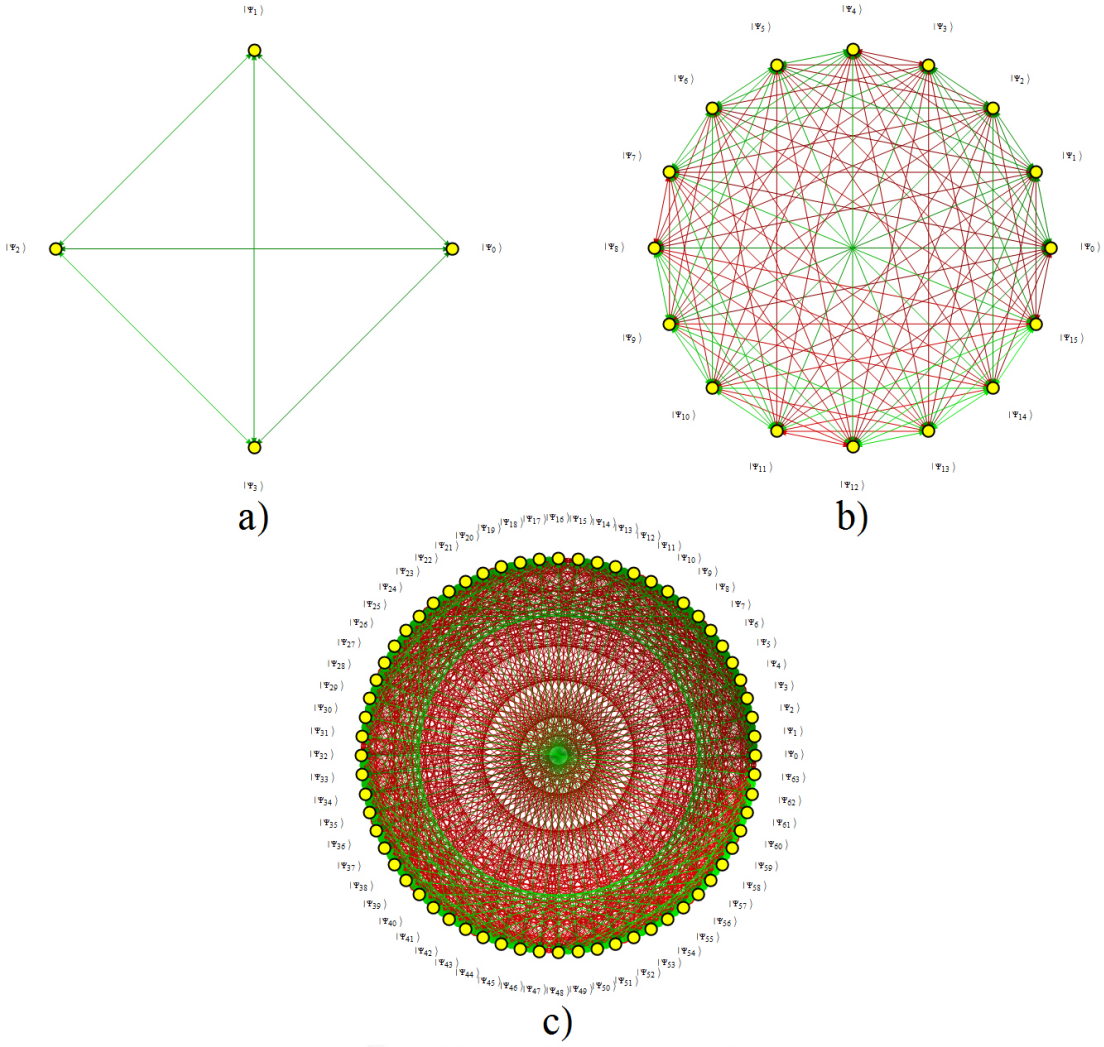

Figure 6. Connectedness graph between states under $SU(2)$ decomposition for one (green) and two (red) exchange scripts for all GBS basis states: a) $d = 1$, b) $d = 2$, and c) $d = 3$.

Connectedness in a finite number of steps by applying some or the whole cases in each type of interaction (piecewise with constant parameters or with time-dependent parameters in each case) warrants the full probability exchange between the occupancy level of each state in terms of the discussion included in the section 3. Nevertheless, not all interactions are able to reach arbitrary evolutions. As is obvious, H_I and H_{III} are not able to generate extended entanglement out of the correspondent pair on which they operate (this fact assumes no rearrangements made in the correspondent pairs and their elements). We discuss this aspect in the next subsection.

6.2. Notable quantum processing operations achievable under $SU(2)$ decomposition

Departing from $\mathbb{S}_U$, then by fixing in (32) $\omega t = \frac{2n+1}{2}\pi$, $\frac{h_{11}-h_{22}}{2\hbar\omega} = \epsilon$, $\frac{h_{12}}{\hbar\omega} = i^c\delta$, $\frac{h_{11}+h_{22}}{2\hbar\omega} = 2(m - \frac{1}{2}); n, m \in \mathbb{Z}, c \in \{0,1\}, \delta \in \mathbb{R}$, where $\epsilon^2 + \delta^2 = 1$. Note that parameter c depends on each kind of interaction in the terms discussed in the previous section. Then, we get the $\mathbb{S}_{U\mathcal{I},\mathcal{I}'}$ block [26]:

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$$\mathbb{H}^c_m(\delta,\epsilon)_{\mathcal{I},\mathcal{I}'} \equiv (-1)^m \begin{pmatrix} \epsilon & i^c\delta \\ (-i)^c\delta & -\epsilon \end{pmatrix} \tag{41}$$

operating on the GBS basis. Note this form not always can be achieved independently in all blocks in terms of the free parameters and the possible restriction $h_{0,0,\ldots,0}=0$ (here $\det(\mathbb{H}^c_m(\delta,\epsilon)_{\mathcal{I},\mathcal{I}'})=-1$, despite it is not decisive in the following development), nevertheless we need achieve it only in some blocks in the immediate discussion. We are using the time non-dependent case, but other more practical cases with time dependent Hamiltonian coefficients can be implemented. Last form is highly versatile. If $s_\epsilon|\epsilon|=\delta=\frac{1}{\sqrt{2}}$ ($s_\epsilon=\text{sign}(\epsilon)$, referred in the notation as $-,+$), we get a Hadamard-like gate $H^{m,c,\text{sign}(\epsilon)}_{\mathcal{I},\mathcal{I}'}\equiv\mathbb{S}_{U\mathcal{I},\mathcal{I}'}$ (in particular if $c=0$, but this condition can be relaxed). When $\delta=1$, we get an exchange-like gate [9,23] for the pair in the $SU(2)$ block, $E^{m,c}_{\mathcal{I},\mathcal{I}'}\equiv\mathbb{S}_{U\mathcal{I},\mathcal{I}'}$. We should be advised that this case is a limit case for the time non-dependent case (32) when $h_{12}\gg h_{11}-h_{22}$. Otherwise, it can be achieved in two steps of time non-dependent piecewise Hamiltonians (as in [9]) or as a continuous time dependent Hamiltonian. These gates are:

$$H^{m,c,s_\epsilon}_{\mathcal{I},\mathcal{I}'}=\frac{(-1)^m}{\sqrt{2}}\begin{pmatrix} s_\epsilon & i^c \\ (-i)^c & -s_\epsilon \end{pmatrix},\quad E^{m,c}_{\mathcal{I},\mathcal{I}'}=(-1)^m\begin{pmatrix} 0 & i^c \\ (-i)^c & 0 \end{pmatrix} \tag{42}$$

