Article

A New Analytical Procedure to solve Two phase Flow in Tubes

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Abstract: A new formulation for a proposed solution to the 3D Navier-Stokes Equations in cylindrical co-ordinates coupled to the continuity and level set convection equation is presented in terms of an additive solution of the three principle directions in the radial, azimuthal and z directions of flow and a connection between the level set function and composite velocity vector for the additive solution is shown. For the case of a vertical tube configuration with small inclination angle, results are obtained for the level set function defining the interface in both the radial and azimuthal directions. It is found that the curvature dependent part of the problem alone induces sinusoidal azimuthal interfacial waves wheras when the curvature together with the equation for the composite velocity is considered oscillating radial interfacial waves occur. The implications of two extremes indicate the importance of looking at the full equations including curvature.

Keywords: Fluid dynamics, Two phase flow, Level set function, Cylindrical coordinates, Continuity equation

1. Introduction

The level set method, has been used originally as a numerical technique for tracking interfaces and shapes [3],[4] and has been increasingly applied to various areas of engineering and applied mathematics. In the level set method, contours or surfaces are represented as the zero level set of a higher dimensional function called a level set function. This can be the distance from the particular phase of material to the interface. For example in fracture mechanics level set methods have been used to track the shape around a crack in two and three dimensions that is propagating with a sharp kink [7].Also various applications in image segmentation have been used with corresponding active curve evolution algorithms [2],[6].Reachability analysis is frequently used to study the safety of control systems. Using exact reachability operators for nonlinear hybrid systems is presented in [9]. An algorithm for determining reachable sets and synthesizing control laws is implemented using level set methods in [9]. Various models used to compute the interaction of 3D incompressible fluids with elastic membranes or bodies, rely on the use of level set functions [10], to capture the fluid-solid interfaces and to measure elastic stresses that have been used. In [10] the computation of equilibrium shapes of biological vesicles is presented and numerical simulations of spontaneous cardiomoyocyte contractions is presented. A conservative method of level set type for moving interfaces in divergence free velocity fields is presented in [5], [8]. The method in [8] was coupled to a Navier-Stokes solver for incompressible two phase flow with surface tension. Wave phenomena is known to exist at the interface of two phase immiscible flows [11]. In the present paper we present a level set method for moving interfaces for such velocity fields which are coupled to Navier-Stokes equations for two phase flows in tubes.The novelty of the present work is to reveal an analytical approach in solving the 3D

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cylindrical Navier-Stokes equations where the three principle directions of flow, in radial, azimuthal and longitudinal directions are summed to form a new composite vector velocity expression. In this light we propose to solve a curvature only formulation of the governing equation for the level set function and one in which the curvature is added to the governing equation for the composite velocity expression.

1.1. Level Sets in Cylindrical Co-ordinates

Let ϕ be a level set function.[1],[3]. The gradient of the level set function in cylindrical co-ordinates is defined as:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right) \tag{1}$$

The mean curvature, κ, of the interface defined by the zero isocontour of the level set function ϕ, [1],[3], is the divergence of the normal to the interface given by

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} \tag{2}$$

Thus it can be expressed as:

$$\kappa = -\nabla\cdot\vec{n} \tag{3}$$

The mean curvature κ of a dynamic surface $\phi(r,\theta,z,t) = \psi(r,\theta,t) + z$ in cylindrical co-ordinates is,

$$\begin{aligned}\kappa = \frac{1}{S}\Bigg[&\left(\frac{\partial}{\partial r}\phi\left(r,\theta,z,t\right)\right)\left(r^2\left(\left(\frac{\partial}{\partial r}\phi\left(r,\theta,z,t\right)\right)^2+1\right)+\right.\\
&2\left(\frac{\partial}{\partial\theta}\phi\left(r,\theta,z,t\right)\right)\left(\frac{\partial}{\partial\theta}\phi\left(r,\theta,z,t\right)-r\frac{\partial^2}{\partial\theta\partial r}\phi\left(r,\theta,z,t\right)\right)\Bigg)+\\
&r\left(\left(\frac{\partial}{\partial\theta}\phi\left(r,\theta,z,t\right)\right)^2+r^2\right)\frac{\partial^2}{\partial r^2}\phi\left(r,\theta,z,t\right)+\\
&r\left(\frac{\partial^2}{\partial\theta^2}\phi\left(r,\theta,z,t\right)\right)\left(\left(\frac{\partial}{\partial r}\phi\left(r,\theta,z,t\right)\right)^2+1\right)\Bigg]\end{aligned} \tag{4}$$

where

$$S = \left(2\,r^2\left(\left(\frac{\partial}{\partial r}\phi\left(r,\theta,z,t\right)\right)^2+1\right)+2\left(\frac{\partial}{\partial\theta}\phi\left(r,\theta,z,t\right)\right)^2\right)^{3/2} \tag{5}$$

