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On Games and Cost of Change

Sjur Didrik Flåm*

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Abstract. Most microeconomic and game theoretic models of individual choice overlook adjustment costs. Rather often, the modeler's concern is just with improvement of objectives. This optic doesn't quite fit agents somewhat tied to status quo. If rational, any such agent reasons whether moving to another state be worth his while. For that, the realized gains must outweigh the inconveniences of the move. This note offers some observations as to the fact that change usually entails cost.

Key words: adjustment costs · proximal methods · stationary states · games · equilibria.

1. Introduction

Game theory and microeconomics - henceforth just called *theory* - abounds in agent-based models of decision problems. Most instances tend, however, to emphasize *three* questionable features. *First*, each concerned agent should, with little or no hesitation, leap directly to a best choice. *Second*, his behavior ought be totally goal-oriented. *Third*, he is often depicted as fully detached from history, precedent or status quo.

As modelled, these aspects of behavior invite objections. Choice may emerge step-wise; cost to change can be considerable; and clearly, each arrival comes from some point of departure.

It's comforting therefore, that algorithms geared toward best or better choice, have - at least since Cauchy (1847) - been coached as iterative processes. Typically, these require more than just one step. It's also comforting that recent decades have brought forward procedures that expressly account for adjustment costs.[1]

In contrast, much *theory* sidesteps such procedures. It rather moves straight to terminal outcomes, if any, called equilibria [18], [25]. Thereby, pressing queries as to attainment, emergence, selection and stability of equilibrium easily escape attention.

For good reasons, various concepts of steady-state solutions exert considerable attraction. Each describes how parties behave, communicate or fare *in* equilibrium. However, *out* of equilibrium, the underlying concept rarely provides much guidance.[2]

*Informatics Departement, University of Bergen, Norway; sjur.flaam@uib.no. Many thanks for support are due the Informatics Department and Røwdes Fond.

[1] Most of these come with the label "*proximal point*" algorithm. References include [1], [13], [21] and [24].

[2] For examle, in markets, from where might prices come? And in noncooperative games, how could best responses and rational foresights eventually emerge?

Two different hypotheses have been invoked to fill that void; either is tempting, but neither quite attractive. One posits that agents, even out-of-equilibrium, behave as in equilibrium. The other presumes that each party acts throughout *as though* fully foresighted, marvelously competent, perfectly rational.[3]

More realistic approaches ought tolerate imperfections in agents' capacity to choose, foresee or know. Accordingly, here below, local perceptions replace global views - and improvements substitute for full optimization. While seeking own betterment, agents adapt - usually in somewhat moderate or myopic manner [15]. If so, *might they eventually come to a halt? And then, where?*

These questions motivate the paper. For preparation, Section 2 considers just one agent, isolated from others. In contrast, Section 3 lets him play normal-form games among non-cooperative strategists. Section 4 concludes by briefly considering extensive-form games of Stackelberg sort.

2. Preliminaries concerning the single agent

This section introduces notations and preliminaries. To begin with, and to simplify, it considers just *one* agent. Actually, he holds a "position" x^0. If departing from x^0 to x^1, that transition gives him *net benefit* $b(x^1 \,|x^0)$. His improvement or *betterment*

$$(x^1, x^0) \mapsto b(x^1 \,\big|x^0) \in \mathbb{R} \cup \{-\infty\}$$

equals $-\infty$ if $(x^1, x^0) \notin X \times X$ for some non-empty viability subset X in the ambient space $\mathbb{X}$ of alternatives. The "probabilistic" notation $b(x^1 \,|x^0)$ emphasizes that the agent, while *conditioned* by his departure point x^0, seeks a suitable arrival point x^1. In particular, given $x^0 \in X$, he might

$$\text{maximize } b(x^1 \,\big|x^0) \text{ subject to } x^1 \in X. \tag{1}$$

Many formalized decision problems mention no point of departure - or implicitly, the latter is of negligible importance. Moreover, it seems that the agent, upon leaping directly to a very best choice, incurs no cost for "dislodging" himself.[4]

Classical and customary instances let

$$b(x^1 \,\big|x^0) = \beta(x^1) - \beta(x^0) \tag{2}$$

for some *gross benefit function* $\beta : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$, having *effective domain* $X := \beta^{-1}(\mathbb{R})$. This case reports no adjustment costs. The agent is fully goal-driven - and never troubled by friction or inertia. More realistically, proximal point methods [21], [24] posit

$$b(x^1 \,\big|x^0) = \beta(x^1) - \beta(x^0) - C(x^1 \,\big|x^0) \tag{3}$$

for some (adjustment) cost function $C : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+ \cup \{+\infty\}$ which vanishes on the diagonal: $C(x^0 \,|x^0) = 0 \; \forall x^0 \in X$. No symmetry is presumed; it may well happen

[3] Assuming so might be justifiable in equilibrium but hardly out of it.