Note additionally, when $\frac{h_{11}+h_{22}}{2\hbar\omega}=(\frac{\alpha}{m}-1)\pi,\omega t=m\pi;n,m\in\mathbb{Z}$, we get the quasi-identity gate $\mathbb{S}_{U\mathcal{I},\mathcal{I}'}=e^{i\alpha\pi}\mathbb{I}_{\mathcal{I},\mathcal{I}'}\equiv I^\alpha_{\mathcal{I},\mathcal{I}'}$. Combination of these blocks (allowed because the block independence previously discussed) lets to set important quantum processing operations.

6.3. Generating superposition and entanglement

6.3.1. Generating 2-separable superposition

By using the general block operations $\mathbb{S}_{U\mathcal{I},\mathcal{I}'}\in U(1)\times SU(2)$ (32), it is possible to arrive into a state exhibiting complete superposition through the whole basis elements. In the following, we will use an arrow to depict certain group of quantum operations. In the top, we set the type of interaction being used. While, in the bottom, we set the subspace on which they apply or the generic form of each operation, together with their prescriptions. Thus, as instance, departing from the simple state $|\Psi_0\rangle^{2d}=|\Psi_0\rangle_1|\Psi_0\rangle_2\ldots|\Psi_0\rangle_d$ (easily obtained from $|00\ldots0\rangle$), a couple of local operations $H^{(i,k)}_I$ on each correspondent pair k are sufficient to generate a state containing representatives from each basis element:

$$\begin{aligned} |\Psi_0\rangle^{2d} &\xrightarrow[\substack{\oplus_{s,s'}\mathbb{S}_{U_{s,s'}}\\ s'-s=4^{k-1},\\ s^d_{4,k}=0}]{\substack{H^{(1,k)}_I\\ k=1,2,\ldots,d}} \bigotimes_{k=1}^{d}\sum_{i=0}^{1}\alpha^k_{i,0}\,|\Psi_i\rangle_k \\ &\xrightarrow[\substack{\oplus_{s,s'}\mathbb{S}_{U_{s,s'}\oplus\oplus_{s'',s'''}\mathbb{S}_{U_{s'',s'''}}}\\ s'-s=3\cdot4^{k-1},s'''-s''=4^{k-1}\\ s^d_{4,k}=0,s''^d_{4,k}=1}]{\substack{H^{(3,k)}_I\\ k=1,2,\ldots,d}} \bigotimes_{k=1}^{d}\sum_{i=0}^{1}\sum_{j(i)\in\{i,3-i\}}^{1}\alpha^k_{i,0}\beta^k_{j(i),i}\left|\Psi_{j(i)}\right\rangle_k \equiv \sum_{\mathcal{I}=0}^{4^d-1}\gamma_{\mathcal{I}}\,|\Psi_{\mathcal{I}}\rangle \\ &\text{with}: \quad \gamma_{\mathcal{I}}=\prod_{k=1}^{d}\alpha^k_{j^{-1}(\mathcal{I}^d_{4,k}),0}\beta^k_{\mathcal{I}^d_{4,k},j^{-1}(\mathcal{I}^d_{4,k})} \end{aligned} \tag{43}$$

where $j^{-1}(i)$ is the inverse of $j(i)$ and directions $i = 1, 3$ have been used as instance. In addition, $\alpha^k_{i,j}$ are the components of $\mathbb{S}_{U_{s,s'}}$ in the first operations, and $\beta^k_{j(i),i}$ are the components of $\mathbb{S}_{U_{s,s'}}$, $\mathbb{S}_{U_{s'',s'''}}$ for the second operations with $i = 0, 1$ respectively. Figure 7 depicts each step of the process, using the local operations (alternatively, crossed interactions in H_{III} could be considered).

The last process is a particular case of more general operations by considering $O_J^{(i,\{s\})} = \mathbb{S}_{U_{\mathcal{I},\mathcal{I}'}}$ to mix the states through the momentary associated blocks changing the indexes $\{s\}$ with some interaction $H_J, J \in \{I, IIa, IIb, III\}$ in the associated direction i. We coin the term $k-$local operation when $\mathbb{S}_{U_{\mathcal{I},\mathcal{I}'}}$ generates entanglement utmost in k parts. In our basic interactions scheme, there are only $2-$local and $4-$local operations as was discussed before. Thus, following the notation introduced previously, we set a family of procedures to develop superposition containing the last presented. Departing from the $|\Psi_0\rangle^{2d}$, it is possible to apply several alternate $2-$local operations to generate superpositions involving all GBS basis states. By defining a sequence of paired directions for the H_I evolutions involving all pairs $s = 1, 2, ..., d$ (this process can be achieved alternatively by H_{III} similarly): $\{\{i_s, k_s(i_s)\} | \{1, 2, 3\} \ni i_s \neq k_s(i_s) \in \{1, 2, 3\} \setminus \{i_s\}; s = 1, 2, ..., d\}$. Additionally, $j_s(i_s) \in \{1, 2, 3\}, i_s \neq j_s(i_s) \neq k_s(i_s)$. Then, following the evolution process:

