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Varieties of coarse spaces

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Abstract. A class $\mathfrak{M}$ of coarse spaces is called a variety if $\mathfrak{M}$ is closed under formation of subspaces, coarse images and products. We classify the varieties of coarse spaces and, in particular, show that if a variety $\mathfrak{M}$ contains an unbounded metric space then $\mathfrak{M}$ is the variety of all coarse spaces.

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1 Introduction

Following [9], we say that a family $\mathcal{E}$ of subsets of $X \times X$ is a *coarse structure* on a set X if

- each $\varepsilon \in \mathcal{E}$ contains the diagonal $\triangle_X$, $\triangle_X = \{(x,x) : x \in X\}$;
- if $\varepsilon, \delta \in \mathcal{E}$ then $\varepsilon \circ \delta \in \mathcal{E}$ and $\varepsilon^{-1} \in \mathcal{E}$ where $\varepsilon \circ \delta = \{(x,y) : \exists z((x,z) \in \varepsilon, (z,y) \in \delta)\}$, $\varepsilon^{-1} = \{(y,x) : (x,y) \in \varepsilon\}$;
- if $\varepsilon \in \mathcal{E}$ and $\triangle_X \subseteq \varepsilon' \subseteq \varepsilon$ then $\varepsilon' \in \mathcal{E}$.

Each $\varepsilon \in \mathcal{E}$ is called an *entourage* of the diagonal. A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* for $\mathcal{E}$ if, for every $\varepsilon \in \mathcal{E}$ there exists $\varepsilon' \in \mathcal{E}'$ such that $\varepsilon \subseteq \varepsilon'$.

The pair $(X, \mathcal{E})$ is called a *coarse space*. For $x \in X$ and $\varepsilon \in \mathcal{E}$, we denote $B(x, \varepsilon) = \{y \in X : (x,y) \in \varepsilon\}$ and say that $B(x,\varepsilon)$ is a *ball of radius* ε *around* x. We note that a coarse space can be considered as an asymptotic counterpart of a uniform topological space and could be defined in terms of balls, see [6], [8]. In this case a coarse space is called a *ballean*.

A coarse space $(X, \mathcal{E})$ is called *connected* if, for any $x, y \in X$, there exists $\varepsilon \in \mathcal{E}$ such that $y \in B(x, \varepsilon)$. A subset Y of X is called *bounded* if there exist $x \in X$ and $\varepsilon \in \mathcal{E}$ such that $Y \subseteq B(x, \varepsilon)$. The coarse structure $\mathcal{E} = \{\varepsilon \in X \times X : \triangle_X \subseteq \varepsilon\}$ is the unique coarse structure such that $(X, \mathcal{E})$ is connected and bounded.

In what follows, all coarse spaces under consideration are supposed to be **connected**.

Given a coarse space $(X, \mathcal{E})$, each subset $Y \subseteq X$ has the natural coarse structure $\mathcal{E}|_Y = \{\varepsilon \cap (Y \times Y) : \varepsilon \in \mathcal{E}\}$, $(Y, \mathcal{E}|_Y)$ is called a *subspace* of $(X, \mathcal{E})$. A subset Y of X is called *large* (or *coarsely dense*) if there exists $\varepsilon \in \mathcal{E}$ such that $X = B(Y, \varepsilon)$ where $B(Y, \varepsilon) = \cup_{y \in Y} B(Y, \varepsilon)$.

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Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \longrightarrow X'$ is called *coarse* (or *bornologous* in terminology of [9]) if, for every $\varepsilon \in \mathcal{E}$ there exists $\varepsilon' \in \mathcal{E}'$ such that, for every $x \in X$, we have $f(B(x, \varepsilon)) \subseteq (B(f(x), \varepsilon'))$. If f is surjective and coarse then $(X', \mathcal{E}')$ is called a *coarse image* of $(X, \mathcal{E})$. If f is a bijection such that f and f^{-1} are coarse mappings then f is called an *asymorphism*. The coarse spaces $(X, \mathcal{E})$, $(X', \mathcal{E}')$ are called *coarsely equivalent* if there exist large subsets $Y \subseteq X$, $Y' \subseteq X$ such that $(Y, \mathcal{E}|_Y)$ and $(Y', \mathcal{E}'|_{Y'})$ are asymorphic.

