Type of the Paper (Article.)

The zeros of the Dirichlet *Beta* Function encode the odd primes and have real part 1/2

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Abstract: It is well known that the primes and prime powers have a deep relationship with the nontrivial zeros of Riemann's *zeta* function. This is a reciprocal relationship. The zeros and the primes are encoded in each other and are reciprocally recoverable. Riemann's *zeta* is an extended or continued version of Euler's *zeta* function which in turn equates with Euler's product formula over the primes. This paper shows that the *zeros* of the converging Dirichlet or Catalan *beta* function, which requires no continuation to be valid in the critical strip, can be easily determined. The imaginary parts of these zeros have a deep and reciprocal relationship with the odd primes and odd prime powers. This relationship separates the odd primes into those having either 1 or 3 as a remainder after division by 4. The vector pathway of the *beta* function is such that the real part of its zeros has to be a half.

Keywords: Catalan *beta* function; Riemann's *zeta* function; primes; Dirichlet L-function

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1. Introduction

Euclid's lemma, states that if p is prime and $p|ab$, where a and b are integers then $p|a$ or $p|b$ [1]. This leads us to the Fundamental Theorem of Arithmetic (FTA) that every $n \in \mathbb{N}_{>1}$ is the product of a unique set of primes, each prime being raised to a specific power;

$$n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_i^{a_i}, \text{ where } p_1 = 2,\ p_2 = 3,\ p_3 = 5, \ldots \in \mathbb{P} \text{ and } a_1, a_2, \ldots \in \mathbb{N}_0. \quad (1)$$

Euler's *zeta* function equates to Euler's product formula over the primes and this encodes the FTA;

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x} = \prod_p (1 - p^{-x})^{-1}$$
$$= \prod_p (1 + p^{-x} + p^{-2x} + \cdots) \text{ with } x \in \mathbb{R}. \quad (2)$$

This important relationship is easily extracted by multiplying out the right-hand side of Equation (2);

$$\zeta(x) = 1 + p_1^{-x} + p_2^{-x} + p_1^{-2x} + p_3^{-x} + p_1^{-x}p_2^{-x} + p_4^{-x} + p_1^{-3x} + p_2^{-2x} + p_1^{-x}p_3^{-x} + \cdots. \quad (3)$$

Euler's *Zeta* converges when $x > 1$ but Riemann's *zeta* $\zeta(s)$, with $s \in \mathbb{C}$, is in a form that "remains valid for all s". Riemann sought to extend Euler's *zeta* in analogy to the well-known extension of the factorial function for $s > -1$ in $\Gamma(s+1) = \lim_{n\to\infty} (n+1)^s \frac{1.2.3\ldots n}{(s+1)(s+2)(s+3)\ldots(s+n)}$.

Riemann's derivation of $\zeta(s)$ substitutes nx for x in the integral $\Gamma(s+1) = \int_0^{\infty} e^{-x} x^s dx$ to give $\Gamma(s) n^{-s} = \int_0^{\infty} e^{-nx} x^{s-1} dx$ for $s > 0$, $n \in \mathbb{N}_1$. This is then summed over n to give $\Gamma(s) \sum_{n=1}^{\infty} n^{-s} = \int_0^{\infty} (e^x - 1)^{-1} x^{s-1}\, dx$ for $s > 1$. Riemann proceeds with a contour integral expressed as $\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$. This integral is considered to run down a line just above the positive real axis before circling the origin anticlockwise, and then running back up a line just below the positive real axis.

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Then, after some manipulation, *zeta* is defined by $\zeta(s) = \frac{\Gamma(s-1)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\frac{dx}{x}$ which for real values, $s = x$, returns Euler's *zeta* $\zeta(x) = \sum_{n=1}^{\infty} n^{-x}$ [2].

The number of primes less than x, designated $\pi(x)$, is asymptotic to $x/\ln(x)$. This relationship was conjectured by Gauss in the 1790's and proved by Hadamard [3] and de la Vallée Poussin in 1896 [4]. The relationship is known as the Prime Number Theorem and makes use of Riemann's famous paper of 1859 entitled "On the Number of Primes Less Than a Given Magnitude" [5]. Riemann's explicit formula for $\pi_0(x) = \lim_{\varepsilon\to 0}\left(\pi(x+\varepsilon)+\pi(x-\varepsilon)\right)/2$ is given in terms of $\Pi_0(x)$ which counts the primes and prime powers up to x, counting a prime power p^n as $1/n$ of a prime;

$$\Pi_0(x) = \sum_{n=1}^{\infty} n^{-1}\pi_0\left(x^{1/n}\right). \tag{4}$$

The Möbius function $\mu(n)$, is related to the inverse of $\zeta(s)$ being $1/\zeta(s) = \sum_{n=1}^{\infty}\mu(n)n^{-s}$, which enables the number of primes to be recovered, $\pi_0(x) = \sum_{n=1}^{\infty}\mu(n)n^{-1}\Pi_0\left(x^{1/n}\right)$. Riemann's formula then becomes

$$\Pi_0(x) = \mathrm{Li}(x) - \sum_{\rho}\mathrm{Li}(x^{\rho}) - \ln(2) + \int_x^{\infty}\left(t(t^2-1)\ln(t)\right)^{-1}dt. \tag{5}$$

In Equation (5) $\mathrm{Li}(x) = \int_0^x (\ln(t))^{-1}\,dt$ is the un-offset logarithmic integral and $\{\rho\}$ are the nontrivial zeros.

