A New Analytical Procedure to solve Two phase Flow in Tubes

Terry Moschandreou [1,2,*,†] (iD)

1   Fanshawe College; tmoschandreou@fanshawec.ca
2   Western University; tmoschan@uwo.ca
*   Correspondence: tmoschandreou@fanshawec.ca; Tel.: 1-519-452-4430
†   1001 Fanshawe College Blvd,London Ontario, Canada

Abstract: A new formulation for a proposed solution to the 3D Navier-Stokes Equations in cylindrical co-ordinates coupled to the continuity and level set convection equation is presented in terms of an additive solution of the three principle directions in the radial, azimuthal and z directions of flow and a connection between the level set function and composite velocity vector for the additive solution is shown. For the case of a vertical tube configuration with small inclination angle, results are obtained for the level set function defining the interface in both the radial and azimuthal directions. It is found that the curvature dependent part of the problem alone induces sinusoidal azimuthal interfacial waves wheras when the curvature is small oscillating radial interfacial waves occur. The implications of two extremes indicate the importance of looking at the full equations including curvature.

Keywords: Fluid dynamics, Two phase flow, Level set function, Cylindrical coordinates, Continuity equation

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1. Introduction

The level set method, has been used originally as a numerical technique for tracking interfaces and shapes [3],[4] and has been increasingly applied to various areas of engineering and applied mathematics. In the level set method, contours or surfaces are represented as the zero level set of a higher dimensional function called a level set function. This can be the distance from the particular phase of material to the interface. For example in fracture mechanics level set methods have been used to track the shape around a crack in two and three dimensions that is propagating with a sharp kink [7].Also various applications in image segmentation have been used with corresponding active curve evolution algorithms [2],[6].Reachability analysis is frequently used to study the safety of control systems. Using exact reachability operators for nonlinear hybrid systems is presented in [9]. An algorithm for determining reachable sets and synthesizing control laws is implemented using level set methods in [9]. Various models used to compute the interaction of 3D incompressible fluids with elastic membranes or bodies, rely on the use of level set functions [10], to capture the fluid-solid interfaces and to measure elastic stresses that have been used. In [10] the computation of equilibrium shapes of biological vesicles is presented and numerical simulations of spontaneous cardiomoyocyte contractions is presented. A conservative method of level set type for moving interfaces in divergence free velocity fields is presented in [5], [8]. The method in [8] was coupled to a Navier-Stokes solver for incompressible two phase flow with surface tension. Wave phenomena is known to exist at the interface of two phase immiscible flows [11]. In the present paper we present a level set method for moving interfaces for such velocity fields which are coupled to Navier-Stokes equations for two phase flows in tubes.The novelty of the present work is to reveal an analytical approach in solving the 3D cylindrical Navier-Stokes equations where the three principle directions of flow, in radial, azimuthal and longitudinal directions are summed to form a new composite vector velocity expression. In this light we propose to solve a curvature only formulation of the governing equation for the level set function and one in which the curvature is removed from the governing equation.

1.1. Level Sets in Cylindrical Co-ordinates

Let ϕ be a level set function.[1],[3]. The gradient of the level set function in cylindrical co-ordinates is defined as:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z} \right) \tag{1}$$

The mean curvature, κ, of the interface defined by the zero isocontour of the level set function ϕ, [1],[3], is the divergence of the normal to the interface given by

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} \tag{2}$$

Thus it can be expressed as:

$$\kappa = -\nabla\cdot\vec{n} \tag{3}$$

The mean curvature κ of a dynamic surface $\phi(r,\theta,z,t) = \psi(r,\theta,t) + z$ in cylindrical co-ordinates is,