[4] At most, such costs are construed as computational.

that $C(x^1\,|x^0) \neq C(x^0\,|x^1)$; the forward fare can differ from the backward one. It often appears natural though, that C satisfies the triangle inequality: $C(x^1\,|x^0) + C(x^2\,|x^1) \geq C(x^2\,|x^0)$. Then (3) makes a direct move $x^0 \to x^2$ preferable to any indirect one $x^0 \to x^1 \to x^2$.[5]

Both instances (2), (3) support the standing interpretation that $b(x^1\,|x^0)$ denotes additional benefit *in* arrival state x^1, net of costs incurred upon departing (directly) from x^0.

If $x^1 = x^0$, the agent *stays put*; otherwise, he *moves*. A move from x^0 to x^1 is declared (strictly) *improving* iff $b(x^1\,|x^0) > 0$. Naturally, suppose that staying put entails no improvement; that is, $b(x^0\,|x^0) \leq 0$ for all $x^0 \in X$.

Stationary states stand out by allowing no improvements. They solve problem (1) by bringing up contingent fixed points:

Definition 2.1 (stationary states). *$x \in X$ is declared **stationary** for the bivariate mapping $(x^1, x) \in X \mapsto b(x^1\,|x) \in \mathbb{R}$ iff*

$$x \in \arg\max \left\{ b(x^1\,|x) : \ x^1 \in X \right\}. \tag{4}$$

This framing of the agent's decision problem begs the question: Is there some stationary state? The following positive (albeit particular) answer is just a restatement of Ky Fan's inequality [3], [11]:

Theorem 2.1 (on existence of stationary states). *Suppose X is a non-empty compact convex subset of a topological vector space $\mathbb{X}$. Also suppose $b(x^1\,|x^0)$ be quasi-concave in $x^1 \in X$, lower semicontinuous in $x^0 \in X$, and $b(x^0\,|x^0) \leq 0\ \forall x^0$. Then there exists at least one stationary state.* □

Theorem 2.1 points to topological vector spaces as tractable settings. It also emphasizes the roles of closed convex preferences.

Granted existence of at least one stationary state, how might the agent eventually reach one of those - and come to rest there? As in [19], [26] it's convenient to model his step-wise adjustments in terms of a point-to-set correspondence $A : X \rightrightarrows X$. From some accidental or historical point $x^0 \in X$, there emanates an iterative process

$$x^{k+1} \in A(x^k), \quad k = 0, 1, ... \tag{5}$$

Process (5) would be self-defeating if it stops prior to stationarity. Put differently, each *fixed point* $x \in A(x)$ should be a stationary state (4). Conversely, (5) ought halt if it reaches a stationary state. In synthesis, for any fixed or limiting correspondence, say $A : X \rightrightarrows X$, considered in the sequel, it's tacitly required that

$$x \in A(x) \Longleftrightarrow x \text{ is stationary.} \tag{6}$$

[5] If moreover, $C(x^1\,|x^0) = 0$ iff $x^1 = x^0$, then adjustment cost is an *asymmetric distance* [7].

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A leading instance takes the form

$$A(x) := \left\{x^1 \in X : b(x^1 \,|x\,) \geq c(x^1 \,|x\,)\right\} \tag{7}$$

where $c : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+ \cup \{+\infty\}$ reports transitional costs. Reasonably, posit $c(x \,|x\,) = 0$ for all $x \in X$.

Proposition 2.1 (on appropriate cluster points). *Let X be a closed subset of a topological space $\mathbb{X}$. Suppose* (5), (7) *generate a summable sequence $k \mapsto b(x^{k+1} \left|x^k\right.)$, meaning $\sum_{k=0}^{\infty} b(x^{k+1} \left|x^k\right.) < +\infty$. Further suppose that each non-stationary point $x \in X$ has some neighborhood $\mathcal{N}$ and number $\delta > 0$ such that*

$$c(\chi^{+1} \,|\chi\,) \geq \delta \quad \textit{for all} \quad \chi^{+1} \in A(\chi) \quad \textit{when} \quad \chi \in X \cap \mathcal{N}.$$

Then, either the sequence (x^k) is finite with a stationary last point - or, every cluster point of the infinite sequence must be stationary.

