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SYMMETRIC IDENTITIES FOR FUBINI POLYNOMIALS

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Abstract. In this paper, we represent the generating function of w-torsion Fubini polynomials by fermionic p-adic integral on $\mathbb{Z}_p$. Then we consider a quotient of such fermionic p-adic integrals on $\mathbb{Z}_p$, representing generating functions of three w-torsion Fubini polynomials and derive some new symmetric identities for the w-torsion Fubini and two variable w-torsion Fubini polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim (see [6]) as

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} f(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = 2f(0), \quad \text{(see [6, 7])}. \tag{1.2}$$

By (1.2), we easily get

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [1-13])}, \tag{1.3}$$

where $E_n(x)$ are the ordinary Euler polynomials.

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It is known that the two variable Fubini polynomials are defined by the generating function

$$\sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!} = \frac{1}{1-y(e^t-1)}e^{xt}, \quad \text{(see [10])}. \tag{1.4}$$

When $x=0$, $F_n(y)=F_n(0,y)$, $(n \geq 0)$, are called Fubini polynomials.

From (1.3) and (1.4), we note that $F_n(x,-1/2)=E_n(x)$, $(n \geq 0)$. By (1.4), we easily get

$$F_n(y)=\sum_{k=0}^{n} S_2(n,k)k!y^k, \ (n \geq 0), \quad \text{(see [10])}, \tag{1.5}$$

where $S_2(n,k)$ are the Stirling numbers of the second kind.

For $w \in \mathbb{N}$, we define the two variable w-torsion Fubini polynomials given by

$$\frac{1}{1-y^w(e^t-1)^w}e^{xt} = \sum_{n=0}^{\infty} F_{n,w}(x,y)\frac{t^n}{n!}. \tag{1.6}$$

In particular, for $x=0$, $F_{n,w}(y)=F_{n,w}(0,y)$ are called the w-torsion Fubini polynomials. It is clear that $F_{n,1}(x,y)=F_n(x,y)$.

In this paper, we represent the generating function of w-torsion Fubini polynomials by fermionic p-adic integral on $\mathbb{Z}_p$. Then we consider a quotient of such fermionic p-adic integrals on $\mathbb{Z}_p$, representing generating functions of three w-torsion Fubini polynomials and derive some new symmetric identities for the w-torsion Fubini and two variable w-torsion Fubini polynomials. Recently, several researchers have studied symmetric identities for some special polynomials (see [1-17]).

2. Symmetric identities for w-torsion Fubini and two variable w-torsion Fubini polynomials

From (1.2), we note that

$$\int_{\mathbb{Z}_p} (-1)^x(y(e^t-1))^x d\mu_{-1}(x) = \frac{2}{1-y(e^t-1)} = 2\sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}, \tag{2.1}$$

and

$$e^{xt}\int_{\mathbb{Z}_p} (-1)^z(y(e^t-1))^z d\mu_{-1}(z) = \frac{2}{1-y(e^t-1)}e^{xt} = 2\sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!}. \tag{2.2}$$

From (2.1) and (2.2), we note that

$$\begin{aligned}\left(\sum_{l=0}^{\infty} x^l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} 2F_m(y)\frac{t^m}{m!}\right) &= e^{xt}\int_{\mathbb{Z}_p}(-1)^z(y(e^t-1))^z d\mu_{-1}(z)\\ &= \sum_{n=0}^{\infty} 2F_n(x,y)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Thus, by (2.3), we easily get

$$\sum_{l=0}^{n}\binom{n}{l}x^l F_{n-l}(y) = F_n(x,y),\ \ (n \geq 0). \tag{2.4}$$

Now, we observe that

$$\begin{aligned}\frac{1-y^k(e^t-1)^k}{1-y(e^t-1)} &= \sum_{i=0}^{k-1} y^i(e^t-1)^i = \sum_{i=0}^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}y^i e^{lt}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{k-1}\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}y^i l^n\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{k-1} y^i\Delta^i 0^n\right)\frac{t^n}{n!},\end{aligned} \tag{2.5}$$

where $\Delta f(x) = f(x+1) - f(x)$.

For $w \in \mathbb{N}$, w-torsion Fubini polynomials are represented by the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}(-y^w(e^t-1)^w)^x d\mu_{-1}(x) = \frac{2}{1-y^w(e^t-1)^w} = \sum_{n=0}^{\infty} 2F_{n,w}(y)\frac{t^n}{n!}, \tag{2.6}$$

From (2.1) and (2.6), we have

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-y(e^t-1))^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)} &= \frac{1-y^{w_1}(e^t-1)^{w_1}}{1-y(e^t-1)} = \sum_{i=0}^{w_1-1} y^i(e^t-1)^i\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{w_1-1} y^i\Delta^i 0^n\right)\frac{t^n}{n!},\ \ (w_1 \in \mathbb{N}).\end{aligned} \tag{2.7}$$

For $w_1, w_2 \in \mathbb{N}$, we let

$$I = \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^{x_1}(-y^{w_2}(e^t-1)^{w_2})^{x_2} d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}. \tag{2.8}$$

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Then, by (2.8), we get

$$I=\left(\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)\right)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}\right). \tag{2.9}$$

First, we observe that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}&=\frac{1-y^{w_1w_2}(e^t-1)^{w_1w_2}}{1-y^{w_2}(e^t-1)^{w_2}}=\sum_{i=0}^{w_1-1}y^{w_2i}(e^t-1)^{w_2i}\\&=\sum_{i=0}^{w_1-1}y^{w_2i}\sum_{l=0}^{w_2i}\binom{w_2i}{l}(-1)^{w_2i-l}e^{lt}\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{w_1-1}y^{w_2i}\Delta^{w_2i}0^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