$$
\begin{aligned}
\kappa = \frac{1}{S}\Bigg[&\left(\frac{\partial}{\partial r}\phi\,(r,\theta,z,t)\right)\left(r^2\left(\left(\frac{\partial}{\partial r}\phi\,(r,\theta,z,t)\right)^2 + 1\right) + \right. \\
&2\left(\frac{\partial}{\partial\theta}\phi\,(r,\theta,z,t)\right)\left(\frac{\partial}{\partial\theta}\phi\,(r,\theta,z,t) - r\frac{\partial^2}{\partial\theta\partial r}\phi\,(r,\theta,z,t)\right)\Bigg) + \\
&r\left(\left(\frac{\partial}{\partial\theta}\phi\,(r,\theta,z,t)\right)^2 + r^2\right)\frac{\partial^2}{\partial r^2}\phi\,(r,\theta,z,t) + \\
&r\left(\frac{\partial^2}{\partial\theta^2}\phi\,(r,\theta,z,t)\right)\left(\left(\frac{\partial}{\partial r}\phi\,(r,\theta,z,t)\right)^2 + 1\right)\Bigg]
\end{aligned}
\tag{4}
$$

where

$$S = \left(2r^2\left(\left(\frac{\partial}{\partial r}\phi\,(r,\theta,z,t)\right)^2 + 1\right) + 2\left(\frac{\partial}{\partial\theta}\phi\,(r,\theta,z,t)\right)^2 \right)^{3/2} \tag{5}$$

The geometric trace of a dynamic closed surface that bounds an open set can be represented implicitly as

$$S = \{(r,\theta,z,t) \mid \phi(r,\theta,z,t) = 0\} \tag{6}$$

and for two phase flow has two separate regions where $\phi > 0$ and $\phi < 0$ respectively.[1] The surface evolution is determined by:

$$\phi_t = \mathbf{u}\cdot\nabla\phi \tag{7}$$

The Navier–Stokes Equations are:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \mu\nabla^2\mathbf{u} + \frac{1}{\rho}\mathbf{F}$$

where for cylindrical (r, θ, z) coordinate system, Laplace operator has the form

$$\nabla^2 = \left(\underbrace{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right)$$

42 and the gradient is is given by Eq.(1)

43 *1.2. A new composite velocity formulation*

The 3D cylindrical incompressible unsteady Navier-Stokes equations coupled to interface convection equation are written in expanded form, for each component, ur, $u\theta$ and uz, where the remaining non linear terms appearing in full N.S equations are suppressed for the time being:

$$ur_t + ur\frac{\partial ur}{\partial r} + \frac{u\theta}{r}\frac{\partial ur}{\partial \theta} + uz\frac{\partial ur}{\partial z} - \frac{\mu}{\rho}\left(-\frac{ur}{r^2} + \frac{\partial^2 ur}{\partial r^2} + \frac{1}{r}\frac{\partial ur}{\partial r} + \frac{1}{r^2}\frac{\partial^2 ur}{\partial \theta^2} + \frac{\partial^2 ur}{\partial z^2} \right) +$$
$$\frac{1}{\rho}\frac{\partial p}{\partial r} - Fg_r - \frac{1}{\rho}\sigma\kappa\frac{\partial \phi}{\partial r} = 0 \tag{8}$$

$$u\theta_t + ur\frac{\partial u\theta}{\partial r} + \frac{u\theta}{r}\frac{\partial u\theta}{\partial \theta} + uz\frac{\partial u\theta}{\partial z} - \frac{\mu}{\rho}\left(-\frac{u\theta}{r^2} + \frac{\partial^2 u\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u\theta}{\partial \theta^2} + \frac{\partial^2 u\theta}{\partial z^2} \right) +$$
$$\frac{1}{\rho}\frac{\partial p}{\partial \theta} - Fg_\theta - \frac{1}{\rho}\sigma\kappa\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \tag{9}$$

$$uz_t + ur\frac{\partial uz}{\partial r} + \frac{u\theta}{r}\frac{\partial uz}{\partial \theta} + uz\frac{\partial uz}{\partial z} - \frac{\mu}{\rho}\left(\frac{\partial^2 uz}{\partial r^2} + \frac{1}{r}\frac{\partial uz}{\partial r} + \frac{1}{r^2}\frac{\partial^2 uz}{\partial \theta^2} + \frac{\partial^2 uz}{\partial z^2} \right) +$$
$$\frac{1}{\rho}\frac{\partial p}{\partial z} - Fg_z - \frac{1}{\rho}\sigma\kappa\frac{\partial \phi}{\partial z} = 0 \tag{10}$$