Proof. In the viable set X, let $x = \lim_{k \in K} x^k$ for some infinite subsequence K of natural numbers. Suppose x isn't stationary. With no loss of generality, take $x^k \in \mathcal{N}$ for all $k \in K$. Then, it obtains the contradiction

$$+\infty > \sum_{k=0}^{\infty} b(x^{k+1} \left|x^k\right.) \geq \sum_{k \in K} b(x^{k+1} \left|x^k\right.) \geq \sum_{k \in K} c(x^{k+1} \left|x^k\right.) = +\infty. \quad \square$$

Remark (*on upper bounded criteria*). Prop. 2.1 fits instance (3) with β bounded above and $c, C := C/2$.

For greater flexibility one may replace the time-invariant A in (5) with stage-dependent correspondences $A^k : \mathbb{X} \rightrightarrows \mathbb{X}$ to have

$$x^{k+1} \in A^k(x^k), \quad k = 0, 1, ... \tag{8}$$

Definition 2.2 (on asymptotic closure and regularity). *When the space $\mathbb{X}$ is topological, a limiting correspondence $A : X \rightrightarrows X$ **closes** the sequence (A^k) if*

$$(\chi^k, x^k) \to (\chi, x) \text{ with } \chi^k \in A^k(x^k) \text{ implies } \chi \in A(x). \tag{9}$$

*If the space $(\mathbb{X}, d)$ is metric, (A^k) is declared **asymptotically regular** if $x^{k+1} \in A^k(x^k)$ yields $d(x^{k+1}, x^k) \to 0$.*

In these terms, the following result is immediate - and it it structures some subsequent arguments:

Proposition 2.2 (on stationary cluster points). *Suppose $(\mathbb{X}, d)$ is metric and that $A : X \rightrightarrows X$ closes an asymptotically regular sequence (A^k). Then, each cluster point*

x of the sequence (x^k) is stationary. □

For illustration of (8), replace benefit-cost functions b, c with stage-dependent versions $b^k : \mathbb{X} \times \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ and $c^k : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+ \cup \{+\infty\}$. Then, (7) takes the generalized form

$$A^k(x) := \left\{ x^1 \in X : b^k(x^1 \,|x) \geq c^k(x^1 \,|x) \right\}. \tag{10}$$

Proposition 2.3 (on convergence). *Let $(\mathbb{X}, d)$ be a complete metric space and $X \subseteq \mathbb{X}$ non-empty closed. Suppose (A^k), as defined in (10), be closed by A. Also suppose that for any initial $x^0 \in \mathbb{X}$, some number $\delta > 0$ yields*

$$+\infty > \sum_{k=0}^{\infty} c^k(x^{k+1} \,|x^k) \geq \delta \sum_{k=0}^{\infty} d(x^{k+1}, x^k). \tag{11}$$

Then (A^k) is asymptotically regular, and sequence (x^k) generated by (10) converges to a stationary point.

Proof. Since the metric space is complete, (11) implies that $x^k \to x$ for some unique limit x. Also by (11), there is asymptotic regularity: $d(x^{k+1}, x^k) \to 0$. Hence, by closure (9), $x \in A(x)$, and stationarity derives from (6). □

3. NON-COOPERATIVE GAMES

Accommodated henceforth is a fixed finite ensemble I of "players", at least two of them.

By a *strategy profile* $x = (x_i)$ is meant a mapping $i \in I \mapsto x_i \in X_i$ where X_i is a non-empty "viability set" in some ambient space $\mathbb{X}_i$ of alternatives. Given a strategy-profile $x^0 = (x_i^0)$, suppose member $i \in I$ anticipates net benefit $b_i(x_i^1 \,|x^0) \in \mathbb{R}$ upon deviating unilaterally - within his viability set X_i - from strategy x_i^0 to x_i^1. In terms of $x_{-i}^0 := (x_j^0)_{j \neq i}$, he act as though the updated profile equals (x_i^1, x_{-i}^0). That belief is justified iff he alone deviates.

