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A hypothetical upper bound on the heights of the solutions of a Diophantine equation with a finite number of solutions

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Abstract

Let $f(1)=1$, and let $f(n+1)=2^{2^{f(n)}}$ for every positive integer n. We consider the following hypothesis: if a system $\mathcal{S} \subseteq \{x_i \cdot x_j = x_k: i,j,k \in \{1,\ldots,n\}\} \cup \{x_i+1=x_k: i,k \in \{1,\ldots,n\}\}$ has only finitely many solutions in non-negative integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(2n)$. We prove: (1) the hypothesis implies that there exists an algorithm which takes as input a Diophantine equation, returns an integer, and this integer is greater than the heights of integer (non-negative integer, positive integer, rational) solutions, if the solution set is finite; (2) the hypothesis implies that there exists an algorithm for listing the Diophantine equations with infinitely many solutions in non-negative integers; (3) the hypothesis implies that the question whether or not a given Diophantine equation has only finitely many rational solutions is decidable by a single query to an oracle that decides whether or not a given Diophantine equation has a rational solution; (4) the hypothesis implies that the question whether or not a given Diophantine equation has only finitely many integer solutions is decidable by a single query to an oracle that decides whether or not a given Diophantine equation has an integer solution; (5) the hypothesis implies that if a set $\mathcal{M} \subseteq \mathbb{N}$ has a finite-fold Diophantine representation, then $\mathcal{M}$ is computable.

Key words and phrases: computable upper bound on the heights of rational solutions, computable upper bound on the moduli of integer solutions, Diophantine equation with a finite number of solutions, finite-fold Diophantine representation, single query to an oracle that decides whether or not a given Diophantine equation has an integer solution, single query to an oracle that decides whether or not a given Diophantine equation has a rational solution.

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1 Introduction and basic lemmas

The height of a rational number $\frac{p}{q}$ is denoted by $h\left(\frac{p}{q}\right)$ and equals $\max(|p|,|q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple $(x_1,\ldots,x_n)$ is denoted by $h(x_1,\ldots,x_n)$ and equals $\max(h(x_1),\ldots,h(x_n))$. In this article, we present a hypothesis which positively solves the following two open problems:

Open Problem 1. *Is there an algorithm which takes as input a Diophantine equation, returns an integer, and this integer is greater than the moduli of integer (non-negative integer, positive integer) solutions, if the solution set is finite?*

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Open Problem 2. *Is there an algorithm which takes as input a Diophantine equation, returns an integer, and this integer is greater than the heights of rational solutions, if the solution set is finite?*

Lemma 1. *For every non-negative integers b and c, $b+1=c$ if and only if* $2^{2^b} \cdot 2^{2^b} = 2^{2^c}$.

2 A hypothesis on the arithmetic of non-negative integers

Let

$$G_n = \{x_i \cdot x_j = x_k : \ i,j,k \in \{1,\ldots,n\}\} \cup \{x_i + 1 = x_k : \ i,k \in \{1,\ldots,n\}\}$$

Let $f(1)=1$, and let $f(n+1) = 2^{2^{f(n)}}$ for every positive integer n. Let $\theta(1)=0$, and let $\theta(n+1) = 2^{2^{\theta(n)}}$ for every positive integer n.

Hypothesis 1. *If a system $\mathcal{S} \subseteq G_n$ has only finitely many solutions in non-negative integers $x_1,\ldots,x_n$, then each such solution $(x_1,\ldots,x_n)$ satisfies $x_1,\ldots,x_n \leqslant f(2n)$.*

Observations 1 and 2 justify Hypothesis 1.

Observation 1. *For every system $\mathcal{S} \subseteq G_n$ which involves all the variables $x_1,\ldots,x_n$, the following new system*

$$\left(\bigcup_{x_i \cdot x_j = x_k \in \mathcal{S}} \{x_i \cdot x_j = x_k\}\right) \cup \left\{2^{2^{x_k}} = y_k : \ k \in \{1,\ldots,n\}\right\} \cup \bigcup_{x_i+1=x_k \in \mathcal{S}} \{y_i \cdot y_i = y_k\}$$

is equivalent to $\mathcal{S}$. If the system $\mathcal{S}$ has only finitely many solutions in non-negative integers $x_1,\ldots,x_n$, then the new system has only finitely many solutions in non-negative integers $x_1,\ldots,x_n,y_1,\ldots,y_n$.

