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Article

Some Normal Intuitionistic Fuzzy Heronian Mean Operators Using Hamacher Operation and Their Application

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Abstract: Hamacher operation which is generalization of the Algebraic and Einstein operation, can widely provide a large number of arithmetical operation with respect to uncertainty information, and Heronian mean can deal with correlations of the input arguments or different criteria and don't make calculation redundancy, meanwhile, the normal intuitionistic fuzzy numbers (NIFNs) can depict distinctively normal distribution information in practical decision making. In this paper, a multi-criteria group decision-making (MCGDM) problem is researched under the NIFNs environment, and a new MCGDM approach is introduced on the basis of the Hamacher operation. Firstly, according to Hamacher t-conorm and t-norm, some operational laws of NIFNs are presented. Secondly, it is noticed that Heronian mean can' t only once take into account mutual relation between attribute values once, but also consider the correlation between input argument and itself. Therefore, we develop some operators and study their properties in order to aggregate normal intuitionistic fuzzy numbers information, these operators include Hamacher Heronian mean (NIFHHM), Hamacher weighted Heronian mean (NIFHWHM), Hamacher geometric Heronian mean (NIFHGHM) and Hamacher weighted geometric Heronian mean (NIFHWGHM). Furthermore, we apply the proposed operators to the MCGDM problem and present a new method. The main characteristics of this new method involve that: (1) it is suitable to make decision under the normal intuitionistic fuzzy numbers environment and more reliable and reasonable to aggregate the normal distribution information. (2) it utilizes Hamacher operation which can provide more reliable and flexible decision-making results and offer an effective and powerful mathematic tool for the MAGDM under uncertainty. (3) it uses the Heronian mean operator which can considers relationships between the input arguments or the attributes and don't brings subsequently about redundancy. Lastly, an application is given for showing the feasibility and effectiveness of the presented method in this paper.

Keywords: Normal intuitionistic fuzzy numbers; Heronian mean; Hamacher t-conorm; Hamacher t-norm

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1. Introduction

In multi-criteria group decision making, because a lot of problems are uncertain or fuzzy, the value of input argument is not real number at times and may be more effectively described as fuzzy value. Zadeh's fuzzy set (FS) plays an important role to deal with fuzzy number information [1], and the FS theory has been a good tool which is suitable to various fields including decision analysis, machine learning, information retrieval, etc.

However, the FS theory only defines a membership function in domain of discourse, so it is very difficult to comprehensively describe the vagueness and uncertainty of the objective elements

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in the real society [4]. The intuitionistic fuzzy set (denoted as IFS) theory was given [3], which involves a non-membership function besides a membership function. Recently, the IFS theory has been widely applied to the decision-making field. Xu [2] and Yager [5] presented respectively some aggregation method under intuitionistic fuzzy number circumstance to settle MCGDM problem. Atanassov [7] and Gargov [6] extended the IFS to the interval-valued intuitionistic fuzzy set (denoted as IVIFS). Xu and Yager [8] proposed a new measure method of the similarity in regard to the IVIFSs for solving the decision-making problem. Tan and Zhang [9] gave a new TOPSIS approach which can resolve the decision-making problem with IVIFNs. Wang et al. [10] utilized the prospect value function and introduced a score method which can settle the MCGDM problem under the IVIFNs environment. Gomathi Nayagam et al. [11] defined an accuracy function of the IVIFNs. Fuzzy number intuitionistic fuzzy set (denoted as FNIFS) theory was proposed by Liu and Yuan [14]. Some basic operational laws were defined and the relationships among the FNIFS, the IFS and IVFS were discussed. Shu et al. [13] proposed the triangular intuitionistic fuzzy numbers (TriIFNs) and presented the operational rules which are utilized to the fault-tree analysis. Lin et al. [12] introduced some prioritized aggregation operators under the FNIF circumstance. Furthermore, they gave an approach to deal with fuzzy number intuitionistic fuzzy MCGDM problems. Wang [15] extended the TriIFNs and originally defined the TriIFNs and IVTriIFNs. The trapezoidal intuitionistic fuzzy numbers (TraIFNs) [38,44,48-50,55] are the other form of the IFNs. Some MCGDM process are studied for the TriIFNs or TraIFNs information [36,43,45,51-54,56].

In the reality of social life, a lot of social and economic phenomena conform to normal distribution, such as random measurement error, average annual rainfall in a region, etc. Yang and Ko [16] proposed the concept with respect to the normal fuzzy numbers (NFNs) which are very suitable to depict the normal distribution information. Wang et al. [17, 18] gave the definition of the NIFNs, and they introduced some operations and score function of the NIFNs. Some MCGDM problems are discussed in order to deal with NIFNs information [35,37,43], and some new operators were developed. Nevertheless, these aggregation operators only concern the impact of input data and ordered result, and they cannot present the interrelationship of input data. Some normal intuitionistic fuzzy Bonferroni mean operators were proposed by Liu and Liu [19] for describing the correlation of input data. Bonferroni [23] was originally proposed Bonferroni mean (BM) in 1950 which considers the mutuality of attribute values [21]. The BM operators were studied with respect to the circumstance under which the attribute values may be other types, such as IFNs [21,47], IVIFNs [22], HFNs [20] and NIFNs [19].

Compared with Bonferroni mean, Heronian mean (denoted as HM) [24,40,46] is another aggregation technique which can also depict the relationship between aggregated arguments objectively. From Bonferroni mean and Heronian mean, the BM mainly indicates correlation between any pairs of criteria c_i and c_j $(i \neq j)$. However, the correlation between c_i and c_j is equal to the correlation between c_j and c_i $(i \neq j)$. In other words, the BM operator takes it into account twice and it generates subsequently the redundant calculation. Furthermore, it ignores the correlation between criteria c_i and itself. Although the HM operator owns similar structure to the BM operator, it can solve the stated two problems of the BM operator. At present, the HM has been applied to aggregate the input information with the IFNs [25], the IVIFN [27] and the hesitant fuzzy set (HFS) information [26].

However, the HM is not applied to aggregate the normal intuitionistic fuzzy number (NIFN) arguments. From above, all mentioned operators only consider the algebraic operation of the IFNs, hesitant fuzzy set (HFS), IVIFNs or NIFNs where the algebraic product and sum are important and they can define union and intersection of the NIFNs, IVIFNs or HFS. In conclusion, a generalized t-conorm and t-norm can simulate union and intersection between the IFNs or IVIFNs [28, 29]. According to Archimedean t-conorm and t-norm, Xia et al. [28] generalized the Hamacher and Frank operational laws and presented some operational laws of the IFNs. Furthermore, they introduced the weighted average and geometric operator for intuitionistic fuzzy MCGDM problem. Because these operators do not integrate the weight of the attribute and order, they can't be utilized to

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IVIFNs. Wang and Liu [30, 31] applied Einstein operation to develop some operators including the IFEOWG, IFEWG, IFEOWA and IFEWA.

Hamacher operation generalizes algebraic and Einstein operation and plays an important role in studying aggregation method for the MCGDM problem [32,42]. Hamacher operation has been applied to aggregate the IFNs information, IVIFNs information [19, 33] or hesitant fuzzy information [26].

From the above analysis, we notice that there is no method proposed for aggregating NIFNs with respect to Hamacher operation and considering the interrelationship between normal intuitionistic fuzzy arguments at the same time. Therefore, in this paper, Hamacher operation is extended to the normal intuitionistic fuzzy number, and some normal intuitionistic fuzzy Heronian mean operators are introduced. These operators are based on Hamacher operation that can extend the choice scope of decision-makers and significantly investigate some generalized normal intuitionistic fuzzy HM operator.

This paper will be presented: In Section 2, we review the normal intuitionistic fuzzy number, Hamacher operation and Heronian mean operators. In Section 3, we establish Hamacher operational rules and their characteristics with respect to the normal intuitionistic fuzzy numbers, furthermore, we propose the normal intuitionistic fuzzy Heronian mean operators based on Hamacher operation of NIFNs, and discuss some properties of the developed operators. According to the new Hamacher operations, the geometric Heronian mean is extended to the NIFNs and its weighted version is presented in Section 4. In Section 5, we utilize new operators to make a MCGDM procedure for the NIFNs information and present the decision making steps. In Section 6, an application is introduced to show new approach and the evidence of practicality and effectiveness. In Section 7, we make a conclusion and present some remarks.

2. Preliminaries

Some notion and operational rules with respect to NIFNs are introduced, and the definition of Heronian mean operator is given.

2.1. Normal intuitionistic fuzzy number

Definition 1 [16]. If R is a real number set, and $x,\alpha,\sigma \in R$, $A=(\alpha,\sigma)$ is a normal fuzzy number (NFN) whose membership function is presented:

$$A(x)=e^{-\left(\frac{x-\alpha}{\sigma}\right)^2},\ \sigma>0 \tag{1}$$

Definition 2 [18]. If X is a finite nonempty set, a normal intuitionistic fuzzy number (NIFN) $A=\langle(\alpha,\sigma),\mu_A,v_A\rangle$ is defined in the following, where μ_A and v_A are respectively the membership and non-membership function, and they satisfy $0\le\mu_A\le 1$, $0\le v_A\le 1$, $0\le \mu_A+v_A\le 1$.

$$\mu_A(x)=\mu_A e^{-\left(\frac{x-\alpha}{\sigma}\right)^2},\ x\in X \tag{2}$$

$$v_A(x)=1-(1-v_A)e^{-\left(\frac{x-\alpha}{\sigma}\right)^2},\ x\in X \tag{3}$$

Compared with the classical IFNs, the non-membership function of the NIFNs can more comprehensively capture the fuzziness and uncertainty of objects. From the definition of NIFNs, its universe of discourse expands from discrete to continuous, and it can effectively describe a large number of normal distribution under the socioeconomic environment. In the following, some operational rules of the NIFNs were defined in [18].

Let $A_i=\langle(\alpha_i,\sigma_i),\mu_i,v_i\rangle$, $i=1,2$, and $A=\langle(\alpha,\sigma),\mu,v\rangle$, $\gamma>0$, $\lambda>0$, then

$$A_1\oplus A_2=\langle(\alpha_1+\alpha_2,\sigma_1+\sigma_2),\mu_1+\mu_2-\mu_1\mu_2,v_1v_2\rangle \tag{4}$$

$$A_1\otimes A_2=\left\langle\left(\alpha_1\alpha_2,\alpha_1\alpha_2\sqrt{\frac{\sigma_1^2}{\alpha_1^2}+\frac{\sigma_2^2}{\alpha_2^2}}\right),\mu_1\mu_2,v_1+v_2-v_1v_2\right\rangle \tag{5}$$

$$\lambda A=\left\langle\left(\lambda\alpha,\lambda\sigma\right),1-\left(1-\mu\right)^{\lambda},\upsilon^{\lambda}\right\rangle \tag{6}$$

$$A^{\lambda}=\left\langle\left(\alpha^{\lambda},\lambda^{\frac{1}{2}}\alpha^{\lambda-1}\sigma\right),\mu^{\lambda},1-\left(1-\upsilon\right)^{\lambda}\right\rangle \tag{7}$$

The score and accuracy function with respect to NIFNs were given as follows [18]:

$$S_1\left(A\right)=\alpha\left(\mu_A-\upsilon_A\right),\ S_2\left(A\right)=\sigma\left(\mu_A-\upsilon_A\right) \tag{8}$$

$$H_1\left(A\right)=\alpha\left(\mu_A+\upsilon_A\right),\ H_2\left(A\right)=\sigma\left(\mu_A+\upsilon_A\right) \tag{9}$$

In order to rank any two NIFNs, the following method was introduced by Wang and Li [18].

(1) If $S_1\left(A_1\right)>S_1\left(A_2\right)$, then $A_1>A_2$

(2) If $S_1\left(A_1\right)=S_1\left(A_2\right)$, and $H_1\left(A_1\right)>H_1\left(A_2\right)$, then $A_1>A_2$

(3) If $S_1\left(A_1\right)=S_1\left(A_2\right)$ and $H_1\left(A_1\right)=H_1\left(A_2\right)$, then

If $S_2\left(A_1\right)<S_2\left(A_2\right)$ then $A_1>A_2$

If $S_2\left(A_1\right)=S_2\left(A_2\right)$ and $H_2\left(A_1\right)<H_2\left(A_2\right)$ then $A_1>A_2$

If $S_2\left(A_1\right)=S_2\left(A_2\right)$ and $H_2\left(A_1\right)=H_2\left(A_2\right)$ then $A_1=A_2$

2.2. Hamacher t-norm and Hamacher t-conorm

In FS theory, t-conorm (T^*) and t-norm (T) play an active role in the generalization of intersection and union of fuzzy sets [34, 35].

Definition 3 [34]. If A and B are IFSs, then the union and intersection of A and B are expressed:

$$A\cup_{TT^*}B=\left\{\left\langle x,T^*\left(\mu_A\left(x\right),\mu_B\left(x\right)\right),T\left(\upsilon_A\left(x\right),\upsilon_B\left(x\right)\right)\right\rangle,x\in X\right\} \tag{10}$$

$$A\cap_{TT^*}B=\left\{\left\langle x,T\left(\mu_A\left(x\right),\mu_B\left(x\right)\right),T^*\left(\upsilon_A\left(x\right),\upsilon_B\left(x\right)\right)\right\rangle,x\in X\right\} \tag{11}$$

Hamacher defined the Hamacher t-norm and Hamacher t-conorm [10]:

$$T_\gamma\left(x,y\right)=\frac{xy}{\gamma+\left(1-\gamma\right)\left(x+y-xy\right)},\ \gamma>0 \tag{12}$$

$$T_\gamma^*\left(x,y\right)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy},\ \gamma>0 \tag{13}$$

Especially, when $\gamma=1$, Hamacher t-norm and Hamacher t-conorm transform into the Algebraic t-norm and t-conorm:

$$T\left(x,y\right)=xy,\ T^*\left(x,y\right)=x+y-xy \tag{14}$$

If $\gamma=2$, Hamacher t-norm and Hamacher t-conorm are respectively equal to the Einstein t-norm and t-conorm [20].

$$T\left(x,y\right)=\frac{xy}{1+\left(1-x\right)\left(1-y\right)}\ T^*\left(x,y\right)=\frac{x+y}{1+xy} \tag{15}$$

2.3. Heronian mean

Definition 4 [24]. Let $a_i\left(i=1,2,\cdots,n\right)\in R$, which is greater than zero. The Basic Heronian mean (BHM) is defined as follows:

$$BHM\left(a_1,a_2,\cdots,a_n\right)=\frac{2}{n\left(n+1\right)}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{a_ia_j} \tag{16}$$

Based on the BHM, Yu and Wu [32] extended the BHM to a more generalized form by introducing two parameters p and q, and they proposed the geometric Heronian mean [34].

Definition 5 [32]. Let $p,q>0$ and $a_i\in R,a_i\geq 0$ $\left(i=1,2,\cdots,n\right)$, then the Heronian mean(HM) operator is defined as follows:

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$$HM^{p,q}\left(a_1,a_2,\cdots,a_n\right)=\left(\frac{2}{n\left(n+1\right)}\sum_{i=1}^{n}\sum_{j=i}^{n}a_i^p a_j^q\right)^{\frac{1}{p+q}} \tag{17}$$

It is easy to notice that the HM reduces to the BHM whenever $p=q=\frac{1}{2}$.

Definition 6 [32]. If $p,q\geq 0$ and p,q are not equal to zero at the same time, $a_i\in R,\ a_i\geq 0,\ i=1,2,\cdots,n$, then the geometric Heronian mean (denoted as GHM) has the following form:

$$GHM^{p,q}\left(a_1,a_2,\cdots,a_n\right)=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(pa_i+qa_j\right)^{\frac{2}{n(n+1)}}\right) \tag{18}$$

3. Normal intuitionistic fuzzy Hamacher Heronian mean operator and its weighted form

3.1. Hamacher operational laws of the NIFNs

From Definition 3, Hamacher t-norm and Hamacher t-conorm, Hamacher intersection and sum between NIFNs are introduced.

Suppose that $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ $i=1,2$, $A=\left\langle\left(\alpha,\sigma\right),\mu,v\right\rangle$, $\gamma>0$ $\lambda>0$. The Hamacher operational laws of the NIFNs are presented as follows:

$$A_1\oplus_H A_2=\left\langle\left(\alpha_1+\alpha_2,\sigma_1+\sigma_2\right),\frac{\mu_1+\mu_2-\mu_1\mu_2-\left(1-\gamma\right)\mu_1\mu_2}{1-\left(1-\gamma\right)\mu_1\mu_2},\frac{v_1v_2}{\gamma+\left(1-\gamma\right)\left(v_1+v_2-v_1v_2\right)}\right\rangle \tag{19}$$

$$A_1\otimes_H A_2=\left\langle\left(\alpha_1\alpha_2,\alpha_1\alpha_2\sqrt{\frac{\sigma_1^2}{\alpha_1^2}+\frac{\sigma_2^2}{\alpha_2^2}}\right),\frac{\mu_1\mu_2}{\gamma+\left(1-\gamma\right)\left(\mu_1+\mu_2-\mu_1\mu_2\right)},\frac{v_1+v_2-v_1v_2-\left(1-\gamma\right)v_1v_2}{1-\left(1-\gamma\right)v_1v_2}\right\rangle \tag{20}$$

$$\lambda A=\left\langle\left(\lambda\alpha,\lambda\sigma\right),\frac{\left(1+\left(\gamma-1\right)\mu\right)^{\lambda}-\left(1-\mu\right)^{\lambda}}{\left(1+\left(\gamma-1\right)\mu\right)^{\lambda}+\left(\gamma-1\right)\left(1-\mu\right)^{\lambda}},\frac{\gamma v^{\lambda}}{\left(1+\left(\gamma-1\right)\left(1-v\right)\right)^{\lambda}+\left(\gamma-1\right)v^{\lambda}}\right\rangle \tag{21}$$

$$A^{\lambda}=\left\langle\left(\alpha^{\lambda},\lambda^{\frac{1}{2}}\alpha^{\lambda-1}\sigma\right),\frac{\gamma\mu^{\lambda}}{\left(1+\left(\gamma-1\right)\left(1-\mu\right)\right)^{\lambda}+\left(\gamma-1\right)\mu^{\lambda}},\frac{\left(1+\left(\gamma-1\right)v\right)^{\lambda}-\left(1-v\right)^{\lambda}}{\left(1+\left(\gamma-1\right)v\right)^{\lambda}+\left(\gamma-1\right)\left(1-v\right)^{\lambda}}\right\rangle \tag{22}$$

Furthermore, according to Definition 3 and the properties of Hamacher t-norm and t-conorm, the above operation results are all NIFNs. Especially, when $\gamma=1$, the above operational rules reduce to the operational laws of Definition 2. Therefore, the Hamacher operational laws of NIFNs extend the Algebraic operational rules of NIFNs.

Theorem 1. If $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ $i=1,2,3$, $A=\left\langle\left(\alpha,\sigma\right),\mu,v\right\rangle$ and $\gamma>0,\lambda>0,\lambda_i>0\left(i=1,2\right)$, then

$$A_1\oplus_H A_2=A_2\oplus_H A_1 \tag{23}$$

$$A_1\otimes_H A_2=A_2\otimes_H A_1 \tag{24}$$

$$\left(A_1\oplus_H A_2\right)\oplus_H A_3=A_1\oplus_H\left(A_2\oplus_H A_3\right) \tag{25}$$

$$\left(A_1\otimes_H A_2\right)\otimes_H A_3=A_1\otimes_H\left(A_2\otimes_H A_3\right) \tag{26}$$

$$\lambda\left(A_1\oplus_H A_2\right)=\left(\lambda A_1\right)\oplus_H\left(\lambda A_2\right) \tag{27}$$

$$\left(\lambda_1 A\right)\oplus_H\left(\lambda_2 A\right)=\left(\lambda_1+\lambda_2\right)A \tag{28}$$

$$A^{\lambda_1}\otimes_H A^{\lambda_2}=A^{(\lambda_1+\lambda_2)} \tag{29}$$

$$A_1^{\lambda}\otimes_H A_2^{\lambda}=\left(A_1\otimes_H A_2\right)^{\lambda} \tag{30}$$

$$\left(A^{\lambda_1}\right)^{\lambda_2}=A^{\lambda_1\lambda_2} \tag{31}$$

$$\lambda_1\left(\lambda_2 A\right)=\left(\lambda_1\lambda_2\right)A \tag{32}$$

Proof.

(1) In light of the operation laws of NIFNs, the Equations (23) and (24) hold.

(2) For the Equation (25)

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$$\left(A_1 \oplus_H A_2\right) \oplus_H A_3 =$$

$$\left\langle \left(\alpha_1+\alpha_2, \sigma_1+\sigma_2\right), \frac{\mu_1+\mu_2-\mu_1\mu_2-(1-\gamma)\mu_1\mu_2}{1-(1-\gamma)\mu_1\mu_2}, \frac{v_1 v_2}{\gamma+(1-\gamma)\left(v_1+v_2-v_1 v_2\right)} \right\rangle \oplus_H \left\langle \left(\alpha_3, \sigma_3\right), \mu_3, v_3 \right\rangle$$

$$=\left\langle \left(\sum_{i=1}^{3}\alpha_i, \sum_{i=1}^{3}\sigma_i\right), \frac{\prod_{i=1}^{3}\left(1+(\gamma-1)\mu_i\right)-\prod_{i=1}^{3}\left(1-\mu_i\right)}{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_i\right)+(\gamma-1)\prod_{i=1}^{n}\left(1-\mu_i\right)}, \frac{\gamma\prod_{i=1}^{3} v_i}{\prod_{i=1}^{3}\left(1+(\gamma-1)\left(1-v_i\right)\right)+(\gamma-1)\prod_{i=1}^{3} v_i} \right\rangle$$

$$A_1 \oplus_H \left(A_2 \oplus_H A_3\right) =$$

$$\left\langle \left(\alpha_1, \sigma_1\right), \mu_1, v_1 \right\rangle \oplus_H \left\langle \left(\alpha_2+\alpha_3, \sigma_2+\sigma_3\right), \frac{\mu_2+\mu_3-\mu_2\mu_3-(1-\gamma)\mu_2\mu_3}{1-(1-\gamma)\mu_2\mu_3}, \frac{v_2 v_3}{\gamma+(1-\gamma)\left(v_2+v_3-v_2 v_3\right)} \right\rangle$$

$$=\left\langle \left(\sum_{i=1}^{3}\alpha_i, \sum_{i=1}^{3}\sigma_i\right), \frac{\prod_{i=1}^{3}\left(1+(\gamma-1)\mu_i\right)-\prod_{i=1}^{3}\left(1-\mu_i\right)}{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_i\right)+(\gamma-1)\prod_{i=1}^{n}\left(1-\mu_i\right)}, \frac{\gamma\prod_{i=1}^{3} v_i}{\prod_{i=1}^{3}\left(1+(\gamma-1)\left(1-v_i\right)\right)+(\gamma-1)\prod_{i=1}^{3} v_i} \right\rangle$$

Therefore, the Equation (25) holds.

(3) It is easily noticed that the Equation (26) has the same characteristic as the Equation (25), thus the proof of the Equation (26) is omitted.

(4) For the Equation (27)

$$\lambda\left(A_1 \oplus_H A_2\right) = \lambda\left\langle \left(\alpha_1+\alpha_2, \sigma_1+\sigma_2\right), \frac{\mu_1+\mu_2-\mu_1\mu_2-(1-\gamma)\mu_1\mu_2}{1-(1-\gamma)\mu_1\mu_2}, \frac{v_1 v_2}{\gamma+(1-\gamma)\left(v_1+v_2-v_1 v_2\right)} \right\rangle$$

$$=\left\langle \left(\lambda\left(\alpha_1+\alpha_2\right), \lambda\left(\sigma_1+\sigma_2\right)\right), \frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\mu_i\right)^{\lambda}-\prod_{i=1}^{2}\left(1-\mu_i\right)^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\mu_i\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}\left(1-\mu_i\right)^{\lambda}}, \frac{\gamma\prod_{i=1}^{2} v_i^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(1-v_i\right)\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2} v_i^{\lambda}} \right\rangle$$

$$\left(\lambda A_1\right) \oplus_H \left(\lambda A_2\right) = \left\langle \left(\lambda\alpha_1, \lambda\sigma_1\right), \frac{\left(1+(\gamma-1)\mu_1\right)^{\lambda}-\left(1-\mu_1\right)^{\lambda}}{\left(1+(\gamma-1)\mu_1\right)^{\lambda}+(\gamma-1)\left(1-\mu_1\right)^{\lambda}}, \frac{\gamma v_1^{\lambda}}{\left(1+(\gamma-1)\left(1-v_1\right)\right)^{\lambda}+(\gamma-1)v_1^{\lambda}} \right\rangle$$

$$\oplus_H \left\langle \left(\lambda\alpha_2, \lambda\sigma_2\right), \frac{\left(1+(\gamma-1)\mu_2\right)^{\lambda}-\left(1-\mu_2\right)^{\lambda}}{\left(1+(\gamma-1)\mu_2\right)^{\lambda}+(\gamma-1)\left(1-\mu_2\right)^{\lambda}}, \frac{\gamma v_2^{\lambda}}{\left(1+(\gamma-1)\left(1-v_2\right)\right)^{\lambda}+(\gamma-1)v_2^{\lambda}} \right\rangle$$

$$=\left\langle \left(\lambda\left(\alpha_1+\alpha_2\right), \lambda\left(\sigma_1+\sigma_2\right)\right), \frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\mu_i\right)^{\lambda}-\prod_{i=1}^{2}\left(1-\mu_i\right)^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\mu_i\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}\left(1-\mu_i\right)^{\lambda}}, \frac{\gamma\prod_{i=1}^{2} v_i^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\left(1-v_i\right)\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2} v_i^{\lambda}} \right\rangle$$

Therefore, the Equation (27) holds.

(5) It is easily noticed that the Equation (28) has the same characteristic as the Equation (27), thus the proof of the Equation (28) is omitted.

(6) In view of the Equation (29)

$$A^{\lambda_1} \otimes_H A^{\lambda_2} = \left\langle \left(\alpha^{\lambda_1}, \lambda_1^{\frac{1}{2}}\alpha^{\lambda_1-1}\sigma\right), \frac{\gamma\mu^{\lambda_1}}{\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_1}+(\gamma-1)\mu^{\lambda_1}}, \frac{\left(1+(\gamma-1)v\right)^{\lambda_1}-(1-v)^{\lambda_1}}{\left(1+(\gamma-1)v\right)^{\lambda_1}+(\gamma-1)(1-v)^{\lambda_1}} \right\rangle$$

$$\otimes_H \left\langle \left(\alpha^{\lambda_2}, \lambda_2^{\frac{1}{2}}\alpha^{\lambda_2-1}\sigma\right), \frac{\gamma\mu^{\lambda_2}}{\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_2}+(\gamma-1)\mu^{\lambda_2}}, \frac{\left(1+(\gamma-1)v\right)^{\lambda_2}-(1-v)^{\lambda_2}}{\left(1+(\gamma-1)v\right)^{\lambda_1}+(\gamma-1)(1-v)^{\lambda_2}} \right\rangle$$

$$=\left\langle \left(\alpha^{\lambda_1+\lambda_2}, \alpha^{\lambda_1+\lambda_2}\sqrt{\frac{\lambda_1\alpha^{2\lambda_1-2}\sigma^2}{\alpha^{2\lambda_1}}+\frac{\lambda_2\alpha^{2\lambda_2-2}\sigma^2}{\alpha^{2\lambda_2}}}\right), \frac{\gamma\prod_{i=1}^{2}\mu^{\lambda_i}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{2}\mu^{\lambda_i}}, \frac{\prod_{i=1}^{2}\left(1+(\gamma-1)v\right)^{\lambda_i}-\prod_{i=1}^{2}(1-v)^{\lambda_i}}{\prod_{i=1}^{2}\left(1+(\gamma-1)v\right)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{2}(1-v)^{\lambda_i}} \right\rangle$$

$$=\left\langle \left(\alpha^{\lambda_1+\lambda_2}, \alpha^{\lambda_1+\lambda_2}\sqrt{\left(\lambda_1+\lambda_2\right)\alpha^{-2}\sigma^2}\right), \frac{\gamma\prod_{i=1}^{2}\mu^{\lambda_i}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{2}\mu^{\lambda_i}}, \frac{\prod_{i=1}^{2}\left(1+(\gamma-1)v\right)^{\lambda_i}-\prod_{i=1}^{2}(1-v)^{\lambda_i}}{\prod_{i=1}^{2}\left(1+(\gamma-1)v\right)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{2}(1-v)^{\lambda_i}} \right\rangle$$

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$$=\left\langle\left(\alpha^{\lambda_1+\lambda_2},(\lambda_1+\lambda_2)^{\frac{1}{2}}\alpha^{\lambda_1+\lambda_2-1}\sigma\right),\frac{\gamma\prod_{i=1}^{2}\mu^{\lambda_i}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{2}\mu^{\lambda_i}},\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu\right)^{\lambda_i}-\prod_{i=1}^{2}(1-\nu)^{\lambda_i}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu\right)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{2}(1-\nu)^{\lambda_i}}\right\rangle=A^{(\lambda_1+\lambda_2)}$$

Therefore, the Equation (29) holds.