From (2.9) and (2.10), we can derive the following equation.

$$\begin{aligned}I&=\left(\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)\right)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}\right)\\&=\left(\sum_{m=0}^{\infty}2F_{m,w_1}(y)\frac{t^m}{m!}\right)\times\left(\sum_{k=0}^{\infty}(\sum_{i=0}^{w_1-1}y^{w_2i}\Delta^{w_2i}0^k\right)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}\sum_{i=0}^{w_1-1}y^{w_2i}\Delta^{w_2i}0^kF_{n-k,w_1}(y)\binom{n}{k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.11}$$

On the other hand, by (2.8), we get

$$I=\left(\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)\right)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}\right). \tag{2.12}$$

We note that

$$\begin{aligned}\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}&=\frac{1-y^{w_1w_2}(e^t-1)^{w_1w_2}}{1-y^{w_1}(e^t-1)^{w_1}}=\sum_{i=0}^{w_2-1}y^{w_1i}(e^t-1)^{w_1i}\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{w_2-1}y^{w_1i}\Delta^{w_1i}0^n\right)\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

Thus, by (2.12) and (2.13), we get

$$\begin{aligned}I &= \left(\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)\right)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}\right)\\ &= \left(\sum_{m=0}^{\infty}2F_{m,w_2}(y)\frac{t^m}{m!}\right)\times\left(\sum_{k=0}^{\infty}(\sum_{i=0}^{w_2-1}y^{w_1i}\Delta^{w_1i}0^k\right)\frac{t^k}{k!}\\ &= \sum_{n=0}^{\infty}\left(2\sum_{k=0}^{n}\sum_{i=0}^{w_2-1}y^{w_1i}\Delta^{w_1i}0^kF_{n-k,w_2}(y)\binom{n}{k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

Therefore, by (2.11) and (2.14), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have*

$$\sum_{k=0}^{n}\sum_{i=0}^{w_1-1}\binom{n}{k}F_{n-k,w_1}(y)y^{w_2i}\Delta^{w_2i}0^k = \sum_{k=0}^{n}\sum_{i=0}^{w_2-1}\binom{n}{k}F_{n-k,w_2}(y)y^{w_1i}\Delta^{w_1i}0^k. \tag{2.15}$$

Remark. In particular, for $w_1 = 1$, we have

$$F_n(y) = \sum_{k=0}^{n}\sum_{i=0}^{w_2-1}\binom{n}{k}F_{n-k,w_2}(y)y^i\Delta^i0^k. \tag{2.16}$$

From the symmetry of the fermionic p-adic integral on $\mathbb{Z}_p$, we have

$$\begin{aligned}I &= \left(\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)\right)\times\left(\frac{\int_{\mathbb{Z}_p}(-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)}\right)\\ &= \left(\int_{\mathbb{Z}_p}(-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)\right)\times\left(\frac{1-y^{w_1w_2}(e^t-1)^{w_1w_2}}{1-y^{w_2}(e^t-1)^{w_2}}\right)\\ &= \left(\sum_{i=0}^{w_1-1}y^{w_2i}(e^t-1)^{w_2i}\right)\times\left(\frac{2}{1-y^{w_1}(e^t-1)^{w_1}}\right)\\ &= \sum_{i=0}^{w_1-1}\sum_{l=0}^{w_2i}y^{w_2i}(-1)^l\frac{2}{1-y^{w_1}(e^t-1)^{w_1}}e^{(w_2i-l)t}\\ &= 2\sum_{n=0}^{\infty}\left(\sum_{i=0}^{w_1-1}\sum_{l=0}^{w_2i}y^{w_2i}(-1)^lF_{n,w_1}(w_2i-l,y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.17}$$

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On the other hand, by (2.8), we get

$$\begin{aligned} I &= \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{\int_{\mathbb{Z}_p} (-y^{w_1}(e^t-1)^{w_1})^x d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} (-y^{w_1w_2}(e^t-1)^{w_1w_2})^x d\mu_{-1}(x)} \right) \\ &= \left(\int_{\mathbb{Z}_p} (-y^{w_2}(e^t-1)^{w_2})^x d\mu_{-1}(x) \right) \times \left(\frac{1-y^{w_1w_2}(e^t-1)^{w_1w_2}}{1-y^{w_1}(e^t-1)^{w_1}} \right) \\ &= \left(\sum_{i=0}^{w_2-1} y^{w_1 i}(e^t-1)^{w_1 i} \right) \times \left(\frac{2}{1-y^{w_2}(e^t-1)^{w_2}} \right) \\ &= \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} y^{w_1 i}(-1)^l \frac{2}{1-y^{w_2}(e^t-1)^{w_2}} e^{(w_1 i - l)t} \\ &= 2\sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} y^{w_1 i}(-1)^l F_{n,w_2}(w_1 i - l, y) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.2. *For* $w_1, w_2 \in \mathbb{N}, n \geq 0$, *we have*

$$\sum_{i=0}^{w_1-1} \sum_{l=0}^{w_2 i} y^{w_2 i}(-1)^l F_{n,w_1}(w_2 i - l, y) = \sum_{i=0}^{w_2-1} \sum_{l=0}^{w_1 i} y^{w_1 i}(-1)^l F_{n,w_2}(w_1 i - l, y). \tag{2.19}$$

Remark. In particular, if we take $w_1 = 1$, then by Theorem 2, we get

$$F_n(y) = \sum_{i=0}^{w_2-1} \sum_{l=0}^{i} y^i (-1)^l F_{n,w_2}(i-l, y). \tag{2.20}$$

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