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45 and where ρ is density, μ is dynamic viscosity , Fg_r, Fg_θ, Fg_z are body forces on fluid and σ is surface
46 interfacial tension.
47
48 Multiplying Eqs(8-10) by unit vectors $\vec{e_r}$, $\vec{e_\theta}$ and $\vec{k}$ respectively and adding Equations (8-10)
49 gives the following equation, for $\vec{L} = ur\vec{e_r} + u\theta\vec{e_\theta} + uz\vec{k}$,

$$\vec{L}_t + ur\frac{\partial \vec{L}}{\partial r} + \frac{u\theta}{r}\frac{\partial \vec{L}}{\partial \theta} + uz\frac{\partial \vec{L}}{\partial z} - \frac{\mu}{\rho}\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2} \right) + \frac{1}{\rho}P_T - F_T - \frac{1}{\rho}\sigma\kappa\overline{\overline{L}} = 0 \tag{11}$$

The level set function ϕ is governed by Eq(7), which when expanded becomes in cylindrical

co-ordinates:

$$\phi_t + ur\frac{\partial \phi}{\partial r} + \frac{u\theta}{r}\frac{\partial \phi}{\partial \theta} + uz\frac{\partial \phi}{\partial z} = 0 \tag{12}$$

The continuity equation in cylindrical co-ordinates is

$$\frac{\partial \rho}{\partial t} + ur\frac{\partial \rho}{\partial r} + \frac{u\theta}{r}\frac{\partial \rho}{\partial \theta} + uz\frac{\partial \rho}{\partial z} = -\rho\left(\frac{\partial ur}{\partial r} + \frac{1}{r}\frac{\partial u\theta}{\partial \theta} + \frac{\partial uz}{\partial z}\right) \tag{13}$$

Multiply Eq(11) by $\frac{\rho}{\mu}\phi$:

$$\frac{\rho}{\mu}[\phi\vec{L}_t + \phi ur\frac{\partial \vec{L}}{\partial r} + \phi\frac{u\theta}{r}\frac{\partial \vec{L}}{\partial \theta} + \phi uz\frac{\partial \vec{L}}{\partial z} -$$
$$\frac{\mu}{\rho}\phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\rho}\phi P_T - \phi F_T - \frac{1}{\rho}\sigma\phi\kappa\vec{L}] = 0 \tag{14}$$

Multiply Level set function convection Eq.(12) by $\frac{\rho}{\mu}\vec{L}$:

$$\frac{\rho}{\mu}[\vec{L}\phi_t + \vec{L}ur\frac{\partial \phi}{\partial r} + \vec{L}\frac{u\theta}{r}\frac{\partial \phi}{\partial \theta} + \vec{L}uz\frac{\partial \phi}{\partial z}] = 0 \tag{15}$$

Adding Eqs(14) and (15) gives

$$\frac{\rho}{\mu}[\phi\vec{L}_t + \vec{L}\phi_t + \phi ur\vec{L}_r + \vec{L}ur\phi_r + \phi\frac{u\theta}{r}\vec{L}_\theta + \vec{L}\frac{u\theta}{r}\phi_\theta + \phi uz\vec{L}_z + \vec{L}uz\phi_z -$$
$$\frac{\mu}{\rho}\phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\rho}\phi P_T - \phi F_T - \frac{1}{\rho}\sigma\phi\kappa\vec{L}] = 0 \tag{16}$$

By product rule we rewrite the previous equation as:

$$\frac{\rho}{\mu}[(\phi\vec{L})_t + (\phi ur\vec{L})_r + \frac{1}{r}(\phi u\theta\vec{L})_\theta + (\phi uz\vec{L})_z] -$$
$$\phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2 \vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial \vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \vec{L}}{\partial \theta^2} + \frac{\partial^2 \vec{L}}{\partial z^2}\right) + \frac{1}{\mu}\phi P_T - \frac{\rho}{\mu}\phi F_T - \frac{1}{\mu}\sigma\phi\kappa\overline{L} = 0 \tag{17}$$

It is noted that since $\vec{L}$ is a composite velocity term it must have units of length per time,

and since ϕ is usually taken to be a distance function from the interface we can consider the following expression in terms of ϕ :

$$\vec{L}\,(r,\theta,z,t) = \frac{1}{\phi}\frac{\mu}{\rho}\overline{\vec{L}} \tag{18}$$

where $\frac{\mu}{\rho}$ has SI units of $\frac{m^2}{s}$ and $\overline{\vec{L}} = \phi_r\vec{e_r} + \frac{1}{r}\phi_\theta\vec{e_z}\theta + \phi_z\vec{z}$ is by definition in Navier Stokes equation(Eqs(8-10)) of dimension $\frac{1}{m}$ and ϕ is dimensionless.

