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The results on vertex domination in Fuzzy graphs

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Abstract

We do fuzzification the concept of domination in crisp graph by using membership values of nodes, α-strong arcs and arcs. In this paper, we introduce a new variation on the domination theme which we call vertex domination. We determine the vertex domination number γ_v for several classes of fuzzy graphs, specially complete fuzzy graph and complete bipartite fuzzy graphs. The bounds is obtained for the vertex domination number of fuzzy graphs. Also the relationship between M-strong arcs and α-strong is obtained. In fuzzy graphs, monotone decreasing property and monotone increasing property is introduced. We prove the vizing's conjecture is monotone decreasing fuzzy graph property for vertex domination. we prove also the Grarier-Khelladi's conjecture is monotone decreasing fuzzy graph property for it. We obtain Nordhaus-Gaddum (NG) type results for these parameters. The relationship between several classes of operations on fuzzy graphs with the vertex domination number of them is studied.

Keywords: Fuzzy graph, α-strong arcs, Weight of nodes, vertex domination
AMS Subject Classification: 05C72, 05C69, 03E72, 94D05

1 Introduction

In 1965, Zadeh published his seminal paper "fuzzy sets" (Ref. [77]) which described fuzzy set theory and consequently fuzzy logic. Fuzzy graphs were introduced by Rosenfeld (Ref. [56]) and Yeh and Bang (Ref. [73]) independently in 1975. The concept of domination in fuzzy graphs was introduced by A.Somasundaram and S.Somasundaram (Ref. [63]). They defined domination using effective edges in fuzzy graph (Refs. [63] and [64]). Nagoorgani and Chandrasekharan defined domination in fuzzy graphs using strong arcs (Ref. [47]). Manjusha and Sunitha discussed some concepts in domination and total domination in fuzzy graphs using strong arcs (Refs. [36] and [37]).

We first briefly illustrate our opinion. The rest of this paper is organized as follows. In Section 2, we lay down the preliminary results which recall some basic concepts of fuzzy graph, path, cycle, connectedness, complete fuzzy graph, order, size, complement, types of arcs consists of α-strong, β-strong,δ-strong and M-strong, bipartite fuzzy graph, complete bipartite fuzzy graph, star fuzzy graph, be isolated, domatic partition, Vizing's conjecture, Gravier and Khelladi's conjecture, some operations on fuzzy graphs consists of cartesian product, join and union, Nordhaus-Gaddum (NG) results and finally we conclude this section with Remark (2.1) and In Section 3, The vertex domination number of a fuzzy graph is defined in a classic way, Definition (3.1), (3.3), (3.4). We determine vertex domination number for several classes of fuzzy graphs

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consists of complete fuzzy graph, Proposition (3.10), empty fuzzy graph, Proposition (3.11), star fuzzy graph, Proposition (3.13), complete bipartite fuzzy graph, Proposition (3.14). We give an upper bound for the vertex domination number of fuzzy graphs, Proposition (3.15). For any fuzzy graph the Nordhaus-Gaddum(NG)'result holds, Theorem (3.16). Finding domatic partition of size two in fuzzy graph G of order $n \geq 2$ is studied, Theorem (3.19). We improve upper bound for the vertex domination number of fuzzy graphs without isolated nodes, Theorem (3.20). We also improve Nordhaus-Gaddum(NG)'result for fuzzy graphs without isolated nodes, Corollary (3.21). We give the relationship between M-strong arcs and α-strong arcs, Corollary (3.24). We give a necessary and sufficient condition for vertex domination which is half of order, In fact fuzzy graphs with vertex domination which is half of order is characterized in the special case, Theorem (3.26). The vertex domination of union of two fuzzy graphs is studied, Proposition (3.27). Also the vertex domination of union of fuzzy graphs Family is discussed, Corollary (3.28). The concepts of both monotone increasing fuzzy graph property, Definition (3.29), and monotone decreasing fuzzy graph property, Definition (3.31), are introduced. The result in relation with vizing's conjecture by using α-strong arc and monotone decreasing fuzzy graph property is determined, Theorem (3.34). Some results in relation with vizing's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.35). The vertex domination of join of two fuzzy graphs is studied, Proposition (3.36). Also the vertex domination of join of fuzzy graphs Family is discussed, Corollary (3.37). The result in relation with Gravier and Khelladi's conjecture by using α-strong arc and monotone decreasing fuzzy graph property is determined, Theorem (3.38). We conclude this section with Some result in relation with Gravier and Khelladi's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.39). In Section 4, We give 9 practical applications in relation with these concepts.