The geometric trace of a dynamic closed surface that bounds an open set can be represented implicitly as

$$S = \{(r,\theta,z,t) \mid \phi(r,\theta,z,t) = 0\} \tag{6}$$

and for two phase flow has two separate regions where $\phi > 0$ and $\phi < 0$ respectively.[1] The surface evolution is determined by:

$$\phi_t = \mathbf{u}\cdot\nabla\phi \tag{7}$$

The Navier–Stokes Equations are:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mu\nabla^2\mathbf{u} + \frac{1}{\rho}\mathbf{F}$$

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where for cylindrical (r, θ, z) coordinate system, Laplace operator has the form

$$\nabla^2 = \left(\underbrace{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right)$$

and the gradient is is given by Eq.(1)

1.2. A new composite velocity formulation

The 3D cylindrical incompressible unsteady Navier-Stokes equations coupled to interface convection equation are written in expanded form, for each component, ur,$u\theta$ and uz, where the remaining non linear terms appearing in full N.S equations are suppressed for the time being:

$$ur_t + ur\frac{\partial ur}{\partial r} + \frac{u\theta}{r}\frac{\partial ur}{\partial \theta} + uz\frac{\partial ur}{\partial z} - \frac{\mu}{\rho}\left(-\frac{ur}{r^2} + \frac{\partial^2 ur}{\partial r^2} + \frac{1}{r}\frac{\partial ur}{\partial r} + \frac{1}{r^2}\frac{\partial^2 ur}{\partial \theta^2} + \frac{\partial^2 ur}{\partial z^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial r} - Fg_r - \frac{1}{\rho}\sigma\kappa\frac{\partial \phi}{\partial r} = 0 \quad (8)$$

$$u\theta_t + ur\frac{\partial u\theta}{\partial r} + \frac{u\theta}{r}\frac{\partial u\theta}{\partial \theta} + uz\frac{\partial u\theta}{\partial z} - \frac{\mu}{\rho}\left(-\frac{u\theta}{r^2} + \frac{\partial^2 u\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u\theta}{\partial \theta^2} + \frac{\partial^2 u\theta}{\partial z^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial \theta} - Fg_\theta - \frac{1}{\rho}\sigma\kappa\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \quad (9)$$

$$uz_t + ur\frac{\partial uz}{\partial r} + \frac{u\theta}{r}\frac{\partial uz}{\partial \theta} + uz\frac{\partial uz}{\partial z} - \frac{\mu}{\rho}\left(\frac{\partial^2 uz}{\partial r^2} + \frac{1}{r}\frac{\partial uz}{\partial r} + \frac{1}{r^2}\frac{\partial^2 uz}{\partial \theta^2} + \frac{\partial^2 uz}{\partial z^2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial z} - Fg_z - \frac{1}{\rho}\sigma\kappa\frac{\partial \phi}{\partial z} = 0 \quad (10)$$

and where ρ is density, μ is dynamic viscosity ,Fg_r, Fg_θ, Fg_z are body forces on fluid and σ is surface interfacial tension.

Multiplying Eqs(8-10) by unit vectors $\vec{e_r}$, $\vec{e_\theta}$ and $\vec{k}$ respectively and adding Equations (8-10) gives the following equation, for $\vec{L} = ur\vec{e_r} + u\theta\vec{e_\theta} + uz\vec{k}$,

$$\vec{L}_t + ur\frac{\partial \vec{L}}{\partial r} + \frac{u\theta}{r}\frac{\partial \vec{L}}{\partial \theta} + uz\frac{\partial \vec{L}}{\partial z} - \frac{\mu}{\rho}\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\rho}P_T - F_T - \frac{1}{\rho}\sigma\kappa\overline{\overline{L}} = 0 \quad (11)$$

The level set function ϕ is governed by Eq(7), which when expanded becomes in cylindrical

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co-ordinates:

$$\phi_t + ur\frac{\partial \phi}{\partial r} + \frac{u\theta}{r}\frac{\partial \phi}{\partial \theta} + uz\frac{\partial \phi}{\partial z} = 0 \tag{12}$$