From Level set Eq(7),

$$\vec{L}\left(\vec{L}\cdot\overline{\vec{L}}\right) = -\vec{L}\,(\phi_t) \tag{19}$$

From the first line of Eq.(17) using Eq(18) we have a derivative term in t, first we write:

$$\frac{\mu}{\rho}\overline{\vec{L}} = \frac{(\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t))\,\overline{\vec{L}}\,(r,\theta,z,t)}{\rho_1 + (\rho_2 - \rho_1)\,\phi\,(r,\theta,z,t)} \tag{20}$$

where

$$\rho = \rho_1 + (\rho_2 - \rho_1)\phi, \ \mu = \mu_1 + (\mu_2 - \mu_1)\phi$$

and

$$\left(\frac{\mu}{\rho}\overline{\vec{L}}\right)_t = -\frac{\left(\frac{\partial}{\partial t}\phi\,(r,\theta,z,t)\right)\overline{\vec{L}}\,(r,\theta,z,t)\,((\rho_2 - \rho_1)\,\mu_1 - (\mu_2 - \mu_1)\,\rho_1)}{(\rho_1 + (\rho_2 - \rho_1)\,\phi\,(r,\theta,z,t))^2} + \\ \frac{(\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t))\,\frac{\partial}{\partial t}\overline{\vec{L}}\,(r,\theta,z,t)}{\rho_1 + (\rho_2 - \rho_1)\,\phi\,(r,\theta,z,t)} \tag{21}$$

and for derivative terms in r,θ,z , from the first line of Eq.(17) we have,

$$\frac{N1}{D1} + \frac{N2}{D2} + \frac{N3}{D2}, \tag{22}$$

$$N1 = \overline{\vec{L}}\,(r,\theta,z,t)\,((\rho_2 - \rho_1)\,\mu_1 - (\mu_2 - \mu_1)\,\rho_1) \times \\ \left(ur\left(\frac{\partial}{\partial r}\phi\,(r,\theta,z,t)\right)r + u\theta\frac{\partial}{\partial\theta}\phi\,(r,\theta,z,t) + uz\left(\frac{\partial}{\partial z}\phi\,(r,\theta,z,t)\right)r\right) \tag{23}$$

$$D1 = r\,(\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t))^2) \tag{24}$$

$$N2 = (\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t)) \times$$
$$\left(ur\,\frac{\partial}{\partial r}\overline{\overline{L}}\,(r,\theta,z,t) + \frac{u\theta\,\frac{\partial}{\partial\theta}\overline{\overline{L}}\,(r,\theta,z,t)}{r} + uz\,\frac{\partial}{\partial z}\overline{\overline{L}}\,(r,\theta,z,t) \right) \tag{25}$$

$$N3 = (\mu_1 + (\mu_2 - \mu_1)\,\phi\,(r,\theta,z,t)) \times$$
$$\left(\overline{\overline{L}}\,\frac{\partial}{\partial r}ur\,(r,\theta,z,t) + \frac{\overline{\overline{L}}\,\frac{\partial}{\partial\theta}u\theta\,(r,\theta,z,t)}{r} + \vec{L}\,\frac{\partial}{\partial z}uz\,(r,\theta,z,t) \right) \tag{26}$$

$$D2 = \rho_1 + (\rho_2 - \rho_1)\,\phi\,(r,\theta,z,t) \tag{27}$$

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59   *1.3. Special Case Solution*