To conclude the coarse vocabulary, we take a family $\{(X_\alpha, \mathcal{E}_\alpha) : \alpha < \kappa\}$ of coarse spaces and define the *product* $P_{\alpha<\kappa}(X_\alpha, \mathcal{E}_\alpha)$ as the set $P_{\alpha<\kappa}X_\alpha$ endowed with the coarse structure with the base $P_{\alpha<\kappa}\mathcal{E}_\alpha$. If $\varepsilon_\alpha \in \mathcal{E}_\alpha$, $\alpha < \kappa$ and $x, y \in P_{\alpha<\kappa}X_\alpha$, $x = (x_\alpha)_{\alpha<\kappa}$, $y = (y_\alpha)_{\alpha<\kappa}$ then $(x, y) \in (\varepsilon_\alpha)_{\alpha<\kappa}$ if and only if $(x_\alpha, y_\alpha) \in \varepsilon_\alpha$ for every $\alpha < \kappa$.

Let $\mathfrak{M}$ be a class of coarse spaces closed under asymorphisms. We say that $\mathfrak{M}$ is a *variety* if $\mathfrak{M}$ is closed under formation of subspaces ($\mathbf{S}\mathfrak{M} \subseteq \mathfrak{M}$), coarse images ($\mathbf{Q}\mathfrak{M} \subseteq \mathfrak{M}$) and products ($\mathbf{P}\mathfrak{M} \subseteq \mathfrak{M}$).

For an infinite cardinal κ, we say that a coarse space $(X, \mathcal{E})$ is *κ-bounded* if every subset $Y \subseteq X$ such that $|Y| < \kappa$ is bounded, and denote by $\mathfrak{M}_\kappa$ the variety of all κ-bounded coarse spaces.

We denote by $\mathfrak{M}_{single}$ and $\mathfrak{M}_{bound}$ the variety of singletons and the variety of all bounded coarse spaces.

Then we get the chain of varieties

$$\mathfrak{M}_{single} \subset \mathfrak{M}_{bound} \subset \ldots \subset \mathfrak{M}_\kappa \subset \ldots \subset \mathfrak{M}_\omega.$$

In section 2, we prove that every variety of coarse spaces lies in this chain and, in section 3, we discuss some extensions of this result to coarse spaces endowed with additional algebraic structures.

2 Results

We recall that a family $\mathcal{I}$ of subsets of a set X is an *ideal* in the Boolean algebra $\mathcal{P}_X$ of all subsets of X if $\mathcal{I}$ is closed under finite unions and subsets. Every ideal $\mathcal{I}$ defines the coarse structure with the base $\{\mathcal{E}_A : A \in \mathcal{I}\}$ where $\mathcal{E}_A = (A \times A) \cup \triangle_X$, so $B(x, \mathcal{E}_A) = A$ if $x \in A$ and $B(x, \mathcal{E}_A) = \{x\}$ if $x \in X \setminus A$. We denote the obtained coarse space by $(X, \mathcal{I})$. For a cardinal κ, $[X]^{<\kappa}$ denotes the ideal $\{Y \subseteq X : |Y| < \kappa\}$. If $(X, \mathcal{E})$ is a coarse space, the family $\mathcal{I}$ of all bounded subsets of X is an ideal. The coarse space $(X, \mathcal{I})$ is called the *companion* of $(X, \mathcal{E})$.

Let $\mathcal{K}$ be a class of coarse spaces. We say that a coarse space $(X, \mathcal{E})$ is *free* with respect to $\mathcal{K}$ if, for every $(X', \mathcal{E}') \in \mathcal{K}$ every mapping $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ is coarse. For example, $(X, [X]^{<\kappa})$ is free with respect to the variety $\mathfrak{M}_\kappa$. Since $(\kappa, [\kappa]^{<\kappa}) \in \mathfrak{M}_\kappa$ but $(\kappa, [\kappa]^{<\kappa}) \notin \mathfrak{M}_{\kappa'}$ for each $\kappa' > \kappa$, the inclusion $\mathfrak{M}_{\kappa'} \subset \mathfrak{M}_\kappa$ is strict.

Lemma 1. *If a coarse space $(X, \mathcal{E})$ is free with respect to a class $\mathcal{K}$ then $(X, \mathcal{E})$ is free with respect to* $\mathbf{S}\mathcal{K}$, $\mathbf{Q}\mathcal{K}$, $\mathbf{P}\mathcal{K}$.