(7) For the Equation (30)

$$A_1^{\lambda}\otimes_H A_2^{\lambda}=\left\langle\left(\alpha_1^{\lambda},\lambda^{\frac{1}{2}}\alpha_1^{\lambda-1}\sigma_1\right),\frac{\gamma\mu_1^{\lambda}}{\left(1+(\gamma-1)(1-\mu_1)\right)^{\lambda}+(\gamma-1)\mu_1^{\lambda}},\frac{\left(1+(\gamma-1)\nu_1\right)^{\lambda}-(1-\nu_1)^{\lambda}}{\left(1+(\gamma-1)\nu_1\right)^{\lambda}+(\gamma-1)(1-\nu_1)^{\lambda}}\right\rangle$$

$$\otimes_H\left\langle\left(\alpha_2^{\lambda},\lambda^{\frac{1}{2}}\alpha_2^{\lambda-1}\sigma_2\right),\frac{\gamma\mu_2^{\lambda}}{\left(1+(\gamma-1)(1-\mu_2)\right)^{\lambda}+(\gamma-1)\mu_2^{\lambda}},\frac{\left(1+(\gamma-1)\nu_2\right)^{\lambda}-(1-\nu_2)^{\lambda}}{\left(1+(\gamma-1)\nu_2\right)^{\lambda}+(\gamma-1)(1-\nu_2)^{\lambda}}\right\rangle$$

$$=\left\langle\left(\alpha_1^{\lambda}\alpha_2^{\lambda},\alpha_1^{\lambda}\alpha_2^{\lambda}\sqrt{\frac{\lambda\alpha_1^{2\lambda-2}\sigma_1^2}{\alpha_1^{2\lambda}}+\frac{\lambda\alpha_2^{2\lambda-2}\sigma_2^2}{\alpha_2^{2\lambda}}}\right),\frac{\gamma\prod_{i=1}^{2}\mu_i^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu_i)\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}\mu_i^{\lambda}},\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}-\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}\right\rangle$$

$$=\left\langle\left(\alpha_1^{\lambda}\alpha_2^{\lambda},\lambda^{\frac{1}{2}}\alpha_1^{\lambda}\alpha_2^{\lambda}\sqrt{\frac{\sigma_1^2}{\alpha_1^2}+\frac{\sigma_2^2}{\alpha_2^2}}\right),\frac{\gamma\prod_{i=1}^{2}\mu_i^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu_i)\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}\mu_i^{\lambda}},\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}-\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}\right\rangle$$

$$\left(A_1\otimes_H A_2\right)^{\lambda}=\left\langle\left(\alpha_1\alpha_2,\alpha_1\alpha_2\sqrt{\frac{\sigma_1^2}{\alpha_1^2}+\frac{\sigma_2^2}{\alpha_2^2}}\right),\frac{\mu_1\mu_2}{\gamma+(1-\gamma)(\mu_1+\mu_2-\mu_1\mu_2)},\frac{\nu_1+\nu_2-\nu_1\nu_2-(1-\gamma)\nu_1\nu_2}{1-(1-\gamma)\nu_1\nu_2}\right\rangle^{\lambda}$$

$$=\left\langle\left(\alpha_1^{\lambda}\alpha_2^{\lambda},\lambda^{\frac{1}{2}}\alpha_1^{\lambda-1}\alpha_2^{\lambda-1}\alpha_1\alpha_2\sqrt{\frac{\sigma_1^2}{\alpha_1^2}+\frac{\sigma_2^2}{\alpha_2^2}}\right),\frac{\gamma\prod_{i=1}^{2}\mu_i^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu_i)\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}\mu_i^{\lambda}},\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}-\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}\right\rangle$$

$$=\left\langle\left(\alpha_1^{\lambda}\alpha_2^{\lambda},\lambda^{\frac{1}{2}}\alpha_1^{\lambda}\alpha_2^{\lambda}\sqrt{\frac{\sigma_1^2}{\alpha_1^2}+\frac{\sigma_2^2}{\alpha_2^2}}\right),\frac{\gamma\prod_{i=1}^{2}\mu_i^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)(1-\mu_i)\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}\mu_i^{\lambda}},\frac{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}-\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}{\prod_{i=1}^{2}\left(1+(\gamma-1)\nu_i\right)^{\lambda}+(\gamma-1)\prod_{i=1}^{2}(1-\nu_i)^{\lambda}}\right\rangle$$

Therefore, the Equation (30) holds.

(8) For the Equation (31)

$$\left(A^{\lambda_1}\right)^{\lambda_2}=\left\langle\left(\alpha^{\lambda_1},\lambda_1^{\frac{1}{2}}\alpha^{\lambda_1-1}\sigma\right),\frac{\gamma\mu^{\lambda_1}}{\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_1}+(\gamma-1)\mu^{\lambda_1}},\frac{\left(1+(\gamma-1)\nu\right)^{\lambda_1}-(1-\nu)^{\lambda_1}}{\left(1+(\gamma-1)\nu\right)^{\lambda_1}+(\gamma-1)(1-\nu)^{\lambda_1}}\right\rangle^{\lambda_2}$$

$$=\left\langle\left(\alpha^{\lambda_1\lambda_2},\lambda_2^{\frac{1}{2}}\left(\alpha^{\lambda_1}\right)^{\lambda_2-1}\lambda_1^{\frac{1}{2}}\alpha^{\lambda_1-1}\sigma\right),\frac{\gamma\mu^{\lambda_1\lambda_2}}{\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_1\lambda_2}+(\gamma-1)\mu^{\lambda_1\lambda_2}},\frac{\left(1+(\gamma-1)\nu\right)^{\lambda_1\lambda_2}-(1-\nu)^{\lambda_1\lambda_2}}{\left(1+(\gamma-1)\nu\right)^{\lambda_1\lambda_2}+(\gamma-1)(1-\nu)^{\lambda_1\lambda_2}}\right\rangle$$

$$=\left\langle\left(\alpha^{\lambda_1\lambda_2},(\lambda_1\lambda_2)^{\frac{1}{2}}\alpha^{\lambda_1\lambda_2-1}\sigma\right),\frac{\gamma\mu^{\lambda_1\lambda_2}}{\left(1+(\gamma-1)(1-\mu)\right)^{\lambda_1\lambda_2}+(\gamma-1)\mu^{\lambda_1\lambda_2}},\frac{\left(1+(\gamma-1)\nu\right)^{\lambda_1\lambda_2}-(1-\nu)^{\lambda_1\lambda_2}}{\left(1+(\gamma-1)\nu\right)^{\lambda_1\lambda_2}+(\gamma-1)(1-\nu)^{\lambda_1\lambda_2}}\right\rangle=A^{\lambda_1\lambda_2}$$

Therefore, the Equation (31) holds.

(9) For the Equation (32)

$$\lambda_1(\lambda_2A)=\lambda_1\left\langle(\lambda_2\alpha,\lambda_2\sigma),\frac{\left(1+(\gamma-1)\mu\right)^{\lambda_2}-(1-\mu)^{\lambda_2}}{\left(1+(\gamma-1)\mu\right)^{\lambda_2}+(\gamma-1)(1-\mu)^{\lambda_2}},\frac{\gamma\nu^{\lambda_2}}{\left(1+(\gamma-1)(1-\nu)\right)^{\lambda_2}+(\gamma-1)\nu^{\lambda_2}}\right\rangle$$
$$=\lambda_1\left\langle(\lambda_1\lambda_2\alpha,\lambda_1\lambda_2\sigma),\frac{\left(1+(\gamma-1)\mu\right)^{\lambda_1\lambda_2}-(1-\mu)^{\lambda_1\lambda_2}}{\left(1+(\gamma-1)\mu\right)^{\lambda_1\lambda_2}+(\gamma-1)(1-\mu)^{\lambda_1\lambda_2}},\frac{\gamma\nu^{\lambda_1\lambda_2}}{\left(1+(\gamma-1)(1-\nu)\right)^{\lambda_1\lambda_2}+(\gamma-1)\nu^{\lambda_1\lambda_2}}\right\rangle=(\lambda_1\lambda_2)A$$

Therefore, the Equation (31) holds.

As a result, all the equations of Theorem 1 are right.

3.2. Normal intuitionistic fuzzy Hamacher Heronian mean operator and its weighted form

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Definition 7. Let $A_i=\langle(\alpha_i,\sigma_i),\mu_i,\upsilon_i\rangle$ $(i=1,2,\cdots,n)$ be a collection of NIFNs, and $p,q\geq 0$, then the normal intuitionistic fuzzy Hamacher Heronian mean (NIFHHM) operator is defined in the following formula.

$$NIFHHM^{p,q}(A_1,A_2,\cdots,A_n)=\left(\frac{2}{n(n+1)}\left(\mathop{\oplus_H}_{i=1,j=i}^{n}\left(A_i^p\otimes_H A_j^q\right)\right)\right)^{\frac{1}{p+q}} \tag{33}$$

Lemma 1: Let $A_i=\langle(\alpha_i,\sigma_i),\mu_i,\upsilon_i\rangle,(i=1,2,\cdots,n)$ be NIFNs, then

$$A_i^p\otimes_H A_j^q=\left\langle\left(\alpha_i^p\alpha_j^q,\alpha_i^p\alpha_j^q\sqrt{p\frac{\sigma_i^2}{\alpha_i^2}+q\frac{\sigma_j^2}{\alpha_j^2}}\right),\mu_{ij}^{pq},\upsilon_{ij}^{pq}\right\rangle \tag{34}$$

$$\mu_{ij}^{pq}=\frac{\gamma\mu_i^p\mu_j^q}{\left((1+(\gamma-1)(1-\mu_i))^p\times(1+(\gamma-1)(1-\mu_j))^q\right)+(\gamma-1)\mu_i^p\mu_j^q} \tag{35}$$

$$\upsilon_{ij}^{pq}=\frac{\left((1+(\gamma-1)\upsilon_i)^p\times(1+(\gamma-1)\upsilon_j)^q\right)-\left((1-\upsilon_i)^p\times(1-\upsilon_j)^q\right)}{\left((1+(\gamma-1)\upsilon_i)^p\times(1+(\gamma-1)\upsilon_j)^q\right)+(\gamma-1)\left((1-\upsilon_i)^p\times(1-\upsilon_j)^q\right)} \tag{36}$$

Proof. By (22), we have $A_i^p=\left\langle\left(\alpha_i^p,p^{\frac{1}{2}}\alpha_i^{p-1}\sigma_i\right),\mu_{p_i},\upsilon_{p_i}\right\rangle$, $\mu_{p_i}=\frac{\gamma\mu_i^p}{(1+(\gamma-1)(1-\mu_i))^p+(\gamma-1)\mu_i^p}$,

$$\upsilon_{p_i}=\frac{(1+(\gamma-1)\upsilon_i)^p-(1-\upsilon_i)^p}{(1+(\gamma-1)\upsilon_i)^p+(\gamma-1)(1-\upsilon_i)^p},\quad A_j^q=\left\langle\left(\alpha_j^q,q^{\frac{1}{2}}\alpha_j^{q-1}\sigma_j\right),\mu_{q_j},\upsilon_{q_j}\right\rangle$$

$$\mu_{q_j}=\frac{\gamma\mu_j^q}{(1+(\gamma-1)(1-\mu_j))^q+(\gamma-1)\mu_j^q},\quad \upsilon_{q_j}=\frac{(1+(\gamma-1)\upsilon_j)^q-(1-\upsilon_j)^q}{(1+(\gamma-1)\upsilon_j)^q+(\gamma-1)(1-\upsilon_j)^q}$$

From the formula (20), we obtain $A_i^p\otimes_H A_j^q=\left\langle\left(\alpha_i^p\alpha_j^q,\alpha_i^p\alpha_j^q\sqrt{p\frac{\sigma_i^2}{\alpha_i^2}+q\frac{\sigma_j^2}{\alpha_j^2}}\right),\mu_{ij}^{pq},\upsilon_{ij}^{pq}\right\rangle$

$$\mu_{ij}^{pq}=\frac{\mu_{p_i}\cdot\mu_{q_j}}{\gamma+(1-\gamma)(\mu_{p_i}+\mu_{q_j}-\mu_{p_i}\cdot\mu_{q_j})}=\frac{\gamma\mu_i^p\mu_j^q}{\left((1+(\gamma-1)(1-\mu_i))^p\times(1+(\gamma-1)(1-\mu_j))^q\right)+(\gamma-1)\mu_i^p\mu_j^q}$$

$$\upsilon_{ij}^{pq}=\frac{\upsilon_{p_i}+\upsilon_{q_j}-\upsilon_{p_i}\cdot\upsilon_{q_j}-(1-\gamma)\upsilon_{p_i}\cdot\upsilon_{q_j}}{1-(1-\gamma)\upsilon_{p_i}\cdot\upsilon_{q_j}}=\frac{\left((1+(\gamma-1)\upsilon_i)^p\times(1+(\gamma-1)\upsilon_j)^q\right)-\left((1-\upsilon_i)^p\times(1-\upsilon_j)^q\right)}{\left((1+(\gamma-1)\upsilon_i)^p\times(1+(\gamma-1)\upsilon_j)^q\right)+(\gamma-1)\left((1-\upsilon_i)^p\times(1-\upsilon_j)^q\right)}$$

Lemma 2. If $A_i=\langle(\alpha_i,\sigma_i),\mu_i,\upsilon_i\rangle$, $\lambda_i>0(i=1,2,\cdots,n)$ are NIFNs, then

$$\mathop{\oplus_H}_{i=1}^{n}\lambda_iA_i=\left\langle\left(\sum_{i=1}^{n}\lambda_i\alpha_i,\sum_{i=1}^{n}\lambda_i\sigma_i\right),\mu_{\oplus_n},\upsilon_{\oplus_n}\right\rangle \tag{37}$$

$$\mu_{\oplus_n}=\frac{\prod_{i=1}^{n}(1+(\gamma-1)\mu_i)^{\lambda_i}-\prod_{i=1}^{n}(1-\mu_i)^{\lambda_i}}{\prod_{i=1}^{n}(1+(\gamma-1)\mu_i)^{\lambda_i}+(\gamma-1)\prod_{i=1}^{n}(1-\mu_i)^{\lambda_i}} \tag{38}$$

$$\upsilon_{\oplus_n}=\frac{\gamma\prod_{i=1}^{n}\upsilon_i^{\lambda_i}}{\prod_{i=1}^{n}(1+(\gamma-1)(1-\upsilon_i))^{\lambda_i}+(\gamma-1)\prod_{i=1}^{n}\upsilon_i^{\lambda_i}} \tag{39}$$

Proof. When $n=1$, $\mathop{\oplus_H}_{i=1}^{n}\lambda_iA_i=\lambda_1A_1$. According to the formula (19), we have:

$$\lambda_1A_1=\left\langle(\lambda_1\alpha_1,\lambda_1\sigma_1),\frac{(1+(\gamma-1)\mu_1)^{\lambda_1}-(1-\mu_1)^{\lambda_1}}{(1+(\gamma-1)\mu_1)^{\lambda_1}+(\gamma-1)(1-\mu_1)^{\lambda_1}},\frac{\gamma\upsilon_1^{\lambda_1}}{(1+(\gamma-1)(1-\upsilon_1))^{\gamma_1}+(\gamma-1)\upsilon_1^{\lambda_1}}\right\rangle=\langle(\lambda_1\alpha_1,\lambda_1\sigma_1),\mu_{\oplus_1},\upsilon_{\oplus_1}\rangle$$

So the Equation (37) holds for $n=1$.

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Assume that the Equation (37) is true for $n=k$, we have:

$$\overset{k}{\underset{i=1}{\oplus_H}} \lambda_i A_i = \left\langle \left(\sum_{i=1}^{k} \lambda_i \alpha_i, \sum_{i=1}^{k} \lambda_i \sigma_i \right), \mu_{\oplus_k}, \upsilon_{\oplus_k} \right\rangle$$

$$\mu_{\oplus_k} = \frac{\prod_{i=1}^{k}\left(1+(\gamma-1)\mu_i\right)^{\lambda_i} - \prod_{i=1}^{k}\left(1-\mu_i\right)^{\lambda_i}}{\prod_{i=1}^{k}\left(1+(\gamma-1)\mu_i\right)^{\lambda_i} + (\gamma-1)\prod_{i=1}^{k}\left(1-\mu_i\right)^{\lambda_i}}$$

$$\upsilon_{\oplus_k} = \frac{\gamma \prod_{i=1}^{k} \upsilon_i^{\lambda_i}}{\prod_{i=1}^{k}\left(1+(\gamma-1)(1-\upsilon_i)\right)^{\lambda_i} + (\gamma-1)\prod_{i=1}^{k} \upsilon_i^{\lambda_i}}$$

When $n=k+1$, we apply the formulas (19) and (21), and get the following:

$$\overset{k+1}{\underset{i=1}{\oplus_H}} \lambda_i A_i =$$

$$\left(\overset{k}{\underset{i=1}{\oplus_H}} \lambda_i A_i \right) \oplus_H \left(\lambda_{k+1} A_{k+1} \right) = \left\langle \left(\sum_{i=1}^{k} \lambda_i \alpha_i, \sum_{i=1}^{k} \lambda_i \sigma_i \right), \mu_{\oplus_k}, \upsilon_{\oplus_k} \right\rangle \oplus_H$$

$$\left\langle \left(\lambda_{k+1}\alpha_{k+1}, \lambda_{k+1}\sigma_{k+1} \right), \frac{\left(1+(\gamma-1)\mu_{k+1}\right)^{\lambda_{k+1}} - \left(1-\mu_{k+1}\right)^{\lambda_{k+1}}}{\left(1+(\gamma-1)\mu_{k+1}\right)^{\lambda_{k+1}} + (\gamma-1)\left(1-\mu_{k+1}\right)^{\lambda_{k+1}}}, \frac{\gamma \upsilon_{k+1}^{\lambda_{k+1}}}{\left(1+(\gamma-1)(1-\upsilon_{k+1})\right)^{\lambda_{k+1}} + (\gamma-1)\upsilon_{k+1}^{\lambda_{k+1}}} \right\rangle$$

$$= \left\langle \left(\sum_{i=1}^{k+1} \lambda_i \alpha_i, \sum_{i=1}^{k+1} \lambda_i \sigma_i \right), \mu_{\oplus_{k+1}}, \upsilon_{\oplus_{k+1}} \right\rangle$$

So when $n=k+1$, the Equation (37) holds.

Based on steps (1) and (2), we can get that the Equation (37) holds for any n.

Lemma 3. Let $A_i = \left\langle (\alpha_i, \sigma_i), \mu_i, \upsilon_i \right\rangle$ $\lambda_i > 0$ $i=1,2,\cdots,n$ be a set of NIFNs, then

$$\overset{n}{\underset{i=1}{\otimes_H}} A_i^{\lambda_i} = \left\langle \left(\prod_{i=1}^{n} \alpha_i^{\lambda_i}, \left(\prod_{i=1}^{n} \alpha_i^{\lambda_i} \right) \left(\sum_{i=1}^{n} \lambda_i \frac{\sigma_i^2}{\alpha_i^2} \right)^{\frac{1}{2}} \right), \mu_{\otimes_n}, \upsilon_{\otimes_n} \right\rangle \tag{40}$$

$$\mu_{\otimes_n} = \frac{\gamma \prod_{i=1}^{n} \mu_i^{\lambda_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)(1-\mu_i)\right)^{\lambda_i} + (\gamma-1)\prod_{i=1}^{n} \mu_i^{\lambda_i}} \tag{41}$$

$$\upsilon_{\otimes_n} = \frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\upsilon_i\right)^{\lambda_i} - \prod_{i=1}^{n}\left(1-\upsilon_i\right)^{\lambda_i}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\upsilon_i\right)^{\lambda_i} + (\gamma-1)\prod_{i=1}^{n}\left(1-\upsilon_i\right)^{\lambda_i}} \tag{42}$$

From Lemma 2, we apply the mathematical induction to similarly prove Lemma 3 and the proof is omitted here.

Theorem 2. If $A_i = \left\langle (\alpha_i, \sigma_i), \mu_i, \upsilon_i \right\rangle$ $p, \lambda, q > 0$ $(i=1,2,\cdots,n)$ is a set of NIFNs, then the aggregation result of the formula (33) is still a NIFN, and it has the following expression:

$$NIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) = \left(\frac{2}{n(n+1)} \left(\overset{n}{\underset{i=1,j=i}{\oplus_H}} \left(A_i^p \otimes_H A_j^q \right) \right) \right)^{\frac{1}{p+q}} = \left\langle (\alpha, \sigma), \mu, \upsilon \right\rangle \tag{43}$$

$$\alpha = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_i^p \alpha_j^q \right)^{\frac{1}{p+q}} \tag{44}$$

$$\sigma = \left(\left(\frac{1}{p+q} \right)^{\frac{1}{2}} \left(\frac{2}{n(n+1)} \right)^{\frac{1}{p+q}} \left(\sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_i^p \alpha_j^q \right)^{\frac{1}{p+q}-1} \times \left(\sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_i^p \alpha_j^q \sqrt{p \frac{\sigma_i^2}{\alpha_i^2} + q \frac{\sigma_j^2}{\alpha_j^2}} \right) \right) \tag{45}$$

$$\mu = \frac{\gamma \left(\prod_{i=1,j=i}^{n} \left(V_{ij} + (\gamma^2-1) W_{ij} \right)^{\frac{2}{n(n+1)}} - \prod_{i=1,j=i}^{n} \left(V_{ij} - W_{ij} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,j=i}^{n} \left(V_{ij} + (\gamma^2-1) W_{ij} \right)^{\frac{2}{n(n+1)}} + (\gamma^2-1) \prod_{i=1,j=i}^{n} \left(V_{ij} - W_{ij} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}} + (\gamma-1) \left(\prod_{i=1,j=i}^{n} \left(V_{ij} + (\gamma^2-1) W_{ij} \right)^{\frac{2}{n(n+1)}} - \prod_{i=1,j=i}^{n} \left(V_{ij} - W_{ij} \right)^{\frac{2}{n(n+1)}} \right)^{\frac{1}{p+q}}} \tag{46}$$

$$V_{ij} = \left(1+(\gamma-1)(1-\mu_i)\right)^p \left(1+(\gamma-1)(1-\mu_j)\right)^q \quad W_{ij} = \mu_i^p \mu_j^q$$

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$v =$

$$\frac{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^{2}-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^{2}-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)v_{i}\right)^{p}\left(1+\left(\gamma-1\right)v_{j}\right)^{q}\quad W_{ij}=\left(1-v_{i}\right)^{p}\left(1-v_{j}\right)^{q}$$

(47)

Proof.

According to Lemma 2, Lemma 3 and Equation (22), the Equation (43) holds. Consequently, it is proved that $NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)$ is a NIFN. For the ordinal pair $\left(\mu,\ v\right)$, we need prove that it is a IFN . i.e. $0\le\mu\le1,\ 0\le v\le1$ and $0\le\mu+v\le1$.

(1) For the value of μ, according to $\gamma>0$ and $0\le\mu_i\le1,\left(i=1,2,\cdots,n\right)$, we have:

$$1+\left(\gamma-1\right)\left(1-\mu_i\right)\ge\mu_i,\left(i=1,2,\cdots n\right)$$

Moreover, for any i and j, $\left(i,\ j=1,2,\cdots,n\right)$,

$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^{p}\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^{q}\ge\mu_i^{p}\mu_j^{q}=W_{ij}\ge0$$

Thus $\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}\ge\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge0$, and then

$$\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0$$

From the following conditions:

$$\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^{2}-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge0$$

We obtain:

$$\begin{aligned}&\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^{2}-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\&+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge\\&\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0\end{aligned}$$

As a result, $0\le\mu\le1$.

(2) For the value of v, according to $\gamma>0$ and $0\le v_i\le1,\left(i=1,2,\cdots,n\right)$, we have:

$$1+\left(\gamma-1\right)v_i\ge v_i,\left(i=1,2,\cdots n\right)$$

Moreover, for any i and j, $\left(i,\ j=1,2,\cdots,n\right)$,

$$V_{ij}=\left(1+\left(\gamma-1\right)v_i\right)^{p}\left(1+\left(\gamma-1\right)v_j\right)^{q}\ge\left(1-v_i\right)^{p}\left(1-v_j\right)^{q}=W_{ij}\ge0$$

Thus $\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}\ge\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge0$, and then

$$\begin{aligned}&\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^{2}-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\&\ge\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^{2}-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0\end{aligned}$$

By the following conditions:

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$$\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$$
$$+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\geq 0$$

$$\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\geq 0$$

We get $0\leq \upsilon\leq 1$.

(3) For the value of $\mu+\upsilon$.

By the steps (1) and (2), we can obtain $\mu+\upsilon\geq 0$, and we need prove that $\mu+\upsilon\leq 1$ is right in the following.

1) If there exist at least μ_k with $\mu_k=1$, then $\upsilon_k=0$ and $V_{kk}-W_{kk}=0$, thus $\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}=0$, therefore we obtain $\mu=1,\upsilon=0$, i.e. $\mu+\upsilon=1$

2) If there exist at least μ_k with $\mu_k=0$, then $\upsilon_k=1$.

For the equation $\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}$, we have:

$$V_{kj}=\begin{cases}\gamma^p\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^q & in\quad \mu\\ \gamma^p\left(1+\left(\gamma-1\right)\upsilon_j\right)^q & in\quad \upsilon\end{cases},\ \left(k\leq j\leq n\right)\quad V_{ik}=\begin{cases}\gamma^q\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^p & in\quad \mu\\ \gamma^q\left(1+\left(\gamma-1\right)\upsilon_i\right)^p & in\quad \upsilon\end{cases},\ \left(1\leq i\leq k\right)$$

and $W_{kj}=W_{ik}=0,\ \left(1\leq i\leq k\leq j\leq n\right)$, thus

$$\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}=\begin{cases}\left(\prod_{j=k}^{n}\gamma^p\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^q\right)\left(\prod_{i=1}^{k-1}\gamma^q\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^p\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in\quad \mu\\ \left(\prod_{j=k}^{n}\gamma^p\left(1+\left(\gamma-1\right)\upsilon_j\right)^q\right)\left(\prod_{i=1}^{k-1}\gamma^q\left(1+\left(\gamma-1\right)\upsilon_i\right)^p\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in\quad \upsilon\end{cases}$$

$$\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}=\begin{cases}\left(\prod_{j=k}^{n}\gamma^p\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^q\right)\left(\prod_{i=1}^{k-1}\gamma^q\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^p\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in\quad \mu\\ \left(\prod_{j=k}^{n}\gamma^p\left(1+\left(\gamma-1\right)\upsilon_j\right)^q\right)\left(\prod_{i=1}^{k-1}\gamma^q\left(1+\left(\gamma-1\right)\upsilon_i\right)^p\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in\quad \upsilon\end{cases}$$

Therefore

$$\mu=\frac{\gamma\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left\{\begin{array}{l}\left(\prod_{i=1,i\neq k,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,i\neq k,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\ +\left(\gamma-1\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}\right\}}$$
$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^q\quad W_{ij}=\mu_i^p\mu_j^q$$

$$\upsilon=\frac{\left\{\begin{array}{l}\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\ -\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}\right\}}{\left\{\begin{array}{l}\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\ +\left(\gamma-1\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}\right\}}$$
$$V_{ij}=\left(1+\left(\gamma-1\right)\upsilon_i\right)^p\left(1+\left(\gamma-1\right)\upsilon_j\right)^q\quad W_{ij}=\left(1-\upsilon_i\right)^p\left(1-\upsilon_j\right)^q$$

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It is easily noticed that the above equations respectively have the same characteristic as Equation (46) and (47) except for the number of factors in each product part, i.e. the symbol $\prod_{i=1,j=i}^{n}$ shows $n(n+1)/2$ product factors, but $\prod_{i=1,i\neq k,j=i,j\neq k}^{n}$ shows $n(n+1)/2-n$ product factors.

From the aforementioned analysis, without the loss of generality, there is no harm in assuming the conditions with $0<\mu_i<1$ and $0<v_i<1$, $(i=1,2,\cdots,n)$. Thereupon, we can transform μ and v into the following formulation.

$$\mu=\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\left(1-\mu_i\right)/\mu_i\right)\gamma\right)^p\left(1+\left(\left(1-\mu_j\right)/\mu_j\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\gamma-1\right)$$

$$v=1\Bigg/\left(1+\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(v_i/\left(1-v_i\right)\right)\gamma\right)^p\left(1+\left(v_j/\left(1-v_j\right)\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)$$

According to $0\leq\mu_i+v_i\leq 1$, $(i=1,2,\cdots,n)$, we have $(1-\mu_i)/\mu_i\geq v_i/(1-v_i)\geq 0$, $(i=1,2,\cdots,n)$, and then:

$$\begin{aligned}&\left(1+\left(\left(1-\mu_i\right)/\mu_i\right)\gamma\right)^p\left(1+\left(\left(1-\mu_j\right)/\mu_j\right)\gamma\right)^q-1\geq\\&\left(1+\left(v_i/\left(1-v_i\right)\right)\gamma\right)^p\left(1+\left(v_j/\left(1-v_j\right)\right)\gamma\right)^q-1\geq 0,\ \left(\forall i,\ j=1,2,\cdots,n\right)\end{aligned}$$

Thus $\mu+v\leq\mu+1\Bigg/\left(1+\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\left(1-\mu_i\right)/\mu_i\right)\gamma\right)^p\left(1+\left(\left(1-\mu_j\right)/\mu_j\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)=1$

Therefore, we obtain $0\leq\mu+v\leq 1$. As a result, $(\mu,\ v)$ is a IFN.

With respect to the parameter γ, some cases of NIFHHM operator are discussed.

(1) If γ=1, then the formula (46)and(47) follow that:

$$\mu=\left(1-\prod_{i=1,j=i}^{n}\left(1-\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \tag{48}$$

$$v=1-\left(1-\prod_{\substack{i=1\\j=i}}^{n}\left(1-\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \tag{49}$$

We call (43)-(45) and (48)-(49) normal intuitionistic fuzzy Heronian mean (NIFHM) operator.