2 Preliminary

We provide some basic background for the paper in this section.

Some of the books discussing these various themes are Bezdek and Pal [7], Lootsma [35], Morderson and Malik [40], Comelius . T. Leondes [34] and Klir and Bo Yuan [31]. We shall now list below some basic definitions and results from [41], [56]. Also Background on fuzzy graphs and the following definitions can be found in them.

we lay down the preliminary results which recall some basic concepts of fuzzy graph, path, cycle, connectedness, complete fuzzy graph, order, size, complement, types of arcs consists of α-strong, β-strong,δ-strong and M-strong, bipartite fuzzy graph, complete bipartite fuzzy graph, star fuzzy graph, be isolated, domatic partition, Vizing's conjecture, Gravier and Khelladi's conjecture, some operations on fuzzy graphs consists of cartesian product, join and union, Nordhaus-Gaddum (NG) results and finally we conclude this section with Remark (2.1)

We recall that a fuzzy subset of a set S is a function of S into the closed interval [0, 1], [77]. A fuzzy graph is denoted by $G = (V, \sigma, \mu)$ such that $\mu(\{x, y\}) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$ where V is a vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on V. We call σ the fuzzy node set (or fuzzy vertex set) of G and μ the fuzzy arc set (or fuzzy edge set) of G, respectively. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e., $\mu(\{x, x\}) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(\{x, y\}) = \mu(\{y, x\})$, for all $x, y \in V$). In all the examples σ and μ is chosen suitably. In any fuzzy graph, the underlying crisp graph is denoted by $G^* = (V, E)$ where V and E are domain of σ and μ, respectively. This definition of fuzzy graph is essentially the same as the one appearing in [56]. The fuzzy graph $H = (\tau, \nu)$ is called a partial fuzzy subgraph of $G = (\sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. Similarly,

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the fuzzy graph $H = (\tau, \nu)$ is called a fuzzy subgraph of $G = (V, \sigma, \mu)$ induced by P if $P \subseteq V, \tau(x) = \sigma(x)$ for all $x \in P$ and $\nu(\{x, y\}) = \mu(\{x, y\})$ for all $x, y \in P$. For the sake of simplicity, we sometimes call H a fuzzy subgraph of G. We say that the partial fuzzy subgraph (τ, ν) spans the fuzzy graph (σ, μ) if $\sigma = \tau$. In this case, we call (τ, ν) a spanning fuzzy subgraph of (σ, μ).

For the sake of simplicity, we sometimes write xy instead of $\{x, y\}$

A path P of length n is a sequence of distinct vertices $u_0, u_1, \cdots, u_n$ such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \cdots, n$ and the degree of membership of a weakest edge is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest edge. The strength of a cycle is the strength of the weakest edge in it. The strength of connectedness between two vertices x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by μ.

A fuzzy graph $G = (V, \sigma, \mu)$ is connected if for every x, y in V, $CONN_G(x, y) > 0$.

A fuzzy graph G is said complete if $\mu(uv) = \sigma(x) \wedge \sigma(y)$. for all $u, v \in V$.

The order p and size q of a fuzzy graph $G = (V, \sigma, \mu)$ are defined $p = \Sigma_{x \in V} \sigma(x)$ and $q = \Sigma_{x,y \in V} \mu(xy)$.

The complement of a fuzzy graph G, denoted by $\bar{G}$ is defined to $\bar{G} = (V, \sigma, \bar{\mu})$ where $\bar{\mu}(xy) = \sigma(x) \wedge \sigma(y) - \mu(xy)$ for all $x, y \in V$.

An arc of a fuzzy graph is called α-strong if its weights is greater than strength of connectedness of its end nodes when it is deleted. Depending on $CONN_G(x, y)$ of an arc xy in a fuzzy graph G, Mathew and Sunitha [68] defined three types of arcs. Note that $CONN_{G-xy}(x, y)$ is the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc xy. An arc xy in G is α-strong if $\mu(xy) > CONN_{G-xy}(x, y)$. An arc xy in G is β-strong if $\mu(xy) = CONN_{G-xy}(x, y)$. An arc xy in G is δ-arc if $\mu(xy) < CONN_{G-xy}(x, y)$. An arc uv of a fuzzy graph is called an M-strong arc if $\mu(uv) = \sigma(u) \wedge \sigma(v)$. In order to avoid confusion with the notion of strong arcs introduced by Bhutani and Rosenfeld [15], we shall call strong in the sense of Mordeson as M-strong [46].