Proof. It follows from Lemma 1. □

Observation 2. *For every positive integer n, the following system*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1,\ldots,n-1\} \ 2^{2^{x_i}} = x_{i+1} \ (\text{if } n>1) \end{cases}$$

has exactly two solutions in non-negative integers, namely $(\theta(1),\ldots,\theta(n))$ and $(f(1),\ldots,f(n))$. The second solution has greater height.

Observations 1 and 2, in substantially changed forms, remain true for solutions in non-negative rationals, see [6].

3 Algebraic lemmas

Lemma 2. *(cf. [7, p. 100]) For every non-negative real numbers x,y,z, $x+y=z$ if and only if*

$$((z+1)x+1)((z+1)(y+1)+1) = (z+1)^2(x(y+1)+1)+1 \tag{1}$$

Proof. The left side of equation (1) minus the right side of equation (1) equals $(z+1)(x+y-z)$. □

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Let α, β, and γ denote variables.

Lemma 3. *In non-negative integers, the equation $x + y = z$ is equivalent to a system which consists of equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$.*

Proof. It follows from Lemma 2. □

Lemma 4. *Let $D(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$. Assume that $\deg(D, x_i) \geqslant 1$ for each $i \in \{1, \dots, p\}$. We can compute a positive integer $n > p$ and a system $\mathcal{T} \subseteq G_n$ which satisfies the following two conditions:*

`Condition 1.` *For every non-negative integers $\tilde{x}_1, \dots, \tilde{x}_p$,*

$$D(\tilde{x}_1, \dots, \tilde{x}_p) = 0 \Longleftrightarrow \exists \tilde{x}_{p+1}, \dots, \tilde{x}_n \in \mathbb{N}\ (\tilde{x}_1, \dots, \tilde{x}_p, \tilde{x}_{p+1}, \dots, \tilde{x}_n) \textit{ solves } \mathcal{T}$$

`Condition 2.` *If non-negative integers $\tilde{x}_1, \dots, \tilde{x}_p$ satisfy $D(\tilde{x}_1, \dots, \tilde{x}_p) = 0$, then there exists a unique tuple $(\tilde{x}_{p+1}, \dots, \tilde{x}_n) \in \mathbb{N}^{n-p}$ such that the tuple $(\tilde{x}_1, \dots, \tilde{x}_p, \tilde{x}_{p+1}, \dots, \tilde{x}_n)$ solves $\mathcal{T}$.*

Conditions 1 and 2 imply that the equation $D(x_1, \dots, x_p) = 0$ and the system $\mathcal{T}$ have the same number of solutions in non-negative integers.

Proof. We write down the polynomial $D(x_1, \dots, x_p)$ and replace each coefficient by the successor of its absolute value. Let $\widetilde{D}(x_1, \dots, x_p)$ denote the obtained polynomial. The polynomials $D(x_1, \dots, x_p) + \widetilde{D}(x_1, \dots, x_p)$ and $\widetilde{D}(x_1, \dots, x_p)$ have positive integer coefficients. The equation $D(x_1, \dots, x_p) = 0$ is equivalent to

$$D(x_1, \dots, x_p) + \widetilde{D}(x_1, \dots, x_p) + 1 = \widetilde{D}(x_1, \dots, x_p) + 1$$

There exist a positive integer a and a finite non-empty list A such that

$$D(x_1, \dots, x_p) + \widetilde{D}(x_1, \dots, x_p) + 1 = \Bigg(\bigg(\Big(\sum_{(i_1, j_1, \dots, i_k, j_k) \in A} x_{i_1}^{j_1} \cdot \ldots \cdot x_{i_k}^{j_k}\Big) \underbrace{+ 1\bigg) + \ldots\Bigg) + 1}_{a \text{ units}} \tag{2}$$

and all the numbers $k, i_1, j_1, \dots, i_k, j_k$ belong to $\mathbb{N} \setminus \{0\}$. There exist a positive integer b and a finite non-empty list B such that