In this section it is assumed that the tube is in a vertical configuration with a small inclination angle and

$$(\rho_2 - \rho_1)\,\mu_1 - (\mu_2 - \mu_1)\,\rho_1 = 0 \tag{28}$$

60   Equation(26) can be rewritten using the time dependent continuity equation Eq(13),

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$$-\rho \times \left(\overline{\overline{L}}\,\frac{\partial}{\partial r}ur\,(r,\theta,z,t) + \frac{\overline{\overline{L}}\,\frac{\partial}{\partial\theta}u\theta\,(r,\theta,z,t)}{r} + \vec{L}\,\frac{\partial}{\partial z}uz\,(r,\theta,z,t) \right) = \overline{\overline{L}}\left(\frac{\partial\rho}{\partial t} + \vec{L}\cdot\nabla\rho \right) \tag{29}$$

$$\nabla\rho = \frac{1}{\gamma}\nabla\phi \tag{30}$$

Use of Eq.(17), Eqs.(22-30) gives a non-linear PDE

$$\frac{\rho}{\mu}(\phi\vec{L})_t + \frac{\rho}{\mu}\left(\vec{L}\cdot\nabla\vec{L} + \frac{1}{\rho}\overline{\overline{L}}\frac{\partial\rho}{\partial t} \right) + \frac{1}{\rho}\frac{\rho}{\gamma\mu}\overline{\overline{L}}\vec{L}\cdot\nabla\phi -$$
$$\phi\left(-\frac{\vec{L}}{r^2} + \frac{\partial^2\vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial\vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\vec{L}}{\partial\theta^2} + \frac{\partial^2\vec{L}}{\partial z^2} \right) + \frac{1}{\mu}\phi\vec{P}_T - \frac{\rho}{\mu}\phi\vec{F}_T - \frac{1}{\mu}\sigma\phi\kappa\overline{\overline{L}} = 0 \tag{31}$$

$$\phi\vec{L}_t - \phi\left(\frac{\partial^2\vec{L}}{\partial r^2} + \frac{1}{r}\frac{\partial\vec{L}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\vec{L}}{\partial\theta^2} + \frac{\partial^2\vec{L}}{\partial z^2} \right) + \frac{1}{\mu}\phi\vec{P}_T - \frac{\rho}{\mu}\phi\vec{F}_T - \frac{1}{\mu}\sigma\phi\kappa\overline{\overline{L}} +$$
$$\vec{\Psi}(r, ur, u\theta, \frac{\partial ur}{\partial\theta}, \frac{\partial u\theta}{\partial\theta}) = 0 \tag{32}$$

A solution of the algebraic equation (28) gives as one special solution $\mu_1 = 0$ and $\rho_1 = 0$, with μ_2 and ρ_2 general expressions. Using Eq.(4) and incorporating the curvature κ, and using Eq(18) and dot

product in Eq(19), with Ψ defined as the numerator of κ appearing in Eq.(4), Eq.(31) reduces to

$$Fz + \frac{\frac{\partial}{\partial t}\overline{\overline{L}}(r,\theta,t)}{\phi(r,\theta,t)} - \frac{\overline{\overline{L}}(r,\theta,t)}{\phi(r,\theta,t)} \frac{\Psi}{\left(-\frac{\frac{\partial}{\partial t}\phi(r,\theta,t)}{\phi(r,\theta,t)}\right)^{3/2}} + \frac{\overline{\overline{L}}(r,\theta,t)}{r^2\phi(r,\theta,t)} - \frac{\frac{\partial^2}{\partial r^2}\overline{\overline{L}}(r,\theta,t)}{\phi(r,\theta,t)} + 2\times$$

$$\frac{\left(\frac{\partial}{\partial r}\overline{\overline{L}}(r,\theta,t)\right)\frac{\partial}{\partial r}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2} - 2\frac{\overline{\overline{L}}(r,\theta,t)\left(\frac{\partial}{\partial r}\phi(r,\theta,t)\right)^2}{(\phi(r,\theta,t))^3} + \frac{\overline{\overline{L}}(r,\theta,t)\frac{\partial^2}{\partial r^2}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2} -$$

$$\frac{1}{r}\left(\frac{\frac{\partial}{\partial r}\overline{\overline{L}}(r,\theta,t)}{\phi(r,\theta,t)} - \frac{\overline{\overline{L}}(r,\theta,t)\frac{\partial}{\partial r}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2}\right) -$$