$$
\begin{aligned}
|\Psi_0\rangle^{2d} \quad & \xrightarrow[O_I^{(i_1,\{1\})}]{H_I^{(i_1,1)}} \sum_{t\in\{0,i_1\}} \alpha^1_{t,0} |\Psi_{t,0,...,0}\rangle \xrightarrow[O_I^{(k_1(i_1),\{1\})}]{H_I^{(k_1(i_1),1)}} \sum_{\epsilon_1=0}^{3} \alpha^1_{p_1(\epsilon_1),0} \beta^1_{\epsilon_1,p_1(\epsilon_1)} |\Psi_{\epsilon_1,0,...,0}\rangle \\
& \xrightarrow[O_I^{(i_2,\{2\})}]{H_I^{(i_2,2)}} \cdots \xrightarrow[O_I^{(k_d(i_d),\{d\})}]{H_I^{(k_d(i_d),d)}} \sum_{\epsilon_1,...,\epsilon_d=0}^{3} \Big(\prod_{s=1}^{d} \alpha^s_{p_s(\epsilon_s),0} \beta^s_{\epsilon_s,p_s(\epsilon_s)}\Big) |\Psi_{\epsilon_1,\epsilon_2,...,\epsilon_d}\rangle \equiv \left|\Psi_{f_{\text{sep}}}\right\rangle
\end{aligned}
\tag{44}
$$

where $p_s(\epsilon_s)$ are the inverses of the association rules for the one index exchanges depicted in Figure 5a (or 5d for H_{III}): $p_s(i_s) = i_s = p_s(j_s(i_s)), p_s(0) = 0 = p_s(k_s(i_s))$. Additionally, $|\alpha^s_{0,0}|^2 + |\alpha^s_{i_s,0}|^2 = 1, |\beta^s_{0,0}|^2 + |\beta^s_{k_s(i_s),0}|^2 = 1, |\beta^s_{i_s,i_s}|^2 + |\beta^s_{j_s(i_s),i_s}|^2 = 1$. If $\text{Tr}^S(\rho_{\mathcal{IJ}})$ represents the partial trace with respect to the entire system except the $s \in S$ parts. As is expected, $\left|\Psi_{f_{\text{sep}}}\right\rangle$ is $2-$separable:

$$
\text{Tr}^{\{k'\}}\Big(\left|\Psi_{f_{\text{sep}}}\right\rangle \left\langle\Psi_{f_{\text{sep}}}\right|\Big) = \Big(\sum_{\epsilon_{k'}=0}^{3} \alpha^{k'}_{p_{k'}(\epsilon_{k'}),0} \beta^{k'}_{\epsilon_{k'},p_{k'}(\epsilon_{k'})} |\Psi_{\epsilon_{k'}}\rangle\Big)\Big(\sum_{\epsilon_{k'}=0}^{3} \alpha^{k'}_{p_{k'}(\epsilon_{k'}),0} \beta^{k'}_{\epsilon_{k'},p_{k'}(\epsilon_{k'})} |\Psi_{\epsilon_{k'}}\rangle\Big)^{\dagger} \tag{45}
$$

due to the limited nature of operations involved, which cannot be able to generate more extended entanglement. In addition, superposition could be limited into the $SU(2)$ blocks coverage through the number of parameters introduced, $\alpha^s_{p_s(\epsilon_s),0}, \beta^s_{\epsilon_s,p_s(\epsilon_s)}$, and their physical scope. As was shown in [8], a richer superposition coverage on $SU(2^{2d})$ can be achieved with additional $2-$local operations on each part, introducing extra parameters and probability mixing. As there, $\mathbf{n}$ in (32) is limited to take the two forms $(n_x, 0, n_z)$ or $(0, n_y, n_z)$ (for the time-independent case), but by combining both forms we arrive into two general forms with arbitrary $\mathbf{n} = (n_x, n_y, n_z)$ (it also fulfills for the time-dependent case with adequate $h_{ij}(t)$).

Nevertheless this procedure could to include a general full 2-separable state together with entangled segments between correspondent pairs, it cannot to exhibit states with more extended entanglement, requiring more extended entangling operations as H_{IIa} and H_{IIb}. The quest is to obtain general states departing from a simple resource, which is still an open challenge, particularly for the possible entanglement degree there (a more ambitious challenge is the transformation between two general states [24], but it always can be reduced in two steps of this kind). In the remaining subsection we will discuss this issue and we develop some procedures to generate some maximal entangled states of arbitrary size.

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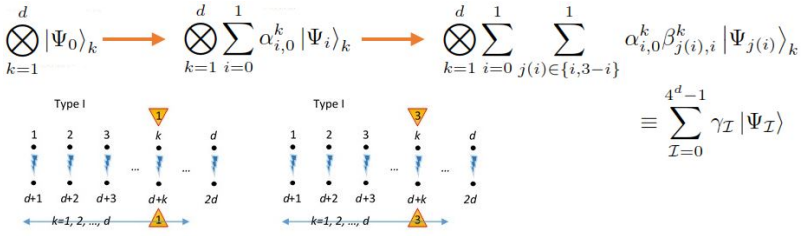

Figure 7. Processes to build 2-separable states with complete superposition.

6.3.2. Entanglement dynamics under interactions

Now, we analyze the entanglement generation under the interactions being considered. We employ the Partial Trace criterion [16] for pure states by considering a single $SU(2)$ combination of two GBS basis states $|\phi_{\mathcal{IJ}}\rangle = \alpha_{\mathcal{I}}|\Psi_{\mathcal{I}}\rangle + \alpha_{\mathcal{J}}|\Psi_{\mathcal{J}}\rangle$. In addition, the explicit form for coefficients will be written as $\alpha_{\mathcal{I}} = \cos\theta/2, \alpha_{\mathcal{J}} = e^{i\phi}\sin\theta/2$. Then, we construct their associated density matrix $\rho_{\mathcal{IJ}} = |\phi_{\mathcal{IJ}}\rangle\langle\phi_{\mathcal{IJ}}|$ to take conveniently partial traces in order to analyze the entanglement of specific subsystems in this quantum state under concrete interactions. Due to the rules in the exchange scripts in the GBS basis states to form the $SU(2)$ blocks are basically the same for the three interactions $H_I, H_{IIa,b}, H_{III}$, the analysis is reduced to only two cases. The first one is for a pair of GBS basis elements $|\Psi_{\mathcal{I}}\rangle, |\Psi_{\mathcal{J}}\rangle$ differing in only one subscript between $\mathcal{I}$ and $\mathcal{J}$: $i_s = j_s \forall s \in \{1,\ldots,d\}, s \neq k'$ (in H_I, H_{III} interactions). Thus, in this case (omitting the base b and the size d for simplicity in the scripts):

$$\left|\phi^1_{\mathcal{IJ}}\right\rangle = \frac{1}{\sqrt{2^d}}\sum_{\mathcal{E},\mathcal{D}=0}^{2^d-1}\Big(\bigotimes_{k'\neq s=1}^{d}\tilde{\sigma}_{i_s}\otimes(\alpha_{\mathcal{I}}\tilde{\sigma}_{i_{k'}} + \alpha_{\mathcal{J}}\tilde{\sigma}_{j_{k'}})\Big)_{\mathcal{E},\mathcal{D}}|\mathcal{E}\rangle\otimes|\mathcal{D}\rangle \tag{46}$$