The continuity equation in cylindrical co-ordinates is

$$\frac{\partial \rho}{\partial t} + ur\frac{\partial \rho}{\partial r} + \frac{u\theta}{r}\frac{\partial \rho}{\partial \theta} + uz\frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial ur}{\partial r} + \frac{1}{r}\frac{\partial u\theta}{\partial \theta} + \frac{\partial uz}{\partial z}\right) \tag{13}$$

Multiply Eq(11) by $\frac{\rho}{\mu}\phi$:

$$\begin{gathered}\frac{\rho}{\mu}[\phi\vec{L}_t + \phi ur\frac{\partial \vec{L}}{\partial r} + \phi\frac{u\theta}{r}\frac{\partial \vec{L}}{\partial \theta} + \phi uz\frac{\partial \vec{L}}{\partial z} - \\ \frac{\mu}{\rho}\phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\rho}\phi P_T - \phi F_T - \frac{1}{\rho}\sigma\phi\kappa\vec{\vec{L}}] = 0\end{gathered} \tag{14}$$

Multiply Level set function convection Eq.(12) by $\frac{\rho}{\mu}\vec{L}$:

$$\frac{\rho}{\mu}[\vec{L}\phi_t + \vec{L}ur\frac{\partial \phi}{\partial r} + \vec{L}\frac{u\theta}{r}\frac{\partial \phi}{\partial \theta} + \vec{L}uz\frac{\partial \phi}{\partial z}] = 0 \tag{15}$$

Adding Eqs(14) and (15) gives

$$\begin{gathered}\frac{\rho}{\mu}[\phi\vec{L}_t + \vec{L}\phi_t + \phi ur\vec{L}_r + \vec{L}ur\phi_r + \phi\frac{u\theta}{r}\vec{L}_\theta + \vec{L}\frac{u\theta}{r}\phi_\theta + \phi uz\vec{L}_z + \vec{L}uz\phi_z - \\ \frac{\mu}{\rho}\phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\rho}\phi P_T - \phi F_T - \frac{1}{\rho}\sigma\phi\kappa\vec{\vec{L}}] = 0\end{gathered} \tag{16}$$

By product rule we rewrite the previous equation as:

$$\begin{gathered}\frac{\rho}{\mu}[(\phi\vec{L})_t + (\phi ur\vec{L})_r + \frac{1}{r}(\phi u\theta\vec{L})_\theta + (\phi uz\vec{L})_z] - \\ \phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\mu}\phi P_T - \frac{\rho}{\mu}\phi F_T - \frac{1}{\mu}\sigma\phi\kappa\vec{\vec{L}} = 0\end{gathered} \tag{17}$$

It is noted that since $\vec{L}$ is a composite velocity term it must have units of length per time,

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and since ϕ is usually taken to be a distance function from the interface we can consider the following expression in terms of ϕ :

$$\vec{L}\left(r,\theta,z,t\right)=\frac{1}{\phi}\frac{\mu}{\rho}\vec{\bar{L}} \tag{18}$$

where $\frac{\mu}{\rho}$ has SI units of $\frac{m^2}{s}$ and $\vec{\bar{L}} = \phi_r\vec{e_r} + \frac{1}{r}\phi_\theta\vec{e_z}\theta + \phi_z\vec{z}$ is by definition in Navier Stokes equation(Eqs(8-10)) of dimension $\frac{1}{m}$ and ϕ is dimensionless.

From Level set Eq(7),

$$\vec{L}\left(\vec{L}\cdot\vec{\bar{L}}\right)=-\vec{L}\left(\phi_t\right) \tag{19}$$

From the first line of Eq.(17) using Eq(18) we have a derivative term in t, first we write:

$$\frac{\mu}{\rho}\vec{\bar{L}}=\frac{\left(\mu_1+\left(\mu_2-\mu_1\right)\phi\left(r,\theta,z,t\right)\right)\vec{\bar{L}}\left(r,\theta,z,t\right)}{\rho_1+\left(\rho_2-\rho_1\right)\phi\left(r,\theta,z,t\right)} \tag{20}$$