Riemann's hypothesis is that the $\mathrm{Re}(\rho) = 1/2$ for all ρ for which there is ample evidence [6], but no proof. The *critical line* is $\sigma = 1/2$ and it lies in the middle of the *critical strip* $(0 < \mathrm{Re}(s) < 1)$. In 1914 Hardy [7] proved the infinitude of $\{\rho\}$ with $\sigma = 1/2$ for which large sets of the $\mathrm{Im}(\rho)$ are now published [8]. Riemann's *zeta* obeys the functional equation,

$$\zeta(s) = \Gamma(1-s)(2\pi)^{s-1}2\sin\left(\frac{\pi s}{2}\right)\zeta(1-s). \tag{6}$$

Accepting the Riemann Hypothesis, the nontrivial zeros of Riemann's *zeta*, $\zeta(s) = 0$, with $s = 1/2 + it$ are captured in the set $\{t_i\}$. It is then the set $\{t_i\}$ that has the deep relationship with all primes. It is easy to show that Dirichlet's *eta* function, which is an alternating *zeta* series

$$\eta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} \ldots \text{ using } s = \sigma + it$$

shares the set $\{t_i\}$ for its zeros $\eta(1/2 + it) = 0$.

Cognisant of the Riemann Hypothesis and reflecting on the famous Madhava-Leibniz series

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 \ldots, \tag{7}$$

one is instantly inquisitive about the zeros of the alternating and converging series

$$\beta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + 9^{-s} \ldots \text{ using } s = \sigma + it. \tag{8}$$

Does $\beta(s) = 0$ require $\mathrm{Re}(s) = 1/2$ and does $\{\mathrm{Im}(s) : \beta(s) = 0\}$ have an easily demonstrable relationship with the primes? The series is known as the Catalan, or Dirichlet *beta* series. Empirically the zeros of *beta* do sit on the *critical line* and we will index them with b as t_b, for which $\beta(1/2 + it_b) = 0$. Using t_b for *beta* distinguishes these zeros from the t_i for *zeta*.

The first aim of this work was to determine some zeros of the *beta* function by estimating the value of *beta* on the *critical line*, seeking solutions of $\beta(1/2 + it) = 0$ over a range in t.

The second aim was to locate a few zeros of *beta* through identifying "spikes" in the amplitude of a wave generated by the odd primes themselves.

The third aim was to demonstrate that the zeros of *beta* can locate the odd primes and odd prime powers; so examining the reciprocal relationship.

The third aim was to plot the vector pathway to convergence of the *beta* function and to illustrate why its zeros require the $\mathrm{Re}(s) = 1/2$.

 doi:10.20944/preprints201804.0305.v1

2. Materials and Methods

Repetitive calculations were performed in Microsoft Excel with Visual Basic. GraphPad Prism 7 was used to prepare figures.

Let $p, P \in \mathbb{P}$ the set of all primes, with $\mathbb{P} = \{2,3,5,7, \ldots p \ldots\}$. The odd primes, using \ to mean the set theoretic difference, is the set $\{\mathbb{P} \backslash \{2\}\}$ which we can separate into $\mathbb{P}_1 = \{p : 4|(p-1)\} = \{5, 13, 17, 29, 37 \ldots\}$ and $\mathbb{P}_3 = \{p : 4|(p-3)\} = \{3, 7, 11, 19, 23, \ldots\}$. Let $n, N, r \in \mathbb{N}_1$ and $x, t, \sigma \in \mathbb{R}$ with $i = \sqrt{-1}$ so that $s \in \mathbb{C}$ and $s = \sigma + it$. Without causing confusion, i will also be used to index the nontrivial zeros in the set $\{t_i\}$. Let the prime P be the largest prime in an ordered subset of $\mathbb{P}$ and let N be the largest natural in an ordered subset of $\mathbb{N}_1$.

This work aimed;

1. To estimate the values of early elements of the set $\{t_b\}$, for which $\beta(1/2 + it_b) = 0$, up to some convenient value of t, by determining minima in $\beta(1/2 + it)$ over a range of t. The infinite series were calculated to a convenient end point, see [9].
2. To prepare, by an alternative means, a subset of $\{t_b\}$ by using two subsets of odd primes, one subset from $\mathbb{P}_1$ and one from $\mathbb{P}_3$, and to compare these with values for $\{t_b\}$, from (1) above.
3. To use the set $\{t_b\}$, from (1), to confirm that it could locate the primes and prime powers of the odd primes; anticipating that the primes would separate into those with residuals of 1 or 3 on division by 4.
4. To illustrate that the vector pathway to convergence of the *beta* function can be equated to the difference between two finite series whose behaviour, under changes in the real part of s, limits the zeros of *beta* to having real part a half.

3. Results

The relationship between $\{t_i\}$ and all primes, and the prime powers, is presented first to set the scene, before turning to the relationship between $\{t_b\}$ and the two sets of odd primes and their powers. The arguments for why the $\mathrm{Re}(s) = 1/2$ if $\beta(s) = 0$ follows from the pathway analysis.

3.1. Riemenn's zeta: its nontrivial zeros and the primes

For *zeta*, we define an amplitude $Z(t)$ which we determine over a finite set of primes $\{2,3,5,7, \ldots p \ldots P\}$ and over the prime powers p^n to a convenient limit n. Our limit could be such that $p^n < P$ for all n such that $N = \lfloor \ln(P)/\ln(p) \rfloor$ for each p, or we can let n rise to some other convenient N; the limit is not materially important in this illustration. Our amplitude is

$$Z(t) = \sum_{p=2}^{P} \left(\ln(p) \sum_{n=1}^{N} p^{-(n/2)} \cos\big(t \ln(p^n)\big) \right). \tag{3}$$

Using Equation (3), summating over the primes, and prime powers, to $P = 4409$, the 600th prime, and letting t rise from 11 to 100, and letting $N = 20$ generates Figure 1. Figure 1 incorporates red lines at the published values of the first 28 nontrivial zeros of *zeta* [8].