$$\frac{1}{r^2}\left(\frac{\frac{\partial^2}{\partial\theta^2}\overline{\overline{L}}(r,\theta,t)}{\phi(r,\theta,t)} - 2\frac{\left(\frac{\partial}{\partial\theta}\overline{\overline{L}}(r,\theta,t)\right)\frac{\partial}{\partial\theta}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2} + 2\frac{\overline{\overline{L}}(r,\theta,t)\left(\frac{\partial}{\partial\theta}\phi(r,\theta,t)\right)^2}{(\phi(r,\theta,t))^3} - \frac{\overline{\overline{L}}(r,\theta,t)\frac{\partial^2}{\partial\theta^2}\phi(r,\theta,t)}{(\phi(r,\theta,t))^2}\right) +$$

$$\frac{\overline{\overline{L}}(r,\theta,t)}{(\phi(r,\theta,t))^2}\left(\frac{\partial}{\partial r}\overline{\overline{L}}(r,\theta,t) + \frac{\frac{\partial}{\partial\theta}\overline{\overline{L}}(r,\theta,t)}{r}\right) = 0 \tag{33}$$

$,Fz$ is force of gravity in inclined tube.

1.3.1. Curvature Term Omitted

In this first part we solve Eq.(33) with the curvature part Ψ ignored. This is because we can subsititute $\overline{\overline{L}} = \nabla\phi$ and set $\phi = e^{\alpha t}F(r,\theta)$ for $\alpha < 0$ and t sufficiently large to cancel terms with α and t. With curvature term dropped it can be proven that $F(r,\theta)$ is multiplicatively separable, $F(r,\theta) = f(r)g(\theta)$ and we obtain the following,

$$\frac{d^3}{dr^3}f(r) = -\frac{\left(\frac{d}{dr}f(r)\right)^2 c_1}{f(r)r} +$$

$$\frac{Fz\,r^2(f(r))^3 + 4\left(\frac{d^2}{dr^2}f(r)\right)r^2\left(\frac{d}{dr}f(r)\right)f(r) - 2\left(\frac{d}{dr}f(r)\right)^3 r^2 - (f(r))^2\left(\frac{d^2}{dr^2}f(r)\right)r + (f(r))^2\frac{d}{dr}f(r)}{(f(r))^2 r^2}$$

$$\tag{34}$$

$$\frac{d}{d\theta}g(\theta) = -g(\theta)c_1 - g(\theta) \tag{35}$$

Setting $f(r) = \exp(G(r))$ and $G(r) = \ln(H(r))$ in Eq.(34) and factoring exponential terms out we obtain,

$$2\frac{\left(\frac{d}{dr}H(r)\right)^3}{(H(r))^3} - 4\frac{\left(\frac{d^2}{dr^2}H(r)\right)\frac{d}{dr}H(r)}{(H(r))^2} + \frac{\frac{d^3}{dr^3}H(r)}{H(r)} + \frac{\frac{d^2}{dr^2}H(r)}{rH(r)} + \frac{\left(\frac{d}{dr}H(r)\right)^2 c_1}{r(H(r))^2} -$$

$$\frac{\frac{d}{dr}H(r)}{r^2 H(r)} - Fz = 0 \tag{36}$$

Let $\Omega(r) = \frac{H'(r)}{H(r)}$ and from Eq.(36) the following linear differential equation emerges, where

$$H(r) = Y(r),$$

$$\frac{d^2}{dr^2}Y(r) - \frac{\left(4\,(\Omega(r))^2\,r^2 - 2\left(\frac{d}{dr}\Omega(r)\right)r^2 - \Omega(r)\,r\right)\frac{d}{dr}Y(r)}{\Omega(r)\,r^2} - $$

$$\frac{\left(-2\,(\Omega(r))^3\,r^2 + Fzr^2 - (\Omega(r))^2\,c_1 r + 4\,\Omega(r)\left(\frac{d}{dr}\Omega(r)\right)r^2 - \left(\frac{d^2}{dr^2}\Omega(r)\right)r^2 - \left(\frac{d}{dr}\Omega(r)\right)r + \Omega(r)\right)Y(r)}{\Omega(r)\,r^2} = 0$$

$$(37)$$

If Eq.(37) has closed form solutions then the pseudo-exact form, [13], of Eq.(37) has solutions which are given by,

$$\frac{2\,(\Omega(r))^3\,r - F\,zr + (\Omega(r))^2\,c_1 + 4\left(\frac{d}{dr}\Omega(r)\right)r\Omega(r)}{r} = 1 \tag{38}$$

