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Article

Weighted negative binomial Poisson-Lindley distribution with actuarial applications

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Abstract: This study introduces a new discrete distribution which is a weighted version of Poisson-Lindley distribution. The weighted distribution is obtained using the negative binomial weight function and can be fitted to count data with over-dispersion. The p.m.f., p.g.f. and simulation procedure of the new weighted distribution, namely weighted negative binomial Poisson-Lindley (WNBPL), are provided. The maximum likelihood method for parameter estimation is also presented. The WNBPL distribution is fitted to several insurance datasets, and is compared to the Poisson and negative binomial distributions in terms of several statistical tests.

Keywords: weighted distribution; Poisson-Lindley distribution; discrete distribution; weighted negative binomial Poisson-Lindley distribution.

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1. Introduction

Mixed Poisson and mixed negative binomial distributions have been considered as alternatives for fitting count data with over-dispersion. Several examples of mixed Poisson and mixed negative binomial distributions can be found in several statistical literatures, such as negative binomial which is obtained as a mixture of Poisson and gamma, Poisson-Lindley (Sankaran 1970; Ghitany et al. 2008), Poisson-lognormal (Bulmer 1974), Poisson-inverse Gaussian (Trembley 1992; Willmot 1987), negative binomial-Pareto (Meng et al. 1999), negative binomial-inverse Gaussian (Gomez-Deniz et al. 2008), negative binomial-Lindley (Zamani and Ismail 2010; Lord and Geedipally 2011), Poisson-exponential (Cancho et al. 2011), Poisson-weighted exponential (Zamani et al, 2014), two parameter Poisson-Lindley (Shanker and Mishra 2014) and Poisson-Janardan distributions (Shanker et al. 2014).

Besides mixed distributions, weighted distributions have also been considered as alternatives for fitting count data with over-dispersion, and can be generally obtained by multiplying a count distribution with a weight function. To derive a new weighted distribution, let X be a count random variable with p.m.f. $P(X=k)$, where $k \in N_0 = \{0,1,2,...\}$. Let $\omega(k)$ be a non-negative function on N_0 having a finite expectation $E[\omega(X)] = \sum_{k=0}^{\infty} \omega(k) P(K=k) < \infty$, where the weight function $\omega(k)$ can be used to adjust the probability when $X = k$ occur. Thus, the weighted version of r.v. X, which is the realization of count r.v. Y, has the following p.m.f:

$$P(Y=k) = p(k;\theta) = \frac{\omega(k) P(K=k)}{E[\omega(X)]}, \quad k \in N_0 . \qquad (1)$$

The most popular weighted count distributions are the weighted Poisson (WP) distributions which are obtained when the initial count r.v., X, follows a Poisson distribution. The initial concept of WP distribution was introduced in Rao (1965), which lead to several more recent and different types of WP

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distributions derived and analyzed in other studies. Examples of a more recent WP distributions can be found in Ridout and Besbeas (2004), Shmueli et al. (2005), and Castillo and Perez-Casany (2005).

In recent studies, some authors used particulars weights for deriving new versions of weighted distributions. Such examples can be found in Neel and Schull (1966) who used the Poisson weight function $\omega(k;\varphi)=\varphi^k e^{-\varphi}(k!)^{-1}$, Kokonendji and Casany (2012) who utilized the binomial weight function $\omega(k;\varphi)=1-(1-\varphi)^k$, and the negative binomial weight function $\omega(k;\varphi)=\binom{\varphi+k-1}{k}$ which was applied by Hussain et. al (2016). A more detailed study of weighted distributions and weight functions can be found in Patil et. al (1986).

The objective of this study is to introduce a new discrete weighted distribution based on the Poisson-Lindley distribution. The weighted distribution, namely the weighted negative binomial Poisson-Lindley (WNBPL), is weighted with negative binomial weight function and can be used as an alternative for fitting count data with over-dispersion. The rest of this paper is organized as follows. Section 2 provides the p.m.f., p.g.f. and simulation procedure for the WNBPL. Maximum likelihood method for parameters estimation is provided in Section 3. Several numerical illustrations are provided in Section 4, where the Poisson, negative binomial and WNBPL are fitted to a few datasets.

2. Weighted Negative Binomial Poisson-Lindely (WNBPL)

2.1. P.m.f., p.g.f., mean and variance

Assume r.v. $Y|\lambda$ follows Poisson distribution with p.m.f:

$$p(y\,|\,\lambda)=\frac{e^{-\lambda}\lambda^y}{y!},\quad y=0,1,2,... \tag{2}$$

and parameter λ is distributed as Lindley with parameter θ :

$$f(y)=\frac{\theta^2}{\theta+1}(1+\lambda)e^{-\theta\lambda},\quad \lambda>0. \tag{3}$$

The Poisson-Lindley (PL) distribution is obtained by mixing Poisson and Lindley distributions, and the p.m.f. is:

$$p(y)=\frac{\theta^2(y+\theta+2)}{(1+\theta)^{y+3}},\quad y=0,1,2,3,.., \tag{4}$$

with mean $E(Y)=\dfrac{\theta+2}{\theta(\theta+1)}$ and variance $Var(Y)=\dfrac{\theta^3+4\theta^2+6\theta+2}{\theta^2(\theta+1)^2}$.