$$\widetilde{D}(x_1, \dots, x_p) + 1 = \Bigg(\bigg(\Big(\sum_{(i_1, j_1, \dots, i_k, j_k) \in B} x_{i_1}^{j_1} \cdot \ldots \cdot x_{i_k}^{j_k}\Big) \underbrace{+ 1\bigg) + \ldots\Bigg) + 1}_{b \text{ units}} \tag{3}$$

and all the numbers $k, i_1, j_1, \dots, i_k, j_k$ belong to $\mathbb{N} \setminus \{0\}$. By Lemma 3, we can equivalently express the equality of the right sides of equations (2) and (3) using only equations of the forms $\alpha + 1 = \gamma$ and $\alpha \cdot \beta = \gamma$. Consequently, we can effectively find the system $\mathcal{T}$. □

Lemma 4 remains true for solutions in non-negative rationals, see [6].

4 Hypothetical upper bounds on the heights of the solutions

Theorem 1. *If we assume Hypothesis 1 and a Diophantine equation $D(x_1, \dots, x_p) = 0$ has only finitely many solutions in non-negative integers, then an upper bound for these solutions can be computed.*

Proof. It follows from Lemma 4. □

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Theorem 2. *If we assume Hypothesis 1 and a Diophantine equation $D(x_1, \ldots, x_p) = 0$ has only finitely many solutions in positive integers, then an upper bound for these solutions can be computed.*

Proof. We apply Theorem 1 to the equation $D(x_1 + 1, \ldots, x_p + 1) = 0$. Next, we increase the computed bound by 1. □

Theorem 3. *If we assume Hypothesis 1 and a Diophantine equation $D(x_1, \ldots, x_p) = 0$ has only finitely many integer solutions, then an upper bound for their moduli can be computed by applying Theorem 1 to the equation*

$$\prod_{(i_1, \ldots, i_p) \in \{1,2\}^p} D((-1)^{i_1} \cdot x_1, \ldots, (-1)^{i_p} \cdot x_p) = 0$$

Lemma 5. *([8, Corollary 2, p. 25]) If a and b are two relatively prime positive integers, then every integer $n > ab$ can be written in the form $n = ax + by$, where x, y are positive integers.*

Lemma 6. *For every non-negative integers c and d, the following system*

$$\begin{cases} cx + (d+1)y & = & (d+1)c + 1 \\ x + y + u & = & (d+1)c + 1 \end{cases} \tag{4}$$

has at most finitely many solutions in non-negative integers x, y, u. For every non-negative integers c and d, system (4) is solvable in non-negative integers x, y, u if and only if c and $d + 1$ are relatively prime.

Proof. The equality $x + y + u = (d + 1)c + 1$ implies that $x, y, u \leqslant (d + 1)c + 1$. Hence, at most finitely many non-negative integers x, y, u satisfy system (4). The equality

$$cx + (d+1)y = (d+1)c + 1 \tag{5}$$

gives $cx + (d + 1)(y - c) = 1$. Hence, the integers c and $d + 1$ are relatively prime. Conversely, assume that c and $d + 1$ are relatively prime. By this, if $c = 0$, then $d = 0$. In this case, system (4) has exactly one solution in non-negative integers, namely $\begin{cases} x & = & 0 \\ y & = & 1 \\ u & = & 0 \end{cases}$. If $c > 0$, then Lemma 5 implies that there exist positive integers x and y that satisfy equation (5). We set $u = (c - 1)x + dy$. Then,

$$x + y + u = x + y + (c-1)x + dy = cx + (d+1)y = (d+1)c + 1$$

□

Theorem 4. *Hypothesis 1 implies that there exists a computable upper bound on the heights of the rationals that solve a Diophantine equation with a finite number of solutions.*

Proof. Let $W(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, and let

$$\widehat{W}(x_1, \ldots, x_n) = \prod_{(i_1, \ldots, i_n) \in \{1,2\}^n} W((-1)^{i_1} \cdot x_1, \ldots, (-1)^{i_n} \cdot x_n)$$