where

$$\rho=\rho_1+(\rho_2-\rho_1)\phi,\ \mu=\mu_1+(\mu_2-\mu_1)\phi$$

and

$$\left(\frac{\mu}{\rho}\vec{\bar{L}}\right)_t=-\frac{\left(\frac{\partial}{\partial t}\phi\left(r,\theta,z,t\right)\right)\vec{\bar{L}}\left(r,\theta,z,t\right)\left(\left(\rho_2-\rho_1\right)\mu_1-\left(\mu_2-\mu_1\right)\rho_1\right)}{\left(\rho_1+\left(\rho_2-\rho_1\right)\phi\left(r,\theta,z,t\right)\right)^2}+$$
$$\frac{\left(\mu_1+\left(\mu_2-\mu_1\right)\phi\left(r,\theta,z,t\right)\right)\frac{\partial}{\partial t}\vec{\bar{L}}\left(r,\theta,z,t\right)}{\rho_1+\left(\rho_2-\rho_1\right)\phi\left(r,\theta,z,t\right)} \tag{21}$$

and for derivative terms in r,θ,z , from the first line of Eq.(17) we have,

$$\frac{N1}{D1}+\frac{N2}{D2}+\frac{N3}{D2}, \tag{22}$$

$$N1=\vec{\bar{L}}\left(r,\theta,z,t\right)\left(\left(\rho_2-\rho_1\right)\mu_1-\left(\mu_2-\mu_1\right)\rho_1\right)\times$$
$$\left(ur\left(\frac{\partial}{\partial r}\phi\left(r,\theta,z,t\right)\right)r+u\theta\frac{\partial}{\partial\theta}\phi\left(r,\theta,z,t\right)+uz\left(\frac{\partial}{\partial z}\phi\left(r,\theta,z,t\right)\right)r\right) \tag{23}$$

$$D1=r\left(\mu_1+\left(\mu_2-\mu_1\right)\phi\left(r,\theta,z,t\right)\right)^2) \tag{24}$$

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$$N2 = (\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t)) \times \left(ur\,\frac{\partial}{\partial r}\overline{\vec{L}}\,(r,\theta,z,t) + \frac{u\theta\,\frac{\partial}{\partial\theta}\overline{\vec{L}}\,(r,\theta,z,t)}{r} + uz\,\frac{\partial}{\partial z}\overline{\vec{L}}\,(r,\theta,z,t) \right) \tag{25}$$

$$N3 = (\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t)) \times \left(\overline{\vec{L}}\,\frac{\partial}{\partial r}ur\,(r,\theta,z,t) + \frac{\overline{\vec{L}}\,\frac{\partial}{\partial\theta}u\theta\,(r,\theta,z,t)}{r} + \overline{\vec{L}}\,\frac{\partial}{\partial z}uz\,(r,\theta,z,t) \right) \tag{26}$$

$$D2 = \rho_1 + (\rho_2 - \rho_1)\,\phi\,(r,\theta,z,t) \tag{27}$$

1.3. Special Case Solution

In this section it is assumed that the tube is in a vertical configuration with a small inclination angle and

$$(\rho_2 - \rho_1)\,\mu_1 - (\mu_2 - \mu_1)\,\rho_1 = 0 \tag{28}$$

Equation(26) can be rewritten using the time dependent continuity equation Eq(13),

$$-\rho \times \left(\overline{\vec{L}}\,\frac{\partial}{\partial r}ur\,(r,\theta,z,t) + \frac{\overline{\vec{L}}\,\frac{\partial}{\partial\theta}u\theta\,(r,\theta,z,t)}{r} + \overline{\vec{L}}\,\frac{\partial}{\partial z}uz\,(r,\theta,z,t) \right) = \overline{\vec{L}}\left(\frac{\partial\rho}{\partial t} + \vec{L}\cdot\nabla\rho \right) \tag{29}$$

$$\nabla\rho = \frac{1}{\gamma}\nabla\phi \tag{30}$$

Use of Eq.(17), Eqs.(22-30) gives a non-linear PDE

$$\frac{\rho}{\mu}(\phi\vec{L})_t + \frac{\rho}{\mu}\left(\vec{L}\cdot\nabla\overline{\vec{L}} + \frac{1}{\rho}\overline{\vec{L}}\frac{\partial\rho}{\partial t} \right) + \frac{1}{\rho}\frac{\rho}{\gamma\mu}\overline{\vec{L}}\vec{L}\cdot\nabla\phi - \phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2\vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial\vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\vec{L}}{\partial\theta^2} + \frac{\partial^2\vec{L}}{\partial z^2} \right) + \frac{1}{\mu}\phi\vec{P}_T - \frac{\rho}{\mu}\phi\vec{F}_T - \frac{1}{\mu}\sigma\phi\kappa\overline{\vec{L}} = 0 \tag{31}$$