Using $\theta+1=\dfrac{1}{p}$ for re-parameterization, the PL p.m.f. in (4) can be re-written as:

$$p(y)=(1-p)^2 p^y(1+p+py),\quad y=0,1,2,3,... \tag{5}$$

A new discrete distribution can be easily obtained by inserting negative binomial weight function $\omega(k;r)=\binom{r+k-1}{k}$ and PL p.m.f. (5) into the weighted equation in (1). The new distribution, namely the WNBPL, has the following p.m.f:

$$P(Y=k)=p(k)=\binom{r+k-1}{k}\frac{(1-p)^{r+1}p^k(1+p+pk)}{(1-p^2+rp^2)},\quad k=0,1,2,3,..,\quad 0<p<1, \tag{6}$$

with mean and variance:

$$\mu = \frac{rp^2 - p^2 - p}{1 + rp^2 - p^2} + \frac{p(r+1)}{1-p}$$
$$\sigma^2 = \frac{(rp^2 - p^2 - p)(p+1)}{(1 + rp^2 - p^2)^2} + \frac{p(r+1)}{(1-p)^2} \tag{7}$$

The p.g.f. can be obtained in a closed form, and is given by:

$$G_Y(t) = E(t^Y) = \frac{(1 - p^2 t + rp^2 t) + p(1-t)}{1 - p^2 + rp^2}\left(\frac{1-p}{1-pt}\right)^{r+1} \tag{8}$$

2.2. Over-dispersion

In statistics, cases of over-dispersion can be determined by comparing the mean and variance, where a distribution is known to be over-dispersed if the variance is greater than the mean. For WNBPL, the variance and mean can be written as:

$$\sigma^2 - \mu = \frac{p^2(r+1)}{(1-p)^2} - \frac{(rp^2 - p^2 - p)^2}{(1 + rp^2 - p^2)^2},$$

so that we can determine whether the term $\frac{(rp^2 - p^2 - p)^2}{(1 + rp^2 - p^2)^2}$ is less than one for all values of p and r. If $\frac{(rp^2 - p^2 - p)^2}{(1 + rp^2 - p^2)^2}$ is less than one, then $\frac{p^2(r+1)}{(1-p)^2}$ is greater than one, indicating that $\sigma^2 - \mu$ is greater than zero. Therefore, the variance of WNBPL is greater than the mean, and the distribution can be used to handle over-dispersed count data.

Figure 1 shows the p.m.f. of WNBPL for different values of (r, p). The graphs indicate that the distribution can be considered as an alternative for over-dispersed count data.

2.3 Random data generation

P.m.f. (6) indicates that $WNBPL(r, p)$ is a mixture of negative binomial distributions, which can be written as:

$$p(k) = \frac{1 - p^2}{1 - p^2 + rp^2} NB(r, 1-p) + \frac{rp^2}{1 - p^2 + rp^2} NB(r+1, 1-p) \ .$$

Therefore, the $WNBPL(r, p)$ random samples can be generated via the weighted negative binomial approach.

We analyze the performance of ML estimates of $WNBPL(r, p)$ based on 1000 simulations. The average estimators, average mean square errors and average standard errors of the ML estimates for several sample sizes, n, and several initial values, (r, p), are provided in Table 1. The results show that increasing the sample size is an effective way of decreasing the standard errors of parameters. As shown in this table, the MSEs decrease when the sample size increase, and thus, suggesting the consistency of the proposed model.