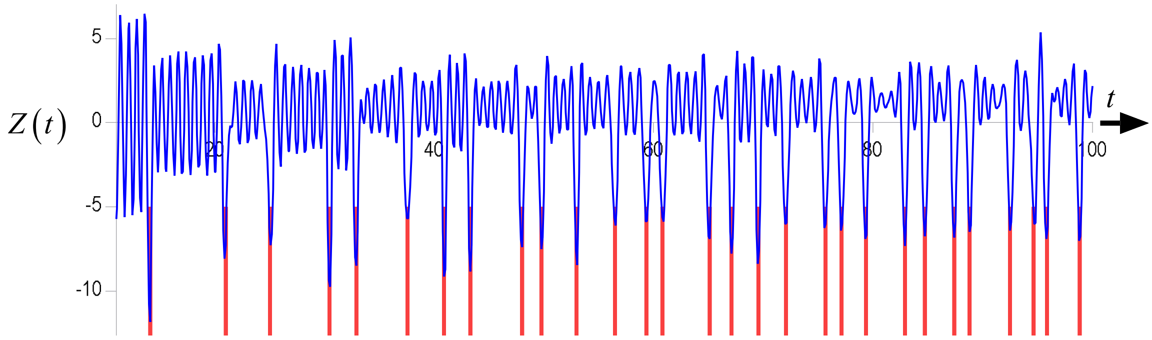

Figure 1. In blue the summated amplitudes of cosine functions, $\ln(p)p^{-(n/2)}\cos\big(t\ln(p^n)\big)$, over a set of primes and prime powers. In red, the first 28 elements of the set $\{t_i\}$.

 doi:10.20944/preprints201804.0305.v1

Using a subset of $\{t_i\}$, the location of the primes p, and the prime powers p^n, can be recovered over an interval in x with $A_{t_i}(x)$ to a convenient limit in i of I

$$A_{t_i}(x) = \sum_{i=1}^{I} \cos\big(t_i \ln(x)\big). \tag{4}$$

Using the first two hundred nontrivial zeros, $I = 200$, we can recover the primes and prime powers up to $x = 50$ as shown in Figure 2.

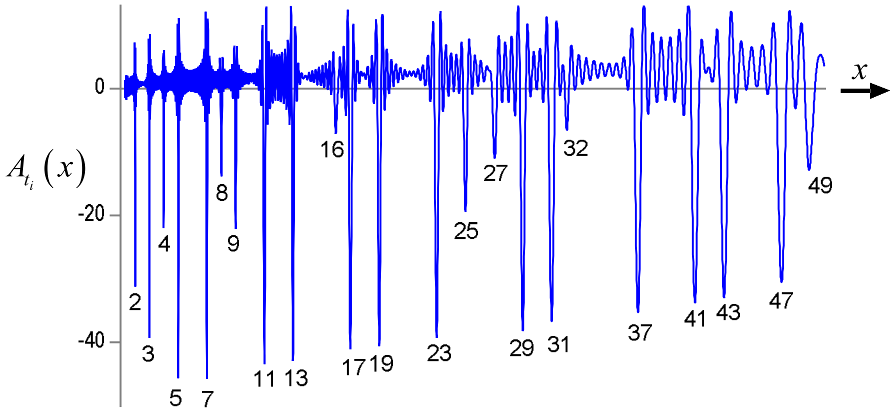

Figure 2. The location of the primes and prime powers up to $x = 50$ as determined from $A_{t_i}(x) = \sum_{i=1} \cos\big(t_i \ln(x)\big)$.

Having reminded ourselves of the deep reciprocal relationship $\{t_i\} \Leftrightarrow \{\mathbb{P}\}$ for *zeta*, we now turn to the *beta* function.

3.2. The beta function and the odd primes

The *beta* function

$$\beta(s) = \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)^{-s} \tag{5}$$

is also recognisable as a Dirichlet L-series defined as $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ with the Dirichlet character χ_1 having modulus 4;

$$L(s,\chi_1) = \beta(s) \ \text{modulus 4}. \tag{6}$$

Since all integers differ from a multiple of 4 by either 0, 1, 2 or 3, all the primes > 2 can be placed into two groups; those where $4|(p-1)$ and those where $4|(p-3)$. Then in a similar way to that in which Euler's *zeta* $\zeta_\infty(s) = \sum_{n=1}^{\infty} n^{-s}$ can be shown to equate to Euler's product over the primes $\prod_p \frac{1}{1-p^{-s}}$ in encoding the FTA, the *beta* series can be factorized as follows:

$$\beta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + 9^{-s} \ldots$$

$$3^{-s}\beta(s) = 3^{-s} - 9^{-s} + 15^{-s} - 21^{-s} + 27^{-s} \ldots$$

$$(1+3^{-s})\beta(s) = 1 + 5^{-s} - 7^{-s} - 11^{-s} + 13^{-s} + 17^{-s} \ldots$$

$$5^{-s}(1+3^{-s})\beta(s) = 5^{-s} + 25^{-s} - 35^{-s} - 55^{-s} + 65^{-s} + 85^{-s} \ldots$$

$$(1-5^{-s})(1+3^{-s})\beta(s) = 1 - 7^{-s} - 11^{-s} + 13^{-s} + 17^{-s} - 19^{-s} - 23^{-s} \ldots$$

proceeding until it is clear that

$$\beta(s) = \prod_{p \equiv 1 \, mod 4} \frac{1}{1-p^{-s}} \prod_{p \equiv 3 \, mod 4} \frac{1}{1+p^{-s}}. \tag{7}$$

In this way Dirichlet's *beta* can be seen to have a relationship with the odd primes, $\{\mathbb{P} \backslash \{2\}\}$.