Proof. We verify only the second statement. Let $(X', \mathcal{E}') \in \mathcal{K}$, $(X'', \mathcal{E}'') \in \mathbf{Q}\mathcal{K}$, and $h : (X', \mathcal{E}') \longrightarrow (X'', \mathcal{E}'')$ be a coarse surjective mapping. We take an arbitrary $f : X \longrightarrow X''$ and choose $h' : X \longrightarrow X'$ such that $f = hh'$. Since $(X, \mathcal{E})$ is free with respect to $\mathcal{K}$, $h' : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ is coarse so f is coarse as the composition of the coarse mappings h, h'. □

Lemma 2. *Let X be a set and let $\mathcal{K}$ be a class of coarse spaces, $\mathcal{K} \neq \mathfrak{M}_{single}$. Then there exists a coarse structure $\mathcal{E}$ on X such that $(X, \mathcal{E}) \in \mathbf{SP}\mathcal{K}$ and $(X, \mathcal{E})$ is free with respect to $\mathcal{K}$.*

Proof. We take a set S of all pairwise non-asymorphic coarse spaces $(X', \mathcal{E}') \in \mathcal{K}$ such that $|X'| \leq |X|$ and enumerate all possible triplets $\{(X_\alpha, \mathcal{E}_\alpha, f_\alpha) : \alpha < \lambda\}$ such that $(X_\alpha, \mathcal{E}_\alpha) \in S$ and $f_\alpha : X \longrightarrow X_\alpha$. Then we consider the product $P_{\alpha<\lambda}(X_\alpha, \mathcal{E}_\alpha)$ and define $f : X \longrightarrow P_{\alpha<\lambda} X_\alpha$ by $f(x) = (f_\alpha(x))_{\alpha<\lambda}$. Since $\mathcal{K} \neq \mathfrak{M}_{single}$, f is injective so we can identify X with $f(X)$ and consider the subspace $(X, \mathcal{E})$ of $P_{\alpha<\lambda}(X_\alpha, \mathcal{E}_\alpha)$. Clearly, $(X, \mathcal{E}) \in \mathbf{SP}\mathcal{K}$.

To see that $(X, \mathcal{E})$ is free with respect to $\mathcal{K}$, it suffices to verify that, for each $(X', \mathcal{E}') \in S$, every mapping $h : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ is coarse. We take $\beta < \lambda$ such that $(X', \mathcal{E}') = (X_\beta, \mathcal{E}_\beta)$ and $h = f_\beta$. Then f_β is the restriction to X of the projection $pr_\beta : P_{\alpha<\lambda}(X_\alpha, \mathcal{E}_\alpha) \longrightarrow (X_\beta, \mathcal{E}_\beta)$ so f_β is coarse. □

Theorem 1. *For every class $\mathcal{K}$ of coarse spaces, the smallest variety $Var\,\mathcal{K}$ containing $\mathcal{K}$ is $\mathbf{QSP}\mathcal{K}$.*

*Proof.*The inclusion $\mathbf{QSP}\mathcal{K} \subseteq \mathcal{K}$ is evident. To prove the inverse inclusion, we suppose that $\mathcal{K} \neq \mathfrak{M}_{single}$ (this case is evident) and take an arbitrary $(X', \mathcal{E}') \in Var\ (\mathcal{K})$. Then $(X', \mathcal{E}')$ can be obtained from $\mathcal{K}$ by means of some finite sequence of operations $\mathbf{S}, \mathbf{P}, \mathbf{Q}$. We use Lemma 2 to choose a coarse space $(X, \mathcal{E}) \in \mathbf{SP}\mathcal{K}$, $|X| = |X'|$ free with respect to $\mathcal{K}$. By Lemma 1, any bijection $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ is coarse so $(X', \mathcal{E}') \in \mathbf{QSP}\mathcal{K}$. □

Theorem 2. *Let $\mathfrak{M}$ be a variety of coarse spaces such that $\mathfrak{M} \neq \mathfrak{M}_{single}$, $\mathfrak{M} \neq \mathfrak{M}_{bound}$. Then there exists a cardinal κ such that $\mathfrak{M} = \mathfrak{M}_\kappa$.*

Proof. Since $\mathfrak{M} \neq \mathfrak{M}_{bound}$ and $\mathfrak{M} \neq \mathfrak{M}_{single}$, there exists the minimal cardinal κ such that each space from $\mathfrak{M}$ is κ-bounded so $\mathfrak{M} \subseteq \mathfrak{M}_\kappa$.