Definition 3.1 (non-cooperative stationary states). *A strategy profile $x \in \Pi_{i\in I} X_i$ is declared **stationary** - and a **Nash equilibrium modulo cost of change** - iff*

$$x_i \in \arg\max \left\{ b_i(x_i^1 \,|x) : \; x_i^1 \in X_i \right\} \quad \text{for all } i \in I. \tag{12}$$

In some special instances, such multi-agent stationarity adds nothing to the customary concept of Nash equilibrium [18]:

Proposition 3.1 (on stationary states as ordinary Nash equilibria). *Suppose player $i \in I$ worships gross benefit $\beta_i : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$, and that*

$$b_i(x_i^1 \,|x^0) = \beta_i(x_i^1, x_{-i}^0) - \beta_i(x_i^0, x_{-i}^0). \tag{13}$$

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Then, a state x is stationary iff it's a customary Nash equilibrium in the non-cooperative game $G := (\beta_i, X_i)_{i\in I}$, meaning

$$x_i \in \arg\max \left\{\beta_i(x_i^1, x_{-i}) : \; x_i^1 \in X_i\right\} \quad \forall i \in I. \; \square$$

Prop. 3.1 mentions no adjustment costs. Each player agent is fully goal-driven. Nobody is ever troubled by friction or inertia. More realistically, following the lead of proximal point methods, one may posit

$$b_i(x_i^1 \,|x^0\,) = \beta_i(x_i^1, x_{-i}^0) - \beta_i(x^0) - C_i(x_i^1 \,|x^0\,) \tag{14}$$

for some cost function $C_i : \mathbb{X}_i \times \mathbb{X} \to \mathbb{R}_+ \cup \{+\infty\}$ which is nil when $x_i^1 = x_i^0$.[6] That function could be asymmetric in the agent's own arguments (x_i^1, x_i^0). The aspect that $C_i(x_i^1 \,|x^0\,)$ depends on the entire profile x^0 fits games featuring congestion [22] or use of common resources [9], [10].

If a Nash solution isn't unique, (14) bears on equilibrium refinement, selection and stability. While $\beta_i(x_i^1, x_{-i})$ is the customary Nash maximand, (14) includes a perturbation - apt to select more robust equilibria. Conversely, cost of change could lock agents into equilibria which otherwise would not withstand minor nudges. This line of inquiry is not pursued here.

In whatever form, $b_i(x_i^1 \,|x\,)$ is meant to measure *cardinal* betterment for player i. Contending with ordinal comparisons, there is a noteworthy link to characterization and existence of stationary points:

Proposition 3.2 (on concave ordinal improvements). *For each $i \in I$, suppose X_i is a non-empty compact convex subset of some topological vector space $\mathbb{X}_i$. Further suppose that*

$$b_i(x_i^1 \,|x^0\,) > 0 \implies \beta_i(x_i^1, x_{-i}^0) - \beta_i(x^0) > 0 \tag{15}$$

with gross benefit function $\beta_i : \mathbb{X} \to \mathbb{R}$ concave in $x_i^1 \in X_i$ and continuous in $x^0 \in X$. Then, there exists at least one Nash equilibrium in the game $G = (\beta_i, X_i)_{i\in I}$. Moreover, each such equilibrium is a stationary state (12).

Proof. When $x^1, x^0 \in X$, posit

$$b(x^1 \,|x^0\,) := \sum_{i\in I}[\beta_i(x_i^1, x_{-i}^0) - \beta_i(x^0)]$$

to have $b(x^1 \,|x^0\,)$ concave in x^1, continuous in x^0, and $b(x^0 \,|x^0\,) = 0$ for all $x^0 \in X$. By Ky Fan's inequality (Theorem 2.1), there exists a point $x \in X$ such that $b(x^1 \,|x\,) \leq 0$ for all $x^1 \in X$. Consequently, $\beta_i(x_i^1, x_{-i}) \leq \beta_i(x)$ for all $x_i^1 \in X_i$ and for each $i \in I$.

[6] Provided (14) be concave in x_i^1 and continuous in x^0, the function $b(x^1 \,|x^0\,) := \sum_{i\in I} b_i(x_i^1 \,|x^0\,)$ fits Theorem 2.1.

So, $x = (x_i)$ is a Nash equilibrium. From (15) follows that x must be stationary. □

Prop. 3.2 fits the instance (14) when all $C_i(x_i^1 \,|x^0\,) \geq 0$. Monderer and Shapley (1996) studied games $G = (\beta_i, X_i)_{i \in I}$ in which

$$\beta_i(x_i^1, x_{-i}^0) - \beta_i(x^0) > 0 \Longrightarrow P(x_i^1, x_{-i}^0) - P(x^0) > 0$$

for some player-independent *generalized ordinal potential* $P : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$. Then, P may replace β_i in (15).