(2) When γ=2, the formula (46) and (47) follow:

$$\mu=\frac{2\left(\prod_{i=1,j=i}^{n}\left(\left(2-\mu_i\right)^p\left(2-\mu_j\right)^q+3\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(\left(2-\mu_i\right)^p\left(2-\mu_j\right)^q-\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\begin{aligned}&\prod_{i=1,j=i}^{n}\left(\left(2-\mu_i\right)^p\left(2-\mu_j\right)^q+3\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}\\&+3\prod_{i=1,j=i}^{n}\left(\left(2-\mu_i\right)^p\left(2-\mu_j\right)^q-\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}\end{aligned}\right)^{\frac{1}{p+q}}+\left(\begin{aligned}&\prod_{i=1,j=i}^{n}\left(\left(2-\mu_i\right)^p\left(2-\mu_j\right)^q+3\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}\\&-\prod_{i=1,j=i}^{n}\left(\left(2-\mu_i\right)^p\left(2-\mu_j\right)^q-\mu_i^p\mu_j^q\right)^{\frac{2}{n(n+1)}}\end{aligned}\right)^{\frac{1}{p+q}}} \tag{50}$$

$$v=\frac{\left(\begin{aligned}&\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q+3\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\\&+3\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q-\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\end{aligned}\right)^{\frac{1}{p+q}}-\left(\begin{aligned}&\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q+3\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\\&-\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q-\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\end{aligned}\right)^{\frac{1}{p+q}}}{\left(\begin{aligned}&\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q+3\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\\&+3\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q-\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\end{aligned}\right)^{\frac{1}{p+q}}+\left(\begin{aligned}&\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q+3\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\\&-\prod_{i=1,j=i}^{n}\left(\left(1+v_i\right)^p\left(1+v_j\right)^q-\left(1-v_i\right)^p\left(1-v_j\right)^q\right)^{\frac{2}{n(n+1)}}\end{aligned}\right)^{\frac{1}{p+q}}} \tag{51}$$

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We call (43)-(45) and (50)-(51) normal intuitionistic fuzzy Einstein Heronian mean (NIFEHM) operator.

(3) When $p=q=\frac{1}{2}$, the NIFHHM operator degenerates into the normal intuitionistic fuzzy evolution Hamacher Heronian mean (NIFEHHM) operator.

$$NIFHHM^{\frac{1}{2},\frac{1}{2}}\left(A_1,A_2,\cdots,A_n\right)=$$

$$\frac{2}{n(n+1)}\left(\oplus_{H\,i=1,j=i}^{n}\left(A_i^{\frac{1}{2}}\otimes_H A_j^{\frac{1}{2}}\right)\right)=\left\langle\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{\alpha_i\alpha_j},\ \frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{\sigma_i^2\alpha_j/2\alpha_i+\sigma_j^2\alpha_i/2\alpha_j}\right)\right),\mu,v\right\rangle$$

$$\mu=\frac{\prod_{i=1,j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)(1-\mu_i)\right)\left(1+(\gamma-1)(1-\mu_j)\right)}+\left(\gamma^2-1\right)\sqrt{\mu_i\mu_j}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)(1-\mu_i)\right)\left(1+(\gamma-1)(1-\mu_j)\right)}-\sqrt{\mu_i\mu_j}\right)^{\frac{2}{n(n+1)}}}{\prod_{i=1,j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)(1-\mu_i)\right)\left(1+(\gamma-1)(1-\mu_j)\right)}+\left(\gamma^2-1\right)\sqrt{\mu_i\mu_j}\right)^{\frac{2}{n(n+1)}}+(\gamma-1)\prod_{i=1,j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)(1-\mu_i)\right)\left(1+(\gamma-1)(1-\mu_j)\right)}-\sqrt{\mu_i\mu_j}\right)^{\frac{2}{n(n+1)}}}$$

$$v=\frac{\gamma\prod_{i=1,\ j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)v_i\right)\left(1+(\gamma-1)v_j\right)}-\sqrt{(1-v_i)(1-v_j)}\right)^{\frac{2}{n(n+1)}}}{\prod_{i=1,\ j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)v_i\right)\left(1+(\gamma-1)v_j\right)}+\left(\gamma^2-1\right)\sqrt{(1-v_i)(1-v_j)}\right)^{\frac{2}{n(n+1)}}+(\gamma-1)\prod_{i=1,\ j=i}^{n}\left(\sqrt{\left(1+(\gamma-1)v_i\right)\left(1+(\gamma-1)v_j\right)}-\sqrt{(1-v_i)(1-v_j)}\right)^{\frac{2}{n(n+1)}}}$$

(4) If $q\to 0$, the NIFHHM operator reduces to generalized normal intuitionistic fuzzy Hamacher Arithmetical mean (NIFEHAM) operator.

$$NIFHHM^{p}\left(A_1,A_2,\cdots,A_n\right)=\left(\frac{1}{n}\left(\oplus_{H\,i=1}^{n}\left(A_i^p\right)\right)\right)^{\frac{1}{p}}=\left\langle\left(\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i^p\right)^{\frac{1}{p}},\left(\frac{1}{n}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\alpha_i^p\right)^{\frac{1}{p}-1}\left(\sum_{i=1}^{n}\alpha_i^{p-1}\sigma_i\right)\right),\mu,v\right\rangle$$

$$\mu=\frac{\gamma\left(\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^p+\left(\gamma^2-1\right)\mu_i^p\right)^{\frac{1}{n}}-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^p-\mu_i^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}}{\left\{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^p+\left(\gamma^2-1\right)\mu_i^p\right)^{\frac{1}{n}}\\+\left(\gamma^2-1\right)\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^p-\mu_i^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}+(\gamma-1)\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^p+\left(\gamma^2-1\right)\mu_i^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^p-\mu_i^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}\right\}}$$

$$v=\frac{\left\{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p+\left(\gamma^2-1\right)(1-v_i)^p\right)^{\frac{1}{n}}\\+\left(\gamma^2-1\right)\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p-(1-v_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}-\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p+\left(\gamma^2-1\right)(1-v_i)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p-(1-v_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}\right\}}{\left\{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p+\left(\gamma^2-1\right)(1-v_i)^p\right)^{\frac{1}{n}}\\+\left(\gamma^2-1\right)\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p-(1-v_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}+(\gamma-1)\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p+\left(\gamma^2-1\right)(1-v_i)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^p-(1-v_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}\right\}}$$

(5) If $p=1,q\to 0$, the NIFHHM operator is transformed into normal intuitionistic fuzzy Hamacher Arithmetical mean (NIFHAM) operator.

$$NIFHHM\left(A_1,A_2,\cdots,A_n\right)=\frac{1}{n}\left(\oplus_{H\,i=1}^{n}\left(A_i\right)\right)=\left\langle\left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i,\left(\frac{1}{n}\right)\left(\sum_{i=1}^{n}\sigma_i\right)\right),\frac{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_i\right)^{\frac{1}{n}}-\prod_{i=1}^{n}\left(1-\mu_i\right)^{\frac{1}{n}}}{\prod_{i=1}^{n}\left(1+(\gamma-1)\mu_i\right)^{\frac{1}{n}}+(\gamma-1)\prod_{i=1}^{n}\left(1-\mu_i\right)^{\frac{1}{n}}},\frac{\gamma\prod_{i=1}^{n}v_i^{\frac{1}{n}}}{\prod_{i=1}^{n}\left(\gamma+(1-\gamma)v_i\right)^{\frac{1}{n}}+(\gamma-1)\prod_{i=1}^{n}v_i^{\frac{1}{n}}}\right\rangle$$

The NIFHHM operator is generalization of the above five operators, and it has some properties as follows.

Theorem 3 (Idempotency). If all $A_i=A=\left\langle(\alpha,\sigma),\mu,v\right\rangle$ $(i=1,2,\cdots,n)$ then:

$$NIIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHHM^{p,q}\left(A,A,\cdots,\ A\right)=A$$

Proof. According to Theorem 1, we have:

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$$NIIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) = NIFHHM^{p,q}\left(A, A, \cdots, A\right)$$
$$= \left(\frac{2}{n(n+1)}\left(\oplus_{H\, i=1, j=i}^{n}\left(A^p \otimes_H A^q\right)\right)\right)^{\frac{1}{p+q}} = \left(\frac{2}{n(n+1)}\left(\oplus_{H\, i=1, j=i}^{n}\left(A^{p+q}\right)\right)\right)^{\frac{1}{p+q}} = \frac{2}{n(n+1)}\frac{n(n+1)}{2}\left(A^{p+q}\right)^{\frac{1}{p+q}} = A$$

Theorem 4 (Permutation). If $\dot{A}_i\left(i=1,2,\cdots,n\right)$ is any permutation of $A_i\left(i=1,2,\cdots,n\right)$ then:

$$NIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) = NIFHHM^{p,q}\left(\dot{A}_1, \dot{A}_2, \cdots, \dot{A}_n\right)$$

Proof. By Theorem 1, we have:

$$NIIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) =$$
$$\left(\frac{2}{n(n+1)}\left(\oplus_{H\, i=1, j=i}^{n}\left(A_i^p \otimes_H A_j^q\right)\right)\right)^{\frac{1}{p+q}} = \left(\frac{2}{n(n+1)}\left(\oplus_{H\, i=1, j=i}^{n}\left(\left(\dot{A}_i\right)^p \otimes_H \left(\dot{A}_j\right)^q\right)\right)\right)^{\frac{1}{p+q}} = NIIFHHM^{p,q}\left(\dot{A}_1, \dot{A}_2, \cdots, \dot{A}_n\right)$$

Theorem 5 (Monotonicity). Let $p \ge 0, q \ge 0$ and p, q don't simultaneously equal the value of zero, $A_i = \left\langle\left(\alpha_{A_i}, \sigma_{A_i}\right), \mu_{A_i}, \upsilon_{A_i}\right\rangle$, $B_i = \left\langle\left(\alpha_{B_i}, \sigma_{B_i}\right), \mu_{B_i}, \upsilon_{B_i}\right\rangle$ $i = 1,2,\cdots,n$, are two sets of NIFNs, if the following conditions hold,

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q = \sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q, \quad \mu_{A_i} \le \mu_{B_i}, \quad \upsilon_{A_i} \ge \upsilon_{B_i} \quad \left(i=1,2,\cdots,n\right) \tag{52}$$

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\alpha_{A_i}^{2p-2}\alpha_{A_j}^{2q}\sigma_{A_i}^2 + q\alpha_{A_i}^{2p}\alpha_{A_j}^{2q-2}\sigma_{A_j}^2} = \sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\alpha_{B_i}^{2p-2}\alpha_{B_j}^{2q}\sigma_{B_i}^2 + q\alpha_{B_i}^{2p}\alpha_{B_j}^{2q-2}\sigma_{B_j}^2} \tag{53}$$

then for $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q = \sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q \ge 0$

$$NIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) \le NIFHHM^{p,q}\left(B_1, B_2, \cdots, B_n\right) \tag{54}$$

for $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q = \sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q < 0$

$$NIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) \ge NIFHHM^{p,q}\left(B_1, B_2, \cdots, B_n\right) \tag{55}$$

Proof. Let $NIFHHM^{p,q}\left(A_1, A_2, \cdots, A_n\right) = \left\langle\left(\alpha_A, \sigma_A\right), \mu_A, \upsilon_A\right\rangle$, $\alpha_A = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q\right)^{\frac{1}{p+q}}$

$$\sigma_A = \left(\tfrac{1}{p+q}\right)^{\frac{1}{2}}\left(\tfrac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left|\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q\right|^{\frac{1}{p+q}-1} \times \left(\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\alpha_{A_i}^{2p-2}\alpha_{A_j}^{2q}\sigma_{A_i}^2 + q\alpha_{A_i}^{2p}\alpha_{A_j}^{2q-2}\sigma_{A_j}^2}\right)$$

$$\mu_A = \frac{\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} + \left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} + \left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij} = \left(1+\left(\gamma-1\right)\left(1-\mu_{A_i}\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-\mu_{A_j}\right)\right)^q \quad W_{ij} = \mu_{A_i}^p\mu_{A_j}^q$$

$$\upsilon_A = \frac{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} + \left(\gamma^2-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} - \left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} + \left(\gamma^2-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} + \left(\gamma-1\right)\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij} = \left(1+\left(\gamma-1\right)\upsilon_{A_i}\right)^p\left(1+\left(\gamma-1\right)\upsilon_{A_j}\right)^q \quad W_{ij} = \left(1-\upsilon_{A_i}\right)^p\left(1-\upsilon_{A_j}\right)^q$$

and

$$NIFHHM^{p,q}\left(B_1, B_2, \cdots, B_n\right) = \left\langle\left(\alpha_B, \sigma_B\right), \mu_B, \upsilon_B\right\rangle,$$

$$\alpha_B = \left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q\right)^{\frac{1}{p+q}}$$

$$\sigma_B = \left(\left(\tfrac{1}{p+q}\right)^{\frac{1}{2}}\left(\tfrac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left|\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q\right|^{\frac{1}{p+q}-1} \times \left(\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\alpha_{B_i}^{2p-2}\alpha_{B_j}^{2q}\sigma_{B_i}^2 + q\alpha_{B_i}^{2p}\alpha_{B_j}^{2q-2}\sigma_{B_j}^2}\right)\right)$$

$$\mu_B=\frac{\gamma\left(\prod\limits_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod\limits_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod\limits_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+(\gamma-1)\left(\prod\limits_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij}=\left(1+(\gamma-1)(1-\mu_{B_i})\right)^p\left(1+(\gamma-1)(1-\mu_{B_j})\right)^q\quad W_{ij}=\mu_{B_i}^p\mu_{B_j}^q$$

$$\upsilon_B=\frac{\left(\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+(\gamma-1)\left(\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij}=\left(1+(\gamma-1)\upsilon_{B_i}\right)^p\left(1+(\gamma-1)\upsilon_{B_j}\right)^q\quad W_{ij}=\left(1-\upsilon_{B_i}\right)^p\left(1-\upsilon_{B_j}\right)^q$$

we consider the following conditions.

(1) If there exist at least μ_{A_k} with $\mu_{A_k}=1$,then according to the condition (52), $\mu_{B_k}=1$, $\upsilon_{A_k}=0$ and $\upsilon_{B_k}=0$. Moreover, $\mu_A=\mu_B=1$, $\upsilon_A=\upsilon_B=0$, and we obtain $\alpha_A=\alpha_B$, thus

$$S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A-\upsilon_A\right)=\alpha_B\left(\mu_B-\upsilon_B\right)=S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A+\upsilon_A\right)=\alpha_B\left(\mu_B+\upsilon_B\right)=H_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

From the conditions (52) and (53), we have $\sigma_A=\sigma_B$, and

$$S_2\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A-\upsilon_A\right)=\sigma_B\left(\mu_B-\upsilon_B\right)=S_2\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_2\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A+\upsilon_A\right)=\sigma_B\left(\mu_B+\upsilon_B\right)=H_2\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

thus for $\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q=\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q\geq 0$ or $\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q=\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q<0$, the following equation holds.

$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

(2) If there exist at least μ_{B_k} with $\mu_{B_k}=0$, then $\mu_{A_k}=0$, $\upsilon_{A_k}=1$, and $\upsilon_{B_k}=1$. For the condition with $\mu_{A_k}=0$ and $\upsilon_{A_k}=1$, we can obtain

$$V_{kj}=\gamma^p\left(1+(\gamma-1)(1-\mu_{A_j})\right)^q,\ \left(k\leq j\leq n\right)$$

$$V_{ik}=\gamma^q\left(1+(\gamma-1)(1-\mu_{A_i})\right)^p,\ \left(1\leq i\leq k\right)$$

$$W_{kj}=W_{ik}=0,\ \left(1\leq i\leq k\leq j\leq n\right)$$

thus

$$\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}=\left(\prod_{j=k}^{n}\gamma^p\left(1+(\gamma-1)(1-\mu_{A_j})\right)^q\right)\left(\prod_{i=1,i\neq k,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)$$

$$\prod_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}=\left(\prod_{j=k}^{n}\gamma^p\left(1+(\gamma-1)(1-\mu_{A_j})\right)^q\right)\left(\prod_{i=1,i\neq k,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}\right)$$

Therefore

$$\mu_A=\frac{\gamma\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\begin{array}{l}\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\+(\gamma-1)\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}}$$
$$V_{ij}=\left(1+(\gamma-1)(1-\mu_{A_i})\right)^p\left(1+(\gamma-1)(1-\mu_{A_j})\right)^q\quad W_{ij}=\mu_{A_i}^p\mu_{A_j}^q$$

$$\upsilon_A=\frac{\begin{array}{l}\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\-\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}}{\begin{array}{l}\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\+(\gamma-1)\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}}$$
$$V_{ij}=\left(1+(\gamma-1)\upsilon_{A_i}\right)^p\left(1+(\gamma-1)\upsilon_{A_j}\right)^q\quad W_{ij}=\left(1-\upsilon_{A_i}\right)^p\left(1-\upsilon_{A_j}\right)^q$$

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The above similar process is applied to calculate the value of $NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$, we have the following result.

$$\mu_B=\frac{\gamma\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-\mu_{B_i}\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-\mu_{B_j}\right)\right)^q \quad W_{ij}=\mu_{B_i}^p\mu_{B_j}^q$$

$$\upsilon_B=\frac{\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\upsilon_{B_i}\right)^p\left(1+\left(\gamma-1\right)\upsilon_{B_j}\right)^q \quad W_{ij}=\left(1-\upsilon_{B_i}\right)^p\left(1-\upsilon_{B_j}\right)^q$$

We can easily notice that the above equations respectively have the same characteristic as Equation (46) and (47) in each product part. The different part of these equations is only that the symbol $\prod\limits_{i=1,j=i}^{n}$ shows $n(n+1)/2$ factors, but $\prod\limits_{i=1,i\neq k,j=i,j\neq k}^{n}$ has $n(n+1)/2-n$ terms.

Based on the analysis, we assume $0<\mu_{A_i}<1$, $0<\mu_{B_i}<1$, $0<\upsilon_{A_i}<1$ and $0<\upsilon_{B_i}<1$ for any $i=1,2,\cdots,n$. Thereupon, μ_A, υ_A, μ_B and υ_B are transformed as follows.

$$\mu_A=\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\left(1-\mu_{A_i}\right)/\mu_{A_i}\right)\gamma\right)^p\left(1+\left(\left(1-\mu_{A_j}\right)/\mu_{A_j}\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\gamma-1\right)$$

$$\upsilon_A=1\Bigg/\left(1+\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\upsilon_{A_i}/\left(1-\upsilon_{A_i}\right)\right)\gamma\right)^p\left(1+\left(\upsilon_{A_j}/\left(1-\upsilon_{A_j}\right)\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)$$

$$\mu_B=\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\left(1-\mu_{B_i}\right)/\mu_{B_i}\right)\gamma\right)^p\left(1+\left(\left(1-\mu_{B_j}\right)/\mu_{B_j}\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\gamma-1\right)$$

$$\upsilon_B=1\Bigg/\left(1+\gamma\Bigg/\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\upsilon_{B_i}/\left(1-\upsilon_{B_i}\right)\right)\gamma\right)^p\left(1+\left(\upsilon_{B_j}/\left(1-\upsilon_{B_j}\right)\right)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)$$

By the conditions $0<\mu_{A_i}\leq\mu_{B_i}<1$, $0<\upsilon_{A_i}\leq\upsilon_{B_i}<1,\left(i=1,2,\cdots,n\right)$, we have $0<\mu_{A_i}+\upsilon_{A_i}<1$ and $0<\mu_{B_i}+\upsilon_{B_i}<1$, $\left(i=1,2,\cdots,n\right)$, and then, for any $\gamma>0$,

$$\left(1-\mu_{A_i}\right)/\mu_{A_i}\geq\left(1-\mu_{B_i}\right)/\mu_{B_i},\quad \upsilon_{A_i}/\left(1-\upsilon_{A_i}\right)\geq\upsilon_{B_i}/\left(1-\upsilon_{B_i}\right),\quad \left(i=1,2,\cdots,n\right)$$

Moreover $\mu_A\leq\mu_B$ and $\upsilon_A\geq\upsilon_B$, and according to the condition (52), $\alpha_A=\alpha_B$ is right. Thus

(a) For $\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q=\sum\limits_{i=1}^{n}\sum\limits_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q\geq 0$ i.e. $\alpha_A=\alpha_B\geq 0$

$$S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A-\upsilon_A\right)\leq\alpha_B\left(\mu_B-\upsilon_B\right)=S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

If $S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)<S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, then

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$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)<NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

If $S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, i.e. $\alpha_A\left(\mu_A-v_A\right)=\alpha_B\left(\mu_B-v_B\right)$. By the conditions $\alpha_A=\alpha_B\geq 0,\mu_A-v_A\leq\mu_B-v_B$, we can get $\mu_A-v_A=\mu_B-v_B$. From $\mu_A\leq\mu_B$ and $v_A\geq v_B$, we have $\mu_A=\mu_B$ and $v_A=v_B$, and then

$$H_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A+v_A\right)=\alpha_B\left(\mu_B+v_B\right)=H_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

According to the conditions (52) and (53), we get $\sigma_A=\sigma_B>0$, thus

$$S_2\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A-v_A\right)=\sigma_B\left(\mu_B-v_B\right)=S_2\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_2\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A+v_A\right)=\sigma_B\left(\mu_B+v_B\right)=H_2\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right),$$

therefore

$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

That is, when the conditions (52) and (53) is satisfied and $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q=\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q\geq 0$,

$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

(b) For $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q=\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q<0$ i.e. $\alpha_A=\alpha_B<0$

$$S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A-v_A\right)\geq\alpha_B\left(\mu_B-v_B\right)=S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

If $S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)>S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, then

$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)>NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

If $S_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=S_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, i.e. $\alpha_A\left(\mu_A-v_A\right)=\alpha_B\left(\mu_B-v_B\right)$. By the conditions $\alpha_A=\alpha_B,\mu_A-v_A\leq\mu_B-v_B$, we can get $\mu_A-v_A=\mu_B-v_B$. From $\mu_A\leq\mu_B$ and $v_A\geq v_B$, we have $\mu_A=\mu_B$ and $v_A=v_B$, and then

$$H_1\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A+v_A\right)=\alpha_B\left(\mu_B+v_B\right)=H_1\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

According to the conditions (52) and (53), we get $\sigma_A=\sigma_B>0$, thus

$$S_2\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A-v_A\right)=\sigma_B\left(\mu_B-v_B\right)=S_2\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_2\left(NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A+v_A\right)=\sigma_B\left(\mu_B+v_B\right)=H_2\left(NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right),$$

therefore

$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

That is, when the conditions (52) and (53) is satisfied and $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{A_i}^p\alpha_{A_j}^q=\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_{B_i}^p\alpha_{B_j}^q<0$,

$$NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\geq NIFHHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

Theorem 6 (Boundary). Let $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ be a set of NIFNs, and

$$A^-=\left\langle\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^p\alpha_j^q\right)^{\frac{1}{p+q}},\ \left(\frac{1}{p+q}\right)^{\frac{1}{2}}\left(\frac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^p\alpha_j^q\right)^{\frac{1}{p+q}-1}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\alpha_i^{2p-2}\alpha_j^{2q}\sigma_i^2+q\alpha_i^{2p}\alpha_j^{2q-2}\sigma_j^2}\right),\ \min_i\{\mu_i\},\ \max_i\{v_i\}\right\rangle$$

$$A^+=\left\langle\left(\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^p\alpha_j^q\right)^{\frac{1}{p+q}},\ \left(\frac{1}{p+q}\right)^{\frac{1}{2}}\left(\frac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^p\alpha_j^q\right)^{\frac{1}{p+q}-1}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\alpha_i^{2p-2}\alpha_j^{2q}\sigma_i^2+q\alpha_i^{2p}\alpha_j^{2q-2}\sigma_j^2}\right),\ \max_i\{\mu_i\},\ \min_i\{v_i\}\right\rangle$$

Then when $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^p\alpha_j^q\geq 0$, $A^-\leq NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq A^+$

when $\sum_{i=1}^{n}\sum_{j=i}^{n}\alpha_i^p\alpha_j^q<0$, $A^+\leq NIFHHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq A^-$

The boundary is distinctly derived by the monotonicity, the proof is omitted.

From the aforementioned analysis, we can see that the NIFHHM operator do not involve the significance of the input data. However, the importance of the attributes should be considered in re-

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alization, so the weighted form of the NIFHHM operator is defined.

Definition 8. If $A_i=\langle(\alpha_i,\sigma_i),\mu_i,v_i\rangle$ $(i=1,2,\cdots,n)$ is a collection of NIFNs and ω_i is the weight of A_i, $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$, then normal intuitionistic fuzzy Hamacher weighted Heronian mean operator (NIFHWHM) is defined:

$$NIFHWHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\left(\frac{2}{n(n+1)}\bigoplus_{H\,i=1,j=i}^{n}\left(\left(\omega_iA_i\right)^p\otimes_H\left(\omega_jA_j\right)^q\right)\right)^{\frac{1}{p+q}} \tag{56}$$

Theorem 7. If $A_i=\langle(\alpha_i,\sigma_i),\mu_i,v_i\rangle$ $(i=1,2,\cdots,n)$ is a collection of NIFNs and $p,q\geq 0$, then the aggregation result of the Equation (56) is a NIFN and

$$NIFHWHM_\omega^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\langle(\alpha,\sigma),\mu,v\rangle \tag{57}$$

$$\alpha=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\omega_i\alpha_i\right)^p\left(\omega_j\alpha_j\right)^q\right)^{\frac{1}{p+q}} \tag{58}$$

$$\sigma=\left(\frac{1}{p+q}\right)^{\frac{1}{2}}\left(\frac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\omega_i\alpha_i\right)^p\left(\omega_j\alpha_j\right)^q\right)^{\frac{1}{p+q}-1}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\omega_i^{2p}\alpha_i^{2p-2}\alpha_j^{2q}\sigma_i^2+q\alpha_i^{2p}\omega_j^{2q}\alpha_j^{2q-2}\sigma_j^2} \tag{59}$$

$$\mu=(46)\quad V_{ij}=\left(\frac{\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+\left(\gamma^2-1\right)\left(1-\mu_i\right)^{\omega_i}}{\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma-1)\left(1-\mu_i\right)^{\omega_i}}\right)^p\left(\frac{\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+\left(\gamma^2-1\right)\left(1-\mu_j\right)^{\omega_j}}{\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+(\gamma-1)\left(1-\mu_j\right)^{\omega_j}}\right)^q\quad W_{ij}=\left(\frac{\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-\left(1-\mu_i\right)^{\omega_i}}{\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma-1)\left(1-\mu_i\right)^{\omega_i}}\right)^p\left(\frac{\left(1+(\gamma-1)\mu_j\right)^{\omega_j}-\left(1-\mu_j\right)^{\omega_j}}{\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+(\gamma-1)\left(1-\mu_j\right)^{\omega_j}}\right)^q \tag{60}$$

$$v=(47)\quad V_{ij}=\left(\frac{\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+\left(\gamma^2-1\right)v_i^{\omega_i}}{\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma-1)v_i^{\omega_i}}\right)^p\left(\frac{\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}+\left(\gamma^2-1\right)v_j^{\omega_j}}{\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}+(\gamma-1)v_j^{\omega_j}}\right)^q\quad W_{ij}=\left(\frac{\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}}{\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma-1)v_i^{\omega_i}}\right)^p\left(\frac{\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}-v_j^{\omega_j}}{\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}+(\gamma-1)v_j^{\omega_j}}\right)^q \tag{61}$$

Proof. We utilize Lemma1, Lemma2, Lemma3 and the operational laws (21) to get the Equation (57). Consequently, we need prove $0\leq\mu\leq 1$, $0\leq v\leq 1$ and $0\leq\mu+v\leq 1$. i.e. For the ordinal pair $(\mu,\ v)$, $NIFHWHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)$ is a NIFN. According to operational law (21), for any $i=1,2,\cdots,n$, ω_iA_i is a NIFN and has the following situation.

$$\omega_iA_i=\langle(\alpha_i,\sigma_i),\mu_i,v_i\rangle=\left\langle(\omega_i\alpha_i,\omega_i\sigma_i),\frac{\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-\left(1-\mu_i\right)^{\omega_i}}{\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma-1)\left(1-\mu_i\right)^{\omega_i}},\frac{\gamma v_i^{\omega_i}}{\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma-1)v_i^{\omega_i}}\right\rangle$$

Thus the Equations (60) and (61) are transformed into the following formulas.

$$\mu=(46)\quad V_{ij}=\left(1+(\gamma-1)\left(1-\mu_i\right)\right)^p\left(1+(\gamma-1)\left(1-\mu_j\right)\right)^q\quad W_{ij}=\tilde{\mu}_i^{\,p}\,\tilde{\mu}_j^{\,q}$$

$$v=(47)\quad V_{ij}=\left(1+(\gamma-1)v_i\right)^p\left(1+(\gamma-1)v_j\right)^q\quad W_{ij}=\left(1-v_i\right)^p\left(1-v_j\right)^q$$

By the similar process of Theorem 2, We can get that $(\mu,\ v)$ is a IFN.

It is prone to notice that NIFHWHM operator involves the monotonicity and boundedness, but has not the property of commutativity and idempotency.