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The series has a Functional Equation [2]

$$\beta(1-s) = 2^{s}\pi^{-s}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\,\beta(s), \tag{8}$$

and so if there is a zero at a t_u with $\mathrm{Re}(s) = \sigma_\alpha$ then, since we can ignore the sign of t, there is also a zero at $\mathrm{Re}(s) = \sigma_\beta$ for t_u with $0 < \sigma_\alpha < 1/2 < \sigma_\beta$ and $\sigma_\alpha + \sigma_\beta = 1$; reminiscent of the arguments around the nontrivial zeros of $\zeta(s)$. Using $m \in \mathbb{N}_1$ a partial series $\beta_n(s)$ has

$$\mathrm{Re}\,\beta_n(s) = \sum_{m=1}^{n} (-1)^{m-1}\,(2m-1)^{-\sigma}\cos(t\ln(2m-1))$$

$$\text{and } \mathrm{Im}\,\beta_n(s) = -\sum_{m=1}^{n} (-1)^{m-1}\,(2m-1)^{-\sigma}\sin(t\ln(2m-1)), \tag{9}$$

which facilitate the plotting of the vector pathway in $\mathbb{R}^2$. We will denote this operation as $\mathrm{P}(\beta_n(s))$ meaning the plotting of the vector pathway [9]. The pathway $\mathrm{P}(\beta_n(s))$ is then a plot of the sequential vectors representing the partial series $\beta_n(s)$;

$$\beta_n(s) = \sum_{m=1}^{n} (-1)^{m-1}(2m-1)^{-s} \text{ and } \mathrm{P}\big(\beta_n(s)\big) \equiv \sum_{m=1}^{n} \vec{m}$$

$$\text{with the vectors } \vec{m} \text{ plotted in } \mathbb{R}^2 \text{ having} \tag{10}$$

$$|\vec{m}| = (2m-1)^{-\sigma} \text{ and } \arg(\vec{m}) = -t\ln(2m-1) + \phi$$

$$\text{with } \phi = 0 \text{ if } 2 \nmid m \text{ and } \phi = \pi \text{ if } 2|m.$$

The pathway has *proximal* and *distal* parts separated at a vector *kappa*. The *distal* pathway has paired pseudo-spiral structures $\mathrm{P}(\mathcal{R}_r)$ which can be counted from convergence in a retrograde fashion starting with $\mathrm{P}(\mathcal{R}_1)$. These structures have a *principal-axis* whose argument has a stability under changes in σ which arises from a relationship with Euler's spiral, see [9]. The magnitude of the *principal-axis* of the $\mathrm{P}(\mathcal{R}_r)$ is also derivable from the relationship with Euler's spiral [9]. These relationships allow the $\mathrm{P}(\mathcal{R}_r)$ to be represented by novel vectors $\vec{\mathcal{R}}_r$.

A positive integer ν_β, locates the $\vec{m}$ nearest the *inflection point* in the pathway of a smooth curve that follows the final paired pseudo-spiral $\mathrm{P}(\mathcal{R}_1)$ of $\mathrm{P}\big(\beta_n(s)\big)$ with

$$\nu_\beta = \left[\frac{1}{2}\left(\left(e^{\pi/t}-1\right)^{-1} + \left(1-e^{-\pi/t}\right)^{-1}\right)\right] \tag{11}$$

and that $\vec{m}$ will have a magnitude $2\nu_\beta - 1$.

A second convergent vector series $h_r(s)$ and its pathway of vectors $\mathrm{P}(h_r(s))$ summate an ordered set of $\vec{\mathcal{R}}_r$ using the index r to a final term of r. A vector $\vec{\mathcal{R}}_r$ represents the magnitude of the *principal-axis* of a $\mathrm{P}(\mathcal{R}_r)$ and is an object in its own right. It has a magnitude $|\vec{\mathcal{R}}_i|$ being

$$|\vec{\mathcal{R}}_r| = \left(2\nu_\beta - 1\right)^{(1/2-\sigma)} r^{(\sigma-1)}, \tag{12}$$

with an argument

$$\arg(\vec{\mathcal{R}}_r) = \theta + t\ln\left(\frac{2r-1}{2\nu_\beta - 1}\right)$$

$$\text{with } \theta = \pi/4 \text{ if } 2|\nu_\beta \text{ and } \theta = 5\pi/4 \text{ if } 2 \nmid \nu_\beta. \tag{13}$$

The parameter θ accounts for the alternating sign preceding each term of *beta*. The $\vec{\nu}_\beta$ faces forwards for odd values and is reversed for even values. The series to r terms, $h_r(s)$ is defined as;

$$h_r(s) = \sum_{r=1}^{r} \vec{\mathcal{R}}_r \quad \text{with } \vec{\mathcal{R}}_r \text{ following } r \in \mathbb{N}_1, \tag{14}$$

and the series $h_\kappa(s)$, to *kappa* terms is simply $h_\kappa(s) = \sum_{r=1}^{\kappa} \vec{\mathcal{R}}_r$.

We then have an approximation for the *beta* function as the difference between the two finite series to *kappa* terms

$$\beta(s) \approx \beta_\kappa(s) - h_\kappa(s). \qquad (15)$$

This difference can be refined by taking account of the intersection of the $\vec{m}_\kappa$ and $\vec{\mathcal{R}}_\kappa$ with a function $x = f(\dot{\kappa})$ having $x \in \{x\colon\ 0 < x \leq 1\}$ but whose finer details are not important. A vector function closely related to $\beta(s)$ which we will designate $\beta(s,x)$ is

$$\beta(s,x) = \left(\beta_{\kappa-1}(s) + x\vec{\kappa}_\beta\right) - \left(h_{\kappa-1}(s) + x\vec{\mathcal{R}}_\kappa\right), \qquad (16)$$

whose symmetry breaking either side of $\sigma = 1/2$ is easily appreciated from $\mathrm{P}\left(\beta_\tau(s)\right)$, $\mathrm{P}\left(\beta_\kappa(s)\right)$ and $\mathrm{P}\left(h_\kappa(s)\right)$. It is this symmetry breaking which forces $\beta(s) = 0$ to have real part a half.