In many games, strategic interaction works via objectives *and* constraints.[7] Besides individual restrictions that $x_i \in X_i \; \forall i$, choice could be subject to collective, coupling constraints in that each strategy profile $x = (x_i)$ must belong to a non-empty, *non-rectangular* subset $X \subsetneq \Pi_{i \in I} X_i$. Then, letting

$$b(x^1 \,|x\,) := \max_{i \in I} b_i(x_i^1 \,|x\,), \tag{16}$$

and modifying (12), x is *stationary* - and declared a *generalized Nash equilibrium* - iff (4) holds.[8] Theorem 2.1 immediately entails

Theorem 3.1 (on stationary states and generalized Nash equilibria). *Suppose X is a non-empty compact convex subset of a topological vector space $\mathbb{X}$. If $(x^1, x) \in X \times X \longmapsto b(x^1 \,|x) \in \mathbb{R}$ (16) is quasi-concave in x^1 and lower semicontinuous in x, then there exists a generalized Nash equilibrium.* □

This solution concept selects among "ordinary" Nash equilibria, satisfying (4).

4. Stackelberg games

Stationarity (12) fits normal-form games, but it's less apt for settings with extensive-form interaction.[9] To illustrate some of the difficulties that emerge, this section concludes by considering two-player, two-move instances of Stackelberg (or principal-agent) sort [14].

A leading player 1 first chooses some $x_1 \in X_1$. Observing that choice, a responding player 2 follows up with some choice $x_2 \in X_2$. Thereafter, given $x = (x_1, x_2)$, they collect upper semicontinuous payoffs $\pi_1(x)$ and $\pi_2(x)$ respectively. Both sets X_1, X_2 are compact.

In principle, the follower reduces to a strategic dummy, just selecting some best response

$$x_2 \in \mathcal{R}(x_1) := \arg\max_{X_2} \pi_2(x_1, \cdot).$$

[7] See [8], [10], [9], [10] and references therein.

[8] Provided $b_i(x_i^1 \,|x\,)$ be quasi-concave in x_i^1 and lower semicontinuous in x, format (16) fits Theorem 2.1.

[9] Following [19], Section 1.2.2, players might memorize the preceding path of play, and history could affect the continuation. This idea is not pursued here.

In contrast, up front, the leader ought

$$\text{maximize } \pi_1(x_1, \mathcal{R}(x_1)) \text{ subject to } x_1 \in X_1.$$

His task is often rather demanding. He had better foresee or guess - or outright be told - the entire response correspondence $\mathcal{R}$. Moreover, if some $\mathcal{R}(x_1)$ isn't a singleton, which selection therein appears appropriate?

To see some prospects for learning to interact, suppose the game be played iteratively. By assumption, entering stage $k = 0, 1, ..$ with most recent choices x_1^k, x_2^k already sunk, the respective players use approximate payoff functions

$$\pi_1^k(x_1 \left| x_1^k \right., x_2) \leq \pi_1(x_1, x_2) \quad \& \quad \pi_2^k(x_1, x_2 \left| x_2^k \right.) \leq \pi_2(x_1, x_2). \tag{17}$$

Inequalities (17) reflect two features. First, either agent incurs some cost of change. Second, approximate payoffs are underestimates. Suppose that

$$x_1^k \to x_1 \Longrightarrow \limsup \pi_1^k(\chi_1 \left| x_1^k \right., \chi_2^k) \geq \liminf \pi_1(\chi_1, \chi_2^k), \text{ and} \tag{18}$$

$$\chi^k \to \chi \Longrightarrow \limsup \pi_2^k(\chi^k \left| x_2^k \right.) \geq \liminf \pi_2(\chi^k). \tag{19}$$

Assumptions (18), (19) capture that ultimately, when play settles, cost of change, becomes negligible.