Theorem 8. Let $p\geq 0,q\geq 0$ and p,q isn't simultaneously equal to zero, $A_i=\langle(\alpha_{A_i},\sigma_{A_i}),\mu_{A_i},v_{A_i}\rangle$, $B_i=\langle(\alpha_{B_i},\sigma_{B_i}),\mu_{B_i},v_{B_i}\rangle$ $i=1,2,\cdots,n$ are NIFNs, and $\omega_i\,(i=1,2,\cdots n)$ is the weight of $A_i\,(i=1,2,\cdots,n)$, $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$. If the following conditions is satisfied as follows,

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_i^p\alpha_{A_i}^p\omega_j^q\alpha_{A_j}^q=\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_i^p\alpha_{B_i}^p\omega_j^q\alpha_{B_j}^q\ ,\quad \mu_{A_i}\leq\mu_{B_i}\ ,\quad v_{A_i}\geq v_{B_i}\quad (i=1,2,\cdots,n) \tag{62}$$

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\omega_i^{2p}\alpha_{A_i}^{2p-2}\alpha_{A_j}^{2q}\sigma_{A_i}^2+q\alpha_{A_i}^{2p}\omega_j^{2q}\alpha_{A_j}^{2q-2}\sigma_{A_j}^2}=\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\omega_i^{2p}\alpha_{B_i}^{2p-2}\alpha_{B_j}^{2q}\sigma_{B_i}^2+q\alpha_{B_i}^{2p}\omega_j^{2q}\alpha_{B_j}^{2q-2}\sigma_{B_j}^2} \tag{63}$$

then for $\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_i^p\alpha_{A_i}^p\omega_j^q\alpha_{A_j}^q-\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_i^p\alpha_{B_i}^p\omega_j^q\alpha_{B_j}^q\geq 0$

$$NIFHWHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHWHM_w^{p,q}\left(B_1,B_2,\cdots,B_n\right) \tag{64}$$

for $\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_i^p\alpha_{A_i}^p\omega_j^q\alpha_{A_j}^q-\sum_{i=1}^{n}\sum_{j=i}^{n}\omega_i^p\alpha_{B_i}^p\omega_j^q\alpha_{B_j}^q<0$

$$NIFHWHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)\geq NIFHWHM_w^{p,q}\left(B_1,B_2,\cdots,B_n\right) \tag{65}$$

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Proof. By the operational law (21), $\omega_i A_i$ and $\omega_i B_i$ $(i=1,2,\cdots,n)$ are NIFNs , and we can obtain:

$$\omega_i A_i=\left\langle\left(\alpha_{A_i},\sigma_{A_i}\right),\mu_{A_i},\upsilon_{A_i}\right\rangle=\left\langle\left(\omega_i\alpha_{A_i},\omega_i\sigma_{A_i}\right),\frac{\left(1+(\gamma-1)\mu_{A_i}\right)^{\omega_i}-\left(1-\mu_{A_i}\right)^{\omega_i}}{\left(1+(\gamma-1)\mu_{A_i}\right)^{\omega_i}+(\gamma-1)\left(1-\mu_{A_i}\right)^{\omega_i}},\frac{\gamma\upsilon_{A_i}^{\omega_i}}{\left(1+(\gamma-1)\left(1-\upsilon_{A_i}\right)\right)^{\omega_i}+(\gamma-1)\upsilon_{A_i}^{\ \omega_i}}\right\rangle$$

$$\omega_i B_i=\left\langle\left(\alpha_{B_i},\sigma_{B_i}\right),\mu_{B_i},\upsilon_{B_i}\right\rangle=\left\langle\left(\omega_i\alpha_{B_i},\omega_i\sigma_{B_i}\right),\frac{\left(1+(\gamma-1)\mu_{B_i}\right)^{\omega_i}-\left(1-\mu_{B_i}\right)^{\omega_i}}{\left(1+(\gamma-1)\mu_{B_i}\right)^{\omega_i}+(\gamma-1)\left(1-\mu_{B_i}\right)^{\omega_i}},\frac{\gamma\upsilon_{B_i}^{\omega_i}}{\left(1+(\gamma-1)\left(1-\upsilon_{B_i}\right)\right)^{\omega_i}+(\gamma-1)\upsilon_{B_i}^{\ \omega_i}}\right\rangle$$

For the part of $NIFHWHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\left\langle\left(\alpha_A,\sigma_A\right),\mu_A,\upsilon_A\right\rangle$, we have:

$$\alpha_A=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\omega_i\alpha_{A_i}\right)^p\left(\omega_j\alpha_{A_j}\right)^q\right)^{\frac{1}{p+q}}$$

$$\sigma_A=\left(\frac{1}{p+q}\right)^{\frac{1}{2}}\left(\frac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\omega_i\alpha_{A_i}\right)^p\left(\omega_j\alpha_{A_j}\right)^q\right)^{\frac{1}{p+q}-1}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\omega_i^{2p}\alpha_{A_i}^{\ 2p-2}\alpha_{A_j}^{\ 2q}\sigma_{A_i}^{\ 2}+q\alpha_{A_i}^{\ 2p}\omega_j^{2q}\alpha_{A_j}^{\ 2q-2}\sigma_{A_j}^{\ 2}}$$

$$\mu_A=(46)\quad V_{ij}=\left(1+(\gamma-1)\left(1-\mu_{A_i}\right)\right)^p\left(1+(\gamma-1)\left(1-\mu_{A_j}\right)\right)^q\quad W_{ij}=\tilde{\mu}_{A_i}^{\,p}\tilde{\mu}_{A_j}^{\,q}$$

$$\upsilon_A=(47)\quad V_{ij}=\left(1+(\gamma-1)\upsilon_{A_i}\right)^p\left(1+(\gamma-1)\upsilon_{A_j}\right)^q\quad W_{ij}=\left(1-\upsilon_{A_i}\right)^p\left(1-\upsilon_{A_j}\right)^q$$

For the part of $NIFHWHM_w^{p,q}\left(B_1,B_2,\cdots,B_n\right)=\left\langle\left(\alpha_B,\sigma_B\right),\mu_B,\upsilon_B\right\rangle$, we have:

$$\alpha_B=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\omega_i\alpha_{B_i}\right)^p\left(\omega_j\alpha_{B_j}\right)^q\right)^{\frac{1}{p+q}}$$

$$\sigma_B=\left(\frac{1}{p+q}\right)^{\frac{1}{2}}\left(\frac{2}{n(n+1)}\right)^{\frac{1}{p+q}}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\omega_i\alpha_{B_i}\right)^p\left(\omega_j\alpha_{B_j}\right)^q\right)^{\frac{1}{p+q}-1}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{p\omega_i^{2p}\alpha_{B_i}^{\ 2p-2}\alpha_{B_j}^{\ 2q}\sigma_{B_i}^{\ 2}+q\alpha_{B_i}^{\ 2p}\omega_j^{2q}\alpha_{B_j}^{\ 2q-2}\sigma_{B_j}^{\ 2}}$$

$$\mu_B=(46)\quad V_{ij}=\left(1+(\gamma-1)\left(1-\mu_{B_i}\right)\right)^p\left(1+(\gamma-1)\left(1-\mu_{B_j}\right)\right)^q\quad W_{ij}=\tilde{\mu}_{B_i}^{\,p}\tilde{\mu}_{B_j}^{\,q}$$

$$\upsilon_B=(47)\quad V_{ij}=\left(1+(\gamma-1)\upsilon_{B_i}\right)^p\left(1+(\gamma-1)\upsilon_{B_j}\right)^q\quad W_{ij}=\left(1-\upsilon_{B_i}\right)^p\left(1-\upsilon_{B_j}\right)^q$$

In the following, we will respectively prove $\mu_i(\mu_i)$ and $\upsilon_i(\upsilon_i)$ are monotonically increasing with respect to independent variable.

(1) For the part of μ_i .

Taking the derivative of μ_i with respect to μ_i, we get:

$$\frac{d\mu_i}{d\mu_i}=\frac{\gamma\left(\omega_i\left(1+(\gamma-1)\mu_i\right)^{\omega_i-1}(\gamma-1)+\omega_i\left(1-\mu_i\right)^{\omega_i-1}+\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-\left(1-\mu_i\right)^{\omega_i}\right)}{\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma-1)\left(1-\mu_i\right)^{\omega_i}\right)^2}$$
$$=\frac{\gamma\left(\omega_i\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i-1}\gamma+\left(1-\mu_i\right)^{\omega_i-1}-\left(1+(\gamma-1)\mu_i\right)^{\omega_i-1}\right)+\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-\left(1-\mu_i\right)^{\omega_i}\right)\right)}{\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma-1)\left(1-\mu_i\right)^{\omega_i}\right)^2}$$

By the conditions $\omega_i\in[0,1]$, $\gamma>0$, $\mu_i\in[0,1]$, the following inequalities are right.

$$\left(1+(\gamma-1)\mu_i\right)^{\omega_i-1}>0,\ \left(1-\mu_i\right)^{\omega_i-1}-\left(1+(\gamma-1)\mu_i\right)^{\omega_i-1}>0,\ \left(1+(\gamma-1)\mu_i\right)^{\omega_i}-\left(1-\mu_i\right)^{\omega_i}>0$$

Therefore $\frac{d\mu_i}{d\mu_i}>0$, in other words, μ_i is monotonically increasing with respect to μ_i .

(2) For the part of υ_i .

Taking the derivative of υ_i with respect to μ_i, we get:

$$\frac{d\upsilon_i}{d\upsilon_i}=\frac{\gamma\omega_i\upsilon_i^{\omega_i-1}\left(1+(\gamma-1)\left(1-\upsilon_i\right)\right)^{\omega_i}+\gamma\omega_i\upsilon_i^{\omega_i}\left(1+(\gamma-1)\left(1-\upsilon_i\right)\right)^{\omega_i-1}(\gamma-1)}{\left(\left(1+(\gamma-1)\left(1-\upsilon_i\right)\right)^{\omega_i}+(\gamma-1)\upsilon_i^{\omega_i}\right)^2}$$
$$=\frac{\gamma\omega_i\upsilon_i^{\omega_i-1}\left(1+(\gamma-1)\left(1-\upsilon_i\right)\right)^{\omega_i-1}\left(1+(\gamma-1)\left(1-\upsilon_i\right)+\upsilon_i(\gamma-1)\right)}{\left(\left(1+(\gamma-1)\left(1-\upsilon_i\right)\right)^{\omega_i}+(\gamma-1)\upsilon_i^{\omega_i}\right)^2}=\frac{\gamma^2\omega_i\upsilon_i^{\omega_i-1}}{\left(\left(1+(\gamma-1)\left(1-\upsilon_i\right)\right)^{\omega_i}+(\gamma-1)\upsilon_i^{\omega_i}\right)^2}>0$$

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Thus, v_i is monotonically increasing with respect to v_i. Therefore, we utilize the similar process in Theorem 5 to get the proof of the Theorem8.

Theorem 9 (Boundary). Let $A_i = \left\langle(\alpha_i,\sigma_i),\mu_i,v_i\right\rangle$ be a set of NIFNs, and $\omega_i\,(i=1,2,\cdots n)$ is the weight of $A_i\,(i=1,2,\cdots,n)$, $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$.

$$A_i^- = \left\langle(\alpha_i,\sigma_i),\min_i\{\mu_i\},\max_i\{v_i\}\right\rangle,\quad A_i^+ = \left\langle(\alpha_i,\sigma_i),\max_i\{\mu_i\},\min_i\{v_i\}\right\rangle$$

Then when $\sum_{i=1}^{n}\sum_{j=i}^{n}(\omega_i\alpha_i)^p(\omega_j\alpha_j)^q\geq 0$,

$$NIFHWHM_w^{p,q}\left(A_1^-,A_2^-,\cdots,A_n^-\right)\leq NIFHWHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHWHM_w^{p,q}\left(A_1^+,A_2^+,\cdots,A_n^+\right)$$

when $\sum_{i=1}^{n}\sum_{i=1}^{n}\sum_{j=i}^{n}(\omega_i\alpha_i)^p(\omega_j\alpha_j)^q<0$,

$$NIFHWHM_w^{p,q}\left(A_1^+,A_2^+,\cdots,A_n^+\right)\leq NIFHWHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHWHM_w^{p,q}\left(A_1^-,A_2^-,\cdots,A_n^-\right)$$

The boundary is distinctly corollary of the monotonicity, the proof is omitted.

The NIFHWHM operator also has some cases.

(1) When the parameter γ=1,

$$\mu=\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-(1-\mu_i)^{w_i}\right)^p\times\left(1-(1-\mu_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \quad (66)$$

$$v=1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-v_i^{w_i}\right)^p\left(1-v_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \quad (67)$$

We call (57)-(59) and (66)-(67) normal intuitionistic fuzzy weighted Heronian mean (NIFWHM) operator.

(2) If γ=2, then

$$\mu=\frac{2\left(\prod_{i=1,j=i}^{n}\left(\left((1+\mu_i)^{\omega_i}+3(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}+3(1-\mu_j)^{\omega_j}\right)^q+3\left((1+\mu_j)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(\left((1+\mu_i)^{\omega_i}+3(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}+3(1-\mu_j)^{\omega_j}\right)^q-\left((1+\mu_i)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,j=i}^{n}\left(\left((1+\mu_i)^{\omega_i}+3(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}+3(1-\mu_j)^{\omega_j}\right)^q+3\left((1+\mu_j)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}+3\prod_{i=1,j=i}^{n}\left(\left((1+\mu_i)^{\omega_i}+3(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}+3(1-\mu_j)^{\omega_j}\right)^q-\left((1+\mu_i)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\prod_{i=1,j=i}^{n}\left(\left((1+\mu_i)^{\omega_i}+3(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}+3(1-\mu_j)^{\omega_j}\right)^q+3\left((1+\mu_j)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(\left((1+\mu_i)^{\omega_i}+3(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}+3(1-\mu_j)^{\omega_j}\right)^q-\left((1+\mu_j)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^p\left((1+\mu_j)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}} \quad (68)$$

$$v=\frac{\left(\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q+3\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}+3\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q-\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q+3\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}-\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q-\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q+3\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}+3\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q-\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q+3\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}-\prod_{\substack{i=1\\j=i}}^{n}\left(\left((2-v_i)^{\omega_i}+3v_i^{\omega_i}\right)^p\left((2-v_i)^{\omega_j}+3v_j^{\omega_j}\right)^q-\left((2-v_i)^{\omega_i}-v_i^{\omega_i}\right)^p\left((2-v_j)^{\omega_j}-v_j^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}} \quad (69)$$

We call (57)-(59) and (68)-(69) normal intuitionistic fuzzy Einstein weighted Heronian mean (NIFEWHM) operator.

(3) When $p=q=\frac{1}{2}$, the NIFHWHM operator degenerates into the normal intuitionistic fuzzy evolution Hamacher weight Heronian mean (NIFEHWHM) operator.

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$$NIFHWHM^{\frac{1}{2},\frac{1}{2}}\left(A_1,A_2,\cdots,A_n\right)=\frac{2}{n(n+1)}\left(\bigoplus_{H,\,i=1,j=i}^{n}\left(\left(\omega_iA_i\right)^{\frac{1}{2}}\otimes_H\left(\omega_jA_j\right)^{\frac{1}{2}}\right)\right)$$
$$=\left\langle\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{\omega_i\alpha_i\omega_j\alpha_j},\ \frac{2}{n(n+1)}\left(\sum_{i=1}^{n}\sum_{j=i}^{n}\sqrt{\sigma_i^2\omega_i\omega_j\alpha_j/2\alpha_i+\sigma_j^2\omega_i\omega_j\alpha_i/2\alpha_j}\right)\right),\mu,v\right\rangle$$

$$\mu=\frac{\begin{gathered}\prod_{i=1,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+(\gamma^2-1)(1-\mu_j)^{\omega_j}\right)}\\+(\gamma^2-1)\sqrt{\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)\left(\left(1+(\gamma-1)\mu_j\right)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}\\-\prod_{i=1,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+(\gamma^2-1)(1-\mu_j)^{\omega_j}\right)}\\-\sqrt{\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)\left(\left(1+(\gamma-1)\mu_j\right)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}\end{gathered}}{\begin{gathered}\prod_{i=1,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+(\gamma^2-1)(1-\mu_j)^{\omega_j}\right)}\\+(\gamma^2-1)\sqrt{\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)\left(\left(1+(\gamma-1)\mu_j\right)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}\\+(\gamma-1)\prod_{i=1,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)\mu_j\right)^{\omega_j}+(\gamma^2-1)(1-\mu_j)^{\omega_j}\right)}\\-\sqrt{\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)\left(\left(1+(\gamma-1)\mu_j\right)^{\omega_j}-(1-\mu_j)^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}\end{gathered}}$$

$$v=\frac{\gamma\prod_{i=1,\,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}+(\gamma^2-1)v_j^{\omega_j}\right)}\\-\sqrt{\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)\left(\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}-v_j^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}}{\begin{gathered}\prod_{i=1,\,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}+(\gamma^2-1)v_j^{\omega_j}\right)}\\+(\gamma^2-1)\sqrt{\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)\left(\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}-v_j^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}\\+(\gamma-1)\prod_{i=1,\,j=i}^{n}\left(\begin{gathered}\sqrt{\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)\left(\gamma\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}+(\gamma^2-1)v_j^{\omega_j}\right)}\\-\sqrt{\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)\left(\left(1+(\gamma-1)(1-v_j)\right)^{\omega_j}-v_j^{\omega_j}\right)}\end{gathered}\right)^{\frac{2}{n(n+1)}}\end{gathered}}$$

(4) If $q\to0$, the NIFHWHM operator reduces to generalized normal intuitionistic fuzzy Hamacher weight Arithmetical mean (GNIFEHWAM) operator.

$$NIFHWHM^{p}\left(A_1,A_2,\cdots,A_n\right)=\left(\frac{1}{n}\left(\bigoplus_{H,\,i=1}^{n}\left(\left(\omega_iA_i\right)^{p}\right)\right)\right)^{\frac{1}{p}}=\left\langle\left(\left(\frac{1}{n}\sum_{i=1}^{n}\left(\omega_i\alpha_i\right)^{p}\right)^{\frac{1}{p}},\left(\frac{1}{n}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\left(\omega_i\alpha_i\right)^{p}\right)^{\frac{1}{p}-1}\left(\sum_{i=1}^{n}\left(\omega_i\alpha_i\right)^{p-1}\omega_i\sigma_i\right)\right),\mu,v\right\rangle$$

$$\mu=\frac{\gamma\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}}{\left\{\begin{gathered}\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}\\+(\gamma-1)\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)\mu_i\right)^{\omega_i}+(\gamma^2-1)(1-\mu_i)^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)\mu_i\right)^{\omega_i}-(1-\mu_i)^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}\end{gathered}\right\}}$$

$$v=\frac{\begin{gathered}\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}\\-\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}\end{gathered}}{\begin{gathered}\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}\\+(\gamma-1)\left(\begin{gathered}\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\gamma\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}+(\gamma^2-1)v_i^{\omega_i}\right)^{p}-\left(\left(1+(\gamma-1)(1-v_i)\right)^{\omega_i}-v_i^{\omega_i}\right)^{p}\right)^{\frac{1}{n}}\end{gathered}\right)^{\frac{1}{p}}\end{gathered}}$$

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(5) If $p=1, q\to 0$, the NIFHWHM operator is transformed into normal intuitionistic fuzzy Hamacher weight Arithmetical mean (NIFHWAM) operator.

$$NIFHWHM\left(A_1,A_2,\cdots,A_n\right)=\frac{1}{n}\left(\overset{n}{\underset{i=1}{\oplus_H}}\left(\omega_i A_i\right)\right)$$
$$=\left\langle\left(\frac{1}{n}\sum_{i=1}^{n}\omega_i\alpha_i,\left(\frac{1}{n}\right)\left(\sum_{i=1}^{n}\omega_i\sigma_i\right)\right),\frac{\prod_{i=1}^{n}\sqrt[n]{\left(1+\left(\gamma-1\right)\mu_i\right)^{\omega_i}}-\prod_{i=1}^{n}\sqrt[n]{\left(1-\mu_i\right)^{\omega_i}}}{\prod_{i=1}^{n}\sqrt[n]{\left(1+\left(\gamma-1\right)\mu_i\right)^{\omega_i}}+\left(\gamma-1\right)\prod_{i=1}^{n}\sqrt[n]{\left(1-\mu_i\right)^{\omega_i}}},\frac{\gamma\prod_{i=1}^{n}\sqrt[n]{v_i^{\omega_i}}}{\prod_{i=1}^{n}\sqrt[n]{\left(1+\left(\gamma-1\right)\left(1-v_i\right)\right)^{\omega_i}}+\left(\gamma-1\right)\prod_{i=1}^{n}\sqrt[n]{v_i^{\omega_i}}}\right\rangle$$

4. Normal intuitionistic fuzzy Hamacher Geometric Heronian mean operator and its weighted form

In this section, the GHM is extended to contain the position where the attribute values are NIFNs, and we introduce the normal intuitionistic fuzzy Hamacher geometric Heronian mean (NIFHGHM) aggregation operator and its weighted form.

Definition 9. Let $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be a collection of NIFNs, and $p,q\ge 0$ cannot take the value of zero at the same time, a normal intuitionistic fuzzy Hamacher geometric Heronian mean operator (NIFHGHM) is defined.

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\frac{1}{p+q}\left(\overset{n}{\underset{i=1,j=i}{\otimes_H}}\left(pA_i\oplus_H qA_j\right)^{\frac{2}{n(n+1)}}\right) \tag{70}$$

Theorem 10. If $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ $\left(i=1,2,\cdots,n\right)$ is a set of NIFNs, and $p,q\ge 0$ cannot take the value of zero at the same time, then the aggregation value of the Equation (70) is a NIFN and

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\left\langle\left(\alpha,\sigma\right),\mu,v\right\rangle \tag{71}$$

$$\alpha=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}\right) \tag{72}$$

$$\sigma=\frac{1}{p+q}\left(\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}\right)\times\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n\left(n+1\right)}\frac{\left(p\sigma_i+q\sigma_j\right)^2}{\left(p\alpha_i+q\alpha_j\right)^2}\right)\right)\right)^{\frac{1}{2}} \tag{73}$$

$$\mu=\frac{\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{\substack{i=1\\j=i}}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}} \tag{74}$$
$$V_{ij}=\left(1+\left(\gamma-1\right)\mu_i\right)^p\left(1+\left(\gamma-1\right)\mu_j\right)^q\quad W_{ij}=\left(1-\mu_i\right)^p\left(1-\mu_j\right)^q$$

$$v=\frac{\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right)} \tag{75}$$
$$V_{ij}=\left(\left(1+\left(\gamma-1\right)\left(1-v_i\right)\right)^p\times\left(1+\left(\gamma-1\right)\left(1-v_j\right)\right)^q\right)\quad W_{ij}=v_i^p v_j^q$$

Proof. According to Lemma2, we get

$$pA_i\oplus_H qA_j=$$
$$\left\langle\left(p\alpha_i+q\alpha_j,p\sigma_i+q\sigma_j\right),\frac{\begin{pmatrix}\left(1+\left(\gamma-1\right)\mu_i\right)^p\times\\\left(1+\left(\gamma-1\right)\mu_j\right)^q\end{pmatrix}-\begin{pmatrix}\left(1-\mu_i\right)^p\times\\\left(1-\mu_j\right)^q\end{pmatrix}}{\begin{pmatrix}\left(1+\left(\gamma-1\right)\mu_i\right)^p\times\\\left(1+\left(\gamma-1\right)\mu_j\right)^q\end{pmatrix}+\left(\gamma-1\right)\begin{pmatrix}\left(1-\mu_i\right)^p\times\\\left(1-\mu_j\right)^q\end{pmatrix}},\frac{\gamma v_i^p v_j^q}{\begin{pmatrix}\left(1+\left(\gamma-1\right)\left(1-v_i\right)\right)^p\times\\\left(1+\left(\gamma-1\right)\left(1-v_j\right)\right)^q\end{pmatrix}+\left(\gamma-1\right)v_i^p v_j^q}\right\rangle$$
$$=\left\langle\left(p\alpha_i+q\alpha_j,p\sigma_i+q\sigma_j\right),\mu_{ij}^{pq},v_{ij}^{pq}\right\rangle$$

From Lemma3, the following formula holds

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$$\otimes_{H}{}_{i=1,j=i}^{n}\left(pA_i \oplus_H qA_j\right)^{\frac{2}{n(n+1)}} = \left\langle \left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}, \left(\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}} \right) \times \left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)} \frac{\left(p\sigma_i+q\sigma_j\right)^2}{\left(p\alpha_i+q\alpha_j\right)^2} \right) \right)^{\frac{1}{2}} \right) \right), \mu_e, v_e \right\rangle$$

$$\mu_e = \frac{\gamma \prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}{\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\mu_i\right)^p\left(1+\left(\gamma-1\right)\mu_j\right)^q \quad W_{ij}=\left(1-\mu_i\right)^p\left(1-\mu_j\right)^q$$

$$v_e = \frac{\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}{\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-v_i\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-v_j\right)\right)^q \quad W_{ij}=v_i^p v_j^q$$

From the formula (21), the Equation (71) holds.

In the following, we prove that $NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)$ is a NIFN.

(1) For the value of μ, according to $\gamma>0$ and $0\le\mu_i\le1,\left(i=1,2,\cdots,n\right)$, we have: $1+\left(\gamma-1\right)\mu_i\ge\mu_i,\left(i=1,2,\cdots n\right)$, and for any i and j $\left(i,\ j=1,2,\cdots,n\right)$

$$V_{ij}=\left(1+\left(\gamma-1\right)\mu_i\right)^p\left(1+\left(\gamma-1\right)\mu_j\right)^q\ge\left(1-\mu_i\right)^p\left(1-\mu_j\right)^q=W_{ij}\ge0$$

Thus $\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}\ge\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge0$, and then

$$\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0$$

From the following conditions:

$$\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge$$

$$\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0$$

We obtain: $0\le\mu\le1$.

(2) For the value of v, from the conditions $\gamma>0$ and $0\le v_i\le1,\left(i=1,2,\cdots,n\right)$, we get:

$$1+\left(\gamma-1\right)\left(1-v_i\right)\ge v_i,\left(i=1,2,\cdots n\right)$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-v_i\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-v_j\right)\right)^q\ge v_i^p v_j^q=W_{ij}\ge0 \quad \forall i,\ j=1,2,\cdots,n$$

Thus

$$\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}\ge\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge0$$

$$\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0$$

By the conditions:

$$\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\ge0$$

We obtain:

$$\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}$$
$$\ge\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\ge0$$

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As a result, $0 \le \upsilon \le 1$.

(3) For the value of $\mu + \upsilon$, by the steps (1) and (2), $\mu + \upsilon \ge 0$, and the inequality $\mu + \upsilon \le 1$ is proved in the following.

1) If there exist at least υ_k with $\upsilon_k = 1$, then $\mu_k = 0$, $V_{kk} - W_{kk} = 0$ and $\prod_{i=1,j=i}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}} = 0$, therefore we have $\mu = 0, \upsilon = 1$, i.e. $\mu + \upsilon = 1$

2) If there exist at least υ_k with $\upsilon_k = 0$, then $\mu_k = 1$. For the equation $\prod_{i=1,j=i}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}$, we have:

$$V_{kj} = \begin{cases} \gamma^p \left(1 + (\gamma - 1)\mu_j\right)^q & in \quad \mu \\ \gamma^p \left(1 + (\gamma - 1)(1 - \upsilon_j)\right)^q & in \quad \upsilon \end{cases} \left(k \le j \le n\right) \quad V_{ik} = \begin{cases} \gamma^q \left(1 + (\gamma - 1)\mu_i\right)^p & in \quad \mu \\ \gamma^q \left(1 + (\gamma - 1)(1 - \upsilon_i)\right)^p & in \quad \upsilon \end{cases} \left(1 \le i \le k\right)$$

and $W_{kj} = W_{ik} = 0$, $\left(1 \le i \le k \le j \le n\right)$, thus

$$\prod_{i=1,j=i}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}} = \begin{cases} \left(\prod_{j=k}^{n} \gamma^p \left(1 + (\gamma - 1)\mu_j\right)^q\right) \left(\prod_{i=1}^{k-1} \gamma^q \left(1 + (\gamma - 1)\mu_i\right)^p\right) \left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in \quad \mu \\ \left(\prod_{j=k}^{n} \gamma^p \left(1 + (\gamma - 1)(1 - \upsilon_j)\right)^q\right) \left(\prod_{i=1}^{k-1} \gamma^q \left(1 + (\gamma - 1)(1 - \upsilon_i)\right)^p\right) \left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in \quad \upsilon \end{cases}$$

$$\prod_{i=1,j=i}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} = \begin{cases} \left(\prod_{j=k}^{n} \gamma^p \left(1 + (\gamma - 1)\mu_j\right)^q\right) \left(\prod_{i=1}^{k-1} \gamma^q \left(1 + (\gamma - 1)\mu_i\right)^p\right) \left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in \quad \mu \\ \left(\prod_{j=k}^{n} \gamma^p \left(1 + (\gamma - 1)(1 - \upsilon_j)\right)^q\right) \left(\prod_{i=1}^{k-1} \gamma^q \left(1 + (\gamma - 1)(1 - \upsilon_i)\right)^p\right) \left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}}\right) & in \quad \upsilon \end{cases}$$

Therefore

$$\mu = \frac{\left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} + (\gamma^2 - 1) \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} - \left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} + (\gamma^2 - 1) \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} + (\gamma - 1)\left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij} = \left(1 + (\gamma - 1)\mu_i\right)^p \left(1 + (\gamma - 1)\mu_j\right)^q \quad W_{ij} = \left(1 - \mu_i\right)^p \left(1 - \mu_j\right)^q$$

$$\upsilon = \frac{\gamma \left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} + (\gamma^2 - 1) \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} + (\gamma - 1)\left(\prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} + (\gamma^2 - 1)W_{ij}\right)^{\frac{2}{n(n+1)}} - \prod_{i=1, i\ne k, j=i, j\ne k}^{n} \left(V_{ij} - W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij} = \left(1 + (\gamma - 1)(1 - \upsilon_i)\right)^p \left(1 + (\gamma - 1)(1 - \upsilon_j)\right)^q \quad W_{ij} = \upsilon_i^p \upsilon_j^q$$

Considering that there are $n(n+1)/2$ factors in the product part $\prod_{i=1,j=i}^{n}$ and $n(n+1)/2 - n$ factors in the part $\prod_{i=1, i\ne k, j=i, j\ne k}$, we suppose that $0 < \mu_i < 1$ and $0 < \upsilon_i < 1$, $\left(i = 1, 2, \cdots, n\right)$ is right. Therefore, μ and υ have the following results.

$$\mu = 1 \Bigg/ \left(1 + \gamma \Bigg/ \left(\left(1 + \gamma^2 \Bigg/ \prod_{i=1,j=i}^{n} \left(1 + \gamma^2 \Big/ \left(\left(1 + \left(\mu_i / (1 - \mu_i)\right)\gamma\right)^p \left(1 + \left(\mu_j / (1 - \mu_j)\right)\gamma\right)^q - 1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} - 1\right)\right)$$

$$\upsilon = \gamma \Bigg/ \left(\left(1 + \gamma^2 \Bigg/ \prod_{i=1,j=i}^{n} \left(1 + \gamma^2 \Big/ \left(\left(1 + \left((1 - \upsilon_i)/\upsilon_i\right)\gamma\right)^p \left(1 + \left((1 - \upsilon_j)/\upsilon_j\right)\gamma\right)^q - 1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} + \gamma - 1\right)$$

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Based on $0 \le \mu_i + v_i \le 1,\ (i = 1,2,\cdots,n)$, we have $(1-v_i)/v_i \ge \mu_i/(1-\mu_i) \ge 0,\ (i = 1,2,\cdots,n)$, and then $\left(1+((1-v_i)/v_i)\gamma\right)^p\left(1+((1-v_j)/v_j)\gamma\right)^q - 1 \ge \left(1+(\mu_i/(1-\mu_i))\gamma\right)^p\left(1+(\mu_j/(1-\mu_j))\gamma\right)^q - 1 \ge 0,\ (\forall i,\ j = 1,2,\cdots,n)$

Thus

$$v+\mu \le v + 1\Big/\left(1+\gamma\Big/\left(\left(1+\gamma^2\Big/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+((1-v_i)/v_i)\gamma\right)^p\left(1+((1-v_j)/v_j)\gamma\right)^q-1\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)=1$$

Therefore, $0 \le \mu + v \le 1$ is right. As a result, $(\mu,\ v)$ is a IFN.