The pathway $\mathrm{P}\left(\beta_\infty(s)\right)$ has an $\vec{m} = \vec{\tau}$ in the *distal* pseudo-spiral of its $\mathrm{P}(\mathcal{R}_1)$ for which $m = \tau$;

$$\tau = \left[\left(e^{\pi/t} - 1\right)^{-1} + \left(1 - e^{-\pi/t}\right)^{-1}\right]. \qquad (18)$$

The vector series $\mathrm{P}\left(\beta_n(s)\right)$ indexed by m and $\mathrm{P}\left(h_r(s)\right)$ indexed by r have *proximal* and *distal* parts separated at terms designated *kappa*. A $\vec{m}$ for which $m = \kappa$ is designated $\vec{\kappa}$ and a $\vec{\mathcal{R}}_r$ for which $r = \kappa$ is designated $\vec{\mathcal{R}}_\kappa$. *Kappa* $\in \mathbb{N}_1$ is defined as

$$\kappa = \left[\frac{1}{2}\left(\left(e^{2\pi/t} - 1\right)^{-1/2} + \left(1 - e^{-2\pi/t}\right)^{-1/2}\right)\right]. \qquad (19)$$

A residual, *kappa dot* written $\dot{\kappa} \in \mathbb{R}$, with $-1/2 \leq \dot{\kappa} \leq 1/2$ is simply,

$$\dot{\kappa} = \frac{1}{2}\left(\left(e^{2\pi/t} - 1\right)^{-1/2} + \left(1 - e^{-2\pi/t}\right)^{-1/2}\right) - \kappa. \qquad (20)$$

Kappa dot is the domain of a function indicated $x = f(\dot{\kappa})$ with $0 < x \leq 1$ being the intersection of $\vec{\kappa}$ and $\vec{\mathcal{R}}_\kappa$ at the zeros.

A set of zeros of $\beta(s)$ for $t \leq 5250$ at $\sigma = 1/2$ were located by finding minima with very low magnitudes in $\beta_\tau(s)$, or at very low values of t, using $\beta_{2\tau}(s)$; a sample appear in [9]. Intervals in t of 0.001 were examined from $t = 6$ to $t = 4600$ and then in intervals of 0.0001 from $t = 4600$ to $t = 5250$. The zeros of *beta* from the set $\{t_b\}$ with $b \in \mathbb{N}_1$ being an index of the zeros of $\beta(s)$. The first 420 elements of $\{t_b\}$ appear in Table A1 in the Appendix.

Symmetrical pathways for a *beta* zero are illustrated in Figure 3.

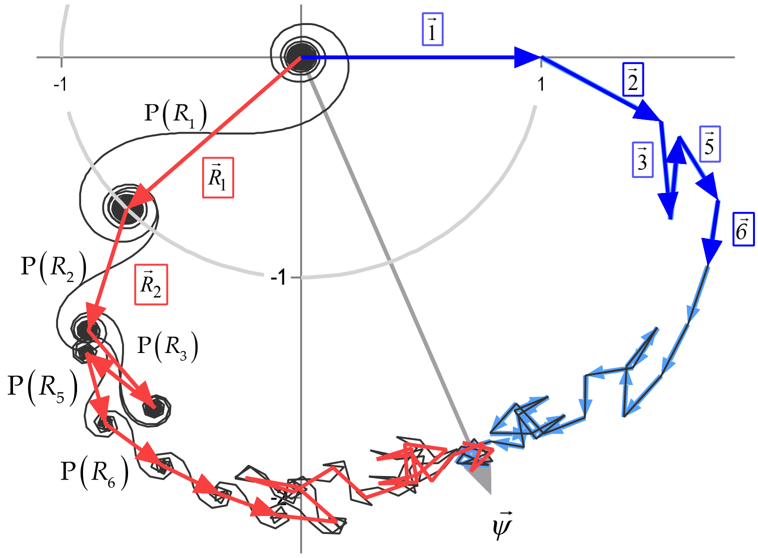

Figure 3. $\mathrm{P}\left(\beta(1/2 + i5247.848260)\right)$ for the 5944th zero of *beta* in black. In blue the $\mathrm{P}\left(\beta_\kappa(s)\right)$ with $\kappa = 29$ with the first 6 vectors emboldened. In red $\mathrm{P}\left(h_\kappa(s)\right)$ retraces the distal $\mathrm{P}\left(\beta(s)\right)$. The line of reflection $\vec{\psi}$ is shown and $\mathrm{P}(\mathcal{R}_1)$ to $\mathrm{P}(\mathcal{R}_3)$ and $\mathrm{P}(\mathcal{R}_5)$ and $\mathrm{P}(\mathcal{R}_6)$ are labelled.

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Figure 3 shows a line of reflection $\vec{\psi}$ made of two rays $\vec{\psi}_0$ and $\vec{\psi}_\pi$ of unspecified magnitude and having arguments ψ_0 and ψ_π. The argument ψ_0 is

$$\psi_0 = \arg(\vec{\psi}_0) = \pi/8 - (t/2)\ln(2\nu_\beta - 1). \tag{17}$$

The principal argument ψ_π is displaced clockwise from ψ_0 and so $\psi_\pi = \psi_0 - \pi$.