The second case is for a pair of elements in the GBS basis differing in two subscripts of $\mathcal{I}$ and $\mathcal{J}$: $i_s = j_s \forall s = 1,\ldots,d, s \neq k', k''$ (in $H_{IIa,b}$ interactions):

$$\left|\phi^2_{\mathcal{IJ}}\right\rangle = \frac{1}{\sqrt{2^d}}\sum_{\mathcal{E},\mathcal{D}=0}^{2^d-1}\Big(\bigotimes_{k',k''\neq s=1}^{d}\tilde{\sigma}_{i_s}\otimes(\alpha_{\mathcal{I}}\tilde{\sigma}_{i_{k'}}\otimes\tilde{\sigma}_{i_{k''}} + \alpha_{\mathcal{J}}\tilde{\sigma}_{j_{k'}}\otimes\tilde{\sigma}_{j_{k''}})\Big)_{\mathcal{E},\mathcal{D}}|\mathcal{E}\rangle\otimes|\mathcal{D}\rangle \tag{47}$$

then, we analyze the entanglement of several subsystems in each case by taking the partial trace with respect to its complement. Calculations are direct and at the end the association rules $0 \leftrightarrow i$ and $j \leftrightarrow k$ should be applied to denote explicitly the viable relations between $\mathcal{I}$ and $\mathcal{J}$ as well as to reduce some traces on parts k', k'' in (46) and (47). We report synthetically in the Table 4 the generalized bipartite concurrence for pure states [25]:

$$\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{IJ}})) \quad = \quad 2(1-\mathrm{Tr}^S(\rho^2_{\mathcal{IJ}})) \tag{48}$$

There, j is assumed as the direction label of interaction involved. If $m = \min(m_1, m_2)$ where m_1, m_2 are the Hilbert space dimensions of each subsystem, then this measure changes smoothly from 0 for separable states to $2(m-1)/m$ for maximally entangled states. Be aware that we take $\tilde{\sigma}_i \equiv e^{i\phi_i}\sigma_i$, despite it is only relevant for σ_2. With this distinction, we introduce $\phi' = \phi + \phi_{i'_k} - \phi_{j'_k}$.

Table 4 includes some obvious results for "local" interactions on parts (H_I) or correspondent pairs (H_{III}): a) any part is maximally entangled with respect to the remaining system (through its correspondent pair) if on it is not currently acting local or non-local crossed interactions in H_I and H_{III}, so $\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{IJ}})) = 1$; b) nevertheless, if these local or non-local crossed interactions act on the

Table 4. Bipartite concurrence $\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{IJ}}))$ for several subsystems in the $SU(2)$ mixing of some pairs of GBS basis states

Case		S	$\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{IJ}}))$
a)	$\left\vert\phi^1_{\mathcal{IJ}}\right\rangle$	$[s \notin \{k', k'+d\}]$	1
b)	$\left\vert\phi^1_{\mathcal{IJ}}\right\rangle$	$[s \in \{k', k'+d\}]$	$1-\sin^2\theta(\cos\phi'\delta_{0,i_{k'}\cdot j_{k'}}+(-1)^{\epsilon_{i_{k'}j_{k'}j}}(1-\delta_{0,i_{k'}\cdot j_{k'}})\sin\phi')^2$
c)	$\left\vert\phi^1_{\mathcal{IJ}}\right\rangle$	$[k', k'+d]$	0
d)	$\left\vert\phi^2_{\mathcal{IJ}}\right\rangle$	$[k', k'+d]$	$\sin^2\theta$
e)	$\left\vert\phi^2_{\mathcal{IJ}}\right\rangle$	$[k', k'']$	$\frac{3}{2}-\frac{1}{2}\sin^2\theta(\cos^2\phi'\delta_{i_{k'}j_{k'}}\delta_{i_{k''}j_{k''}}+\sin^2\phi'(1-\delta_{i_{k'}j_{k'}}\delta_{i_{k''}j_{k''}}))$

correspondent pair, each part of it could become separable or partially entangled to the remaining system; and c) any correspondent pair (as a subsystem) is separable from the remaining system in any GBS basis state, so $\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{IJ}}))=0$. Note in last cases that subsystem comprises two parts $[k', k'+d]$ being compared with the remaining system, so Hilbert space dimension is four ($m=4$). Similarly, the most important result here: d) shows how interactions between non-correspondent parts (crossed or non-crossed) affect the original separability of the each correspondent pair with respect to the remaining system, letting it becomes entangled with the remainder system. Finally, e) exhibits the change of entanglement between non-correspondent parts. Clearly they are originally entangled with their respective pair out of the subsystem, but that entanglement becomes reduced ($\mathcal{C}^2(\mathrm{Tr}^S(\rho_{\mathcal{IJ}}))\leq 3/2$) due to the non-local interactions.

6.3.3. Generating larger maximal entangled systems

The generation of extended entanglement can be shown with a couple of introductory examples [26]. If $\left\vert\beta_{ij}\right\rangle=\left\vert\Psi_{2i+i\oplus j}\right\rangle$ are the GBS basis elements for $d=1$ corresponding to the Bell states [19], then considering the $\vert GHZ\rangle$ and $\vert W\rangle$ states of size $2d$ expressed in the GBS basis:

$$\vert GHZ\rangle^{2d} = \frac{1}{\sqrt{2}}\sum_{i=0}^{1}\bigotimes_{j=1}^{d}\vert i,i\rangle_j = \frac{1}{2^{\frac{d+1}{2}}}\sum_{i=0}^{1}\bigotimes_{j=1}^{d}\left(\vert\Psi_0\rangle_j+(-1)^i\vert\Psi_3\rangle_j\right) \tag{49}$$

$$\vert W\rangle^{2d} = \frac{1}{\sqrt{2d}}\sum_{i=1}^{2d}\bigotimes_{j=1}^{d}\left\vert\delta_{i,2j-1},\delta_{i,2j}\right\rangle_j = \frac{d^{-\frac{1}{2}}}{2^{\frac{d-1}{2}}}\sum_{i=1}^{d}\bigotimes_{\substack{j=1\\ j\neq i}}^{d}(\vert\Psi_0\rangle_j+\vert\Psi_3\rangle_j)\otimes\vert\Psi_1\rangle_i \tag{50}$$

where j sums over correspondent pairs. Note that we are alternating the notation in the kets by convenience: $\vert\Psi_k\rangle_j$ is the Bell state $\vert\Psi_k\rangle$ on the j^{th} correspondent pair; while, $\left\vert\Psi_{i_1,i_2,\ldots,i_d}\right\rangle=\vert\Psi_{\mathcal{I}}\rangle$ is the $\mathcal{I}=4^{d-1}i_d+\ldots+4i_2+i_1$ element in the GBS basis. For $d=2$, they are simply:

$$\vert GHZ\rangle^4 = \frac{1}{\sqrt{2}}(\vert\Psi_{0,0}\rangle+\vert\Psi_{3,3}\rangle)=\frac{1}{\sqrt{2}}\sum_{\mathcal{I}\in\{0,15\}}\vert\Psi_{\mathcal{I}}\rangle \tag{51}$$

$$\vert W\rangle^4 = \frac{1}{2}(\vert\Psi_{1,0}\rangle+\vert\Psi_{0,1}\rangle+\vert\Psi_{3,1}\rangle+\vert\Psi_{1,3}\rangle)=\frac{1}{2}\sum_{\mathcal{I}\in\{1,4,7,13\}}\vert\Psi_{\mathcal{I}}\rangle \tag{52}$$