A fuzzy graph G is said bipartite if the vertex set V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(v_1 v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Moreover, if $\mu(uv) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a complete bipartite graph and is denoted by $K\sigma_1, \sigma_2$, where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2. In this case, If $|V_1| = 1$ or $|V_2| = 1$ then the complete bipartite graph is said a star fuzzy graph which is denoted by $K_{1,\sigma}$.

A node u is said isolated if $\mu(uv) = 0$ for all $v \neq u$.

A domatic partition is a partition of the vertices of a graph into disjoint dominating sets. The maximum number of disjoint dominating sets in a domatic partition of a graph is called its domatic number.

In graph theory, Vizing's conjecture [17] concerns a relation between the domination number and the cartesian product of graphs. This conjecture was first stated by Vadim G. Vizing (1968), and states that, if $\gamma(G)$ denotes the minimum number of vertices in a dominating set for G, then

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H).$$

Vizing's conjecture from 1968 asserts that the domination number of the. Cartesian product of two graphs is at least as large as the product of their domination numbers.

Gravier and Khelladi (1995) conjectured a similar bound for the domination number of the tensor product of graphs; however, a counterexample was found by Klavžar Zmazek (1996) [30]. Since Vizing proposed his conjecture, many mathematicians have worked on it, with partial results described below. For a more detailed overview of these results, see Brešar et al. (2012) [8]

The cartesian product $G = G_1 \times G_2$ [39] of two fuzzy graphs

$G_i = (V_i, \sigma_i, \mu_i), i = 1, 2$ is defined as a fuzzy graph $G = (V \times V, \sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ where $E = \{\{uu_2, uv_2\}|u \in V_1, u_2v_2 \in E_2\} \cup \{\{u_1w, v_1w\}|u_1v_1 \in E_1, w \in V_2\}$. Fuzzy sets $\sigma_1 \times \sigma_2$ and $\mu_1 \times \mu_2$ are defined as $(\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2)$ and $\forall u \in V_1, \forall u_2v_2 \in E_2, (\mu_1 \times \mu_2)(\{uu_2, uv_2\}) = \sigma_1(u) \wedge \mu_2(u_2v_2)$ and $\forall u_1v_1 \in E_1, \forall w \in V_2, (\mu_1 \times \mu_2)(\{u_1w, v_w\}) = \mu_1(u_1v_1) \wedge \sigma_2(w)$.

The union $G = G_1 \cup G_2$ [39] of two fuzzy graphs $G_i = (V_i, \sigma_i, \mu_i), i = 1, 2$ is defined as a fuzzy graph $G = (V_1 \cup V_2, \sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ where $E = E_1 \cup E_2$. Fuzzy sets $\sigma_1 \cup \sigma_2$ and $\mu_1 \cup \mu_2$ are defined as $(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u)$ if $u \in V_1 - V_2, (\sigma_1 \cup \sigma_2)(u) = \sigma_2(u)$ if $u \in V_2 - V_1$, and $(\sigma_1 \cup \sigma_2)(u) = \sigma_1(u) \vee \sigma_2(u)$ if $u \in V_1 \cap V_2$. Also $(\mu_1 \cup \mu_2)(uv) = \mu_1(uv)$ if $uv \in E_1 - E_2$ and $(\mu_1 \cup \mu_2)(uv) = \mu_2(uv)$ if $uv \in E_2 - E_1$, and $(\mu_1 \cup \mu_2)(uv) = \mu_1(uv) \vee \mu_2(uv)$ if $uv \in E_1 \cap E_2$.

Let $G = G_1 + G_2$ denote the join [39] of two fuzzy graphs $G_i = (V_i, \sigma_i, \mu_i), i = 1, 2$ is defined as a fuzzy graph $G = (V_1 \cup V_2, \sigma_1 + \sigma_2, \mu_1 + \mu_2)$ where $E = E_1 \cup E_2 \cup E^{'}$ and $E^{'}$ is the set of all edges joining vertices of V_1 with the vertices of V_2, and we assume that $V_1 \cap V_2 = \emptyset$. Fuzzy sets $\sigma_1 + \sigma_2$ and $\mu_1 + \mu_2$ are defined as $(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u)$ and $\forall u \in V_1 \cup V_2; (\mu_1 + \mu_2)(uv) = (\mu_1 \cup \mu_2)(uv)$ if $uv \in E_1 \cup E_2$ and $(\mu_1 + \mu_2)(uv) = \sigma_1(u) \wedge \sigma_2(v)$ if $uv \in E^{'}$.