To verify the inclusion $\mathfrak{M}_\kappa \subseteq \mathfrak{M}$, we take a coarse space $(X, \mathcal{E}) \in \mathfrak{M}$ free with respect to $\mathfrak{M}$ and show that $(X, \mathcal{E})$ is free with respect to $\mathfrak{M}_\kappa$. We prove that $(X, \mathcal{E}) = (X, [X]^{<\kappa})$. If $|X| < \kappa$ then $(X, \mathcal{E})$ is bounded and the statement is evident. Assume that $|X| \geq \kappa$ but $(X, \mathcal{E}) \neq (X, [X]^{<\kappa})$. Assume that, for every ε, $\varepsilon = \varepsilon^{-1}$, the set $S_\varepsilon = \{x \in X : |B(x, \varepsilon)| > 1\}$ is bounded in $(X, \mathcal{E})$. By the choice of κ, $|S_\varepsilon| < \kappa$ and $|B(x, \varepsilon)| = 1$ for all $x \in X \setminus S_\varepsilon$. It follows that $(X, \mathcal{E}) = (X, [X]^{<\kappa})$. Then there exists $\varepsilon \in \mathcal{E}$ such that the set S_ε is unbounded in $(X, \mathcal{E})$. We choose a maximal by inclusion subset $Y \subset X$ such that $B(y, \varepsilon) \cap B(y', \varepsilon) = \emptyset$ for all distinct $y, y' \in Y$. We observe that Y is unbounded so $|Y| \geq \kappa$. We take an arbitrary $x_0 \in X$ and choose a mapping $f : X \longrightarrow X$ such that $f(y) = x_0$ for each $y \in Y$ and f is injective on $X \setminus Y$. Since $(X, \mathcal{E})$ is free with respect to $\mathfrak{M}$, the mapping $f : (X, \mathcal{E}) \longrightarrow (X, \mathcal{E})$ must be coarse. Hence, there exists $\varepsilon' \in \mathcal{E}$ such that $f(B(x, \varepsilon)) \subseteq B(f(x), \varepsilon')$ for each $x \in X$. It follows that $f(\cup_{y \in Y} B(y, \varepsilon))$ is bounded in $(X, \mathcal{E})$. We note that $|f(\cup_{y \in Y} B(y, \varepsilon))| \geq \kappa$ so $(X, \mathcal{E})$ contains a bounded subset Z such that $|Z| = \kappa$. Since $(X, \mathcal{E})$ is free with respect to $\mathfrak{M}$, every $(X', \varepsilon') \in \mathfrak{M}$ is a κ^+-bounded and we get a contradiction with the choice of κ. To conclude the proof, we take an arbitrary $(X, \mathcal{E}') \in \mathfrak{M}_\kappa$ and note that the identity mapping $id : (X, [X]^{<\kappa}) \longrightarrow (X, \mathcal{E}')$ is coarse so $(X, \mathcal{E}') \in \mathfrak{M}$. □

Remark 1. We note that $\mathfrak{M}_{single}$ is not closed under coarse equivalence because each bounded coarse space is coarsely equivalent to a singleton. Clearly, $\mathfrak{M}_{bound}$ is closed under coarsely equivalence. We show that the same is true for every variety $\mathfrak{M}_\kappa$. Let $(X, \mathcal{E})$ be a coarse space, Y be a large subset of $(X, \mathcal{E})$. We assume that $(Y, \mathcal{E}|_Y) \in \mathfrak{M}_\kappa$ but $(X, \mathcal{E}) \notin \mathfrak{M}_\kappa$. Then X contains an unbounded subset Z such that $|Z| < \kappa$. We choose $\varepsilon \in \mathcal{E}$ such that $\varepsilon = \varepsilon^{-1}$ and $X = B(Y, \mathcal{E})$. For each $z \in Z$, we pick $y_z \in Y$ such that $z \in B(y_z, \mathcal{E})$. We put $Y' = \{y_z \in Z\}$. Since $|Y'| < \kappa$, Y' is bounded in $(Y, \mathcal{E}|_Y)$. It follows that Z is bounded in $(X, \mathcal{E})$, a contradiction with the choice of Z.