Especially, (1) When γ=1, the NIFHGHM reduce to normal intuitionistic fuzzy geometric Heronian mean (NIFGHM) which is presented by the formulas (71)-(73) and (76).

$$\mu = 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \qquad v = \left(1-\prod_{\substack{i=1\\ j=i}}^{n}\left(1-v_i^p v_j^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \tag{76}$$

(2) If γ=2, then

$$v = \frac{2\left(\prod_{i=1,j=i}^{n}\left((2-v_i)^p(2-v_j)^q+3(v_i)^p(v_j)^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left((2-v_i)^p(2-v_j)^q-(v_i)^p(v_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\begin{array}{l}\left(\prod_{i=1,j=i}^{n}\left((2-v_i)^p(2-v_j)^q+3(v_i)^p(v_j)^q\right)^{\frac{2}{n(n+1)}}+3\prod_{i=1,j=i}^{n}\left((2-v_i)^p(2-v_j)^q-(v_i)^p(v_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\ +\left(\prod_{i=1,j=i}^{n}\left((2-v_i)^p(2-v_j)^q+3(v_i)^p(v_j)^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left((2-v_i)^p(2-v_j)^q-(v_i)^p(v_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}} \tag{77}$$

$$\mu = \frac{\begin{array}{l}\left(\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q+3(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}+3\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q-(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\ -\left(\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q+3(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q-(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}}{\begin{array}{l}\left(\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q+3(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}+3\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q-(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\ +\left(\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q+3(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left((1+\mu_i)^p(1+\mu_j)^q-(1-\mu_i)^p(1-\mu_j)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}} \tag{78}$$

So the NIFHGHM operator reduce to the formulas (71)-(73) and (77)-(78), and it is called normal intuitionistic fuzzy Einstein geometric Heronian mean (NIFEGHM).

(3) When $p = q = \frac{1}{2}$, the NIFHGHM operator degenerates into the normal intuitionistic fuzzy evolution Hamacher geometric Heronian mean (NIFEHGHM) operator.

$$\begin{aligned} NIFHGHM^{\frac{1}{2},\frac{1}{2}}(A_1,A_2,\cdots,A_n) &= \bigotimes_{H\ i=1,j=i}^{n}\left(\tfrac{1}{2}A_i \oplus_H \tfrac{1}{2}A_j\right)^{\frac{2}{n(n+1)}} \\ &= \left\langle\left(\prod_{i=1,j=i}^{n}\left(\tfrac{1}{2}\alpha_i+\tfrac{1}{2}\alpha_j\right)^{\frac{2}{n(n+1)}},\left(\prod_{i=1,j=i}^{n}\left(\tfrac{1}{2}\alpha_i+\tfrac{1}{2}\alpha_j\right)^{\frac{2}{n(n+1)}}\right)\times\sqrt{\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{(\sigma_i+\sigma_j)^2}{(\alpha_i+\alpha_j)^2}\right)}\right),\mu,v\right\rangle \end{aligned}$$

$$\mu = \frac{\gamma\prod_{i=1,\ j=i}^{n}\left(\sqrt{(1+(\gamma-1)\mu_i)(1+(\gamma-1)\mu_j)}-\sqrt{(1-\mu_i)(1-\mu_j)}\right)^{\frac{2}{n(n+1)}}}{\begin{array}{l}\prod_{i=1,\ j=i}^{n}\left(\sqrt{(1+(\gamma-1)\mu_i)(1+(\gamma-1)\mu_j)}+(\gamma^2-1)\sqrt{(1-\mu_i)(1-\mu_j)}\right)^{\frac{2}{n(n+1)}}\\ +(\gamma-1)\prod_{i=1,\ j=i}^{n}\left(\sqrt{(1+(\gamma-1)\mu_i)(1+(\gamma-1)\mu_j)}-\sqrt{(1-\mu_i)(1-\mu_j)}\right)^{\frac{2}{n(n+1)}}\end{array}}$$

$$v = \frac{\begin{array}{l}\prod_{i=1,j=i}^{n}\left(\sqrt{(1+(\gamma-1)(1-v_i))(1+(\gamma-1)(1-v_j))}+(\gamma^2-1)\sqrt{v_i\mu_j}\right)^{\frac{2}{n(n+1)}}\\ -\prod_{i=1,j=i}^{n}\left(\sqrt{(1+(\gamma-1)(1-v_i))(1+(\gamma-1)(1-v_j))}-\sqrt{v_iv_j}\right)^{\frac{2}{n(n+1)}}\end{array}}{\begin{array}{l}\prod_{i=1,j=i}^{n}\left(\sqrt{(1+(\gamma-1)(1-v_i))(1+(\gamma-1)(1-v_j))}+(\gamma^2-1)\sqrt{v_iv_j}\right)^{\frac{2}{n(n+1)}}\\ +(\gamma-1)\prod_{i=1,j=i}^{n}\left(\sqrt{(1+(\gamma-1)(1-v_i))(1+(\gamma-1)(1-v_j))}-\sqrt{v_iv_j}\right)^{\frac{2}{n(n+1)}}\end{array}}$$

(4) If $q \to 0$, the NIFHGHM operator reduces to generalized normal intuitionistic fuzzy Hamacher geometric Arithmetical mean (GNIFHGAM) operator.

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$$NIFHGHM^{p}\left(A_1,A_2,\cdots,A_n\right)=\frac{1}{p}\left(\underset{i=1}{\overset{n}{\otimes_H}}\left(pA_i\right)^{\frac{1}{n}}\right)=\left\langle\left(\frac{1}{p}\left(\prod_{i=1}^{n}\left(p\alpha_i\right)^{\frac{1}{n}}\right),\frac{1}{p}\left(\left(\prod_{i=1}^{n}\left(p\alpha_i\right)^{\frac{1}{n}}\right)\times\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\sigma_i)^2}{(\alpha_i)^2}\right)}\right)\right),\mu,v\right\rangle$$

$$v=\frac{\gamma\left(\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-v_i)\right)^p+(\gamma^2-1)v_i^p\right)^{\frac{1}{n}}-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-v_i)\right)^p-v_i^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}}{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-v_i)\right)^p+(\gamma^2-1)v_i^p\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-v_i)\right)^p-v_i^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}+(\gamma-1)\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-v_i)\right)^p+(\gamma^2-1)v_i^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-v_i)\right)^p-v_i^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}$$

$$\mu=\frac{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p+(\gamma^2-1)(1-\mu_i)^p\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p-(1-\mu_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}-\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p+(\gamma^2-1)(1-\mu_i)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p-(1-\mu_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p+(\gamma^2-1)(1-\mu_i)^p\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p-(1-\mu_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}+(\gamma-1)\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p+(\gamma^2-1)(1-\mu_i)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(1+(\gamma-1)\mu_i\right)^p-(1-\mu_i)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}$$

(5) If $p=1,q\to 0$, the NIFHGHM operator is transformed into normal intuitionistic fuzzy Hamacher geometric Arithmetical mean (NIFHGAM) operator.

$$NIFHGHM\left(A_1,A_2,\cdots,A_n\right)=\left(\underset{i=1}{\overset{n}{\otimes_H}}\left(A_i\right)^{\frac{1}{n}}\right)$$
$$=\left\langle\left(\prod_{i=1}^{n}\left(\alpha_i\right)^{\frac{1}{n}},\left(\prod_{i=1}^{n}\left(\alpha_i\right)^{\frac{1}{n}}\right)\times\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\sigma_i)^2}{(\alpha_i)^2}\right)}\right),\frac{\gamma\prod_{i=1}^{n}\mu_i^{\frac{1}{n}}}{\prod_{i=1}^{n}\left(\gamma+(1-\gamma)\mu_i\right)^{\frac{1}{n}}+(\gamma-1)\prod_{i=1}^{n}\mu_i^{\frac{1}{n}}},\frac{\prod_{i=1}^{n}\left(1+(\gamma-1)v_i\right)^{\frac{1}{n}}-\prod_{i=1}^{n}\left(1-v_i\right)^{\frac{1}{n}}}{\prod_{i=1}^{n}\left(1+(\gamma-1)v_i\right)^{\frac{1}{n}}+(\gamma-1)\prod_{i=1}^{n}\left(1-v_i\right)^{\frac{1}{n}}}\right\rangle$$

The NIFHGHM operator has some properties considered in the following.

Theorem 11(Idempotency). If all $A_i=A=\left\langle(\alpha,\sigma),\mu,v\right\rangle$ $(i=1,2,\cdots,n)$ then

$$NIIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHGHM^{p,q}\left(A,A,\cdots A\right)=A$$

Proof. According to Theorem 1, we have:

$$NIIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHGHM^{p,q}\left(A,A,\cdots,\ A\right)$$
$$=\frac{1}{p+q}\left(\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left(pA\oplus_H qA\right)^{\frac{2}{n(n+1)}}\right)=\frac{1}{p+q}\left(\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left((p+q)A\right)^{\frac{2}{n(n+1)}}\right)=\frac{1}{p+q}\left((p+q)A\right)^{\frac{n(n+1)}{2}\cdot\frac{2}{n(n+1)}}=A$$

Theorem 12(Permutation). If $\dot{A}_i\,(i=1,2,\cdots,n)$ is any permutation of $A_i\,(i=1,2,\cdots,n)$ then

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHGHM^{p,q}\left(\dot{A}_1,\dot{A}_2,\cdots,\dot{A}_n\right)$$

Proof. Based on the Theorem 1, we obtain:

$$NIIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=$$
$$\frac{1}{p+q}\left(\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left(pA_i\oplus_H qA_j\right)^{\frac{2}{n(n+1)}}\right)=\frac{1}{p+q}\left(\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left(p\,\dot{A}_i\oplus_H q\,\dot{A}_i\right)^{\frac{2}{n(n+1)}}\right)=NIIFHGHM^{p,q}\left(\dot{A}_1,\dot{A}_2,\cdots,\dot{A}_n\right)$$

Theorem 13 (Monotonicity). Let $p\geq 0,q\geq 0$, p,q don't simultaneously take the value of zero, and $A_i=\left\langle\left(\alpha_{A_i},\sigma_{A_i}\right),\mu_{A_i},v_{A_i}\right\rangle$, $B_i=\left\langle\left(\alpha_{B_i},\sigma_{B_i}\right),\mu_{B_i},v_{B_i}\right\rangle$ $i=1,2,\cdots,n$, are NIFNs which satisfy the following conditions:

$$\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right),\ \ \mu_{A_i}\leq\mu_{B_i},\ \ v_{A_i}\geq v_{B_i}\ \ \left(i=1,2,\cdots,n\right)\qquad(79)$$

$$\sum_{i=1,j=i}^{n}\left(\left(p\sigma_{A_i}+q\sigma_{A_j}\right)^2/\left(p\alpha_{A_i}+q\alpha_{A_j}\right)^2\right)=\sum_{i=1,j=i}^{n}\left(\left(p\sigma_{B_i}+q\sigma_{B_j}\right)^2/\left(p\alpha_{B_i}+q\alpha_{B_j}\right)^2\right)\qquad(80)$$

then for $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)\geq 0$

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\qquad(81)$$

for $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)<0$

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$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\geq NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right) \tag{82}$$

Proof. Let $NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\left\langle\left(\alpha_A,\sigma_A\right),\mu_A,\upsilon_A\right\rangle$

$$\alpha_A=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)^{\frac{2}{n(n+1)}}\right)$$

$$\sigma_A=\frac{1}{p+q}\left(\left(\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)^{\frac{2}{n(n+1)}}\right)\times\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{\left(p\sigma_{A_i}+q\sigma_{A_j}\right)^2}{\left(p\alpha_{A_i}+q\alpha_{A_j}\right)^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\mu_A=\frac{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\mu_{A_i}\right)^p\left(1+\left(\gamma-1\right)\mu_{A_j}\right)^q\quad W_{ij}=\left(1-\mu_{A_i}\right)^p\left(1-\mu_{A_j}\right)^q$$

$$\upsilon_A=\frac{\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-\upsilon_{A_i}\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-\upsilon_{A_j}\right)\right)^q\ W_{ij}=\upsilon_{A_i}^p\upsilon_{A_j}^q$$

and $NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)=\left\langle\left(\alpha_B,\sigma_B\right),\mu_B,\upsilon_B\right\rangle$

$$\alpha_B=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)^{\frac{2}{n(n+1)}}\right)$$

$$\sigma_B=\frac{1}{p+q}\left(\left(\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)^{\frac{2}{n(n+1)}}\right)\times\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{\left(p\sigma_{B_i}+q\sigma_{B_j}\right)^2}{\left(p\alpha_{B_i}+q\alpha_{B_j}\right)^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\mu_B=\frac{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,\ j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,\ j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\mu_{B_i}\right)^p\left(1+\left(\gamma-1\right)\mu_{B_j}\right)^q\quad W_{ij}=\left(1-\mu_{B_i}\right)^p\left(1-\mu_{B_j}\right)^q$$

$$\upsilon_B=\frac{\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left\{\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right\}}$$

$$V_{ij}=\left(1+\left(\gamma-1\right)\left(1-\upsilon_{B_i}\right)\right)^p\left(1+\left(\gamma-1\right)\left(1-\upsilon_{B_j}\right)\right)^q\quad W_{ij}=\upsilon_{B_i}^p\upsilon_{B_j}^q$$

(1) When $\upsilon_{B_k}=1$ is at least satisfied, $\upsilon_{A_k}=1$, $\mu_{A_k}=0$, and $\mu_{B_k}=0$. Thus, $\upsilon_A=\upsilon_B=1$, $\mu_A=\mu_B=0$. According to the condition (79), the equation $\alpha_A=\alpha_B$ is obtained. Therefore,

$$S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A-\upsilon_A\right)=\alpha_B\left(\mu_B-\upsilon_B\right)=S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A+\upsilon_A\right)=\alpha_B\left(\mu_B+\upsilon_B\right)=H_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

If the conditions (79) and (80) are right, then $\sigma_A=\sigma_B$, and

$$S_2\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A-\upsilon_A\right)=\sigma_B\left(\mu_B-\upsilon_B\right)=S_2\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_2\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A+\upsilon_A\right)=\sigma_B\left(\mu_B+\upsilon_B\right)=H_2\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

Consequently, for $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)\geq 0$ or $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)<0$, we have $NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$.

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(2) Supposing at least $\upsilon_{A_k}=0$, then $\mu_{A_k}=1$, $\upsilon_{B_k}=0$, and $\mu_{B_k}=1$. From the equations $\mu_{A_k}=1$ and $\upsilon_{A_k}=0$, we have:

$$V_{kj}=\gamma^p\left(1+(\gamma-1)\left(1-\mu_{A_j}\right)\right)^q,\ (k\le j\le n)$$

$$V_{ik}=\gamma^q\left(1+(\gamma-1)\left(1-\mu_{A_i}\right)\right)^p,(1\le i\le k)$$

$$W_{kj}=W_{ik}=0,\ (1\le i\le k\le j\le n)$$

Thus

$$\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}=\left(\prod_{j=k}^{n}\gamma^p\left(1+(\gamma-1)\left(1-\mu_{A_j}\right)\right)^q\right)\left(\prod_{i=1}^{k-1}\gamma^q\left(1+(\gamma-1)\left(1-\mu_{A_i}\right)\right)^p\right)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)$$

$$\prod_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}=\left(\prod_{j=k}^{n}\gamma^p\left(1+(\gamma-1)\left(1-\mu_{A_j}\right)\right)^q\right)\left(\prod_{i=1}^{k-1}\gamma^q\left(1+(\gamma-1)\left(1-\mu_{A_i}\right)\right)^p\right)\left(\prod_{i=1,i\neq k,j=i,j\neq i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}\right)$$

Therefore

$$\mu_A=\frac{\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+(\gamma-1)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij}=\left(1+(\gamma-1)\mu_{A_i}\right)^p\left(1+(\gamma-1)\mu_{A_j}\right)^q\quad W_{ij}=\left(1-\mu_{A_i}\right)^p\left(1-\mu_{A_j}\right)^q$$

$$\upsilon_A=\frac{\gamma\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod_{i=1,i\neq k,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+(\gamma-1)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij}=\left(1+(\gamma-1)\left(1-\upsilon_{A_i}\right)\right)^p\left(1+(\gamma-1)\left(1-\upsilon_{A_j}\right)\right)^q\quad W_{ij}=\upsilon_{A_i}^p\upsilon_{A_j}^q$$

When $\mu_{B_k}=1$ and $\upsilon_{B_k}=0$, based on the same method as the above, μ_B and υ_B in $NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$ are transformed into the following:

$$\mu_B=\frac{\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+(\gamma-1)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij}=\left(1+(\gamma-1)\mu_{B_i}\right)^p\left(1+(\gamma-1)\mu_{B_j}\right)^q\quad W_{ij}=\left(1-\mu_{B_i}\right)^p\left(1-\mu_{B_j}\right)^q$$

$$\upsilon_B=\frac{\gamma\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma^2-1)\prod_{i=1,i\neq k,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+(\gamma-1)\left(\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,i\neq k,j=i,j\neq k}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}$$
$$V_{ij}=\left(1+(\gamma-1)\left(1-\upsilon_{B_i}\right)\right)^p\left(1+(\gamma-1)\left(1-\upsilon_{B_j}\right)\right)^q\quad W_{ij}=\upsilon_{B_i}^p\upsilon_{B_j}^q$$

We can easily noticed the symbol $\prod_{i=1,j=i}^{n}$ contains $n(n+1)/2$ terms, but $\prod_{i=1,i\neq k,j=i,j\neq k}^{n}$ involves $n(n+1)/2-n$ product terms. Without the loss of generality, we suppose $0<\mu_{A_i}<1$, $0<\mu_{B_i}<1$ and $0<\upsilon_{A_i}<1$, $0<\upsilon_{B_i}<1$, $(i=1,2,\cdots,n)$. Therefore, we have:

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$$\mu_A = 1\Bigg/\left(1+\gamma\Bigg/\left(\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\mu_{A_i}/\left(1-\mu_{A_i}\right)\right)\gamma\right)^p\left(1+\left(\mu_{A_j}/\left(1-\mu_{A_j}\right)\right)\gamma\right)^q-1\right)\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)$$

$$v_A = \gamma\Bigg/\left(\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\left(1-v_{A_i}\right)/v_{A_i}\right)\gamma\right)^p\left(1+\left(\left(1-v_{A_j}\right)/v_{A_j}\right)\gamma\right)^q-1\right)\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\gamma-1\right)$$

$$\mu_B = 1\Bigg/\left(1+\gamma\Bigg/\left(\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\mu_{B_i}/\left(1-\mu_{B_i}\right)\right)\gamma\right)^p\left(1+\left(\mu_{B_j}/\left(1-\mu_{B_j}\right)\right)\gamma\right)^q-1\right)\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}-1\right)\right)$$

$$v_B = \gamma\Bigg/\left(\left(\left(1+\gamma^2\Bigg/\prod_{i=1,j=i}^{n}\left(1+\gamma^2\Big/\left(\left(1+\left(\left(1-v_{B_i}\right)/v_{B_i}\right)\gamma\right)^p\left(1+\left(\left(1-v_{B_j}\right)/v_{B_j}\right)\gamma\right)^q-1\right)\right)\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}+\gamma-1\right)$$

According to the following inequalities

$$0<\mu_{A_i}\le\mu_{B_i}<1,\quad 0<v_{B_i}\le v_{A_i}<1,\quad 0<\mu_{A_i}+v_{A_i}<1 \text{ and } 0<\mu_{B_i}+v_{B_i}<1\ \left(i=1,2,\cdots,n\right)$$

we obtain:

$$\left(1-v_{A_i}\right)/v_{A_i}\le\left(1-v_{B_i}\right)/v_{B_i},\quad \mu_{A_i}/\left(1-\mu_{A_i}\right)\le\mu_{B_i}/\left(1-\mu_{B_i}\right),\quad \left(i=1,2,\cdots,n\right)$$

Thus, $\mu_A\le\mu_B$ and $v_A\ge v_B$, and by the conditions (79) and (80), we have $\alpha_A=\alpha_B$ and $\sigma_A=\sigma_B$.

Therefore, (1) when $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)\ge 0$

$$S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A-v_A\right)\le\alpha_B\left(\mu_B-v_B\right)=S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

If $S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)<S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, then

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)<NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

If $S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, Based on the conditions $\alpha_A=\alpha_B\ge 0,\mu_A-v_A\le\mu_B-v_B,\mu_A\le\mu_B,v_A\ge v_B$, we obtain $\mu_A=\mu_B$ and $v_A=v_B$, and then

$$H_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A+v_A\right)=\alpha_B\left(\mu_B+v_B\right)=H_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$S_2\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A-v_A\right)=\sigma_B\left(\mu_B-v_B\right)=S_2\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_2\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A+v_A\right)=\sigma_B\left(\mu_B+v_B\right)=H_2\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right),$$

thus

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

(2) when $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}+q\alpha_{A_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}+q\alpha_{B_j}\right)<0$

$$S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A-v_A\right)\ge\alpha_B\left(\mu_B-v_B\right)=S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

If $S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)>S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, then

$$NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)>NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$$

If $S_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=S_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$, Based on the conditions $\alpha_A=\alpha_B\ge 0,\mu_A-v_A\le\mu_B-v_B,\mu_A\le\mu_B,v_A\ge v_B$, we obtain $\mu_A=\mu_B$ and $v_A=v_B$, and then

$$H_1\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\alpha_A\left(\mu_A+v_A\right)=\alpha_B\left(\mu_B+v_B\right)=H_1\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$S_2\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A-v_A\right)=\sigma_B\left(\mu_B-v_B\right)=S_2\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right)$$

$$H_2\left(NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\right)=\sigma_A\left(\mu_A+v_A\right)=\sigma_B\left(\mu_B+v_B\right)=H_2\left(NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)\right),$$

thus $NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)=NIFHGHM^{p,q}\left(B_1,B_2,\cdots,B_n\right)$, therefore, the Theorem 13 holds.

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Theorem 14(Boundedness). Let $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle,\left(i=1,2,\cdots,n\right)$ be a set of NIFNs, and

$$A^-=\left\langle\left(\tfrac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}\right),\tfrac{1}{p+q}\left(\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}\right)\sqrt{\sum_{i=1,j=i}^{n}\left(\tfrac{2}{n(n+1)}\left(p\sigma_i+q\sigma_j\right)^2/\left(p\alpha_i+q\alpha_j\right)^2\right)}\right)\right),\min_i\left\{\mu_i\right\},\max_i\left\{v_i\right\}\right\rangle$$

$$A^+=\left\langle\left(\tfrac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}\right),\tfrac{1}{p+q}\left(\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)^{\frac{2}{n(n+1)}}\right)\sqrt{\sum_{i=1,j=i}^{n}\left(\tfrac{2}{n(n+1)}\left(p\sigma_i+q\sigma_j\right)^2/\left(p\alpha_i+q\alpha_j\right)^2\right)}\right)\right),\max_i\left\{\mu_i\right\},\min_i\left\{v_i\right\}\right\rangle$$

Then when $\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)\geq 0$, $A^-\leq NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq A^+$

when $\prod_{i=1,j=i}^{n}\left(p\alpha_i+q\alpha_j\right)<0$, $A^+\leq NIFHGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq A^-$

Theorem 14 is distinctly derived by Theorem 13. the proof is omitted.

Similarly, the weighted form of NIFHGHM operator are defined in the following.