$$\phi\vec{L}_t - \phi\left(\frac{\partial^2\vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial\vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\vec{L}}{\partial\theta^2} + \frac{\partial^2\vec{L}}{\partial z^2} \right) + \frac{1}{\mu}\phi\vec{P}_T - \frac{\rho}{\mu}\phi\vec{F}_T - \frac{1}{\mu}\sigma\phi\kappa\overline{\vec{L}} + \vec{\Psi}(r,ur,u\theta,\frac{\partial ur}{\partial\theta},\frac{\partial u\theta}{\partial\theta}) = 0 \tag{32}$$

A solution of the algebraic equation (28) gives as one special solution $\mu_1 = 0$ and $\rho_1 = 0$, with μ_2 and ρ_2 general expressions. Using Eq.(4) and incorporating the curvature κ, and using Eq(18) and dot

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product in Eq(19), with Ψ defined as the numerator of κ appearing in Eq.(4), Eq.(31) reduces to

$$Fz + \frac{\frac{\partial}{\partial t}\vec{L}(r,\theta,t)}{\phi(r,\theta,t)} - \frac{\vec{L}(r,\theta,t)}{\phi(r,\theta,t)} \frac{\Psi}{\left(-\frac{\frac{\partial}{\partial t}\phi(r,\theta,t)}{\phi(r,\theta,t)^{-1}}\right)^{3/2}} + \frac{\vec{L}(r,\theta,t)}{r^2\phi(r,\theta,t)} - \frac{\frac{\partial^2}{\partial r^2}\vec{L}(r,\theta,t)}{\phi(r,\theta,t)} + 2\times$$
$$\frac{\left(\frac{\partial}{\partial r}\vec{L}(r,\theta,t)\right)\frac{\partial}{\partial r}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2} - 2\frac{\vec{L}(r,\theta,t)\left(\frac{\partial}{\partial r}\phi(r,\theta,t)\right)^2}{(\phi(r,\theta,t))^3} + \frac{\vec{L}(r,\theta,t)\frac{\partial^2}{\partial r^2}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2} -$$
$$\frac{1}{r}\left(\frac{\frac{\partial}{\partial r}\vec{L}(r,\theta,t)}{\phi(r,\theta,t)} - \frac{\vec{L}(r,\theta,t)\frac{\partial}{\partial r}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2}\right) -$$
$$\frac{1}{r^2}\left(\frac{\frac{\partial^2}{\partial\theta^2}\vec{L}(r,\theta,t)}{\phi(r,\theta,t)} - 2\frac{\left(\frac{\partial}{\partial\theta}\vec{L}(r,\theta,t)\right)\frac{\partial}{\partial\theta}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2} + 2\frac{\vec{L}(r,\theta,t)\left(\frac{\partial}{\partial\theta}\phi(r,\theta,t)\right)^2}{(\phi(r,\theta,t))^3} - \frac{\vec{L}(r,\theta,t)\frac{\partial^2}{\partial\theta^2}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2}\right) +$$
$$\frac{\vec{L}(r,\theta,t)}{(\phi(r,\theta,t))^2}\left(\frac{\partial}{\partial r}\vec{L}(r,\theta,t) + \frac{\frac{\partial}{\partial\theta}\vec{L}(r,\theta,t)}{r}\right) = 0 \tag{33}$$

64 ,Fz is force of gravity in inclined tube.

65 1.3.1. Curvature Term with small Angle of Inclination

In this first part we solve Eq.(33) with the curvature part Ψ using the assumption of small inclination of tube. Using Eq.(4) we have that the curvature part of Eq.(33) is

$$\frac{\vec{L}(r,\theta,t)}{\phi(r,\theta,t)} \frac{\Psi}{\left(-\frac{\frac{\partial}{\partial t}\phi(r,\theta,t)}{\phi(r,\theta,t)^{-1}}\right)^{3/2}} = \frac{\left(\frac{\partial}{\partial r}\phi(r,\theta,t)\right)^2}{r\phi(r,\theta,t)\sqrt{-\left(\frac{\partial}{\partial t}\phi(r,\theta,t)\right)\phi(r,\theta,t)}}$$