Equation(38) can be rewritten as an Abel equation of the second kind which can be transformed into an Abel equation of the first kind

$$\frac{d}{dr}y(r) = -1/2 + (B/4 + 1/4)\,(y(r))^3 - 1/4\,\frac{y(r)}{r} \tag{39}$$

where force of gravity in z direction is,

$$B = Fz \tag{40}$$

and

$$\Omega(r) = \frac{1}{y(r)} \tag{41}$$

Equation (39) solved due to [12] can be written as follows,

$$\frac{d}{dr}u(r) = -\frac{2^{2/3}\,(u(r))^3}{r} + \frac{2^{2/3}u(r)\,\beta}{r} + 1/2\,\frac{2^{2/3}u(r)}{r^2} - \frac{2^{2/3}}{r} - \frac{u(r)}{r^2}\sqrt[3]{-(B+1)^{-1}} \tag{42}$$

where the transformation $y(r) = -\frac{2}{B+1}u(r)$, for chosen β is used. If $B > -1$ then we have complex equation and can be written as a real and imaginary part, $F1(r) = \text{Re}(u(r))$ and $F2(r) = \text{Im}(u(r))$. Defining $A = -i\sqrt[3]{-(Fz+1)^{-1}}$, see Figs (1-5) for $\text{Re}(A)$ ranging from low to high values corresponding to various heights in the inclined tube. Here two phases flow upward in tube with greater mass associated with higher positions of fluid column in tube.

1.3.2. Consideration of Curvature Alone

Secondly we solve the curvature pde given in Eq. (4) alone appearing as Ψ in Eq.(33). This is due to large time evolution of Eq(33). For small inclination angle of tube Eq.(4) reduces to,

$$-\frac{\left(\frac{\partial}{\partial r}\phi(r,\theta,t)\right)r^2\frac{\partial}{\partial t}\phi(r,\theta,t)}{\phi(r,\theta,t)} + \left(\frac{\partial^2}{\partial r^2}\phi(r,\theta,t)\right)r^3 - \frac{r\left(\frac{\partial^2}{\partial \theta^2}\phi(r,\theta,t)\right)\frac{\partial}{\partial t}\phi(r,\theta,t)}{\phi(r,\theta,t)} = 0 \tag{43}$$

which is separable into,

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f1\left(r\right) = \frac{f1\left(r\right)c_3 c_2}{r^2} + \frac{\left(\frac{\mathrm{d}}{\mathrm{d}r}f1\left(r\right)\right)c_3}{r} \tag{44}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}f2\left(\theta\right) = c_2 f2\left(\theta\right) \tag{45}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f3\left(t\right) = c_3 f3\left(t\right) \tag{46}$$

74 For constant c_2 negative there is a sinusoidal component of the azimuthal part of
75 $\phi(r, \theta, t) = f1(r)f2(\theta)f3(t)$.