Then, we can depart from the basic state $\vert 0000\rangle=\frac{1}{2}(\vert\Psi_0\rangle_1+\vert\Psi_3\rangle_1)\otimes(\vert\Psi_0\rangle_2+\vert\Psi_3\rangle_2)$ for $d=2$. We arrive to the $\vert GHZ\rangle$ by applying the following operations (interaction Hamiltonian is indicated in the upper position, while the operation is written below):

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$$
\begin{aligned}
|0000\rangle \xrightarrow[H^{0,0,+}_{0,3}\oplus H^{0,0,+}_{12,15}]{H^{(3,1)}_{I}} & \frac{1}{\sqrt{2}}|\Psi_0\rangle_1 \otimes (|\Psi_0\rangle_2 + |\Psi_3\rangle_2) \\
\xrightarrow[H^{0,0,+}_{0,12}]{H^{(3,2)}_{I}} & |\Psi_0\rangle_1 \otimes |\Psi_0\rangle_2 = |\Psi_{0,0}\rangle \\
\xrightarrow[H^{0,0,+}_{0,15}]{H^{(3,1,2)}_{IIa}} & \frac{1}{\sqrt{2}}(|\Psi_{0,0}\rangle + |\Psi_{3,3}\rangle) = \frac{1}{\sqrt{2}}(|\Psi_0\rangle + |\Psi_{15}\rangle) = |GHZ\rangle^4
\end{aligned} \tag{53}
$$

The first operation requires action on two sets of GBS basis states. They are of the same form, so they are easily achieved in terms of prescriptions for $H^{m,c,s_\epsilon}_{\mathcal{I},\mathcal{I}'}$. Note that no more specifications are needed in complementary blocks, they are free because their effect will work on states not included. Similarly, as instance:

$$
\begin{aligned}
|GHZ\rangle^4 \xrightarrow[I^0_{0,10}\oplus E^{0,0}_{5,15}]{H^{(2,1,2)}_{IIa}} & \frac{1}{\sqrt{2}}(|\Psi_{0,0}\rangle + |\Psi_{1,1}\rangle) \\
\xrightarrow[E^{0,0}_{0,4}\oplus E^{0,0}_{1,5}]{H^{(1,2)}_{I}} & \frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 \otimes |\Psi_1\rangle_2 + |\Psi_1\rangle_1 \otimes |\Psi_0\rangle_2) \\
\xrightarrow[H^{0,0,+}_{4,7}\oplus I^{2p}_{1,2}]{H^{(3,1)}_{I}} & \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|\Psi_0\rangle_1 + |\Psi_3\rangle_1) \otimes |\Psi_1\rangle_2 + |\Psi_1\rangle_1 \otimes |\Psi_0\rangle_2) \\
\xrightarrow[I^{2q}_{4,8}\oplus I^{2r}_{7,11}\oplus H^{0,0,+}_{1,13}]{H^{(3,2)}_{I}} & \frac{1}{2}((|\Psi_0\rangle_1 + |\Psi_3\rangle_1) \otimes |\Psi_1\rangle_2 + |\Psi_1\rangle_1 \otimes (|\Psi_0\rangle_2 + |\Psi_3\rangle_2)) \\
& = \frac{1}{2}(|\Psi_4\rangle + |\Psi_7\rangle + |\Psi_1\rangle + |\Psi_{13}\rangle) = |W\rangle^4
\end{aligned} \tag{54}
$$

where $p, q, r \in \mathbb{Z}$. In the last operations, the block independence discussed in the previous section has been applied to justify the construction of some simultaneous operations.

6.3.4. Recursive generation of larger maximal entangled systems

In the last subsection, we obtain the larger maximal entangled states $|GHZ\rangle^4$ and $|W\rangle^4$ departing from the more basic states like $|0000\rangle$. The enlargement of entangled states can be stated in a more overwhelming way as recursive processes. In each case, these processes are based on the control of the parameters involved and the independence among block types generated in each interaction.

Thus, the process shown in Figure 8 combines some of the operations depicted previously to develop $|GHZ\rangle^{2(d+1)}$ departing from $|GHZ\rangle^{2d}$, stating a procedure to get larger versions of these maximal entangled states. The first step begins using the state $|\Psi_0\rangle_{d+1} \otimes |GHZ\rangle^{2d}$. Then, a local operation is applied on each pair in the original state $k = 1, 2, ..., d$ to reduce the factors $(|\Psi_0\rangle_k + |\Psi_3\rangle_k)$ and $(|\Psi_0\rangle_j - |\Psi_3\rangle_j)$ in (49) into $|\Psi_0\rangle_j$ and $|\Psi_3\rangle_j$ respectively. After, we exchange the indexes $30 \leftrightarrow 21$ for the non-correspondent pairs k' and $d+1$ with a non-local operation. This transformation is followed by a couple of local operations changing the indexes $2 \leftrightarrow 3$ for the pair k' and $1 \leftrightarrow 3$ for the pair $d+1$ (which adds a factor i); in this last case we transform the index 0 by itself, but adding the factor i . Finally, we revert for $k = 1, 2, ..., d+1$ the initial transformation between $|\Psi_0\rangle_k$, $|\Psi_3\rangle_k$ and $(|\Psi_0\rangle_k \pm |\Psi_3\rangle_k)$ respectively. All additional index transformations are settled as the identity. The state obtained will be $i\,|GHZ\rangle^{2(d+1)}$. It is notable that only one $4-$entangling operation between the added pair with another one arbitrary pair from the original $2d$-partite system has become needed in this case. It reflects the low-robustness of the genuine entanglement for these states. Considering the expression for $|GHZ\rangle^{2d}$ in (49), the precise prescriptions are:

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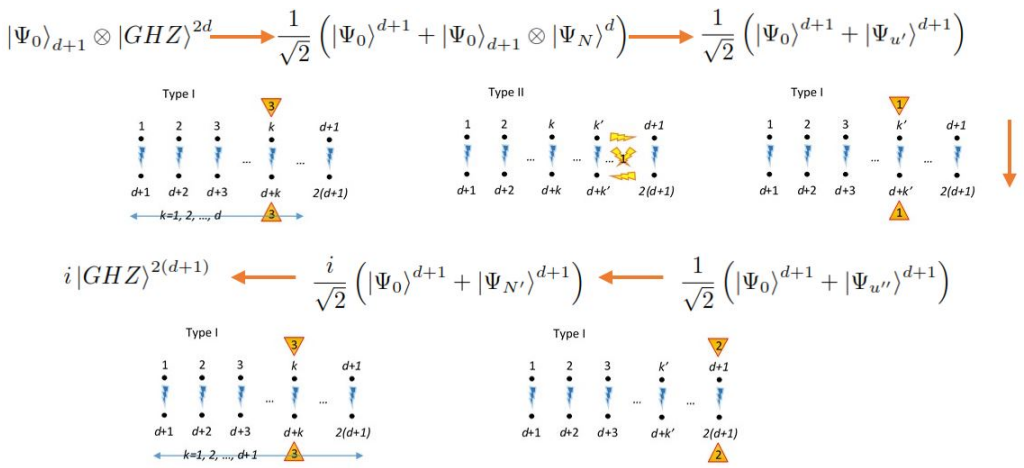

Figure 8. Processes to build recursive enlargement of $|GHZ\rangle$ entangled states.

$$
\begin{aligned}
|\Psi_0\rangle_{d+1} \otimes |GHZ\rangle^{2d} &\xrightarrow[k=1,2,\ldots,d]{H_I^{(3,k)}} \frac{1}{\sqrt{2}}\left(|\Psi_0\rangle^{d+1} + |\Psi_0\rangle_{d+1} \otimes |\Psi_N\rangle^{d}\right) \\
&\xrightarrow[\substack{s'-s=3\cdot 4^{k-1},\\ s,s'\in\{3p\leq N\,|\,p\in\mathbb{N}\}}]{\oplus_{s,s'} H_{s,s'}^{0,0,+}} \\
&\xrightarrow[\substack{u=4^{k'-1}+4^{d}\\ u'=N-4^{k'-1}+4^{d}}]{H_{IIa}^{(1,k',d+1)}} \frac{1}{\sqrt{2}}\left(|\Psi_0\rangle^{d+1} + |\Psi_{u'}\rangle^{d+1}\right) \\
&\xrightarrow[u''=u'+4^{k'-1}]{I_{0,u}^{0}\oplus E_{N,u'}^{0,0}} \\
&\xrightarrow[]{H_I^{(1,k')}} \frac{1}{\sqrt{2}}\left(|\Psi_0\rangle^{d+1} + |\Psi_{u''}\rangle^{d+1}\right) \\
&\xrightarrow[]{I_{0,4^{k'-1}}^{0}\oplus E_{u',u''}^{0,0}} \\
&\xrightarrow[]{H_I^{(2,d+1)}} \frac{i}{\sqrt{2}}\left(|\Psi_0\rangle^{d+1} + |\Psi_{N'}\rangle^{d+1}\right) \\
&\xrightarrow[]{I_{0,2\cdot 4^{d}}^{\frac{1}{2}}\oplus E_{u'',N'}^{0,1}} \\
&\xrightarrow[k=1,2,\ldots,d+1]{H_I^{(3,k)}} i\,|GHZ\rangle^{2(d+1)} \\
&\xrightarrow[\substack{s'-s=3\cdot 4^{k-1},\\ s,s'\in\{3p\leq N'\,|\,p\in\mathbb{N}\}}]{\oplus_{s,s'} H_{s,s'}^{0,0,+}}
\end{aligned}
\tag{55}
$$

where $|\Psi_{\mathcal{I}}\rangle^{n} = |\Psi_{i_1}\rangle_1 \otimes |\Psi_{i_2}\rangle_2 \otimes \ldots \otimes |\Psi_{i_n}\rangle_n$. In addition, $N = 4^d - 1$ and $N' = 4^{d+1} - 1$. Note that first and last operations are really a set of operations for $k = 1, 2, \ldots, d$ and $k = 1, 2, \ldots, d+1$ through several correspondent pairs. They exploit the Hadamard-like block operations for $H_I^{(3,k)}$ to switch first the $|GHZ\rangle^{2d}$ into versions where only the states $|\Psi_0\rangle$ and $|\Psi_3\rangle$ appear. After, operations generated with $H_{IIa}^{(1,k',d+1)}$ between two different correspondent pairs are used as exchange operations entangling the added state $|\Psi_0\rangle_{d+1}$. Then, the additional operations $H_I^{(1,k')}$ and $H_I^{(2,d+1)}$ generate a state expressed only in terms of $|\Psi_0\rangle$ and $|\Psi_3\rangle$, to finally be transformed into $|GHZ\rangle^{2(d+1)}$ with the same kind of initial operations.

To obtain the $|W\rangle^{2(d+1)}$ state, we begin with $|\Psi_0\rangle_{d+1} \otimes |W\rangle^{2d}$, then we use the same local transformation to reduce the factors $(|\Psi_0\rangle_k + |\Psi_3\rangle_k)$ in (49) into $|\Psi_0\rangle_k$ for each $k = 1, 2, \ldots, d$. After, we apply a sequence of non-local transformations between the pairs $k, d+1$ for $k = 1, 2, \ldots, d$ to

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transfer probability between states with indexes $01 \leftrightarrow 10$ there, in such way to reach the coefficient $\frac{1}{\sqrt{d+1}}$ in each term. Finally, we revert the initial transformation for $k = 1, 2, ..., d+1$ changing $|\Psi_0\rangle_k$ into $(|\Psi_0\rangle_k + |\Psi_3\rangle_k)$. The final result is $|W\rangle^{2(d+1)}$ as is shown in Figure 9. Note how the entangling operations need to go through the overall pairs. It reflects the robustness of genuine entanglement in these states. By considering the expression for $|W\rangle^{2d}$ in (49), the following process gives the prescriptions to reach $|W\rangle^{2(d+1)}$ from $|W\rangle^{2d}$:

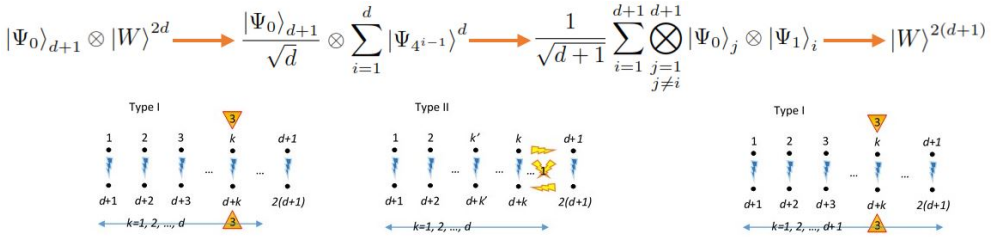

Figure 9. Processes to build recursive enlargement of $|W\rangle$ entangled states.

$$
|\Psi_0\rangle_{d+1} \otimes |W\rangle^{2d} \xrightarrow[\substack{\oplus_{s,s'} I^0_{s,s'} \oplus \oplus_{s'',s'''} H^{0,0,+}_{s'',s'''} \\ s'-s=4^{k-1}, \\ s-4^{k-1}, s'-2\cdot 4^{k-1} \in \{3p \leq N | p \in \mathbb{N}\} \\ s'''-s''=3\cdot 4^{k-1}, \\ s''-4^{i-1}, s'''-4^{i-1} \in \{3p \leq N | p \in \mathbb{N}\}, \\ k \neq i \in \{1,2,...,d\}}}^{H_I^{(3,k)}} \frac{|\Psi_0\rangle_{d+1}}{\sqrt{d}} \otimes \sum_{i=1}^{d} |\Psi_{4^{i-1}}\rangle^d
$$

$$
\xrightarrow[\substack{\oplus_{u,u'} I^0_{u,u'} \oplus \mathbb{H}^0_0(\delta_k, \epsilon_k)_{4^{k-1},4^d} \\ u=4^{i-1}, u'=u+4^{k-1}+4^d \\ k \neq i \in \{1,2,...,d\}}}^{H_{IIa}^{(1,k,d+1)}} \frac{1}{\sqrt{d+1}} \sum_{i=1}^{d+1} \bigotimes_{\substack{j=1 \\ j\neq i}}^{d+1} |\Psi_0\rangle_j \otimes |\Psi_1\rangle_i
$$

$$
\xrightarrow[\substack{\oplus_{s,s'} I^0_{s,s'} \oplus \oplus_{s'',s'''} H^{0,0,+}_{s'',s'''} \\ s'-s=4^{k-1}, \\ s-4^{k-1}, s'-2\cdot 4^{k-1} \in \{3p \leq N' | p \in \mathbb{N}\} \\ s'''-s''=3\cdot 4^{k-1}, \\ s''-4^{i-1}, s'''-4^{i-1} \in \{3p \leq N' | p \in \mathbb{N}\}, \\ k \neq i \in \{1,2,...,d+1\}}}^{H_I^{(3,k)}} |W\rangle^{2(d+1)} \tag{56}
$$

As before, Hadamard-like block operations for $H_I^{(3,k)}$ let to switch, at the beginning and at the end, the states $|W\rangle^{2d}$ and $|W\rangle^{2(d+1)}$ in terms of $|\Psi_0\rangle$ and $|\Psi_1\rangle$. The remarkable set of operations are obtained with $H_{IIa}^{(1,k,d+1)}$ to entangle the added state $|\Psi_0\rangle_{d+1}$ through the operations (41), which transfer progressively the probability to the state $|\Psi_{4^d}\rangle$, completing a state easily transformed into $|W\rangle^{2(d+1)}$ with the final set of operations. Additional exchange in the indexes are settled in the identity. The adequate set of ϵ_k values for each step of operations should fulfill the $d+1$ equations:

$$
\begin{aligned}
g_0 &\equiv 0 \\
\sqrt{d} &= \sqrt{d+1}(\epsilon_j + \delta_j g_{j-1}), \quad j = 1, 2, ..., d \\
g_j &= \delta_j - \epsilon_j g_{j-1}, \quad j = 1, 2, ..., d-1 \\
\sqrt{d} &= \sqrt{d+1}(\delta_d - \epsilon_d g_{d-1})
\end{aligned} \tag{57}
$$

These equations can be solved numerically for any d. Figure 10 shows the $-\log_{10}\epsilon_i$ solutions for $d = 1, 2, ..., 60$ by taking $\epsilon_i, \delta_i > 0$. Note that ϵ_i drops rapidly to zero when d and i grow.

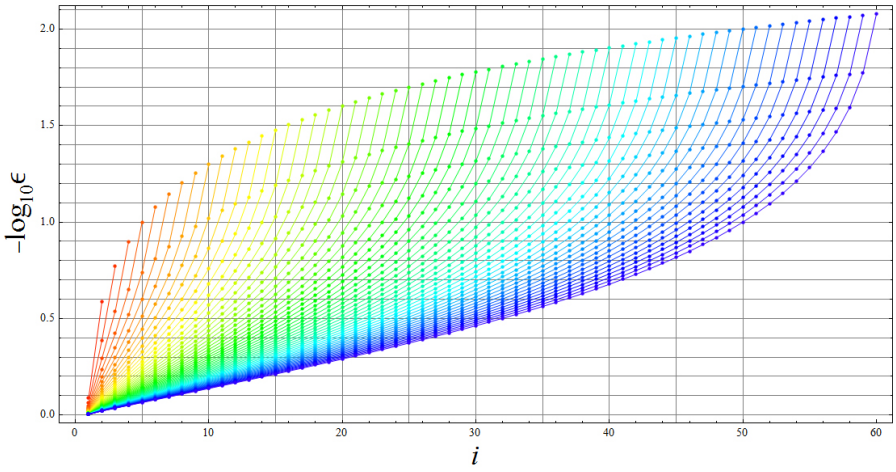

Figure 10. Solutions for ϵ_i in $\mathbb{H}_0^0(\delta_i, \epsilon_i)_{\mathcal{I},\mathcal{I}'}$ involved in the enlargement of $|W\rangle^d$ into $|W\rangle^{d+1}$ for values of $d \in \{2, ..., 60\}$.

6.4. Multipartite entanglement and general states

In the previous subsection we are realized how generate extended superposition using the type I interactions, despite, that process does not reach genuine entangled states. The use of type IIa, IIb interactions is mandatory to extend the entanglement as a set of operations involving elements of two pairs. Nevertheless, it si clear that lots of operations and combinations are necessary and possible.