3.3. Symmetry breaking either side of $\sigma = 1/2$

The terms "bigger" and "smaller" are used here to refer to the non-uniform enlargement and reduction of a pathway whilst accommodating differential scaling along that pathway. Figure 4 shows that for *beta* when $\sigma < 1/2$ the pathway $\mathrm{P}(h_\kappa(s))$ is "bigger" than $\mathrm{P}(\beta_\kappa(s))$, and when $\sigma > 1/2$ the pathway $\mathrm{P}(h_\kappa(s))$ is "smaller" than $\mathrm{P}(\beta_\kappa(s))$. This difference in behaviour is important and underlies why there can be no zero when $\sigma \neq 1/2$.

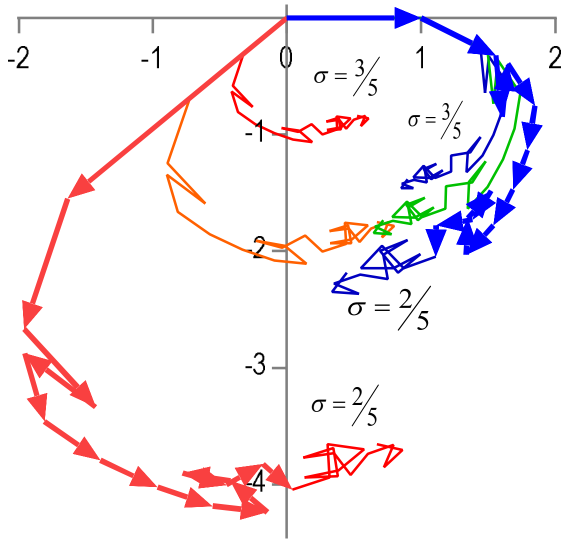

Figure 4. An illustration of how $\mathrm{P}(\beta_\kappa(s))$ and $\mathrm{P}(h_\kappa(s))$ change when σ rises or falls from $1/2$. Symmetry breaks differently either side of $\sigma = 1/2$. In green $\mathrm{P}(\beta_\kappa(s))$ and in orange $\mathrm{P}(h_\kappa(s))$ for $\sigma = 1/2$. In blue $\mathrm{P}(\beta_\kappa(s))$ and in red $\mathrm{P}(h_\kappa(s))$ for two other values of σ as indicated.

The symmetry breaking evident in Figure 4 is consequent upon Equation (9) *proximally*, and Equation (12) *distally*, see [9].

3.4. The beta function: its zeros and the odd primes

We now determine a subset of $\{t_b\}$ by summating $\cos(t\ln(p^n))$ over a subset of the primes and their lower powers. We define $B(t)$

$$B(t) = \sum_{\mathbb{P}_1}^{P_1}\left(\ln(p)\sum_{n=1}^{N} p^{-(n/2)}\cos(t\ln(p^n))\right) - \sum_{\mathbb{P}_3}^{P_3}\left(\ln(p)\sum_{n=1}^{N} p^{-(n/2)}\cos(t\ln(p^n))\right) \tag{21}$$

Figure 5 and Figure 6 were prepared with 325 primes in each group having $\mathbb{P}_1$ running from 5 to $P_1 = 4937$, and $\mathbb{P}_3$ running from 3 to $P_3 = 4751$ and allowing $N = 20$.

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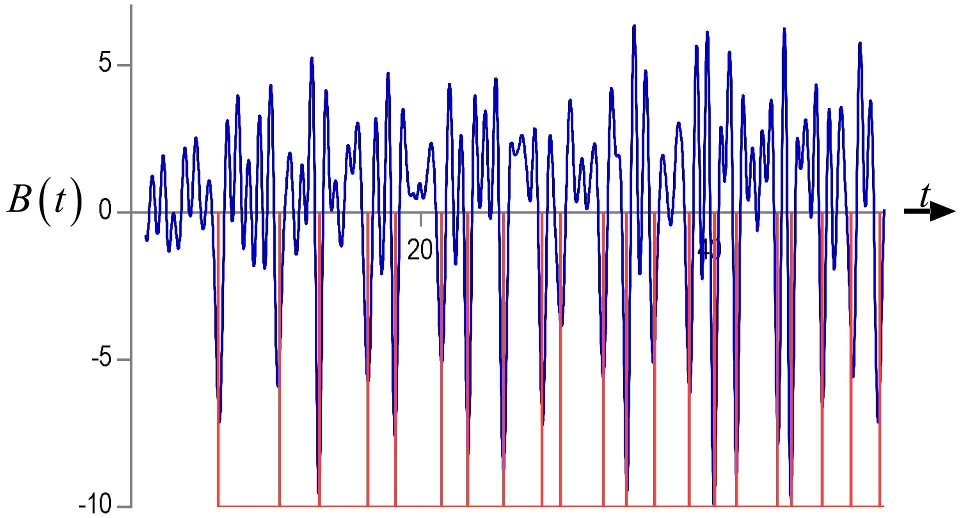

Figure 5. In blue is shown $B(t)$ for $1 \le t \le 52$ prepared with Equation (21) using 325 primes in set $\mathbb{P}_1$ and 325 primes in set $\mathbb{P}_3$. In red are the first 21 zeros of $\beta(s)$, see Table A1 (Appendix) these collocate with the most negative minima in $B(t)$.

The red lines in Figure 5 are zeros of *beta* from the set $\{t_b\}$ in Table A1 (Appendix).