Definition 10. If $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ $\left(i=1,2,\cdots,n\right)$ is a collection of NIFNs, $\omega_i\in[0,1]$ is the weight of $A_i\left(i=1,2,\cdots,n\right)$ with $\sum_{i=1}^{n}\omega_i=1$, then normal intuitionistic fuzzy Hamacher weighted geometric Heronian mean (NIFHWGHM) operator is presented as follows.

$$NIFHWGHM_{\omega}^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\frac{1}{p+q}\left(\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left(pA_i^{\omega_i}\oplus_H qA_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}\right)\tag{83}$$

Theorem 15. Let $A_i=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle$ $\left(i=1,2,\cdots,n\right)$ be a set of NIFNs and $p,q\geq 0$ which cannot simultaneously take the value of zero, the aggregation result of the Equation (83) is a NIFN and

$$NIFHWGHM_{\omega}^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\left\langle\left(\alpha,\sigma\right),\mu,v\right\rangle\tag{84}$$

$$\alpha=\frac{1}{p+q}\prod_{i=1,j=i}^{n}\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}\tag{85}$$

$$\sigma=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}\times\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{\left(p\omega_i^{\frac{1}{2}}\alpha_i^{\omega_i-1}\sigma_i+q\omega_j^{\frac{1}{2}}\alpha_j^{\omega_j-1}\sigma_j\right)^2}{\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j}\right)^2}\right)\right)^{\frac{1}{2}}\right)\tag{86}$$

$$\mu=\frac{\begin{array}{c}\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\-\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}}{\begin{array}{c}\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}}$$

$$V_{ij}=\left(\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^{\omega_i}+\left(\gamma-1\right)\mu_i^{\omega_i}\right)^p\left(\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^{\omega_j}+\left(\gamma-1\right)\mu_j^{\omega_j}\right)^q\tag{87}$$

$$W_{ij}=\left(\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\left(\left(1+\left(\gamma-1\right)\left(1-\mu_j\right)\right)^{\omega_j}-\mu_j^{\omega_j}\right)^q$$

$$v=\frac{\gamma\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}}{\left[\begin{array}{c}\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}+\left(\gamma^2-1\right)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\\+\left(\gamma-1\right)\left(\prod_{i=1,j=i}^{n}\left(V_{ij}+\left(\gamma^2-1\right)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\end{array}\right]}$$

$$V_{ij}=\left(\left(1+\left(\gamma-1\right)v_i\right)^{\omega_i}+\left(\gamma^2-1\right)\left(1-v_i\right)^{\omega_i}\right)^p\left(\left(1+\left(\gamma-1\right)v_j\right)^{\omega_j}+\left(\gamma^2-1\right)\left(1-v_j\right)^{\omega_j}\right)^q\tag{88}$$

$$W_{ij}=\left(\left(1+\left(\gamma-1\right)v_i\right)^{\omega_i}-\left(1-v_i\right)^{\omega_i}\right)^p\left(\left(1+\left(\gamma-1\right)v_j\right)^{\omega_j}-\left(1-v_j\right)^{\omega_j}\right)^q$$

Proof. By the formula (22), we have

$$A_i^{\omega_i}=\left\langle\left(\alpha_i^{\omega_i},\omega_i^{\frac{1}{2}}\alpha_i^{\omega_i-1}\sigma_i\right),\frac{\gamma\mu_i^{\omega_i}}{\left(1+\left(\gamma-1\right)\left(1-\mu_i\right)\right)^{\omega_i}+\left(\gamma-1\right)\mu_i^{\omega_i}},\frac{\left(1+\left(\gamma-1\right)v_i\right)^{\omega_i}-\left(1-v_i\right)^{\omega_i}}{\left(1+\left(\gamma-1\right)v_i\right)^{\omega_i}+\left(\gamma-1\right)\left(1-v_i\right)^{\omega_i}}\right\rangle$$

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$$A_j^{\omega_j}=\left\langle\left(\alpha_j^{\omega_j},\omega_j^{\frac{1}{2}}\alpha_j^{\omega_j-1}\sigma_j\right),\frac{\gamma\mu_j^{\omega_j}}{\left(1+(\gamma-1)(1-\mu_j)\right)^{\omega_j}+(\gamma-1)\mu_j^{\omega_j}},\frac{\left(1+(\gamma-1)v_j\right)^{\omega_j}-(1-v_j)^{\omega_j}}{\left(1+(\gamma-1)v_j\right)^{\omega_j}+(\gamma-1)(1-v_j)^{\omega_j}}\right\rangle$$

From Lemma2, the following equation holds.

$$pA_i^{\omega_i}\oplus_H qA_j^{\omega_j}=\left\langle\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j},p\omega_i^{\frac{1}{2}}\alpha_i^{\omega_i-1}\sigma_i+q\omega_j^{\frac{1}{2}}\alpha_j^{\omega_j-1}\sigma_j\right),\mu_{ij}^{pq},v_{ij}^{pq}\right\rangle$$

$$\mu_{ij}^{pq}=\frac{\left(\left(\beta_{1i}+(\gamma^2-1)\beta_{2i}\right)^p\times\left(\beta_{1j}+(\gamma^2-1)\beta_{2j}\right)^q\right)-\left(\left(\beta_{1i}-\beta_{2i}\right)^p\times\left(\beta_{1j}-\beta_{2j}\right)^q\right)}{\left(\left(\beta_{1i}+(\gamma^2-1)\beta_{2i}\right)^p\times\left(\beta_{1j}+(\gamma^2-1)\beta_{2j}\right)^q\right)+(\gamma-1)\left(\left(\beta_{1i}-\beta_{2i}\right)^p\times\left(\beta_{1j}-\beta_{2j}\right)^q\right)}$$

$$\beta_{1i}=\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}\quad\beta_{2i}=\mu_i^{\omega_i}\quad\beta_{1j}=\left(1+(\gamma-1)(1-\mu_j)\right)^{\omega_j}\quad\beta_{2j}=\mu_j^{\omega_j}$$

$$v_{ij}^{pq}=\frac{\gamma\left(\delta_{1i}-\delta_{2i}\right)^p\left(\delta_{1j}-\delta_{2j}\right)^q}{\left(\left(\delta_{1i}+(\gamma^2-1)\delta_{2i}\right)^p\times\left(\delta_{1j}+(\gamma^2-1)\delta_{2j}\right)^q\right)+(\gamma-1)\left(\left(\delta_{1i}-\delta_{2i}\right)^p\times\left(\delta_{1j}-\delta_{2j}\right)^q\right)}$$

$$\delta_{1i}=\left(1+(\gamma-1)v_i\right)^{\omega_i}\quad\delta_{2i}=(1-v_i)^{\omega_i}\quad\delta_{1j}=\left(1+(\gamma-1)v_j\right)^{\omega_j}\quad\delta_{2j}=(1-v_j)^{\omega_j}$$

By Lemma3, we get:

$$\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left(pA_i^{\omega_i}\oplus_H qA_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}=\left\langle\left(\alpha_e,\sigma_e\right),\mu_e,v_e\right\rangle,\quad\alpha_e=\prod_{i=1,j=i}^{n}\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}$$

$$\sigma_e=\prod_{i=1,j=i}^{n}\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\left(p\omega_i^{\frac{1}{2}}\alpha_i^{\omega_i-1}\sigma_i+q\omega_j^{\frac{1}{2}}\alpha_j^{\omega_j-1}\sigma_j\right)^2\Big/\left(p\alpha_i^{\omega_i}+q\alpha_j^{\omega_j}\right)^2\right)\right)^{\frac{1}{2}}$$

$$\mu_e=\frac{\gamma\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}{\prod_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma-1)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}$$

$$V_{ij}=\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma^2-1)\mu_i^{\omega_i}\right)^p\left(\left(1+(\gamma-1)(1-\mu_j)\right)^{\omega_j}+(\gamma^2-1)\mu_j^{\omega_j}\right)^q$$

$$W_{ij}=\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\left(\left(1+(\gamma-1)(1-\mu_j)\right)^{\omega_j}-\mu_j^{\omega_j}\right)^q$$

$$v_e=\frac{\prod_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}-\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}{\prod_{i=1,j=i}^{n}\left(V_{ij}+(\gamma^2-1)W_{ij}\right)^{\frac{2}{n(n+1)}}+(\gamma-1)\prod_{i=1,j=i}^{n}\left(V_{ij}-W_{ij}\right)^{\frac{2}{n(n+1)}}}$$

$$V_{ij}=\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p\times\left(\left(1+(\gamma-1)v_j\right)^{\omega_j}+(\gamma-1)(1-v_j)^{\omega_j}\right)^q$$

$$W_{ij}=\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\times\left(\left(1+(\gamma-1)v_j\right)^{\omega_j}-(1-v_j)^{\omega_j}\right)^q$$

We can see that the Equation (84) holds by the formula (21). Consequently, we derive $0\le\mu\le1$, $0\le v\le1$ and $0\le\mu+v\le1$. i.e. $NIFHWGHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)$ is a NIFN. According to Equation (22), for any $i=1,2,\cdots,n$, $A_i^{\omega_i}$ is a NIFN and has the following equation.

$$A_i^{\omega_i}=\left\langle\left(\alpha_i,\sigma_i\right),\mu_i,v_i\right\rangle=\left\langle\left(\alpha_i^{\omega_i},\sqrt{\omega_i}\alpha_i^{\omega_i-1}\sigma_i\right),\frac{\gamma\mu_i^{\omega_i}}{\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}},\frac{\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}}{\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma-1)(1-v_i)^{\omega_i}}\right\rangle$$

Therefore, the Equations (87) and (88) have the following situation.

$$\mu=(74)\quad V_{ij}=\left(1+(\gamma-1)\mu_i\right)^p\left(1+(\gamma-1)\mu_j\right)^q\quad W_{ij}=\left(1-\mu_i\right)^p\left(1-\mu_j\right)^q$$

$$v=(75)\quad V_{ij}=\left(1+(\gamma-1)(1-v_i)\right)^p\left(1+(\gamma-1)(1-v_j)\right)^q\quad W_{ij}=v_i^{\,p}v_j^{\,q}$$

Appling the similar derivation of Theorem 10, it is obtained that $(\mu,\ v)$ is a IFN.

From the above, the NIFHWGHM operator has the monotonicity and the boundedness

Theorem 16 (monotonicity). Let $p\ge0,q\ge0$, p,q don't simultaneously equal the zero, and $A_i=\left\langle\left(\alpha_{A_i},\sigma_{A_i}\right),\mu_{A_i},v_{A_i}\right\rangle$, $B_i=\left\langle\left(\alpha_{B_i},\sigma_{B_i}\right),\mu_{B_i},v_{B_i}\right\rangle$ $i=1,2,\cdots,n$ are NIFNs, and the following conditions is right,

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$$\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}^{w_i}+q\alpha_{A_j}^{w_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}^{w_i}+q\alpha_{B_j}^{w_j}\right),\ \mu_{A_i}\le\mu_{B_i},\ \ v_{A_i}\ge v_{B_i}\ \ \left(i=1,2,\cdots,n\right) \quad (89)$$

$$\sum_{i=1,j=i}^{n}\left(\left(p\sqrt{w_i}\alpha_{A_i}^{w_i-1}\sigma_{A_i}+q\sqrt{w_j}\alpha_{A_j}^{w_j-1}\sigma_{A_j}\right)^2\Big/\left(p\alpha_{A_i}^{w_i}+q\alpha_{A_j}^{w_j}\right)^2\right)=\sum_{i=1,j=i}^{n}\left(\left(p\sqrt{w_i}\alpha_{B_i}^{w_i-1}\sigma_{B_i}+q\sqrt{w_j}\alpha_{B_j}^{w_j-1}\sigma_{B_j}\right)^2\Big/\left(p\alpha_{B_i}^{w_i}+q\alpha_{B_j}^{w_j}\right)^2\right) \quad (90)$$

then for $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}^{w_i}+q\alpha_{A_j}^{w_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}^{w_i}+q\alpha_{B_j}^{w_j}\right)\ge 0$

$$NIFHGHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)\le NIFHGHM_w^{p,q}\left(B_1,B_2,\cdots,B_n\right) \quad (91)$$

for $\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}^{w_i}+q\alpha_{A_j}^{w_j}\right)=\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}^{w_i}+q\alpha_{B_j}^{w_j}\right)<0$

$$NIFHGHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)\ge NIFHGHM_w^{p,q}\left(B_1,B_2,\cdots,B_n\right) \quad (92)$$

Proof. From the proof of Theorem 15, we obtain:

$$A_i^{\omega_i}=\left\langle\left(\alpha_{A_i},\sigma_{A_i}\right),\mu_{A_i},v_{A_i}\right\rangle=\left\langle\left(\alpha_{A_i}^{\omega_i},\sqrt{\omega_i}\alpha_{A_i}^{\omega_i-1}\sigma_{A_i}\right),\frac{\gamma\mu_{A_i}^{\omega_i}}{\left(1+(\gamma-1)(1-\mu_{A_i})\right)^{\omega_i}+(\gamma-1){\mu_{A_i}}^{\omega_i}},\frac{\left(1+(\gamma-1)v_{A_i}\right)^{\omega_i}-\left(1-v_{A_i}\right)^{\omega_i}}{\left(1+(\gamma-1)v_{A_i}\right)^{\omega_i}+(\gamma-1)\left(1-v_{A_i}\right)^{\omega_i}}\right\rangle$$

$$B_i^{\omega_i}=\left\langle\left(\alpha_{B_i},\sigma_{B_i}\right),\mu_{B_i},v_{B_i}\right\rangle=\left\langle\left(\alpha_{B_i}^{\omega_i},\sqrt{\omega_i}\alpha_{B_i}^{\omega_i-1}\sigma_{B_i}\right),\frac{\gamma\mu_{B_i}^{\omega_i}}{\left(1+(\gamma-1)(1-\mu_{B_i})\right)^{\omega_i}+(\gamma-1){\mu_{B_i}}^{\omega_i}},\frac{\left(1+(\gamma-1)v_{B_i}\right)^{\omega_i}-\left(1-v_{B_i}\right)^{\omega_i}}{\left(1+(\gamma-1)v_{B_i}\right)^{\omega_i}+(\gamma-1)\left(1-v_{B_i}\right)^{\omega_i}}\right\rangle$$

$$NIFHWGHM_w^{p,q}\left(A_1,A_2,\cdots,A_n\right)=\left\langle\left(\alpha_A,\sigma_A\right),\mu_A,v_A\right\rangle,\ \ \alpha_A=\frac{1}{p+q}\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}^{\omega_i}+q\alpha_{A_j}^{\omega_j}\right)^{\frac{2}{n(n+1)}}$$

$$\sigma_A=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_{A_i}^{\omega_i}+q\alpha_{A_j}^{\omega_j}\right)^{\frac{2}{n(n+1)}}\times\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{\left(p\omega_i^{\frac{1}{2}}\alpha_{A_i}^{\omega_i-1}\sigma_{A_i}+q\omega_j^{\frac{1}{2}}\alpha_{A_j}^{\omega_j-1}\sigma_{A_j}\right)^2}{\left(p\alpha_{A_i}^{\omega_i}+q\alpha_{A_j}^{\omega_j}\right)^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\mu_A=(74)\quad V_{ij}=\left(1+(\gamma-1)\mu_{A_i}\right)^p\left(1+(\gamma-1)\mu_{A_j}\right)^q\quad W_{ij}=\left(1-\mu_{A_i}\right)^p\left(1-\mu_{A_j}\right)^q$$

$$v_A=(75)\quad V_{ij}=\left(1+(\gamma-1)(1-v_{A_i})\right)^p\left(1+(\gamma-1)(1-v_{A_j})\right)^q\quad W_{ij}={v_{A_i}}^p{v_{A_j}}^q$$

$$NIFHWGHM_w^{p,q}\left(B_1,B_2,\cdots,A_n\right)=\left\langle\left(\alpha_B,\sigma_B\right),\mu_B,v_B\right\rangle,\ \ \alpha_B=\frac{1}{p+q}\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}^{\omega_i}+q\alpha_{B_j}^{\omega_j}\right)^{\frac{2}{n(n+1)}}$$

$$\sigma_B=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}\left(p\alpha_{B_i}^{\omega_i}+q\alpha_{B_j}^{\omega_j}\right)^{\frac{2}{n(n+1)}}\times\left(\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{\left(p\omega_i^{\frac{1}{2}}\alpha_{B_i}^{\omega_i-1}\sigma_{B_i}+q\omega_j^{\frac{1}{2}}\alpha_{B_j}^{\omega_j-1}\sigma_{B_j}\right)^2}{\left(p\alpha_{B_i}^{\omega_i}+q\alpha_{B_j}^{\omega_j}\right)^2}\right)\right)^{\frac{1}{2}}\right)$$

$$\mu_B=(74)\quad V_{ij}=\left(1+(\gamma-1)\mu_{B_i}\right)^p\left(1+(\gamma-1)\mu_{B_j}\right)^q\quad W_{ij}=\left(1-\mu_{B_i}\right)^p\left(1-\mu_{B_j}\right)^q$$

$$v_B=(75)\quad V_{ij}=\left(1+(\gamma-1)(1-v_{B_i})\right)^p\left(1+(\gamma-1)(1-v_{B_j})\right)^q\quad W_{ij}={v_{B_i}}^p{v_{B_j}}^q$$

Consequently, it is proved that $\mu_i(\mu_i)$ and $v_i(v_i)$ are monotonically increasing.

(1) Taking the derivative of μ_i with respect to μ_i, we have:

$$\frac{d\mu_i}{d\mu_i}=\frac{\gamma\omega_i v_i^{\omega_i-1}\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+\gamma\omega_i\mu_i^{\omega_i}\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i-1}(\gamma-1)}{\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1){\mu_i}^{\omega_i}\right)^2}$$
$$=\frac{\gamma\omega_i\mu_i^{\omega_i-1}\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i-1}\left(1+(\gamma-1)(1-\mu_i)+\mu_i(\gamma-1)\right)}{\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1){\mu_i}^{\omega_i}\right)^2}=\frac{\gamma^2\omega_i\mu_i^{\omega_i-1}}{\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1){\mu_i}^{\omega_i}\right)^2}>0$$

Thus, v_i is monotonically increasing with respect to v_i.

(2) Taking the derivative of v_i with respect to μ_i, we obtain:

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$$\frac{dv_i}{dv_i}=\frac{\gamma\left(\omega_i\left(1+(\gamma-1)v_i\right)^{\omega_i-1}(\gamma-1)+\omega_i\left(1-v_i\right)^{\omega_i-1}+\left(1+(\gamma-1)v_i\right)^{\omega_i}-\left(1-v_i\right)^{\omega_i}\right)}{\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma-1)\left(1-v_i\right)^{\omega_i}\right)^2}$$
$$=\frac{\gamma\left(\omega_i\left(\left(1+(\gamma-1)v_i\right)^{\omega_i-1}\gamma+\left(1-v_i\right)^{\omega_i-1}-\left(1+(\gamma-1)v_i\right)^{\omega_i-1}\right)+\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-\left(1-v_i\right)^{\omega_i}\right)\right)}{\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma-1)\left(1-v_i\right)^{\omega_i}\right)^2}$$

For any $\omega_i\in[0,1],\ \gamma>0,\ v_i\in[0,1]$, we have:

$$\left(1+(\gamma-1)v_i\right)^{\omega_i-1}>0,\ \left(1-\mu_i\right)^{\omega_i-1}-\left(1+(\gamma-1)v_i\right)^{\omega_i-1}>0,\ \left(1+(\gamma-1)v_i\right)^{\omega_i}-\left(1-v_i\right)^{\omega_i}>0$$

Therefore $\frac{dv_i}{dv_i}>0$, in other words, v_i is monotonically increasing with respect to v_i.

We obtain the proof of the Theorem 16 based on the similar process in Theorem 13.

Theorem 17 (boundedness). Let $A_i=\left\langle(\alpha_i,\sigma_i),\mu_i,v_i\right\rangle$ be a set of NIFNs, and $\omega_i\,(i=1,2,\cdots n)$ is the weight of $A_i\,(i=1,2,\cdots,n)$, $\omega_i\in[0,1]$, $\sum_{i=1}^{n}\omega_i=1$.

$$A_i^-=\left\langle(\alpha_i,\sigma_i),\min_i\{\mu_i\},\max_i\{v_i\}\right\rangle,\quad A_i^+=\left\langle(\alpha_i,\sigma_i),\max_i\{\mu_i\},\min_i\{v_i\}\right\rangle$$

Then when $\prod_{i=1,j=i}^{n}\left(p\alpha_i^{w_i}+q\alpha_j^{w_j}\right)\geq 0$,

$$NIFHWGHM_w^{p,q}\left(A_1^-,A_2^-,\cdots,A_n^-\right)\leq NIFHWGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHWGHM_w^{p,q}\left(A_1^+,A_2^+,\cdots,A_n^+\right)$$

when $\prod_{i=1,j=i}^{n}\left(p\alpha_i^{w_i}+q\alpha_j^{w_j}\right)<0$,

$$NIFHWGHM_w^{p,q}\left(A_1^+,A_2^+,\cdots,A_n^+\right)\leq NIFHWGHM^{p,q}\left(A_1,A_2,\cdots,A_n\right)\leq NIFHWGHM_w^{p,q}\left(A_1^-,A_2^-,\cdots,A_n^-\right)$$

The boundary is distinctly corollary of the monotonicity, the proof is omitted.

The NIFHWGHM operator has also some special cases discussed in the following.

(1) When γ=1, the formulas (87) and (88) reduce to the following equation.

$$\mu=1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-\mu_i^{w_i}\right)^p\left(1-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \tag{93}$$

$$v=\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-\left(1-v_i\right)^{\omega_i}\right)^p\times\left(1-\left(1-v_j\right)^{\omega_j}\right)^q\right)^{\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}} \tag{94}$$

So the normal intuitionistic fuzzy weighted geometric Heronian mean (NIFWGHM) operator can be defined by the formulas (84)-(86) and (93)-(94).

(2) When γ=2, then

$$\mu=\frac{\left(\begin{array}{l}\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q+3\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\+3\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q-\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}-\left(\begin{array}{l}\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q+3\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\-\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q-\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}}{\left(\begin{array}{l}\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q+3\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\+3\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q-\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}+\left(\begin{array}{l}\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q+3\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\-\prod_{i=1,j=i}^{n}\left(\left((2-\mu_i)^{w_i}+\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}+\mu_j^{w_j}\right)^q-\left((2-\mu_i)^{w_i}-\mu_i^{w_i}\right)^p\left((2-\mu_j)^{w_j}-\mu_j^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}} \tag{95}$$

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$$v=\frac{2\left(\begin{array}{l}\prod\limits_{i=1,j=i}^{n}\left(\left((1+v_i)^{w_i}+3(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}+3(1-v_j)^{w_j}\right)^q+3\left((1+v_i)^{w_i}-(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}-(1-v_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\-\prod\limits_{i=1,j=i}^{n}\left(\left((1+v_i)^{w_i}+3(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}+3(1-v_j)^{w_j}\right)^q-\left((1+v_i)^{w_i}-(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}-(1-v_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}}{\left(\begin{array}{l}\left(\begin{array}{l}\prod\limits_{i=1,j=i}^{n}\left(\left((1+v_i)^{w_i}+3(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}+3(1-v_j)^{w_j}\right)^q+3\left((1+v_i)^{w_i}-(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}-(1-v_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\+3\prod\limits_{i=1,j=i}^{n}\left(\left((1+v_i)^{w_i}+3(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}+3(1-v_j)^{w_j}\right)^q-\left((1+v_i)^{w_i}-(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}-(1-v_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}\\+\left(\begin{array}{l}\prod\limits_{i=1,j=i}^{n}\left(\left((1+v_i)^{w_i}+3(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}+3(1-v_j)^{w_j}\right)^q+3\left((1+v_i)^{w_i}-(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}-(1-v_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\\-\prod\limits_{i=1,j=i}^{n}\left(\left((1+v_i)^{w_i}+3(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}+3(1-v_j)^{w_j}\right)^q-\left((1+v_i)^{w_i}-(1-v_i)^{w_i}\right)^p\left((1+v_j)^{w_j}-(1-v_j)^{w_j}\right)^q\right)^{\frac{2}{n(n+1)}}\end{array}\right)^{\frac{1}{p+q}}\end{array}\right)} \tag{96}$$

We can see that the NIFHWGHM operator transform into the normal intuitionistic fuzzy Einstein weighted geometric Heronian mean (NIFEWGHM) operator which is defined by the formulas (84)-(86) and (95)-(96).

(3) When $p=q=\frac{1}{2}$, the NIFHWGHM operator degenerates into the normal intuitionistic fuzzy evolution weight Hamacher geometric Heronian mean (NIFEHWGHM) operator.

$$NIFHGHM_w^{\frac{1}{2}\frac{1}{2}}(A_1,A_2,\cdots,A_n)=\underset{i=1,j=i}{\overset{n}{\otimes_H}}\left(\tfrac{1}{2}A_i^{\omega_i}\oplus_H\tfrac{1}{2}A_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}$$

$$=\left\langle\left(\left(\prod_{i=1,j=i}^{n}\left(\tfrac{1}{2}\alpha_i^{\omega_i}+\tfrac{1}{2}\alpha_j^{\omega_j}\right)^{\frac{2}{n(n+1)}},\left(\prod_{i=1,j=i}^{n}\left(\tfrac{1}{2}\alpha_i^{\omega_i}+\tfrac{1}{2}\alpha_j^{\omega_j}\right)^{\frac{2}{n(n+1)}}\right)\times\sqrt{\sum_{i=1,j=i}^{n}\left(\frac{2}{n(n+1)}\frac{\left(\omega_i\frac{1}{2}\alpha_i^{\omega_i-1}\sigma_i+\omega_j\frac{1}{2}\alpha_j^{\omega_j-1}\sigma_j\right)^2}{\left(\alpha_i^{\omega_i}+\alpha_j^{\omega_j}\right)^2}\right)}\right),\mu,v\right\rangle$$

$$v=\frac{\prod\limits_{i=1,j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}+(\gamma^2-1)(1-v_j)^{\omega_j}\right)}\\+(\gamma^2-1)\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}-(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}-(1-v_j)^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}-\prod\limits_{i=1,j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}+(\gamma^2-1)(1-v_j)^{\omega_j}\right)}\\-\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}-(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}-(1-v_j)^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}}{\prod\limits_{i=1,j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}+(\gamma^2-1)(1-v_j)^{\omega_j}\right)}\\+(\gamma^2-1)\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}-(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}-(1-v_j)^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}+(\gamma-1)\prod\limits_{i=1,j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}+(\gamma^2-1)(1-v_j)^{\omega_j}\right)}\\-\sqrt{\left((1+(\gamma-1)v_i)^{\omega_i}-(1-v_i)^{\omega_i}\right)\left((1+(\gamma-1)v_j)^{\omega_j}-(1-v_j)^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}}$$

$$\mu=\frac{\gamma\prod\limits_{i=1,\ j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)(1-\mu_i))^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)\left((1+(\gamma-1)(1-\mu_j))^{\omega_j}+(\gamma-1)\mu_j^{\omega_j}\right)}\\-\sqrt{\left((1+(\gamma-1)(1-\mu_i))^{\omega_i}-\mu_i^{\omega_i}\right)\left((1+(\gamma-1)(1-\mu_j))^{\omega_j}-\mu_j^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}}{\prod\limits_{i=1,\ j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)(1-\mu_i))^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)\left((1+(\gamma-1)(1-\mu_j))^{\omega_j}+(\gamma-1)\mu_j^{\omega_j}\right)}\\+(\gamma^2-1)\sqrt{\left((1+(\gamma-1)(1-\mu_i))^{\omega_i}-\mu_i^{\omega_i}\right)\left((1+(\gamma-1)(1-\mu_j))^{\omega_j}-\mu_j^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}+(\gamma-1)\prod\limits_{i=1,\ j=i}^{n}\left(\begin{array}{l}\sqrt{\left((1+(\gamma-1)(1-\mu_i))^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)\left((1+(\gamma-1)(1-\mu_j))^{\omega_j}+(\gamma-1)\mu_j^{\omega_j}\right)}\\-\sqrt{\left((1+(\gamma-1)(1-\mu_i))^{\omega_i}-\mu_i^{\omega_i}\right)\left((1+(\gamma-1)(1-\mu_j))^{\omega_j}-\mu_j^{\omega_j}\right)}\end{array}\right)^{\frac{2}{n(n+1)}}}$$

(4) If $q\to 0$, the NIFHWGHM operator reduces to generalized normal intuitionistic fuzzy Hamacher geometric weight Arithmetical mean (NIFEHWGAM) operator.

$$NIFHWGHM_w^{p}(A_1,A_2,\cdots,A_n)=\frac{1}{p}\left(\underset{i=1}{\overset{n}{\otimes_H}}\left(pA_i^{\omega_i}\right)^{\frac{1}{n}}\right)=\left\langle\left(\frac{1}{p}\left(\prod_{i=1}^{n}\left(p\alpha_i^{\omega_i}\right)^{\frac{1}{n}}\right),\frac{1}{p}\left(\left(\prod_{i=1}^{n}\left(p\alpha_i^{\omega_i}\right)^{\frac{1}{n}}\right)\times\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\omega_i\sigma_i^2}{\alpha_i^2}\right)}\right)\right),\mu,v\right\rangle$$

$$v=\frac{\gamma\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}+(\gamma-1)\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}+(\gamma^2-1)(1-v_i)^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}-(1-v_i)^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}$$

$$\mu=\frac{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}-\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}{\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\+(\gamma^2-1)\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}+(\gamma-1)\left(\begin{array}{l}\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p+(\gamma^2-1)\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\\-\prod_{i=1}^{n}\left(\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}+(\gamma-1)\mu_i^{\omega_i}\right)^p-\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}-\mu_i^{\omega_i}\right)^p\right)^{\frac{1}{n}}\end{array}\right)^{\frac{1}{p}}}$$

(5) If $p=1,q\to 0$, the NIFHWGHM operator is transformed into normal intuitionistic fuzzy Hamacher geometric weight Arithmetical mean (NIFHWGAM) operator.

$$NIFHWGHM_w^1\left(A_1,A_2,\cdots,A_n\right)=\left(\bigotimes_{H\,i=1}^{n}\left(A_i^{\omega_i}\right)^{\frac{1}{n}}\right)=$$

$$\left\langle\left(\prod_{i=1}^{n}\left(\alpha_i^{\omega_i}\right)^{\frac{1}{n}},\left(\prod_{i=1}^{n}\left(\alpha_i^{\omega_i}\right)^{\frac{1}{n}}\right)\times\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{(\sigma_i)^2}{(\alpha_i)^2}\right)}\right),\frac{\gamma\prod_{i=1}^{n}\mu_i^{\frac{\omega_i}{n}}}{\prod_{i=1}^{n}\left(\left(1+(\gamma-1)(1-\mu_i)\right)^{\omega_i}\right)^{\frac{1}{n}}+(\gamma-1)\prod_{i=1}^{n}\mu_i^{\frac{\omega_i}{n}}},\frac{\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}\right)^{\frac{1}{n}}-\prod_{i=1}^{n}\left((1-v_i)^{\omega_i}\right)^{\frac{1}{n}}}{\prod_{i=1}^{n}\left(\left(1+(\gamma-1)v_i\right)^{\omega_i}\right)^{\frac{1}{n}}+(\gamma-1)\prod_{i=1}^{n}\left((1-v_i)^{\omega_i}\right)^{\frac{1}{n}}}\right\rangle$$

5. New methods based on Hamacher Heronian mean operator for normal intuitionistic fuzzy information.

We utilize the developed operators to the MCGDM problem where the input arguments are some NIFNs.