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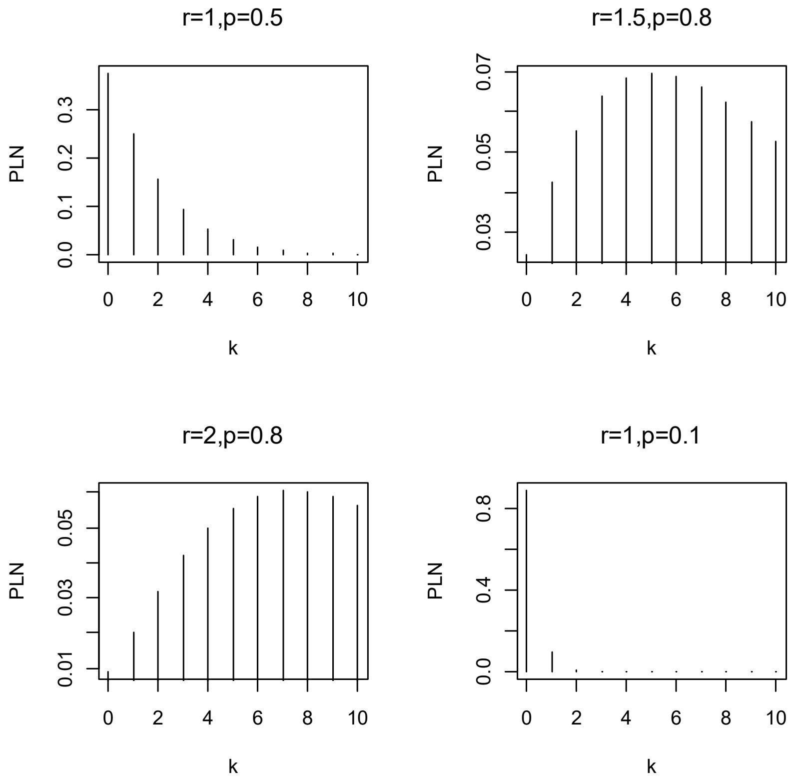

Figure 1. P.m.f. of WNBPLN distribution for different values of (r, p)

Table 1. Average estimates, average MSE and average standard error (1000 simulation).

	Initial values		Average estimates		Average mse		Average std	
n	r	p	$\hat{r}$	$\hat{p}$	$mse(\hat{r})$	$mse(\hat{p})$	$se(\hat{r})$	$se(\hat{p})$
	0.3	0.1	1.227	0.130	1.773	0.046	0.955	0.213
50	0.6	0.6	0.629	0.567	0.105	0.010	0.323	0.094
	0.2	0.8	5.924	0.179	105.247	0.404	8.513	0.147
	0.3	0.1	2.051	0.161	8.413	0.048	2.312	0.211
75	0.6	0.6	0.594	0.572	0.062	0.006	0.249	0.076
	0.2	0.8	4.081	0.189	58.593	0.389	6.597	0.131
	0.3	0.1	1.948	0.177	7.543	0.050	2.197	0.210
100	0.6	0.6	0.581	0.575	0.041	0.005	0.203	0.066
	0.2	0.8	3.490	0.184	43.061	0.389	5.677	0.118
	0.3	0.1	1.577	0.196	5.541	0.094	0.970	0.200
125	0.6	0.6	0.567	0.577	0.027	0.004	0.163	0.058
	0.2	0.8	2.864	0.191	31.381	0.382	4.927	0.112
	0.3	0.1	1.402	0.198	4.614	0.044	1.844	0.186
150	0.6	0.6	0.555	0.581	0.021	0.003	0.140	0.050
	0.2	0.8	2.694	0.190	27.342	0.382	4.595	0.107

3. Parameter Estimation

Let $Y_1, Y_2, ..., Y_n$ be an i.i.d. random sample drawn from WNBPL distribution, with observed values $k_1, k_2, ..., k_n$. The log-likelihood is:

$$\ln L(r,p) = \ell(r,p) = n(r+1)\ln(1-p) + \sum_{i=1}^{n} k_i \ln p + \sum_{i=1}^{n} \ln(1+p+pk_i) - n\ln(1-p^2+rp^2) + \sum_{i=1}^{n} \ln\binom{r+k_i-1}{k_i}$$

By partially differentiating the log-likelihood with respect to p and r, we obtained:

$$\frac{\partial \ell(r,p)}{\partial p} = -\frac{n(r+1)}{1-p} + \frac{n\bar{k}}{p} + \sum_{i=1}^{n} \frac{1+k_i}{1+p+pk_i} - \frac{2np(r-1)}{1-p^2+rp^2}$$

$$\frac{\partial \ell(r,p)}{\partial r} = n\ln(1-p) - \frac{np^2}{1-p^2+rp^2} + \frac{\partial}{\partial r}\sum_{i=1}^{n} \ln\binom{r+k_i-1}{k_i}$$

Klugman et.al (2012) showed that the term $\frac{\partial}{\partial r}\sum_{x=0}^{k} \ln\binom{r+x-1}{x}$ can be simplified into:

$$\frac{\partial}{\partial r}\sum_{x=0}^{k} \ln\binom{r+x-1}{x} = \sum_{x=0}^{k}\sum_{m=0}^{x-1} \ln(r+m).$$

Therefore, the partial differentiation $\frac{\partial \ell(r,p)}{\partial r}$ can be written in a simpler form, which is:

$$\frac{\partial \ell(r,p)}{\partial r} = n\ln(1-p) - \frac{np^2}{1-p^2+rp^2} + \sum_{i=1}^{n}\sum_{m=0}^{k_i-1} \ln(r+m).$$