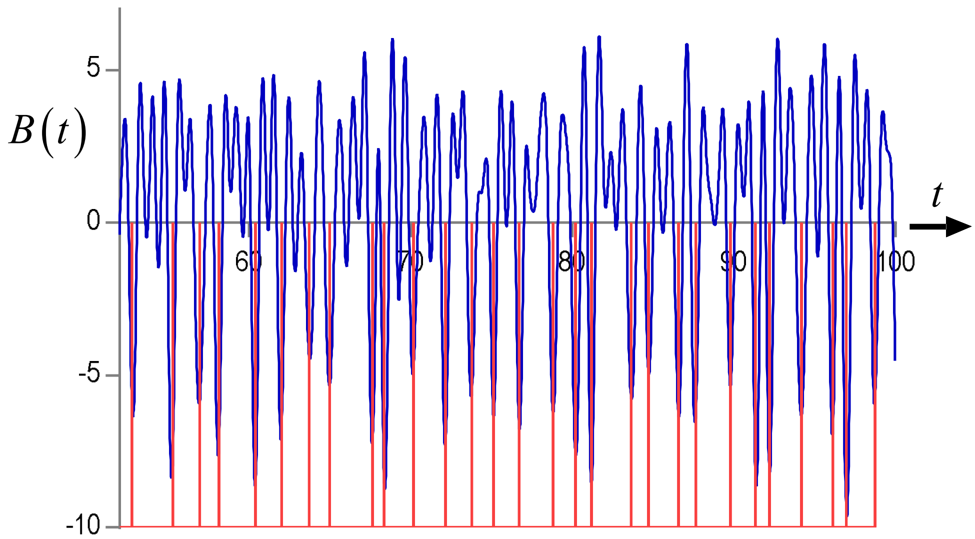

Figure 6. In blue is shown $B(t)$ for $52 \le t \le 100$ prepared with Equation (21) using 325 primes in each set $\mathbb{P}_1$ and $\mathbb{P}_3$. In red are the 28 zeros from the 23rd to the 50th zeros of $\beta(s)$, see Table 1 (Appendix); these collocate with the most negative minima in $B(t)$.

Our next task is to see how well a subset of $\{t_b\}$ perform in $A_{t_b}(x)$, with

$$A_{t_b}(x) = \sum_{b=1} \cos\left(t_b \ln(x)\right). \tag{22}$$

We anticipate that $A_{t_b}(x)$ can identify and separate the odd primes. Figure 7 was prepared using 5000 zeros of *beta*. The 5000th zero has $t = 4511.272\ldots$.

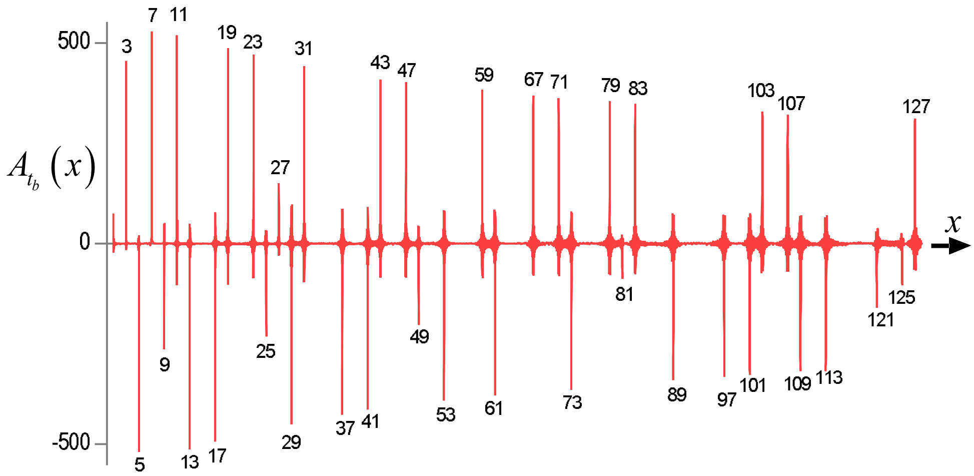

Figure 7. Shows positive spikes in $A_{t_b}(x)$ coinciding with $x \in \mathbb{P}_3$ and with 3^3 and it is noted that $4|(3^3-1)$, whilst negative spikes coincide with $x \in \mathbb{P}_1$ and with $3^2, 5^2, 7^2, 3^4, 11^2, 5^3$ each of which satisfies $4|(p^n-1)$.

Strictly, Figure 7 shows that $[x] \in \mathbb{P}$ rather than $x \in \mathbb{P}$, but the localization of the centre of the spikes to the integers is remarkably good and the errors are very small, (data not shown).

4. Discussion

The deep reciprocal relationship between the nontrivial zeros, $\{t_i\}$, of Riemann's *zeta* function $\zeta(s)$ and the distribution of the primes $\mathbb{P}$—accommodating all of its offspring as integral exponents $\{p^n : n \in \mathbb{N}_2\}$—is beautiful, profound and well-known. The sets $\mathbb{P}$ and $\{t_i\}$ are encoded in each other—like the perfect marriage. Or are they? The literature often implies that the set $\{t_i\}$ is telling $\mathbb{P}$ how to behave—this is erroneous—the relationship is reciprocal, but the driver is $\mathbb{P}$. However, the relationship is not monogamous.

This work shows that a similar reciprocal relationship connects the zeros of *beta*, $\{t_b\}$, with the distribution of the odd primes $\{\mathbb{P}\backslash\{2\}\}$, accommodating all of its integral exponents $\{p^n : p > 2, n \in \mathbb{N}_2\}$, whilst dividing those primes and their offspring according to the remainder after dividing by 4. The set $\{\mathbb{P}\backslash\{2\}\}$ and $\{t_b\}$ are entwined in each other—once $\mathbb{P}$ drops its only even element—but now $\{\mathbb{P}\backslash\{2\}\}$ has a relationship with the beautiful and spontaneously converging $\beta(s)$.