We note also that every variety of coarse spaces is closed under formations of companions. For $\mathfrak{M}_{single}$ and $\mathfrak{M}_{bound}$, this is evident. Let $(X, \mathcal{E}) \in \mathfrak{M}_\kappa$ and $\mathcal{I}$ is the ideal of all bounded subsets of $(X, \mathcal{E})$. Since $(X, [X]^{<\kappa})$ is free with respect to $\mathfrak{M}_\kappa$, the identity mapping $id : (X, [X]^{<\kappa}) \longrightarrow (X, \mathcal{E})$ is coarse so $[X]^{<\kappa} \subseteq \mathcal{I}$ and $(X, \mathcal{E}) \in \mathfrak{M}_\kappa$.

Remark 2. Every metric d on a set X defines the coarse structure $\mathcal{E}_d$ on X with the base $\{(x, y) : d(x, y) \leq n\}$, $n \in \omega$. A coarse structure $\mathcal{E}$ on X is called *metrizable* if there exists a metric d on X such that $\mathcal{E} = \mathcal{E}_d$. By [8, Theorem 2.1.1], $\mathcal{E}$ is metrizable if and only if $\mathcal{E}$ has a countable base. From the coarse point of view, metric spaces are studding in *Asymptotic Topology*, see [1].

We assume that a variety $\mathfrak{M}$ of coarse space contains an unbounded metric space (X, d) and show that $\mathfrak{M} = \mathfrak{M}_\omega$. We choose a countable unbounded subset Y of X and note that $(Y, d) \notin \mathfrak{M}_\kappa$ for $\kappa > \omega$ so $(Y, d) \in \mathfrak{M}_\omega \setminus \mathfrak{M}_\kappa$ and the variety generated by (X, d) is $\mathfrak{M}_\omega$.

3 Comments

1. Let G be a group with the identity e. An ideal $\mathcal{I}$ in $\mathcal{P}_G$ is called a *group ideal* if $[G]^{<\omega} \subseteq \mathcal{I}$ and $AB^{-1} \in \mathcal{I}$ for all $A, B \in \mathcal{I}$.

Let X be a G-space with the action $G \times X \longrightarrow X$, $(g, x) \longmapsto gx$. We assume that G acts on X transitively, take a group ideal $\mathcal{I}$ on G and consider the coarse structure $\mathcal{E}(G, \mathcal{I}, X)$ on X with the base $\{\varepsilon_A : A \in \mathcal{I}, e \in A\}$, $\varepsilon_A = \{(x, gx) : x \in X, g \in A\}$. Then $B(x, \varepsilon_A) = Ax$, $Ax = \{gx : g \in A\}$.

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By [4, Theorem 1], for every coarse structure $\mathcal{E}$ on X, there exist a group G of permutations of X and a group ideal $\mathcal{I}$ in $\mathcal{P}_G$ such that $\mathcal{E} = \mathcal{E}(G, \mathcal{I}, X)$.

Now let $X = G$ and G acts on X by the left shifts $x \longmapsto gx$, $g \in G$. We denote $(G, \mathcal{E}(G, \mathcal{I}, G))$ by $(G, \mathcal{I})$ and say that $(G, \mathcal{I})$ is a *right coarse group*. If $\mathcal{I} = [G]^{<\omega}$ then $(G, \mathcal{I})$ is called *a finitary right coarse group*. In the metric form, these structures on finitely generated groups play an important role in *Geometric Group Theory*, see [2, Chapter 4].

A group G endowed with a coarse structure $\mathcal{E}$ is a right coarse group if and only if, for every $\varepsilon \in \mathcal{E}$, there exists $\varepsilon' \in \mathcal{E}$ such that $(B(x, \varepsilon))g \subseteq B(xg, \varepsilon')$ for all $x, g \in G$. For group ideals and coarse structures on groups see [7] or [8, Chapter 6].

2. A class $\mathfrak{M}$ of right coarse groups is called a *variety* if $\mathfrak{M}$ is closed under formation of subgroups, coarse homomorphic images and products.

Let $\mathcal{K}$ be a class of right coarse groups, G be a group generated by a subset $X \subset G$. We say that a right coarse group $(G, \mathcal{I})$ is *free* with respect to $\mathcal{K}$ if, for every $(G', \mathcal{I}') \in \mathcal{K}$, any mapping $X \longrightarrow G'$ extends to the coarse homomorphism $(G, \mathcal{I}) \longrightarrow (G', \mathcal{I}')$. Then Lemma 1, Lemma 2 and Theorem 1 hold for right coarse groups in place of coarse spaces.