76

2. Discussion

78   It is worthy to note that the interfacial oscillations occurring as extremes of two problems one
79 for no curvature and the other for curvature alone presents the daunting problem of solving the full
80 equation of Eq.(33). There is a plethora of results for computational multiphase flow using level set
81 methods. The advantage of the present work lies in that analytical results are possible for the two
82 extreme cases presented. It is conjectured at this point that the combination or full Eq(33) without the
83 small inclination angle, ie approaching a horizontal tube configuration of flow, is non separable due to
84 the inherent complexity of Eq(4) and Eq(33) combined. We can expect that there be a very complex
85 relationship between azimuthal and radial components of $\overline{\overline{L}}$. Work on the complete problem for this
86 complex relationship is in progress for future studies.
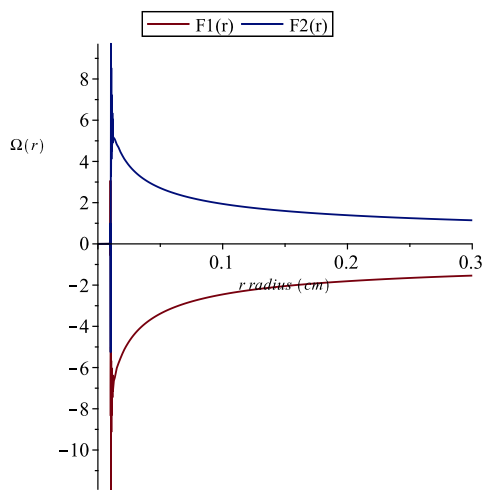
Figure 1. A=2.0 , mass of fluid column in vertical tube *m*=93.7 grams.
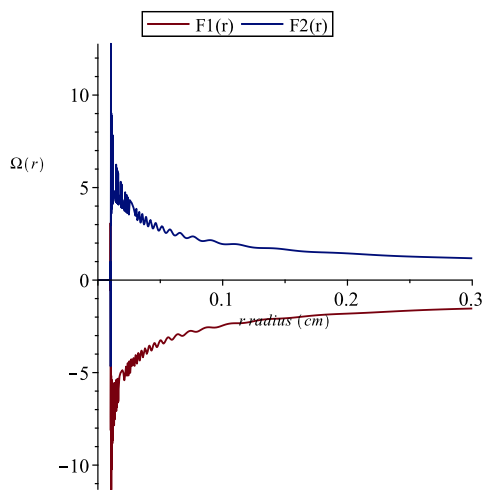
Figure 2. A=2.075, mass of fluid column in vertical tube m=94.5 grams.
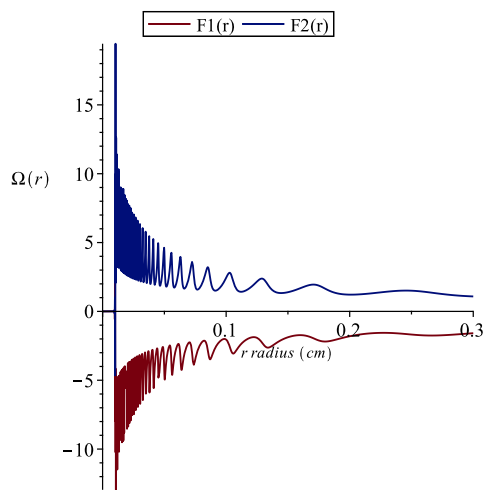
Figure 3. A=2.1375, mass of fluid column in vertical tube *m*=95.2 grams.
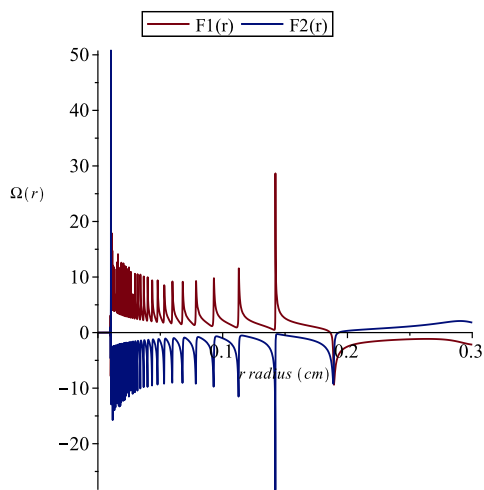
Figure 4. A=2.1820, mass of fluid column in vertical tube *m*=96 grams.
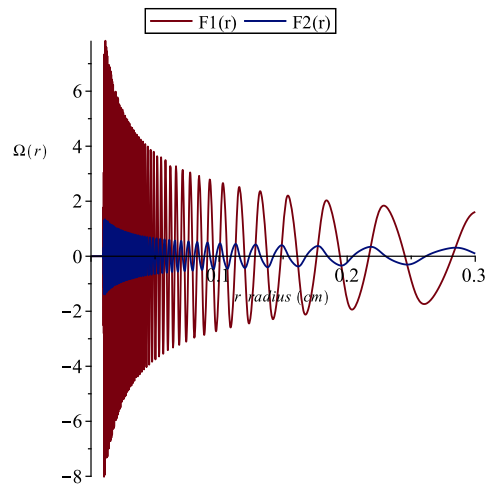
Figure 5. A=34.1375, mass of fluid column in vertical tube *m*=102 grams.

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115   **Conflicts of Interest:** The authors declare no conflict of interest.