where we have subsititued $\vec{L} = \nabla\phi$ and then set $\phi = e^{\alpha t}F(r,\theta)$ for small values of α, where $\alpha<0$. The time dependent term $e^{\alpha t}$ cancels leaving only α. With curvature term it can be proven that $F(r,\theta)$ is multiplicatively separable, $F(r,\theta) = f(r)g(\theta)$ and we obtain the following,

$$\frac{d^3}{dr^3}f(r) = -\frac{\left(\frac{d}{dr}f(r)\right)^2 c_1}{f(r)\,r} +$$
$$\frac{Fz\,r^2(f(r))^3 + 4\left(\frac{d^2}{dr^2}f(r)\right)r^2\left(\frac{d}{dr}f(r)\right)f(r) - 2\left(\frac{d}{dr}f(r)\right)^3 r^2 - (f(r))^2\left(\frac{d^2}{dr^2}f(r)\right)r + (f(r))^2\frac{d}{dr}f(r)}{(f(r))^2 r^2} \tag{34}$$

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$$\frac{d}{d\theta}g(\theta) = -g(\theta)c_1 - g(\theta) \tag{35}$$

Setting $f(r) = \exp(G(r))$ and $G(r) = \ln(H(r))$ in Eq.(34) and factoring exponential terms out we

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obtain,

$$2\frac{\left(\frac{d}{dr}H(r)\right)^3}{(H(r))^3}-4\frac{\left(\frac{d^2}{dr^2}H(r)\right)\frac{d}{dr}H(r)}{(H(r))^2}+\frac{\frac{d^3}{dr^3}H(r)}{H(r)}+\frac{\frac{d^2}{dr^2}H(r)}{rH(r)}+\frac{\left(\frac{d}{dr}H(r)\right)^2c_1}{r(H(r))^2}-\frac{\frac{d}{dr}H(r)}{r^2H(r)}-Fz=0 \tag{36}$$

Let $\Omega(r) = \frac{H'(r)}{H(r)}$ and from Eq.(36) the following linear differential equation emerges,where $H(r) = Y(r)$,

$$\frac{d^2}{dr^2}Y(r)-\frac{\left(4(\Omega(r))^2r^2-2\left(\frac{d}{dr}\Omega(r)\right)r^2-\Omega(r)r\right)\frac{d}{dr}Y(r)}{\Omega(r)r^2}-\frac{\left(-2(\Omega(r))^3r^2+Fzr^2-(\Omega(r))^2c_1r+4\Omega(r)\left(\frac{d}{dr}\Omega(r)\right)r^2-\left(\frac{d^2}{dr^2}\Omega(r)\right)r^2-\left(\frac{d}{dr}\Omega(r)\right)r+\Omega(r)\right)Y(r)}{\Omega(r)r^2}=0 \tag{37}$$

If Eq.(37) has closed form solutions then the pseudo-exact form, [13], of Eq.(37) has solutions which are given by,

$$\frac{2(\Omega(r))^3r-Fzr+(\Omega(r))^2c_1+4\left(\frac{d}{dr}\Omega(r)\right)r\Omega(r)}{r}=1 \tag{38}$$

Equation(38) can be rewritten as an Abel equation of the second kind which can be transformed into an Abel equation of the first kind

$$\frac{d}{dr}y(r)=-1/2+(B/4+1/4)(y(r))^3-1/4\frac{y(r)}{r} \tag{39}$$

where force of gravity in z direction is,

$$B = Fz \tag{40}$$

and

$$\Omega(r)=\frac{1}{y(r)} \tag{41}$$

Equation (39) solved due to [12] can be written as follows,

$$\frac{d}{dr}u(r)=-\frac{2^{2/3}(u(r))^3}{r}+\frac{2^{2/3}u(r)\beta}{r}+1/2\frac{2^{2/3}u(r)}{r^2}-\frac{2^{2/3}}{r}-\frac{u(r)}{r^2}\sqrt[3]{-(B+1)^{-1}} \tag{42}$$

where the transformation $y(r) = -\frac{2}{B+1}u(r)$, for chosen $\beta = \beta_1 + i\beta_2$ is used. If $B > -1$ then we have complex equation and can be written as a real and imaginary part, $F1(r) = \mathrm{Re}(u(r))$ and $F2(r) = \mathrm{Im}(u(r))$. Defining $A = -i\sqrt[3]{-(Fz+1)^{-1}} = A_1 + iA_2$, see Figs (1-5) for $\mathrm{Re}(A)$ ranging from low to high values corresponding to various heights in the inclined tube. Here two phases flow upward in tube (see Fig.1) with greater mass associated with higher positions of fluid column in tube.