At stage k the leader expects that the follower will apply a single-valued response function $r^k : X_1 \to X_2$. His expectation is approximately rational in so far as

$$\pi_2^k(\chi_1, r^k(\chi_1)) \geq \max_{\chi_2} \pi_2^k(\chi_1, \chi_2) - \varepsilon^k \text{ for all } \chi_1 \in X_1 \text{ with } \varepsilon^k \to 0^+. \tag{20}$$

On these premises, at stage k, the leader chooses an update

$$x_1^{k+1} \in A_1^k(x_1^k) := \arg\max \pi_1^k(\cdot \left| x_1^k \right., r^k(\cdot)). \tag{21}$$

After observing x_1^{k+1}, the follower comes up with a best response

$$x_2^{k+1} \in A_2^k(x_1^{k+1}, x_2^k) := \arg\max \pi_2^k(x_1^{k+1}, \cdot \left| x_2^k \right.).$$

Note that because of the sequential mode of play, the coupled updatings

$$x_1^{k+1} \in A_1^k(x_1^k) \quad \& \quad x_2^{k+1} \in A_2^k(x_1^{k+1}, x_2^k)$$

do not fit (10). Nonetheless, it holds:

Proposition 4.1 (convergence in Stackelberg games). *Suppose each function $x \in X \mapsto \pi_i(x)$ is upper semicontinuous, and that the leader's objective $\pi_1(x_1, x_2)$ is lower semicontinuous in $x_2 \in X_2$. Also suppose that for any point $\chi = (\chi_1, \chi_2) \in X$ and sequence $\chi_1^k \in X_1 \to \chi_1$, there exists a sequence $\chi_2^k \in X_2 \to \chi_2$ such that*

$$\liminf \pi_2(\chi^k) \geq \pi_2(\chi). \tag{22}$$

Then, if r^k converges continuously to r, meaning

$$x_1^k \to x_1 \Longrightarrow r^k(x_1^k) \to r(x_1), \tag{23}$$

it holds for each limit point $x_1 = \lim x_1^k$ that

$$x_1 \in \arg\max \pi_1(\cdot, r(\cdot)) \quad \text{and} \quad r(x_1) \in \arg\max \pi_2(x_1, \cdot).$$

Proof. Player 1 chooses x_1^k at stage k. Suppose $x_1 = \lim x_1^k$. By continuous convergence (23)

$$x_2 := \lim r^k(x_1^{k+1}) = r(x_1).$$

With $x = (x_1, x_2)$, it holds for any $\chi_1 \in X_1$ that

$$\begin{aligned} \pi_1(x) &\geq \limsup \pi_1(x_1^{k+1}, r^k(x_1^{k+1})) \geq^{(17)} \limsup \pi_1^k(x_1^{k+1} \,|x_1^k, r^k(x_1^{k+1})) \\ &\geq^{(21)} \limsup \pi_1^k(\chi_1 \,|x_1^k, r^k(\chi_1)) \\ &\geq^{(18)} \liminf \pi_1(\chi_1, r^k(\chi_1)) \geq \pi_1(\chi_1, r(\chi_1)). \end{aligned}$$

The first inequality derives from the upper semicontinuity of π_1. The last follows from the lower semicontinuity of $\pi_1(\chi_1, \cdot)$ and (23). Thus, $x_1 \in \arg\max \pi_1(\cdot, r(\cdot))$.

Further, for the same sequence $x_1^k \to x_1$ and *any* $\chi_2 \in X_2$ there exists a sequence $\chi_2^k \to \chi_2$ such that (22) holds with limit point (x_1, χ_2). So,

$$\begin{aligned} \pi_2(x) &\geq \limsup \pi_2(x_1^{k+1}, r^k(x_1^{k+1})) \geq^{(17)} \limsup \pi_2^k(x_1^{k+1}, r^k(x_1^{k+1}) \,|x_2^k) \\ &\geq^{(20)} \limsup[\pi_2^k(x_1^{k+1}, \chi_2^{k+1} \,|x_2^k) - \varepsilon^k] \quad (\text{with } \varepsilon^k \to 0) \\ &\geq^{(19)} \limsup \pi_2(x_1^{k+1}, \chi_2^{k+1}) \geq \liminf \pi_2(x_1^{k+1}, \chi_2^{k+1}) \geq \pi_2(x_1, \chi_2). \end{aligned}$$

The first inequality derives from the upper semicontinuity of π_2; the last from (22). Thus, $x_2 \in \arg\max \pi_2(x_1, \cdot)$, and the proof is complete. □

Proposition 4.1 leaves several open ends. In particular, what sort of approximations π_i^k might be expedient? What learning scheme, if any, could justify which response functions r^k? And, when will these converge continuously? These questions go beyond the scope of this paper. Suffice it to say that, for finite-action games, *fictitious play* may offer insights [4], [20], [23]; for games with continuous actions spaces, see the proximal point procedures in [5].

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