Let $\mathfrak{M}$ be a variety of right coarse groups. We take an arbitrary $(G, \mathcal{I}) \in \mathfrak{M}$, delete the coarse structure on G and the class $\mathfrak{M}^\flat$ of groups. If $(G, \mathcal{I}) \in \mathfrak{M}$ then $(G, \mathcal{P}_G) \in \mathfrak{M}$. It follows that $\mathfrak{M}^\flat$ is a variety of groups.

Now let $\mathcal{G}$ be a variety of group different from the variety of singletons. We denote by $\mathcal{G}_{bound}$ the variety of right coarse groups $(G, \mathcal{P}_G), G \in \mathcal{G}$. For an infinite cardinal κ, we denote by $\mathcal{G}_\kappa$ the variety of all κ-bounded right coarse groups $(G, \mathcal{I})$, $G \in \mathcal{G}$.

Let $\mathfrak{M}$ be a variety of right coarse groups such that $\mathfrak{M}^\flat = \mathcal{G}$. Repeating arguments proving Theorem 2, we conclude that $\mathfrak{M}$ lies in the chain

$$\mathcal{G}_{bound} \subset \ldots \subset \mathcal{G}_\kappa \subset \ldots \subset \mathcal{G}_\omega.$$

If G is a group of cardinality κ and $G \in \mathcal{G}$ then $(G, [G]^{<\kappa}) \in \mathcal{G}_\kappa \setminus \mathcal{G}_{\kappa'}$ for each $\kappa' > \kappa$. Hence, all inclusions in above chain are strict.

3. let Ω be a signature, A be an Ω-algebra and $\mathcal{E}$ be a coarse structure on A. We say that A is a *coarse Ω-algebra* if every n-ary operation from Ω is coarse as the mapping $(A, \mathcal{E})^n \longrightarrow (A, \mathcal{E})$. We note that each coarse group is a right coarse group but the converse statement needs not to be true, see [8, Section 6.1].

A class $\mathfrak{M}$ of coarse Ω-algebra is called a *variety* if $\mathfrak{M}$ is closed under formation of subalgebras, coarse homomorphic images and products. Given a variety $\mathfrak{M}$ of coarse algebras, the class $\mathfrak{M}^\flat$ of all Ω-algebras A such that $(A, \mathcal{E}) \in \mathfrak{M}$ is a variety of Ω-algebras.

Let A be a variety of Ω-algebras different from the variety of singletons. We denote by $\mathcal{A}_{bound}$ the variety of coarse algebras $(A, \mathcal{P}_A)$, $A \in \mathcal{A}$. For an infinite cardinal κ we denote by $\mathcal{A}_\kappa$ the variety of all κ-bounded Ω-algebras $(A, \mathcal{E})$ such that $A \in \mathcal{A}$.

If $\mathfrak{M}$ be a variety of coarse algebras such that $\mathfrak{M}^\flat = \mathcal{A}$ then $\mathfrak{M}$ lies in the chain

$$\mathcal{A}_{bound} \subseteq \ldots \subseteq \mathcal{A}_\kappa \subseteq \ldots \subseteq \mathcal{A}_\omega,$$

but, we can not state that all inclusions are strict. In the case of course groups, this is so because each non-trivial variety of groups contains some Abelian group A of cardinality κ and the coarse group $(A, [A]^{<\kappa)}$ is κ-bounded but not κ^+-bounded.

4. A class $\mathfrak{M}$ of topological Ω-algebras (with regular topologies) is called a *variety* (*a wide variety*) if $\mathfrak{M}$ is closed under formation of closed subalgebras (arbitrary subalgebras), continuous homomorphic images and products. The wide varieties and varieties are characterized syntactically by the limit laws [10] and filters [5]. In our coarse case, the part of filters play the ideals $[X]^{<\kappa}$.

There are only two wide varieties of topological spaces, the variety of singletons and the variety of all topological spaces, but there is a plenty of varieties of topological spaces. The variety of coarse spaces $\mathfrak{M}_\kappa$ is a twin of the varieties of topological spaces in which every subset of cardinality $< \kappa$ is compact. We note also that $\mathcal{G}_\kappa$ might be considered as a counterpart of the variety $T(\kappa)$ of topological groups from [3], $G \in T(\kappa)$ if and only if each neighborhood of e contains a normal subgroup of index strictly less then κ.

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