(1) Description of a MCGDM problem.

a collection of decision makers: $E=\{e_1,e_2,\cdots,e_q\}$

a collection of alternatives: $A=\{A_1,A_2,\cdots,A_m\}$

attributes set: $C=\{C_1,C_2,\cdots,C_n\}$

weight of attributes set: $\omega=\{\omega_1,\omega_2,\cdots,\omega_n\}$, $\omega_j\in[0,1]$ $\sum_{j=1}^{n}\omega_j=1$

weight of decision makers: $\lambda=\{\lambda_1,\lambda_2,\cdots,\lambda_q\}$, $\lambda_k\in[0,1]$ $\sum_{k=1}^{q}\lambda_k=1$

a decision matrix given by e_k for A with respect to C:

$$E^k=\left[e_{ij}^k\right]_{mxn}\quad k=1,2,\cdots,q\ e_{ij}^k=\left\langle\left(\alpha_{ij}^k,\sigma_{ij}^k\right),\mu_{ij}^k,v_{ij}^k\right\rangle$$

(2) the method based on NIFHWHM operator (or NIFHWGHM)

Step 1. Normalization of decision making information

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In realization, the bigger values of some benefit attributes (I_1) are better and the smaller values of some cost attributes (I_2) are better, therefore, decision-making information should be normalized for the unity of input data. Hence, decision matrices $E^k=\left[e_{ij}^k\right]_{mxn}$ can be transformed into matrices $R^k=\left[r_{ij}^k\right]_{mxn},(k=1,2,\cdots,q)$ by Wang's method [16] where

$$r_{ij}^k=\left\langle\left(\alpha_{N_{ij}}^k,\sigma_{N_{ij}}^k\right),\mu_{N_{ij}}^k,\mathrm{v}_{N_{ij}}^k\right\rangle=\begin{cases}\left\langle\left(\dfrac{\alpha_{ij}}{\max\limits_i\{\alpha_{ij}\}},\dfrac{\sigma_{ij}}{\max\limits_i\{\sigma_{ij}\}},\dfrac{\sigma_{ij}}{\alpha_{ij}}\right),\mu_{ij}^k,\upsilon_{ij}^k\right\rangle & C_j\in I_1\\ \left\langle\left(\dfrac{\min\limits_i\{\alpha_{ij}\}}{\alpha_{ij}},\dfrac{\sigma_{ij}}{\max\limits_i\{\sigma_{ij}\}},\dfrac{\sigma_{ij}}{\alpha_{ij}}\right),\mu_{ij}^k,\upsilon_{ij}^k\right\rangle & C_j\in I_2\end{cases} \tag{97}$$

Step 2. Apply the NIFHWHM or NIFHWGHM operator to integrate $R^k=\left[\gamma_{ij}^k\right]_{mxn}$ $(k=1,2,\cdots,q)$ into the collection of the matrix $R=\left[r_{ij}\right]_{mxn}$, where

$$r_{ij}=NIFHWHM_{\omega}^{p,q}\left(r_{ij}^1,r_{ij}^2,\cdots,r_{ij}^q\right)\text{or } r_{ij}=NIFHWGHM_{\omega}^{p,q}\left(r_{ij}^1,r_{ij}^2,\cdots,r_{ij}^q\right)$$

Step 3. Utilize the NIFHWHM operator or NIFHWGHM operator to get the value $r_i\left(i=1,2,\cdots,m\right)$ of alternative $A_i\left(i=1,2,\cdots,m\right)$ where

$$r_i=NIFHWHM_{\omega}^{p,q}\left(r_{i1},r_{i2},\cdots,r_{in}\right)\ \text{ or }\ r_i=NIFHWGHM_{\omega}^{p,q}\left(r_{i1},r_{i2},\cdots,r_{in}\right).$$

Step 4. Utilize the ranking method of the NIFNs to rank $r_i\left(i=1,2,\cdots,m\right)$ in the descending order and derive the priority of each alternative $A_i\left(i=1,2,\cdots,m\right)$ according to $r_i\left(i=1,2,\cdots,m\right)$.

Step 5. the end.

6. an application example

A MCGDM problem will be presented to demonstrate the application of the developed approach (a stock value evaluation problem), which is adapted from [26]. In the intricate stock market, a real problem is how to analyze the stock investment value and choose the stock. Therefore, an effective stock evaluation approach is very significant. However, most of the financial indicators approximately obey normal distribution, and the NIFNs can effectively describe the phenomenon of normal distribution and evaluate the stock investment value information. To evaluate the stock alternatives, we suppose that there are four stocks (alternatives) denoted as $\{A_1,A_2,A_3,A_4\}$, and we extract the four key financial attributes described as undistributed profits per share (C_1), Net asset value per share (C_2), Earnings per share (C_3), and Equity ratio (C_4), whose weight vector is $\omega=(0.33,0.26,0.16,0.25)^T$.

Obviously, these attributes are all benefit attributes under which three decision workers $e_k\left(k=1,2,3\right)$ utilize NIFNs to evaluate the four alternatives. Three decision makers can evaluate the four alternatives under the four attributes (C_1,C_2,C_3,C_4), and Three decision matrices $E^k=\left[e_{ij}^k\right]_{4\times4}$ are set out in the following tables (see Tables1-3).

Table 1. decision matrix E^1 from e^1.

	C_1	C_2	C_3	C_4
A_1	$\langle(3.07,2.14),0.7,0.15\rangle$	$\langle(0.94,0.69),0.7,0.15\rangle$	$\langle(1.82,0.90),0.6,0.2\rangle$	$\langle(1.76,3.67),0.65,0.15\rangle$
A_2	$\langle(2.12,1.21),0.7,0.2\rangle$	$\langle(0.42,0.35),0.6,0.15\rangle$	$\langle(2.16,0.98),0.55,0.2\rangle$	$\langle(2.35,3.23),0.6,0.1\rangle$
A_3	$\langle(1.55,1.63),0.7,0.2\rangle$	$\langle(0.73,0.41),0.6,0.2\rangle$	$\langle(1.55,0.79),0.7,0.2\rangle$	$\langle(4.25,2.54),0.7,0.2\rangle$
A_4	$\langle(1.23,0.96),0.75,0.25\rangle$	$\langle(0.63,0.50),0.6,0.15\rangle$	$\langle(1.14,0.66),0.6,0.15\rangle$	$\langle(4.96,2.93),0.75,0.2\rangle$

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Table 2. decision matrix E^2 from e^2 .

	C_1	C_2	C_3	C_4
A_1	$\langle(3.07,2.14),0.6,0.15\rangle$	$\langle(0.94,0.69),0.65,0.1\rangle$	$\langle(1.82,0.90),0.7,0.2\rangle$	$\langle(1.76,3.67),0.65,0.2\rangle$
A_2	$\langle(2.12,1.21),0.7,0.25\rangle$	$\langle(0.42,0.35),0.7,0.2\rangle$	$\langle(2.16,0.98),0.6,0.2\rangle$	$\langle(2.35,3.23),0.65,0.2\rangle$
A_3	$\langle(1.55,1.63),0.6,0.2\rangle$	$\langle(0.73,0.41),0.6,0.15\rangle$	$\langle(1.55,0.79),0.6,0.2\rangle$	$\langle(4.25,2.54),0.65,0.15\rangle$
A_4	$\langle(1.23,0.96),0.8,0.2\rangle$	$\langle(0.63,0.50),0.65,0.15\rangle$	$\langle(1.14,0.66),0.7,0.15\rangle$	$\langle(4.96,2.93),0.7,0.2\rangle$

Table 3. decision matrix E^3 from e^3 .

	C_1	C_2	C_3	C_4
A_1	$\langle(3.07,2.14),0.65,0.2\rangle$	$\langle(0.94,0.69),0.60,0.2\rangle$	$\langle(1.82,0.90),0.65,0.2\rangle$	$\langle(1.76,3.67),0.7,0.15\rangle$
A_2	$\langle(2.12,1.21),0.65,0.2\rangle$	$\langle(0.42,0.35),0.75,0.15\rangle$	$\langle(2.16,0.98),0.7,0.15\rangle$	$\langle(2.35,3.23),0.7,0.15\rangle$
A_3	$\langle(1.55,1.63),0.8,0.2\rangle$	$\langle(0.73,0.41),0.65,0.15\rangle$	$\langle(1.55,0.79),0.7,0.25\rangle$	$\langle(4.25,2.54),0.6,0.1\rangle$
A_4	$\langle(1.23,0.96),0.7,0.1\rangle$	$\langle(0.63,0.50),0.7,0.2\rangle$	$\langle(1.14,0.66),0.7,0.2\rangle$	$\langle(4.96,2.93),0.6,0.2\rangle$

6.1. a new method related to the NIFHWHM and NIFHWGHM operator

According to the following steps, all of the alternatives are ranked in order to get the best alternatives.

Step 1. Normalizing the input data

Utilizing Equation (97) to integrate decision matrix into the normalized decision matrix $R^k=\left[r_{ij}^k\right]_{4\times4}$ which is shown Tables 4-6.

Step 2. Applying the NIFHWHM and NIFHWGHM operator to integrate normalization matrices $R^k=\left[r_{ij}^k\right]_{4\times4}$ $(k=1,2,3)$ into a matrix $R^k=\left[r_{ij}\right]_{4\times4}$ (see Table 7 and Table 8) (without the loss of generality, let $\gamma=2,p=q=1$)

Step 3. Utilizing the NIFHWHM and NIFHWGHM operator to derive preference values of $R=\left[r_{ij}\right]_{4\times4}$ and calculate the collection preference value $r_i\,(i=1,2,3,4)$ of alternative $A_i\,(i=1,2,3,4)$ (see Table 9 and Table 10).

Step 4. Calculating the value of score function $S(r_i)(i=1,2,3,4)$ (see Table 9 and Table10).

Step 5. Arranging all of the alternatives $A_i\,(i=1,2,3,4)$ as follows (see Table 9 and Table10).

Table 4. normalization matrix R^1 given by e^1 .

	C_1	C_2	C_3	C_4
A_1	$\langle(1.0,0.697),0.7,0.15\rangle$	$\langle(1.0,0.734),0.7,0.15\rangle$	$\langle(0.843,0.454),0.6,0.2\rangle$	$\langle(0.355,2.085),0.65,0.15\rangle$
A_2	$\langle(0.69,0.323),0.7,0.2\rangle$	$\langle(0.447,0.423),0.6,0.15\rangle$	$\langle(1.0,0.454),0.55,0.2\rangle$	$\langle(0.474,1.21),0.6,0.1\rangle$
A_3	$\langle(0.505,0.801),0.7,0.2\rangle$	$\langle(0.777,0.334),0.6,0.12\rangle$	$\langle(0.718,0.411),0.7,0.2\rangle$	$\langle(0.857,0.414),0.7,0.2\rangle$
A_4	$\langle(0.401,0.35),0.75,0.25\rangle$	$\langle(0.67,0.575),0.6,0.15\rangle$	$\langle(0.528,0.39),0.6,0.15\rangle$	$\langle(1.0,0.472),0.75,0.2\rangle$

Table 5. normalization matrix R^2 given by e^2 .

	C_1	C_2	C_3	C_4
A_1	$\langle(1.0,0.697),0.6,0.15\rangle$	$\langle(1.0,0.734),0.65,0.1\rangle$	$\langle(0.843,0.454),0.7,0.2\rangle$	$\langle(0.355,2.085),0.65,0.2\rangle$
A_2	$\langle(0.69,0.323),0.7,0.25\rangle$	$\langle(0.447,0.423),0.7,0.2\rangle$	$\langle(1.0,0.454),0.6,0.2\rangle$	$\langle(0.474,1.21),0.65,0.2\rangle$
A_3	$\langle(0.505,0.801),0.6,0.2\rangle$	$\langle(0.777,0.334),0.6,0.15\rangle$	$\langle(0.718,0.411),0.6,0.2\rangle$	$\langle(0.857,0.414),0.65,0.15\rangle$
A_4	$\langle(0.401,0.35),0.8,0.2\rangle$	$\langle(0.67,0.575),0.65,0.15\rangle$	$\langle(0.528,0.39),0.7,0.15\rangle$	$\langle(1.0,0.472),0.7,0.2\rangle$

Table 6. normalization matrix R^3 given by e^3 .

	C_1	C_2	C_3	C_4
A_1	$\langle(1.0,0.697),0.65,0.2\rangle$	$\langle(1.0,0.734),0.6,0.2\rangle$	$\langle(0.843,0.454),0.65,0.2\rangle$	$\langle(0.355,2.085),0.7,0.15\rangle$
A_2	$\langle(0.69,0.323),0.65,0.2\rangle$	$\langle(0.447,0.423),0.75,0.15\rangle$	$\langle(1.0,0.454),0.7,0.15\rangle$	$\langle(0.474,1.21),0.7,0.15\rangle$
A_3	$\langle(0.505,0.801),0.8,0.2\rangle$	$\langle(0.777,0.334),0.65,0.15\rangle$	$\langle(0.718,0.411),0.7,0.25\rangle$	$\langle(0.857,0.414),0.6,0.1\rangle$
A_4	$\langle(0.401,0.35),0.7,0.1\rangle$	$\langle(0.67,0.575),0.7,0.2\rangle$	$\langle(0.528,0.39),0.7,0.2\rangle$	$\langle(1.0,0.472),0.6,0.2\rangle$

Table 7. The collective decision matrix R of NIFHWHM operator.

	C_1	C_2	C_3	C_4
A_1	$\langle(0.664,0.073),0.872,0.064\rangle$	$\langle(0.664,0.085),0.863,0.082\rangle$	$\langle(0.605,0.049),0.872,0.038\rangle$	$\langle(0.413,0.002),0.851,0.049\rangle$
A_2	$\langle(0.555,0.069),0.881,0.050\rangle$	$\langle(0.443,0.059),0.989,0.031\rangle$	$\langle(0.660,0.072),0.849,0.071\rangle$	$\langle(0.463,0.082),0.842,0.047\rangle$
A_3	$\langle(0.480,0.898),0.893,0.056\rangle$	$\langle(0.575,0.065),0.850,0.048\rangle$	$\langle(0.562,0.096),0.859,0.073\rangle$	$\langle(0.612,0.061),0.862,0.050\rangle$
A_4	$\langle(0.431,0.113),0.868,0.062\rangle$	$\langle(0.538,0.062),0.879,0.043\rangle$	$\langle(0.481,0.102),0.83,0.062\rangle$	$\langle(0.662,0.063),0.912,0.053\rangle$

Table8. The collective decision matrix R of NIFHWGHM operator.

	C_1	C_2	C_3	C_4
A_1	$\langle(1.00,0.402),0.786,0.0.202\rangle$	$\langle(1.00,0.423),0.787,0.20\rangle$	$\langle(0.945,0.293),0.78,0.207\rangle$	$\langle(0.708,2.397),0.792,0.202\rangle$
A_2	$\langle(0.884,0.239),0.805,0.209\rangle$	$\langle(0.765,0.417),0.795,0.202\rangle$	$\langle(1.00,0.262),0.757,0.205\rangle$	$\langle(0.78,1.147),0.779,0.199\rangle$
A_3	$\langle(0.797,0.729),0.81,0.207\rangle$	$\langle(0.919,0.228),0.762,0.198\rangle$	$\langle(0.896,0.296),0.794,0.209\rangle$	$\langle(0.95,0.265),0.787,0.201\rangle$
A_4	$\langle(0.738,0.371),0.843,0.206\rangle$	$\langle(0.875,0.433),0.779,0.202\rangle$	$\langle(0.808,0.344),0.788,0.202\rangle$	$\langle(1.00,0.272),0.807,0.207\rangle$

Table9. aggregation result of the NIFHWHM operator.

Alternatives A_i	Collection Preference Value r_i	Score Function $S(r_i)$	Ranking
A_1	$r_1=\langle(0.0451,0.1054),0.0537,0.9268\rangle$	$S(r_1)=-0.0394$	2
A_2	$r_2=\langle(0.0440,0.0568),0.0525,0.9235\rangle$	$S(r_2)=-0.0383$	1
A_3	$r_3=\langle(0.0506,0.0335),0.0547,0.9297\rangle$	$S(r_3)=-0.0443$	4
A_4	$r_4=\langle(0.0489,0.0308),0.0581,0.9166\rangle$	$S(r_4)=-0.0420$	3

Table10. aggregation result of the NIFHWGHM operator.

Alternatives A_i	Collection Preference Value r_i	Score Function $S(r_i)$	Ranking
A_1	$r_1=\langle(0.8463,0.6660),0.8967,0.1569\rangle$	$S(r_1)=0.6261$	1
A_2	$r_2=\langle(0.6460,0.2203),0.8977,0.1572\rangle$	$S(r_2)=0.4784$	4
A_3	$r_3=\langle(0.6539,0.0832),0.8979,0.1571\rangle$	$S(r_3)=0.4844$	3
A_4	$r_4=\langle(0.6543,0.0725),0.9073,0.1572\rangle$	$S(r_4)=0.4908$	2

6.2. Sensitivity analysis

In the proposed method of this paper, three parameters γ , p and q are involved and influence aggregation result. Therefore, we make a sensitivity analysis in order to study the influence of the generalized parameters with respect to the ordering results of the above example. In other words, we choose the different parameters γ , p and q in Steps (2) and (3) to rank all the alternatives, and discuss the effect of the parameter values change on the ordering results. The aggregation results are provided in Table 11 and Figure 1-5.

From Table 11 and Figure 1-5, we can observe that different parameter values have a certain influence on the ordering results. In general, the best alternative is A_1 with respect to the NIFWHM operator, A_2 is the best one with respect to the NIFHWGHM operator.

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(1) From Table 11 and Figure 1-2 , best alternative and the ordering results of the alternatives are concordant when $\gamma > 1.33$ and p,q are given in the NIFHWHM or NIFHWGHM operators. When $\gamma \le 1.33$ and there is at least equal to zero in p and q, the rankings of the alternatives are different.
(2) From Figure 3 and Figure 4, we can see that when γ,q are given in the NIFHWHM or NIFHWGHM operators and p takes the values of the different intervals, the best resolution and the orderings are different. For example of the NIFHWHM operator, when $\gamma = 1, q = 1$, $p \in (0, 0.83]$, the ranking is $A_1 \succ A_2 \succ A_4 \succ A_3$, $p \in (0.83, 0.1.36]$, the ranking is $A_2 \succ A_1 \succ A_4 \succ A_3$, $p \in (1.36, 1.78]$, the ranking is $A_2 \succ A_4 \succ A_1 \succ A_3$ and $p \in (1.78, 5]$, the ranking is $A_2 \succ A_4 \succ A_3 \succ A_1$, and we notice that when $p \in (0.83, 0.1.36]$, the best one is A_1, $p \in (0.83, 5]$, the best one is A_2.
(3) From Figure 5 shows that sensitivity of the parameter q is similar to the parameter p, but the influence of the value q is less in the NIFHWGHM operator. As long as $q > 0.11$, the rankings are concordant $A_1 \succ A_4 \succ A_3 \succ A_2$

In a word, based on the generalized parameters γ, p and q in NIFHWHM operator and NIFHWGHM operator, the new method of this paper can provide more reliable and flexible decision-making results. Moreover, the reasonable and best alternative can be properly obtained on the basis of actual situation of the practical MAGDM problems, namely the new method can offer an effective and powerful mathematic tool for the MAGDM under uncertainty.

Table11. sensitivity analysis with respect to γ in the NIFHWHM operator.

	γ	$S(r_1)$	$S(r_2)$	$S(r_3)$	$S(r_4)$	**Ranking**
$p=1$ $q=0$	0.1	−0.0388	−0.0298	−0.0336	−0.0288	$A_4 \succ A_2 \succ A_3 \succ A_1$
	0.5	−0.0440	−0.0336	−0.0378	−0.0333	$A_4 \succ A_2 \succ A_3 \succ A_1$
	1.0	−0.0469	−0.0358	−0.0403	−0.0359	$A_2 \succ A_4 \succ A_3 \succ A_1$
	1.5	−0.0486	−0.0371	−0.0417	−0.0374	$A_2 \succ A_4 \succ A_3 \succ A_1$
	2.0	−0.0497	−0.0380	−0.0427	−0.0384	$A_2 \succ A_4 \succ A_3 \succ A_1$
	5.0	−0.0528	−0.0403	−0.0453	−0.0410	$A_2 \succ A_4 \succ A_3 \succ A_1$
	30	−0.0560	−0.0428	−0.0480	−0.0439	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=0.5$ $q=0.5$	0.1	−0.0283	−0.0280	−0.0326	−0.0291	$A_2 \succ A_1 \succ A_4 \succ A_3$
	0.5	−0.0311	−0.0309	−0.0357	−0.0324	$A_2 \succ A_1 \succ A_4 \succ A_3$
	1.0	−0.0328	−0.0326	−0.0376	−0.0313	$A_2 \succ A_1 \succ A_4 \succ A_3$
	1.5	−0.0337	−0.0336	−0.0387	−0.0355	$A_2 \succ A_1 \succ A_4 \succ A_3$
	2.0	−0.0344	−0.0343	−0.0394	−0.0362	$A_2 \succ A_1 \succ A_4 \succ A_3$
	5.0	−0.0361	−0.0360	−0.0413	−0.0383	$A_2 \succ A_1 \succ A_4 \succ A_3$
	30	−0.0381	−0.0379	−0.0434	−0.0404	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p=1$ $q=1$	0.1	−0.0326	−0.0315	−0.0368	−0.0340	$A_2 \succ A_1 \succ A_4 \succ A_3$
	0.5	−0.0354	−0.0344	−0.0400	−0.0374	$A_2 \succ A_1 \succ A_4 \succ A_3$
	1.0	−0.0374	−0.0363	−0.0421	−0.0397	$A_2 \succ A_1 \succ A_4 \succ A_3$
	1.5	−0.0385	−0.0375	−0.0434	−0.0411	$A_2 \succ A_1 \succ A_4 \succ A_3$
	2.0	−0.0394	−0.0383	−0.0443	−0.0420	$A_2 \succ A_1 \succ A_4 \succ A_3$
	5.0	−0.0416	−0.0406	−0.0468	−0.0447	$A_2 \succ A_1 \succ A_4 \succ A_3$
	30	−0.0441	−0.0430	−0.0495	−0.0476	$A_2 \succ A_1 \succ A_4 \succ A_3$
$p=2$ $q=1$	0.1	−0.0398	−0.0341	−0.0390	−0.0362	$A_2 \succ A_4 \succ A_3 \succ A_1$
	0.5	−0.0433	−0.0372	−0.0424	−0.0398	$A_2 \succ A_4 \succ A_3 \succ A_1$
	1.0	−0.0459	−0.0395	−0.0448	−0.0425	$A_2 \succ A_4 \succ A_3 \succ A_1$
	1.5	−0.0475	−0.0410	−0.0464	−0.0442	$A_2 \succ A_4 \succ A_3 \succ A_1$
	2.0	−0.0487	−0.0420	−0.0475	−0.0454	$A_2 \succ A_4 \succ A_3 \succ A_1$
	5.0	−0.0518	−0.0448	−0.0505	−0.0487	$A_2 \succ A_4 \succ A_3 \succ A_1$
	30	−0.0553	−0.0478	−0.0539	−0.0524	$A_2 \succ A_4 \succ A_3 \succ A_1$
$p=2$ $q=5$	0.1	−0.0334	−0.0342	−0.0434	−0.0458	$A_1 \succ A_2 \succ A_3 \succ A_4$
	0.5	−0.0361	−0.0372	−0.0466	−0.0494	$A_1 \succ A_2 \succ A_3 \succ A_4$

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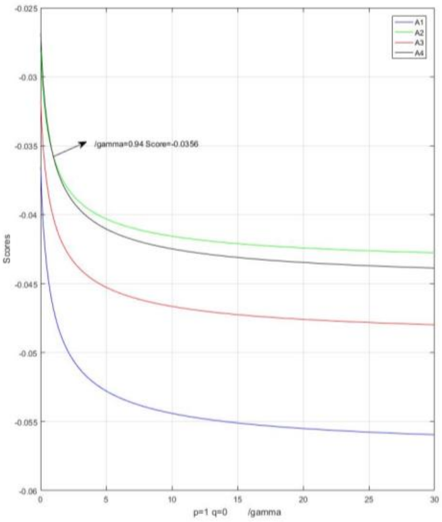
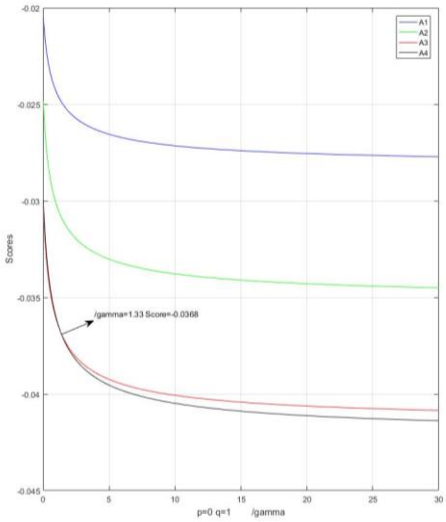
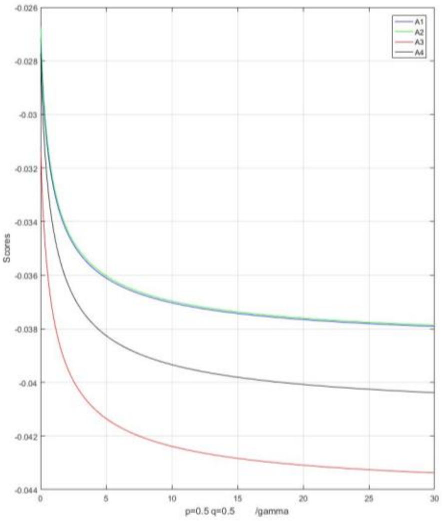
	1.0	–0.0382	–0.0396	–0.0492	–0.0523	$A_1 \succ A_2 \succ A_3 \succ A_4$
	1.5	–0.0395	–0.0411	–0.0509	–0.0542	$A_1 \succ A_2 \succ A_3 \succ A_4$
	2.0	–0.0404	–0.0422	–0.0520	–0.0556	$A_1 \succ A_2 \succ A_3 \succ A_4$
	5.0	–0.0431	–0.0450	–0.0553	–0.0596	$A_1 \succ A_2 \succ A_3 \succ A_4$
	30	–0.0472	–0.0491	–0.0608	–0.0667	$A_1 \succ A_2 \succ A_3 \succ A_4$
$p=20$ $q=5$	0.1	–0.0420	–0.0376	–0.0445	–0.0466	$A_2 \succ A_1 \succ A_3 \succ A_4$
	0.5	–0.0454	–0.0409	–0.0478	–0.0503	$A_2 \succ A_1 \succ A_3 \succ A_4$
	1.0	–0.0483	–0.0437	–0.0507	–0.0537	$A_2 \succ A_1 \succ A_3 \succ A_4$
	1.5	–0.0501	–0.0455	–0.0526	–0.0560	$A_2 \succ A_1 \succ A_3 \succ A_4$
	2.0	–0.0515	–0.0468	–0.0540	–0.0577	$A_2 \succ A_1 \succ A_3 \succ A_4$
	5.0	–0.0613	–0.0556	–0.0647	–0.0712	$A_2 \succ A_1 \succ A_3 \succ A_4$
	30	–0.0613	–0.0556	–0.0647	–0.0712	$A_2 \succ A_1 \succ A_3 \succ A_4$



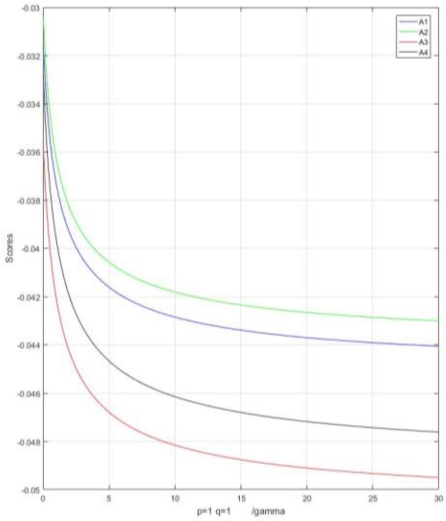
(a)



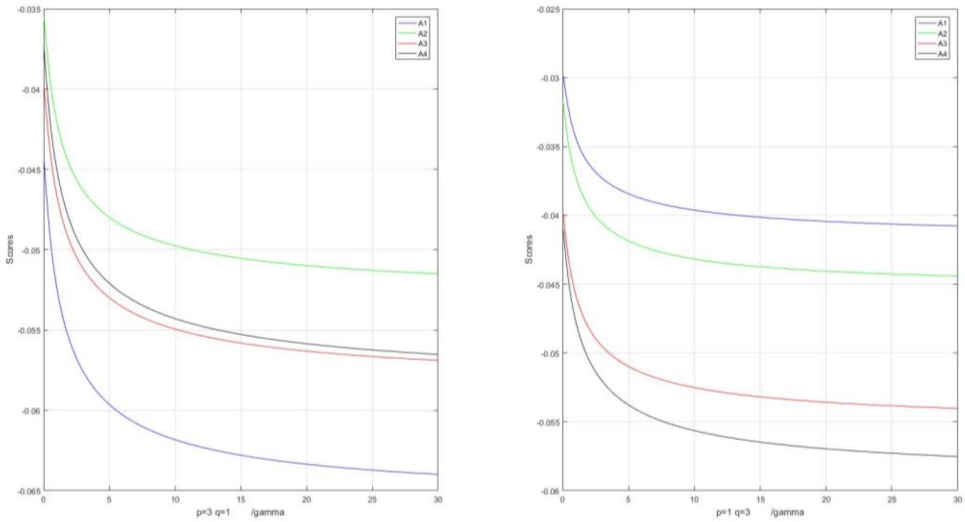
(b)



(c)

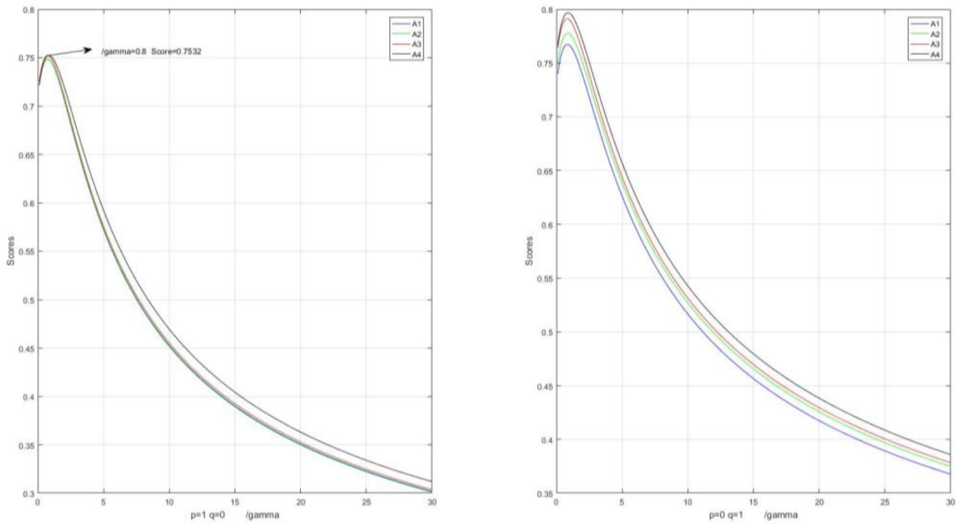


(d)

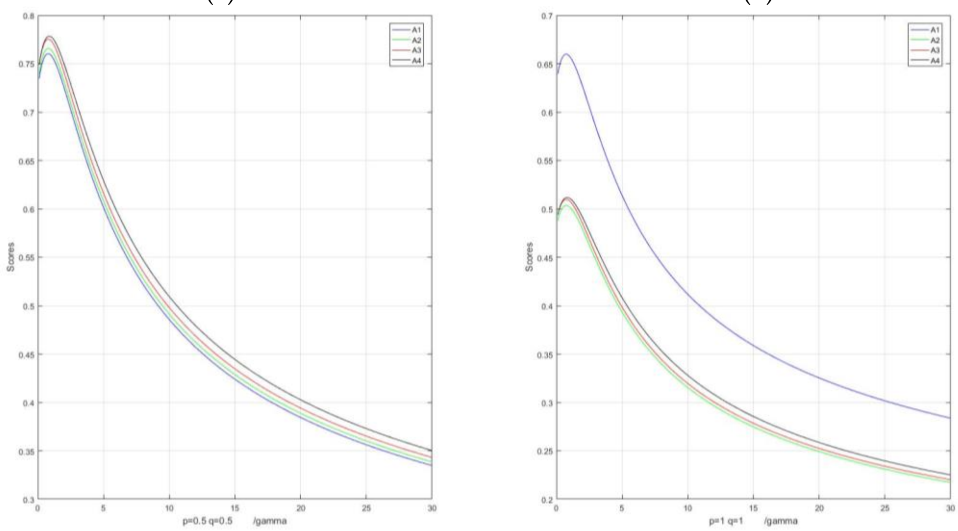


(e) (f)

Figure 1. Sensitivity analysis with respect to γ in NIFHWHM operator based on different p,q: (**a**) Variation tendency of score function value when $p=1,q=0$; (**b**) Variation tendency of score function value when $p=0,q=1$; (**c**) Variation tendency of score function value when $p=0.5,q=0.5$; (**d**) Variation tendency of score function value when $p=1,q=1$; (**e**) Variation tendency of score function value when $p=3,q=1$; (**f**) Variation tendency of score function value when $p=1,q=3$.