1.3.2. Consideration of Curvature Alone in Equation (33)

Secondly we solve the curvature pde given in Eq. (4) alone appearing as Ψ in Eq.(33). For small inclination angle of tube Eq.(4) reduces to,

$$-\left(\frac{\partial}{\partial r}\phi\left(r,\theta,t\right)\right)r^{2}\left(\frac{\partial}{\partial t}\phi\left(r,\theta,t\right)\right)\phi\left(r,\theta,t\right)+r^{3}\frac{\partial^{2}}{\partial r^{2}}\phi\left(r,\theta,t\right)-$$
$$r\left(\frac{\partial^{2}}{\partial\theta^{2}}\phi\left(r,\theta,t\right)\right)\left(\frac{\partial}{\partial t}\phi\left(r,\theta,t\right)\right)\phi\left(r,\theta,t\right)=0 \tag{43}$$

which is separable into,

$$\frac{\mathrm{d}}{\mathrm{d}r}f1\left(r\right)=\frac{c_{1}f1\left(r\right)}{r} \tag{44}$$

$$\frac{\mathrm{d}^{2}}{\mathrm{d}\theta^{2}}f2\left(\theta\right)=c_{2}f2\left(\theta\right) \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f3\left(t\right)=\frac{c_{3}}{f3\left(t\right)} \tag{46}$$

where c_3 is sufficiently large. For constant c_2 negative there is a sinusoidal component of the azimuthal part of $\phi(r,\theta,t)=f1(r)f2(\theta)f3(t)$.

2. Discussion

It is worthy to note that the interfacial oscillations occurring as extremes of two problems one for curvature coupled to composite velocity and the other for curvature alone presents the daunting problem of solving the full equation of Eq.(33) without the assumption of small inclination angle. There is a plethora of results for computational multiphase flow using level set methods. The advantage of the present work lies in that analytical results are possible for the two extreme cases presented. It is conjectured at this point that the combination or full Eq(33) without the small inclination angle, ie approaching a horizontal tube configuration of flow, is non separable due to the inherent complexity of Eq(4) and Eq(33) combined. We can expect that there be a very complex relationship between azimuthal and radial components of $\vec{L}$. Work on the complete problem for this complex relationship is in progress for future studies.