As instance, by considering the permutation i, j, k from $1, 2, 3$ and departing from the state $|\Psi_0\rangle^{2d}$, the process to reach an entangled state based on a complete combination from the basis elements for two correspondent pairs is as follows (note the process is not unique). First, we apply a $2-$local operation on the pair s and direction i followed by another on the pair s' in the direction j. A linear combination from four basis elements is obtained. Then, we apply a $4-$local operation in the direction k and for pairs s, s', obtaining a state of eight terms. Finally, we apply again a $2-$local operation on pair s in the direction j. At the end, we obtain the desired state of sixteen terms with the pairs s, s' genuinely entangled as was seen.

$$
\begin{aligned}
|\Psi_0\rangle^{2d} \xrightarrow[O_I^{(i,\{s\})}]{H_I^{(i,s)}} & \sum_{t\in\{0,i\}} \alpha^s_{t,0} \left|\Psi_{0,...,t,...,0,...,0}\right\rangle \xrightarrow[O_I^{(j,\{s'\})}]{H_I^{(j,s')}} \sum_{\substack{t\in\{0,i\}\\ t'\in\{0,j\}}} \alpha^s_{t,0}\alpha^{s'}_{t',0} \left|\Psi_{0,...,t,...,t',...,0}\right\rangle \\
& \xrightarrow[O_{IIa}^{(k,\{s,s'\})}]{H_{IIa}^{(k,s,s')}} \sum_{\substack{\epsilon,\epsilon'\in C_4\\ t\in\{0,i\}\\ t'\in\{0,j\}}} \alpha^s_{t,0}\alpha^{s'}_{t',0}\beta^{s,s'}_{\epsilon,\epsilon';t,t'}\delta_{t,p^{t,\epsilon}_{s,k}}\delta_{t',p^{t',\epsilon'}_{s',k}} \left|\Psi_{0,...,\epsilon,...,\epsilon',...,0}\right\rangle \\
& \xrightarrow[O_I^{(j,\{s\})}]{H_I^{(j,s)}} \sum_{\substack{\chi,\epsilon,\epsilon'\in C_4\\ t\in\{0,i\}\\ t'\in\{0,j\}}} \alpha^s_{t,0}\alpha^{s'}_{t',0}\beta^{s,s'}_{\epsilon,\epsilon';t,t'}\alpha^s_{\chi,\epsilon}\delta_{t,p^{t,\epsilon}_{s,k}}\delta_{t',p^{t',\epsilon'}_{s',k}}\delta_{\epsilon,p^{t,\chi}_{s,j}} \left|\Psi_{0,...,\chi,...,\epsilon',...,0}\right\rangle
\end{aligned}
\tag{58}
$$

where $C_4 = \{0, 1, 2, 3\}$ and $p^{t,\epsilon}_{s,j}$ is the extension of the inverse exchange rule presented before $p_s(\epsilon)$, but specifying the rule j as function of the direction of the interaction involved and the script $t \in \{0, i\}$

is a label specifying each possible inverse. It means, if j is the characteristic direction of the interaction, then: $p^{0,i}_{s,j} = k = p^{0,k}_{s,j}, p^{0,0}_{s,j} = 0 = p^{0,j}_{s,j}$ and $p^{i,i}_{s,j} = i = p^{i,k}_{s,j}, p^{i,0}_{s,j} = j = p^{i,j}_{s,j}$. This single process could be improved using additional interactions to grow the spectrum of effective coefficients $\alpha^{s}_{\beta,\alpha}, \beta^{s,s'}_{\beta,\beta';\alpha,\alpha'}$ in order to have a wider coverage of SU(4). In addition, it is clear the last process (or alternative) should be repeated varying one or two pairs in order to generate more complex entanglement. The question about how to generate a specific state under this procedure or to generate certain kind or level of entanglement is clearly open mainly due to the non well understood complexity to measure this property for large states in general.

7. Conclusions

Quantum gate array computation is based on transformation of quantum states under certain universal operations. These operations are used to manipulate the information settled on quantum systems to simulate or reproduce computer processing and normally uses separable states as primary resources. Quantum systems involved, light or matter, are manipulated around of entanglement generation in this kind of processing. Then, interactions involved commonly are non-local, implying that their parts become entangled when they are being manipulated. In the process, several interactions slightly differentiated are applied, each one with a different set of eigenvalues. It does not let to set a common grammar to develop the quantum information processing problem. $SU(2)$ decomposition states not only a reduction in the quantum manipulation states, together it states a common language to address the evolution through several kind interactions in order to reach a wider processing. The process is able to be extended to other architectures (the interaction arrangement of qubits) and Hamiltonians upon the adequate selection of the basis in which the dynamics could be expressed. Inclusively, it is advised other configurations based on $q-$dits are possible using other group decompositions than $SU(2^{2d})$ and $SU(2)$.

Some applications of $SU(2)$ decomposition are foreseen. It can be exploited in quantum control of bigger systems in which control schemes are not well developed as those of $SU(2)$ dynamics. Decomposition presented before lets establish exact control when blocks are reduced to standard forms I, NOT, H, etc., nevertheless, the success of last strategy for exact control depends on the number of free parameters involved, which could be reached using a sequence of pulses instead of a single one or otherwise time-dependent controlled parameters in the Hamiltonian, in spite the block structure is conserved. Similarly, optimal control in terms of energy or time can be achieved when procedures as those in [6,7] are adapted to each block in the depicted structure. More ambitious ideas about the control of quantum processing as the use of wave traveling, ion traps, resonant cavities or superconducting circuits [27–30] could be adapted to the architectures presented here.

Note that the selectivity of pairing is related with the non-diagonal elements arisen, it means, with the local interactions involved in all interaction type more than only with the interactions generating diagonal-off entries in all cases. This approach to quantum evolution will let analytically to control the flow of quantum information in different adaptive geometrical arrangements. The use of more feasible external fields (other than stepwise fields) is compulsory, which is completely compatible with the current $SU(2)$ reduction scheme [31].

In a related but not necessarily equivalent direction, selective block decomposition could be useful for unitary factorization in quantum gate design (as that developed for the $SU(4)$ case[23]), particularly for large dedicated gates involving the processing of several qubits. In the current context, it is a quest in the mathematical arena, how to express a general $SU(2^{2d})$ matrix as a finite product of $U(1) \times SU(2)$ block matrices as those developed here.

Finally, other applications in quantum processing could be engineered for multichannel quantum information storage, using certain subspaces to storage differentiated information which could be processed simultaneously in other subspaces, by example in quantum image processing or quantum machine learning. Additional research should be developed to adapt this procedure to specific gate

operations, necessities in specific quantum processing and translation of the most common algorithms into equivalent ones based on entangled resources as those shown here.

Acknowledgments: I gratefully acknowledge the support from Escuela de Ingeniería y Ciencias of Tecnológico de Monterrey to develop this research work.

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