(a) (b)



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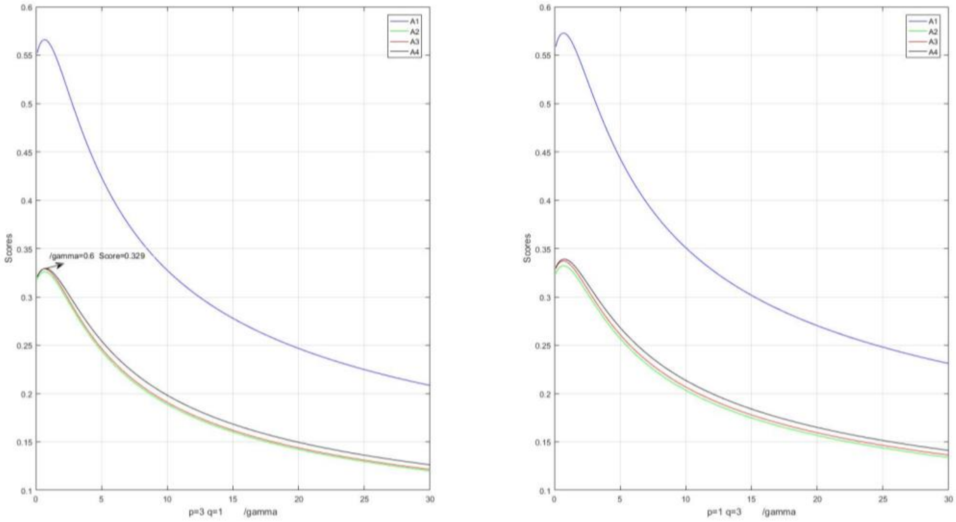
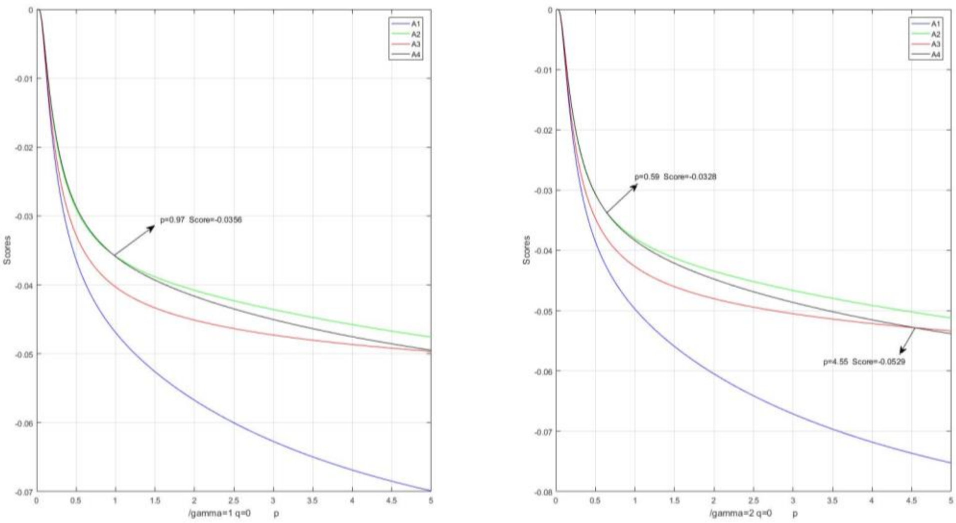
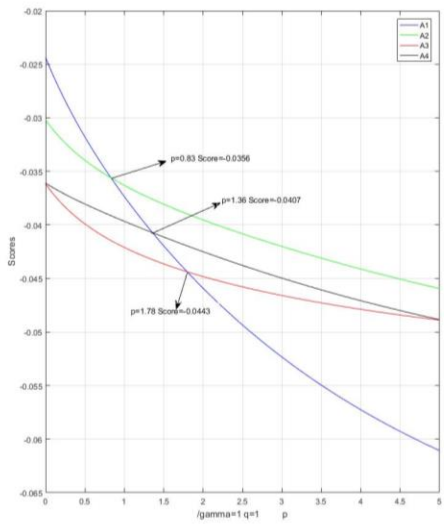


(e) (f)

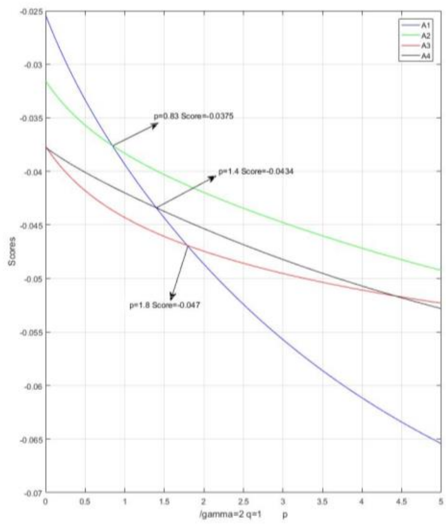
Figure 2. Sensitivity analysis with respect to γ in NIFHWGHM operator based on different p,q: (**a**) Variation tendency of score function value when $p=1,q=0$; (**b**) Variation tendency of score function value when $p=0,q=1$; (**c**) Variation tendency of score function value when $p=0.5,q=0.5$; (**d**) Variation tendency of score function value when $p=1,q=1$; (**e**) Variation tendency of score function value when $p=3,q=1$; (**f**) Variation tendency of score function value when $p=1,q=3$.



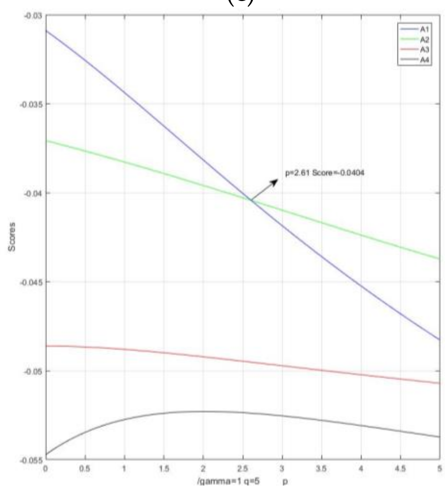
(a) (b)



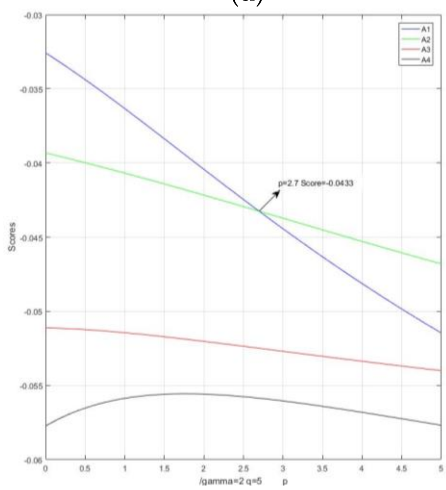
(c)



(d)

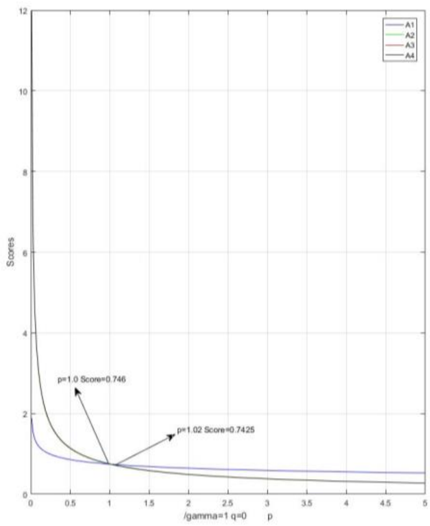


(e)

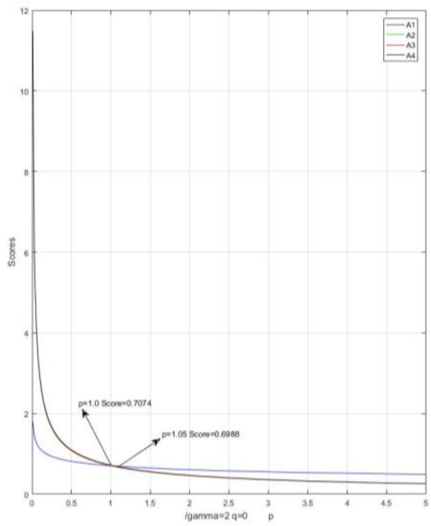


(f)

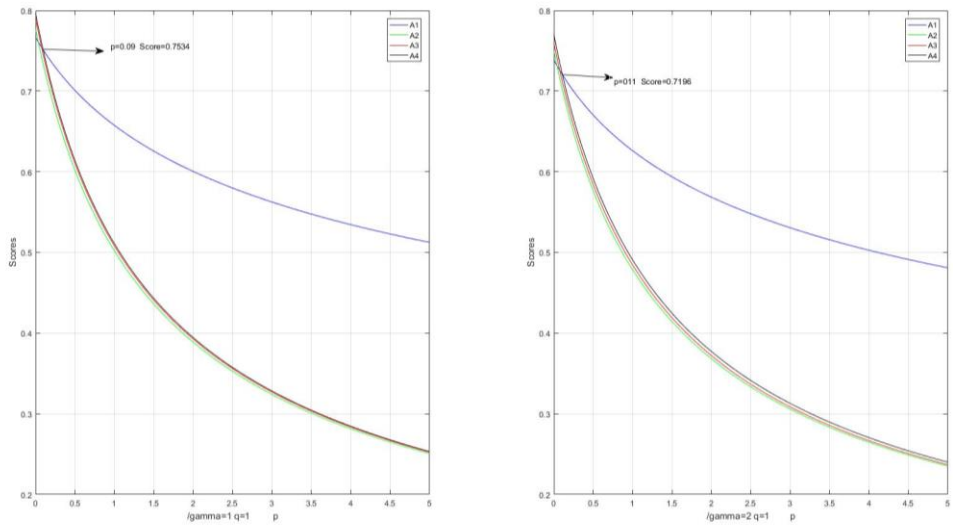
Figure 3. Sensitivity analysis with respect to p in NIFHWHM operator based on different γ, q: (**a**) Variation tendency of score function value when $\gamma=1, q=0$; (**b**) Variation tendency of score function value when $\gamma=2, q=0$; (**c**) Variation tendency of score function value when $\gamma=1, q=1$; (**d**) Variation tendency of score function value when $\gamma=2, q=1$; (**e**) Variation tendency of score function value when $\gamma=1, q=5$; (**f**) Variation tendency of score function value when $\gamma=2, q=5$.



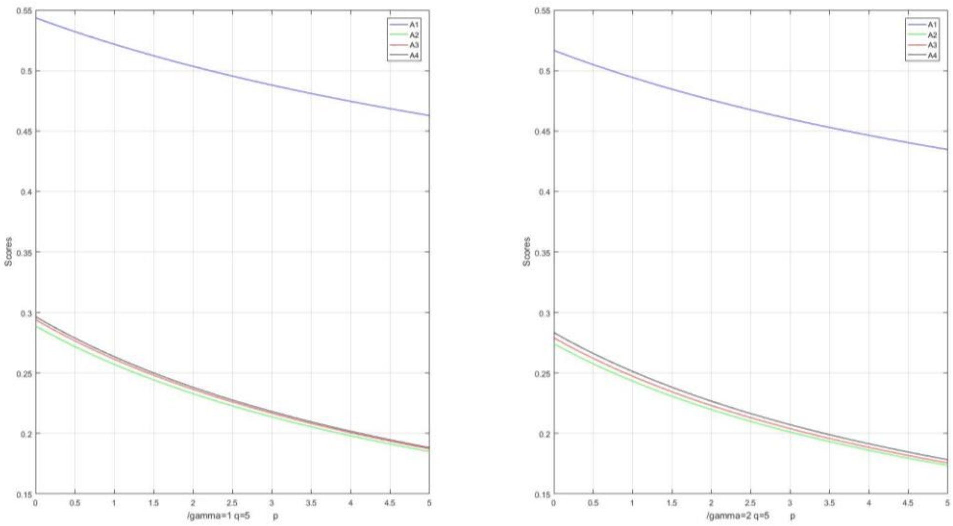
(a)



(b)

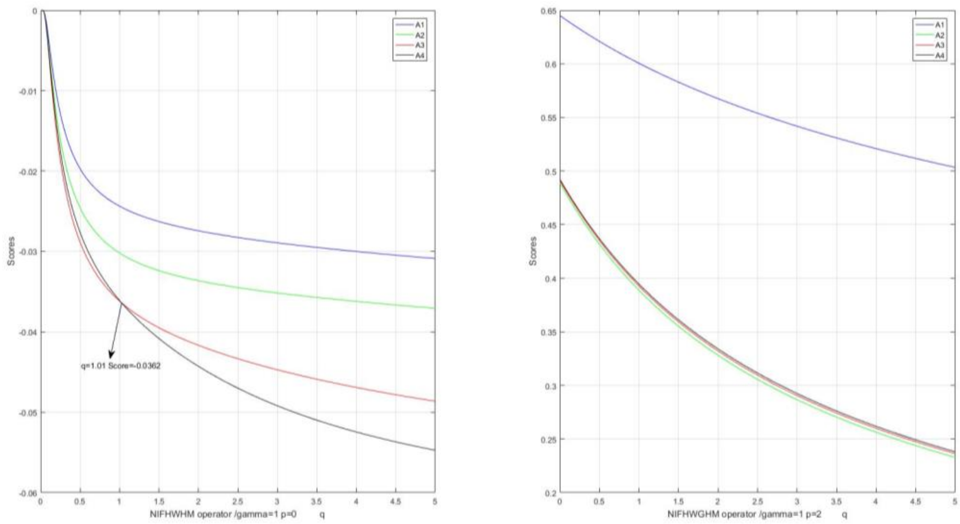


(c) (d)



(e) (f)

Figure 4. Sensitivity analysis with respect to p in NIFHWGHM operator based on different γ,q: (**a**) Variation tendency of score function value when $\gamma=1,q=0$; (**b**) Variation tendency of score function value when $\gamma=2,q=0$; (**c**) Variation tendency of score function value when $\gamma=1,q=1$; (**d**) Variation tendency of score function value when $\gamma=2,q=1$; (**e**) Variation tendency of score function value when $\gamma=1,q=5$; (**f**) Variation tendency of score function value when $\gamma=2,q=5$.



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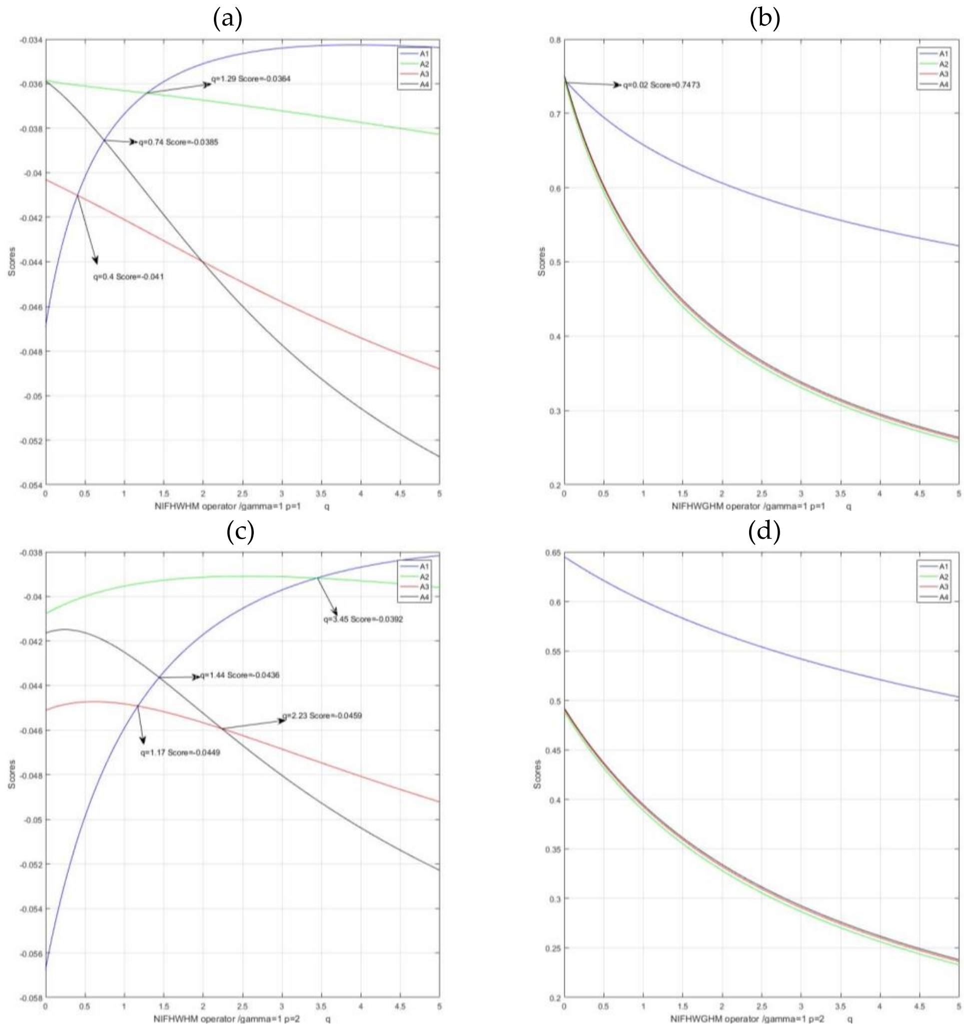

(e) (f)

Figure 5. Sensitivity analysis with respect to q in NIFHWHM and NIFHWGHM operator based on different γ,p : (**a**) Variation tendency of score function value when $\gamma=1,p=0$ in NIFHWHM operator; (**b**) Variation tendency of score function value when $\gamma=1,p=2$ in NIFHWGHM operator; (**c**) Variation tendency of score function value when $\gamma=1,p=1$ in NIFHWHM operator; (**d**) Variation tendency of score function value when $\gamma=1,p=1$ in NIFHWGHM operator; (**e**) Variation tendency of score function value when $\gamma=1,p=2$ in NIFHWHM operator; (**f**) Variation tendency of score function value when $\gamma=1,p=2$ in NIFHWGHM operator.

6.3. Comparison analysis

6.3.1 A comparison with decision-making methods using triangular and trapezoidal intuitionistic fuzzy information A comparison analysis with the exiting method using NIFNs

For further comparison of the rationality and comprehensiveness of the proposed method in this paper, a prospect value determination method with the TraIFNs[44] and an extend TODIM method with TriIFNs[45] are applied to deal with the aforementioned example. Thus we need transform the TraIFNs and TriIFNs by the transformation method in [18] which is shown in Table 12. According to Table 12, the information $r_{ij}^k = \langle (a_{ij}^k, \sigma_{ij}^k); \mu_{ij}^k, \upsilon_{ij}^k \rangle$ from each expert is also transformed into the TraIFNs and the TriIFNs, moreover, the normalization method of the TraIFN and TriIFN decision matrix is presented as follows.

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$$r_{ij}^{k}=\begin{cases}\left\langle\left(\frac{a_{ij}^{k}}{c_{j}^{k+}},\frac{b_{ij}^{k}}{c_{j}^{k+}},\frac{c_{ij}^{k}}{c_{j}^{k+}}\right);\mu_{ij}^{k},v_{ij}^{k}\right\rangle \\ \left\langle\left(\frac{a_{1}{}_{ij}^{k}}{a_{4}{}_{j}^{k+}},\frac{a_{2}{}_{ij}^{k}}{a_{4}{}_{j}^{k+}},\frac{a_{3}{}_{ij}^{k}}{a_{4}{}_{j}^{k+}},\frac{a_{4}{}_{ij}^{k}}{a_{4}{}_{j}^{k+}}\right),\left(\frac{b_{1}{}_{ij}^{k}}{b_{4}{}_{j}^{k+}},\frac{b_{2}{}_{ij}^{k}}{b_{4}{}_{j}^{k+}},\frac{b_{3}{}_{ij}^{k}}{b_{4}{}_{j}^{k+}},\frac{b_{4}{}_{ij}^{k}}{b_{4}{}_{j}^{k+}}\right);\mu_{ij}^{k},v_{ij}^{k}\right\rangle\end{cases} \quad j\in B$$

Where B is benefit attributes set.

From Table 13, we can see that the best alternative of three methods are A_2, but the ordering results are different for these methods. The reason is that TriIFNs and TraIFNs can't well depict the laws of normal distribution and accurately express corresponding normal random phenomena. There are many normal random factors under the social and economic environment. Furthermore, in light of Central Limit Theorem, the limit distribution of the sum of random variables is normal distribution. Therefore, compared with the TriIFNs and TraIFNs, the NIFNs can better depict the decision problems with normal distribution information.

Consequently, the NIFNs can more realistically express the uncertainty information than the TriIFNs and TraIFNs, and the decision-making method of this paper is more reliable and reasonable to aggregate the normal distribution information. Moreover, the proposed method of this paper take into account the interrelationship between input arguments and it is more practical than the method in [19] and [44-45].

Table 12. Transformation method of the NIFN, TraIFN and TriIFN.

NIFN	TraIFN	TriIFN
$\langle(\alpha,\sigma),\mu,v\rangle$	$\langle(a_1,a_2,a_3,a_4),(b_1,b_2,b_3,b_4);\mu,v\rangle$ $a_1=\alpha-2.5\sigma$ $b_1=\alpha-3\sigma$ $a_2=\alpha-1.5\sigma$ $b_2=\alpha-2\sigma$ $a_3=\alpha+1.5\sigma$ $b_3=\alpha+2\sigma$ $a_4=\alpha+2.5\sigma$ $b_4=\alpha+3\sigma$	$\langle(a,b,c);\mu,v\rangle$ $a=\alpha-3\sigma,b=\alpha,c=\alpha+3\sigma$

Table 13. Comparison of the ranking results by methods in [56] and [44-45].

Method	Measure	A_1	A_2	A_3	A_4	Ranking
Tended TODIM in [45]	Closeness coefficient	0.786	0.279	0.491	0.431	$A_2\succ A_4\succ A_3\succ A_1$
TOPSIS in [56]	Closeness coefficient	0.489	0.282	0.798	0.427	$A_2\succ A_4\succ A_1\succ A_3$
Value determination in [44]	Prospect value function	-0.291	-0.137	-0.185	-0.243	$A_2\succ A_3\succ A_4\succ A_1$
Method in this paper	Score function	-0.035	-0.034	-0.040	-0.037	$A_2\succ A_1\succ A_4\succ A_3$ [1]
		-0.037	-0.036	-0.042	-0.038	$A_2\succ A_1\succ A_4\succ A_3$ [2]
		-0.039	-0.038	-0.044	-0.042	$A_2\succ A_1\succ A_4\succ A_3$ [3]

[1] Parameter $\gamma=0.5, p=q=1$ by NIFHWHM operator. [2] Parameter $\gamma=1, p=q=1$ by NIFHWHM operator. [3] Parameter $\gamma=2, p=q=1$ by NIFHWHM operator.

6.3.2 A comparison with decision-making methods using the NIFNs.

In order to study the advantages of the method in this paper, we apply three methods in [35], [41] and [19] to solve MAGDM problems in the aforementioned example, and the aggregation results are shown in Table 14.

From Table 14, we can observe that the best alternatives are all A_2, but the solution ordering results are completely different for four methods which can all tackle NIF information. Compared with two methods in [35] and [41], the new method of this paper considers also the interrelationship factor between input arguments and between input argument and itself. In practical MAGDM problems, there widely exist the interrelationships among the attributes or relationships between input arguments and between input argument and itself, and the ranking results from the new method of this paper is more effective and more reasonable.

In addition, if the parameter $q=0$ or $p=0$, then the interrelationships didn't exist in the new method of this paper. In the aforementioned example, we can obtain the aggregation ordering results, namely, the relationships between arguments or among the attributes are not considered in the proposed method of this paper. The solution ordering results is the same as the method in [35] and [41], consequently, this verify the different ordering results.

In addition, for a group of attributes $c_i(i=1,2,\cdots,n)$ and a collection of input arguments $a_i(i=1,2,\cdots,n)$, the method in [19] also take into account the relationships between any pair of attributes c_i and c_j $(i \neq j)$ or between any pair of input arguments a_i and a_j $(i \neq j)$, but it neglects the correlation between input argument a_i and itself or between the attribute c_i and itself. Considering that the correlation between a_i and a_j $(i \neq j)$ or between c_i and c_j $(i \neq j)$ equals the correlation between a_i and a_j $(i \neq j)$ or between c_i and c_j $(i \neq j)$, the method in [19] deal with it separately and brings subsequently about redundancy. Compared with the method in [19], the new method of this paper not only considers relationships between the input arguments or the attributes, but also take into account the correlation between input argument and itself or attribute and itself, furthermore, interrelationships between and or between and are tackled once.

Table 14. Comparison of the ranking results by methods in [19], [35]and [41].

Method	Measure	A_1	A_2	A_3	A_4	Ranking
MADM method by NIFI operator in [35]	Score function	0.486	0.779	0.491	0.531	$A_2 \succ A_4 \succ A_3 \succ A_1$
MADM method by NIFBM operator in [19]	Score function	0.476	0.482	0.471	0.467	$A_2 \succ A_1 \succ A_3 \succ A_4$
VIKOR-based dynamic method in [41]	Compromise value function	0.998	0.003	0.775	0.698	$A_2 \succ A_4 \succ A_3 \succ A_1$
Method in this paper	Score function	-0.035	-0.034	-0.040	-0.037	$A_2 \succ A_1 \succ A_4 \succ A_3$[1]
		-0.037	-0.036	-0.042	-0.038	$A_2 \succ A_1 \succ A_4 \succ A_3$[2]
		-0.039	-0.038	-0.044	-0.042	$A_2 \succ A_1 \succ A_4 \succ A_3$[3]
		-0.044	-0.035	-0.041	-0.037	$A_2 \succ A_4 \succ A_3 \succ A_1$[4]

[1] Parameter $\gamma=0.5, p=q=1$ by NIFHWHM operator. [2] Parameter $\gamma=1, p=q=1$ by NIFHWHM operator. [3] Parameter $\gamma=2, p=q=1$ by NIFHWHM operator. [4] Parameter $\gamma=1, p=0, q=1$ by NIFHWHM operator.

7. Conclusions

In this paper, motivated by the ideal of Heronian mean, we introduce a family of information fusion operators based on Hamacher operation for NIFNs including NIFHHM, NIFHWHM, NIFHGHM, NIFHWGHM operators and discuss various properties of the proposed operators which have the desirable quality that they can not only contain the normal intuitionistic fuzzy information, but also consider the correlations of two input arguments once. Therefore, the new proposed operators don't result subsequently in redundancy, and these operators also take into account the interrelationship between input argument and itself at the same time. Furthermore, we have manifested that the operators related to Hamacher operation generalize the operators ground on algebraic or Einstein operational rules and they are more flexible. Based on the proposed operators, a new multi-criteria group decision-making approach is presented in order to deal with normal intuitionistic fuzzy number information. The advantages of this new method are that: (1) it is more reliable and reasonable to aggregate the normal distribution information under the normal intuitionistic fuzzy numbers environment. (2) it offers an effective and powerful mathematic tool for the MAGDM under uncertainty and can provide more reliable and flexible decision-making results in decision making. (3) it not only considers relationships between the input arguments or the attributes, but also take into account the correlation between input argument and itself or attribute and itself, furthermore, interrelationships between and or between and are tackled once. Lastly, an application example reveals that the developed approach is effective and practical by the comparison with other method. In further research, it is significant and essential to study the application of the proposed operators in the wide fields, such as uncertain programming, cluster analysis or pattern recognition and so on.

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Conflicts of Interest: The author declares that he has no conflict of interest.
This article does not contain any studies with human participants or animals performed by any of the authors.

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