The sets $\{t_i\}$ and $\{t_b\}$, most likely, share characteristics, perhaps in the moments of the distributions of the gaps between their elements, which could be of interest [9–13]. But it is also an interesting question as to whether there is an encoded relationship between $\{t_i\}$ and $\{t_b\}$ which ignores $\mathbb{P}$—this seems unlikely and $\mathbb{P}$ may be quite happy about that.

The set $\mathbb{P}$ remains prime.

Acknowledgments: This study was not funded. No funds have been received for covering the costs to publish in open access.

Author Contributions: The author was the sole contributor.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

TABLE A1. Estimated zeros of $\beta(1/2+it)$: first 320

Prepared using the minima of $\beta_\tau(s)$ or, at very low values of t, using $\beta_{2\tau}(s)$.															
1	6.0114	41	84.732	81	143.33	121	197.115	161	247.996	201	296.902	241	344.085	281	390.17
2	10.264	42	86.578	82	144.979	122	198.808	162	249.185	202	298.08	242	345.211	282	391.083
3	13.002	43	87.631	83	146.523	123	200.163	163	251.087	203	298.871	243	346.264	283	392.853
4	16.353	44	89.802	84	147.935	124	200.903	164	251.638	204	300.436	244	347.926	284	393.444
5	18.285	45	91.351	85	149.189	125	202.261	165	252.625	205	301.921	245	348.983	285	394.821
6	21.445	46	92.239	86	150.298	126	204.222	166	254.315	206	302.919	246	349.435	286	395.806
7	23.274	47	94.168	87	151.963	127	204.993	167	255.839	207	303.662	247	350.978	287	396.736
8	25.726	48	96.138	88	153.701	128	206.413	168	256.506	208	305.066	248	352.393	288	398.251
9	28.358	49	96.963	89	154.577	129	207.318	169	258.166	209	306.796	249	353.694	289	399.566
10	29.662	50	98.756	90	155.651	130	209.229	170	258.836	210	307.287	250	354.303	290	400.694
11	32.596	51	100.136	91	157.749	131	210.104	171	260.432	211	309.123	251	355.745	291	400.827
12	34.204	52	102.142	92	158.706	132	211.834	172	261.915	212	309.741	252	356.629	292	402.819
13	36.145	53	103.288	93	160.238	133	212.539	173	262.885	213	310.825	253	358.289	293	403.908
14	38.512	54	104.334	94	161.408	134	213.763	174	264.055	214	312.557	254	359.098	294	404.947
15	40.321	55	106.695	95	162.567	135	215.795	175	264.934	215	313.879	255	360.712	295	406.162
16	41.806	56	107.691	96	164.732	136	216.705	176	267.002	216	314.434	256	361.199	296	407.146
17	44.618	57	109.261	97	165.402	137	217.583	177	267.802	217	315.736	257	362.278	297	408.04
18	45.599	58	110.501	98	166.755	138	219.174	178	268.784	218	317.013	258	364.421	298	409.589
19	47.742	59	112.369	99	168.045	139	220.407	179	270.279	219	318.457	259	364.805	299	410.873
20	49.725	60	113.816	100	170.052	140	221.93	180	271.256	220	319.581	260	366.051	300	411.662
21	51.688	61	115.144	101	170.736	141	223.004	181	272.757	221	320.407	261	367.126	301	412.732
22	52.771	62	116.194	102	172.282	142	224.123	182	274.172	222	322.003	262	368.435	302	413.67
23	55.27	63	118.539	103	173.444	143	225.294	183	275.034	223	322.54	263	369.503	303	415.43
24	56.937	64	119.454	104	174.916	144	226.989	184	275.86	224	324.519	264	371.07	304	416.372
25	58.117	65	120.732	105	176.598	145	228.406	185	277.773	225	325.476	265	371.771	305	417.13
26	60.423	66	122.448	106	177.703	146	228.96	186	278.805	226	326.459	266	372.886	306	418.558
27	62.009	67	123.795	107	178.363	147	230.331	187	280.158	227	327.261	267	373.936	307	419.488
28	63.714	68	125.769	108	180.57	148	232.101	188	280.794	228	329.201	268	375.548	308	420.53
29	64.976	69	126.299	109	181.616	149	233.049	189	282.381	229	330.037	269	376.685	309	422.511
30	67.636	70	127.96	110	182.918	150	234.353	190	283.606	230	331.212	270	377.56	310	422.688
31	68.369	71	129.886	111	184.116	151	235.837	191	284.927	231	332.574	271	378.388	311	424.068
32	70.188	72	131.094	112	185.375	152	236.243	192	286.081	232	333.183	272	380.058	312	424.824
33	72.157	73	132.145	113	187.07	153	238.538	193	287.15	233	334.77	273	381.05	313	426.733
34	73.77	74	133.745	114	188.272	154	239.34	194	287.979	234	336.154	274	382.241	314	427.385
35	75.145	75	135.492	115	189.493	155	240.628	195	290.253	235	337.141	275	383.486	315	428.699
36	76.697	76	136.549	116	190.372	156	241.479	196	290.678	236	338.227	276	384.374	316	429.59
37	78.812	77	138.459	117	192.362	157	243.23	197	291.833	237	339.012	277	385.216	317	430.619
38	80.211	78	138.751	118	193.798	158	244.514	198	293.203	238	340.843	278	387.15	318	432.062
39	81.214	79	141.255	119	194.233	159	245.566	199	294.328	239	342.027	279	388.083	319	433.038
40	83.668	80	142.395	120	196.133	160	246.725	200	295.804	240	342.687	280	388.779	320	434.44

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