ML estimates $(\hat{r}, \hat{p})$ can be obtained numerically using statistical packages such as R 3.3.1 with nlminb command. Under regularity conditions, the ML estimates $(\hat{r}, \hat{p})$ for WNBPL has a bivariate normal distribution with mean (r, p) and variance-covariance matrix $[I(r,p)]^{-1}$, where $I(r,p)$ is the Fisher information matrix, which is given as:

$$I(\hat{r},\hat{p}) = \begin{pmatrix} E\left[-\frac{\partial^2 \ell(r,p)}{\partial p^2}\right] & E\left[-\frac{\partial^2 \ell(r,p)}{\partial p \partial r}\right] \\ E\left[-\frac{\partial^2 \ell(r,p)}{\partial r \partial p}\right] & E\left[-\frac{\partial^2 \ell(r,p)}{\partial r^2}\right] \end{pmatrix}.$$

4. Data Applications

4.1 Example 1

The observed number of accident claims per contract for a total of 298 contracts are available for the data (Simon 1961, Klugman et al. 2012). Table 2 provides the observed values, fitted values and estimated parameters for the Poisson, NB and WNBPL distributions. The chi-square and log likelihood, which are considered as comparison criteria, are also provided. The results show that the WNBPL provides the largest log likelihood and the smallest chi-square. Even though the NB distribution is a strong competitor, the WNBPL distribution provides better performance because it has a slightly larger log likelihood, but significantly better values for chi-square and p-value of chi-square.

Table 2. Observed values, fitted values and parameter estimates (example 1)

No. Claim	Frequency	Poisson	Negative Binomial	WNBPL
0	99	54.4	95.5	96.4
1	65	92.5	76.2	74.1
2	57	78.7	50.7	50.4
3	35	44.6	31.4	31.2
4	20	19.0	18.7	19.1
5	10	6.5	11.0	11.1
6	4	1.8	6.3	6.4
8	3	0	2.0	2.0
9	5	0	1.1	1.1
parameters		$\hat{\lambda} = 1.701$	$\hat{p} = 0.469$ $\hat{r} = 1.505$	$\hat{p} = 0.494$ $\hat{r} = 1.182$
-ln L		573.36	527.60	527.44
AIC		1148.72	1059.20	1058.88
chi-square		75.45	3.87	3.32
p-value of chi-square		0.00	0.42	0.51

4.2 *Example 2*

Another data from Klugman et al. (2012) is also considered. The data provides the number of medical claims per reported automobile accident. The Poisson, NB and WNBPL distributions are fitted, and the results are provided in Table 3. It can be seen that the WNBPL also provides the largest log likelihood and the smallest chi-square. Compared to the NB distribution, the WNBPL distribution provides a significantly better performance based on its larger log likelihood and smaller chi-square.

Table 3. Observed values, fitted values and parameter estimates (example 2)

No. Claim	Frequency	Poisson	Negative Binomial	WNBPL
0	529	216.2	474.5	486.3
1	146	376.4	274.7	259.3
2	169	327.7	171.2	166.5
3	137	190.2	109.2	109.8
4	99	82.8	70.5	72.7
5	87	28.8	45.8	48.0
6	41	8.4	29.9	31.5
7	25	2.1	19.6	20.6
parameters		$\hat{\lambda} = 1.7412$	$\hat{p} = 0.3324$ $\hat{r} = 0.8670$	$\hat{p} = 0.6202$ $\hat{r} = 0.6218$
-ln L		2532.85	2216.07	2204.73

AIC	4467.70	4436.14	4413.46
chi-square	252.56	1.50	0.92
p-value of chi-square	0.00	0.91	0.96

5. Conclusions

This paper introduces a new weighted Poisson-Lindley distribution which is obtained using negative binomial weight function and can be used for fitting over-dispersed count data. The p.m.f., p.g.f. and simulation procedure are provided for the new weighted distribution, namely the weighted negative binomial Poisson-Lindley (WNBPL). The WNBPL (r,p) can also be shown to be equivalent to a mixture of negative binomial distributions, and thus, allowing the random samples to be generated via weighted approach. The estimation procedures of WNBPL parameters via the maximum likelihood are also shown. For numerical illustrations, the WNBPL distribution is fitted to two sets of insurance count data, and the results are compared to Poisson and negative binomial distributions. Based on chi-square and log likelihood of the fitted models, both negative binomial and WNBPL distributions provide significant improvements over Poisson, but WNBPL provides the largest log likelihood and the smallest chi-square. Considering the straightforward manner of obtaining its MLE estimators, the WNBPL can be considered as an alternative model for fitting over-dispersed count data.

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