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If the equation $W(x_1, \ldots, x_n) = 0$ has only finitely many solutions in rationals $x_1, \ldots, x_n$, then the equation $\widehat{W}(x_1, \ldots, x_n) = 0$ has only finitely many solutions in non-negative rationals $x_1, \ldots, x_n$. By Lemma 6, it means that the system

$$\begin{cases} \widehat{W}\left(\dfrac{y_1}{z_1+1}, \ldots, \dfrac{y_n}{z_n+1}\right) & = & 0 \\ \forall i \in \{1, \ldots, n\} \quad y_i s_i + (z_i+1)t_i & = & (z_i+1)y_i + 1 \\ \forall i \in \{1, \ldots, n\} \quad s_i + t_i + u_i & = & (z_i+1)y_i + 1 \end{cases} \tag{6}$$

has only finitely many solutions in non-negative integers $y_1, z_1, s_1, t_1, u_1, \ldots, y_n, z_n, s_n, t_n, u_n$. System (6) is equivalent to a single Diophantine equation. By Lemma 4, this equation is equivalent to a system of equations of the forms $\alpha \cdot \beta = \gamma$ and $\alpha + 1 = \gamma$. Next, we apply Theorem 1. □

5 Finite-fold Diophantine representations

The Davis-Putnam-Robinson-Matiyasevich theorem states that every recursively enumerable set $\mathcal{M} \subseteq \mathbb{N}^n$ has a Diophantine representation, that is

$$(a_1, \ldots, a_n) \in \mathcal{M} \Longleftrightarrow \exists x_1, \ldots, x_m \in \mathbb{N} \;\; W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0 \tag{R}$$

for some polynomial W with integer coefficients, see [3]. The polynomial W can be computed, if we know the Turing machine M such that, for all $(a_1, \ldots, a_n) \in \mathbb{N}^n$, M halts on $(a_1, \ldots, a_n)$ if and only if $(a_1, \ldots, a_n) \in \mathcal{M}$, see [3]. The representation (R) is said to be finite-fold, if for every $a_1, \ldots, a_n \in \mathbb{N}$ the equation $W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$ has only finitely many solutions $(x_1, \ldots, x_m) \in \mathbb{N}^m$. Yuri Matiyasevich conjectured that each recursively enumerable set $\mathcal{M} \subseteq \mathbb{N}^n$ has a finite-fold Diophantine representation, see [1, pp. 341–342], [4, p. 42], and [5, p. 745]. Currently, he seems agnostic on his conjecture, see [5, p. 749]. In [9, p. 581], the author explains why Matiyasevich's conjecture although widely known is less widely accepted. Matiyasevich's conjecture implies a negative answer to Open Problem 1, see [4, p. 42].

Lemma 7. *Let $W(x, x_1, \ldots, x_m) \in \mathbb{Z}[x, x_1, \ldots, x_m]$. We claim that the function*

$$\mathbb{N} \ni b \mapsto W(b, x_1, \ldots, x_m) \in \mathbb{Z}[x_1, \ldots, x_m]$$

is computable.

Theorem 5. *Hypothesis 1 implies that if a set $\mathcal{M} \subseteq \mathbb{N}$ has a finite-fold Diophantine representation, then $\mathcal{M}$ is computable.*

Proof. Let a set $\mathcal{M} \subseteq \mathbb{N}$ have a finite-fold Diophantine representation. It means that there exists a polynomial $W(x, x_1, \ldots, x_m)$ with integer coefficients such that

$$\forall b \in \mathbb{N} \left(b \in \mathcal{M} \Longleftrightarrow \exists x_1, \ldots, x_m \in \mathbb{N} \;\; W(b, x_1, \ldots, x_m) = 0 \right)$$

and for every $b \in \mathbb{N}$ the equation $W(b, x_1, \ldots, x_m) = 0$ has only finitely many solutions $(x_1, \ldots, x_m) \in \mathbb{N}^m$. By Lemma 7 and Theorem 1, there is a computable function $\xi \colon \mathbb{N} \to \mathbb{N}$ such that for each $b, x_1, \ldots, x_m \in \mathbb{N}$ the equality $W(b, x_1, \ldots, x_m) = 0$ implies $\max(x_1, \ldots, x_m) \leqslant \xi(b)$. Hence, we can decide whether or not a non-negative integer b belongs to $\mathcal{M}$ by checking whether or not the equation $W(b, x_1, \ldots, x_m) = 0$ has an integer solution in the box $[0, \xi(b)]^m$. □

Theorem 5 remains true if we change the bound $f(2n)$ in Hypothesis 1 to any other computable bound $\delta(n)$.