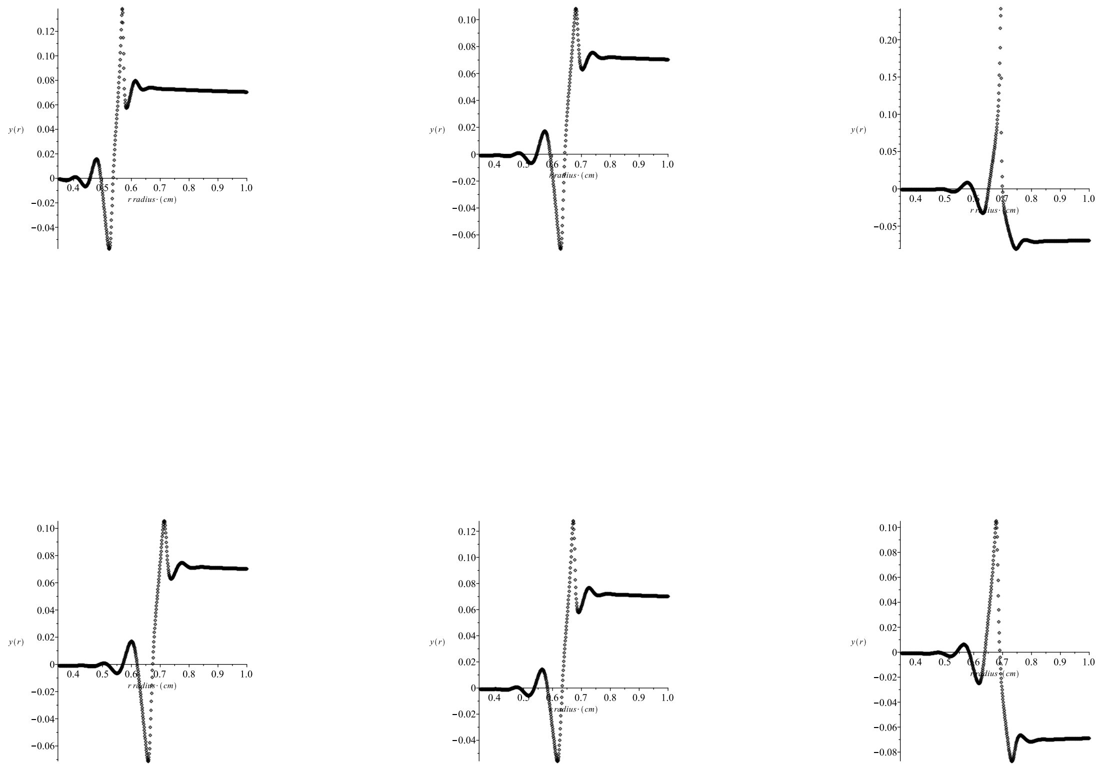

Figure 1. $y(r) = F_2(r)$, $\alpha = 7.7$, $\beta = 18.85$, $A_2 = -3.58$,
Left to right , top to bottom windows , $A_1 = 2.0024, 2.0048, 2.0072, 2.0106, 2.0130, 2.0154$

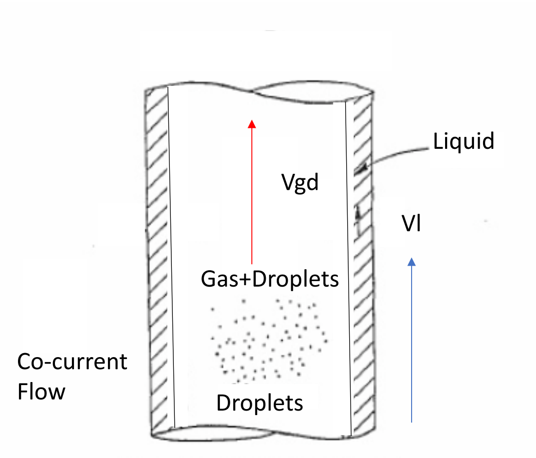

Figure 2. Two Phase Upward Co-current Annular Flow

References

1. S. Osher, R. Fedkiw, Level Set Methods and Dynamic Implicit Surfaces, Springer-Verlag, Berlin, 2003.
2. V. Caselles, F. Catte, T. Coll, and F. Dibos, A geometric model for active contours in image processing, Numer. Math., vol. 66, no. 1, pp. 1-31, Dec. 1993
3. S.Osher and J.Sethian, Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations, J. Comp. Phys., vol. 79, no. 1, pp. 12-49, Nov. 1988.
4. J. Sethian, Level Set Methods and Fast Marching Methods, Cambridge University Press, Cambridge, 1999.
5. M. Sussman, P. Smereka, S. Osher, A level set approach for computing solutions to incompressible two-phase flow, J. Comput. Phys. 114 pp 146-159, 1994.
6. Chunming Li, Rui Huang, Zhaohua Ding, J. Chris Gatenby, Dimitris N. Metaxas, A Level Set Method for Image Segmentation in the Presence of Intensity Inhomogeneities With Application to MRI ,IEEE Transactions on Image Processing, 20, no. 7, July 2011.
7. Marc Duflot , A study of the representation of cracks with level sets, International Journal for Numerical Methods in Engineering Volume 70, Issue 11, pp 1261-1302, 11 June 2007
8. Elin Olsson , Gunilla Kreiss ,A conservative level set method for two phase flow, Journal of Computational Physics 210 (2005) 225-246.
9. Ian Mitchell,Claire J. Tomlin,Level Set Methods for Computation in Hybrid Systems, International Workshop on Hybrid Systems: Computation and Control HSCC 2000: Hybrid Systems: Computation and Control pp 310-323 |
10. Emmanuel Maitre Thomas Milcenta Georges-Henri Cotteta Annie Raoult Yves Usson , Applications of level set methods in computational biophysics, Mathematical and Computer Modelling, Volume 49, Issues 11-12, June 2009, pp 2161-2169
11. A. A. Ayati, P. S. C. Farias, L. F. A. Azevedo, and I. B. de Paula, Characterization of linear interfacial waves in a turbulent gas-liquid pipe flow, Physics of Fluids 29, 062106, pp 1-13, 2017.
12. Differentialgleichungen Losungsmethoden und Losungen, E. Kamke, Springer Fachmedien Wiesbaden, 1977.

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13. M. Saravi, A procedure for solving some second-order linear ordinary differential equations, Applied Mathematics Letters 25, pp 408-411, 2012.

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