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6 Theorems on relative decidability

Conjecture. ([1, p. 372]). *There is no algorithm for listing the Diophantine equations with infinitely many solutions in non-negative integers.*

Theorem 6. *Hypothesis 1 implies that there exists an algorithm for listing the Diophantine equations with infinitely many solutions in non-negative integers.*

Proof. It follows from Theorem 1. □

Corollary 1. *Hypothesis 1 implies that $d(inf, \mathbb{N}) \leqslant 0'$, where $d(inf, \mathbb{N})$ denotes the Turing degree of the set of Diophantine equations with infinitely many solutions in non-negative integers.*

Lemma 8. *A Diophantine equation $D(x_1, \ldots, x_p) = 0$ is solvable in non-negative integers $x_1, \ldots, x_p$ if and only if the equation $D(x_1, \ldots, x_p) + 0 \cdot x_{p+1} = 0$ has infinitely many solutions in non-negative integers $x_1, \ldots, x_{p+1}$.*

Corollary 1, Lemma 8, and the Davis-Putnam-Robinson-Matiyasevich theorem imply the next theorem.

Theorem 7. *Hypothesis 1 implies that $d(inf, \mathbb{N}) = 0'$.*

Theorem 8. *([1, p. 372]) Matiyasevich's conjecture on finite-fold Diophantine representations implies that $d(inf, \mathbb{N}) = 0''$.*

By Theorem 4, Hypothesis 1 implies Hypothesis 2.

Hypothesis 2. *There exists an algorithm which takes as input a Diophantine equation $D(x_1, \ldots, x_p) = 0$ and returns an integer $b \geqslant 2$, where b is greater than the number of rational solutions, if the solution set is finite.*

Guess ([2, p. 16]). *The question whether or not a Diophantine equation has only finitely many rational solutions is decidable with an oracle for deciding whether or not a Diophantine equation has a rational solution.*

Originally, Minhyong Kim formulated the Guess as follows: for rational solutions, the finiteness problem is decidable relative to the existence problem.

Theorem 9. *([6]) Hypothesis 2 implies that the question whether or not a given Diophantine equation has only finitely many rational solutions is decidable by a single query to an oracle that decides whether or not a given Diophantine equation has a rational solution.*

Corollary 2. *([6]) Hypothesis 2 implies that the question whether or not a given Diophantine equation has only finitely many rational solutions is decidable by a single query to an oracle that decides whether or not a given Diophantine equation has an integer solution.*

By Theorem 3, Hypothesis 1 implies Hypothesis 3.

Hypothesis 3. *There exists an algorithm which takes as input a Diophantine equation $D(x_1, \ldots, x_p) = 0$ and returns an integer $r \geqslant 2$, where r is greater than the moduli of integer solutions, if the solution set is finite.*

Hypothesis 3 implies Hypothesis 4.

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Hypothesis 4. *There exists an algorithm which takes as input a Diophantine equation $D(x_1, \ldots, x_p) = 0$ and returns an integer $c \geqslant 2$, where c is greater than the number of integer solutions, if the solution set is finite.*

Theorem 10. *Hypothesis 4 implies Hypothesis 3.*

Proof. Assume that a Diophantine equation $D(x_1, \ldots, x_p) = 0$ has only finitely many integer solutions. Then, the equation

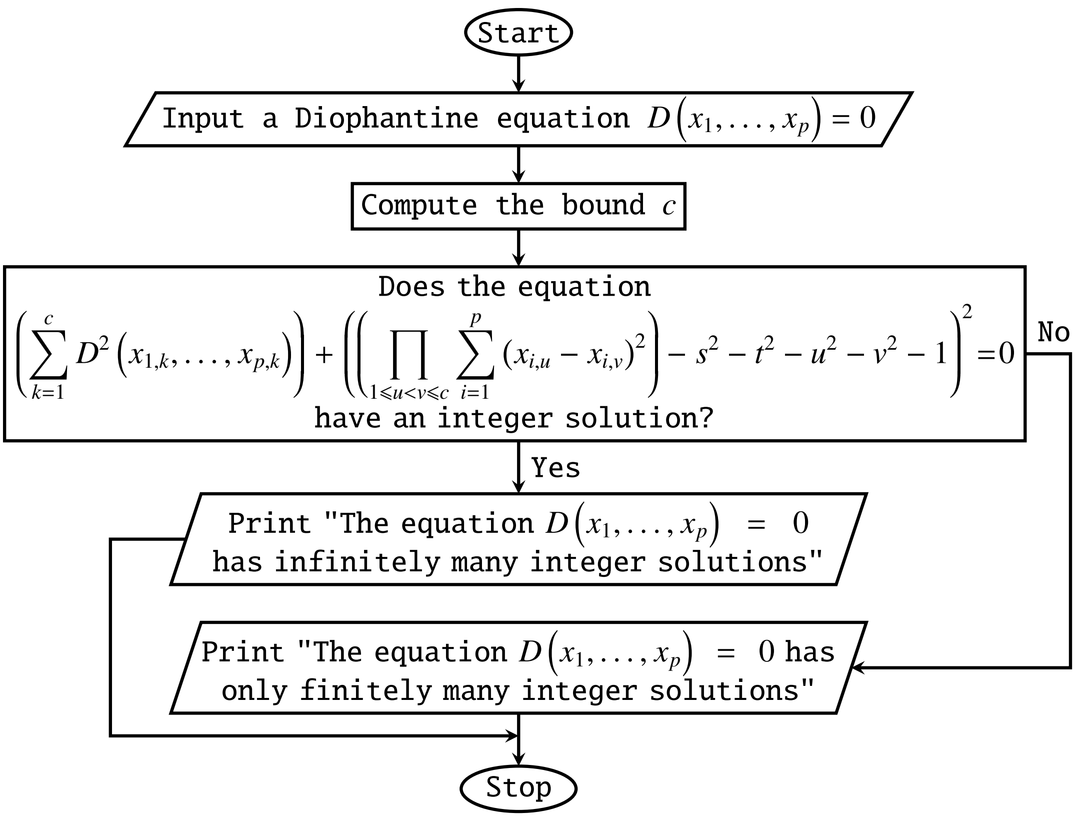
$$D^2\left(x_1, \ldots, x_p\right) + \left(\left(x_1^2 + \ldots + x_p^2\right) - \left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right) - \left(z_1^2 + z_2^2 + z_3^2 + z_4^2\right)\right)^2 = 0 \qquad (7)$$

has only finitely many integer solutions. By Lagrange's four-square theorem, every integer p-tuple $(a_1, \ldots, a_p)$ with $D(a_1, \ldots, a_p) = 0$ implies the existence of $a_1^2 + \ldots + a_p^2 + 1$ integer solutions of equation (7). The inequality $|a_1|, \ldots, |a_p| < a_1^2 + \ldots + a_p^2 + 1$ completes the proof. □

The conclusion of the next theorem contradicts the following conjecture of Yuri Matiyasevich ([2, p. 16]): *the finiteness problem for integral points is undecidable relative to the existence problem.*

Theorem 11. *Hypothesis 4 implies that the question whether or not a given Diophantine equation has only finitely many integer solutions is decidable by a single query to an oracle that decides whether or not a given Diophantine equation has an integer solution.*

Proof. Assuming that Hypothesis 4 holds, Lagrange's four-square theorem guarantees that the execution of the flowchart below decides whether or not a Diophantine equation $D(x_1, \ldots, x_p) = 0$ has only finitely many integer solutions.



□

Corollary 3. *Hypothesis 1 implies that* $d(fin, \mathbb{Z}) \leqslant 0'$*, where* $d(fin, \mathbb{Z})$ *denotes the Turing degree of the set of Diophantine equations which have only finitely many integer solutions.*

Lemma 9. *A Diophantine equation* $D(x_1, \ldots, x_p) = 0$ *is solvable in integers* $x_1, \ldots, x_p$ *if and only if the equation* $D(x_1, \ldots, x_p) + 0 \cdot x_{p+1} = 0$ *has infinitely many solutions in integers* $x_1, \ldots, x_{p+1}$.

Corollary 3, Lemma 9, and the Davis-Putnam-Robinson-Matiyasevich theorem imply the next theorem.

Theorem 12. *Hypothesis 1 implies that* $d(fin, \mathbb{Z}) = 0'$.

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