The classical paper [49] of Nordhaus and Gaddum established the inequalities for the chromatic numbers of a graph $G = (V, E)$ and its complement $\bar{G}$. We are concerned with analogous inequalities involving domination parameters in graphs. We begin with a brief overview of Nordhaus-Gaddum (NG) inequalities for several domination-related parameters. For each generic invariant μ of a graph G, let $\mu = \mu(G)$ and $\bar{\mu} = \mu(\bar{G})$. Inequalities on $\mu + \bar{\mu}$ and $\mu.\bar{\mu}$ exist in the literature for only a few of the many domination-related parameters and most of these results are of the additive form. In 1972 Jaeger and Payan [26] published the first NG results involving domination. Cockayne and Hedetniemi [18] sharpened the upper bound for the sum. Laskar and Peters [33] improved this bound for the case when both G and $\bar{G}$ are connected. A much improved bound was established for the case when neither G nor $\bar{G}$ has isolated nodes by Bollobás and Cockayne [14] and by Joseph and Arumugam [27] independently.

Remark 2.1. For the sake of simplicity, we do sometimes

- writing xy instead of $\{x, y\}$.
- calling x both vertex and node.
- calling xy both edge and arc.
- writing Cartesian product both $\square$ and $\times$.
- saying $\sigma(x)$ and $\mu(xy)$ with different literature, e.g. value, weight, membership value and etc.

3 Main Results

In this section, we provide the main results.

The vertex domination number of a fuzzy graph is defined in a classic way, Definition (3.1), (3.3), (3.4).

Definition 3.1. Let $G = (\sigma, \mu)$ be a fuzzy graph on V. Let $x, y \in V$. We say that x **dominates** y in G as α-strong if the arc $\{x, y\}$ is α-strong.

Example 3.2. By attention to fuzzy graph In Figure (1), the arcs v_2v_5, v_2v_4, v_3v_4 and v_1v_3 are α-strong and the arcs v_1v_4, v_1v_2 and v_4v_5 are not α-strong.

Definition 3.3. A subset S of V is called a **α-strong dominating set** in G if for every $v \notin S$, there exists $u \in S$ such that u dominates v.

Definition 3.4. Let S be the set of all α-strong dominating sets in G, the **vertex domination number** of G is defined as $\min_{D\in S}[\Sigma_{u\in D}(\sigma(u)+\frac{d_s(u)}{d(u)})]$ and it is denoted by $\gamma_{\mathbf{v}}(\mathbf{G})$. If $d(u)=0$, then we consider $\frac{d_s(u)}{d(u)}$ equal with 0. The α-strong dominating set that is correspond to $\gamma_v(G)$ is called by **vertex dominating set**. We also say $\Sigma_{u\in D}(\sigma(u)+\frac{d_s(u)}{d(u)})$, **vertex weight** of D, for every $D \in S$ and it is denoted by $\mathbf{w_v(D)}$.

Example 3.5. By attention to fuzzy graph In Figure (1), the set $\{v_2, v_3\}$ is the α-strong dominating set. This set is also vertex dominating set in fuzzy graph G. Hence $\gamma_v(G) = 1.75 + 0.9 + 0.7 = 3.35$. So $\gamma_v(G) = 3.35$.

Theorem 3.6. *[38] If G is a complete fuzzy graph, then all arcs are strong.*

Theorem 3.7. *[38] If G is a complete bipartite fuzzy graph, then all arcs are strong.*

Remark 3.8. If G is a complete fuzzy graph, then all arcs are α-strong.

Remark 3.9. If G is a complete bipartite fuzzy graph, then all arcs are α-strong.

It is well known and generally accepted that the problem of determining the domination number of an arbitrary graph is a difficult one. Because of this, researchers have turned their attention to the study of classes of graphs for which the domination problem can be solved in polynomial time.

We determine vertex domination number for several classes of fuzzy graphs consists of complete fuzzy graph, Proposition (3.10), empty fuzzy graph, Proposition (3.11), star fuzzy graph, Proposition (3.13), complete bipartite fuzzy graph, Proposition (3.14).

Proposition 3.10 (Complete fuzzy graph)**.** *If $G = (V, \sigma, \mu)$ is a complete fuzzy graph, then $\gamma_v(G) = \min_{u\in V}(\sigma(u)) + 1$.*

Proof. Since G is a complete fuzzy graph, all arcs are α-strong by Remark (3.8) and each node is incident to all other nodes. Hence $D = \{u\}$ is a α-strong dominating set and $d_s(u) = d(u)$ for each $u \in V$. Hence the result follows. □

Proposition 3.11 (Empty fuzzy graph)**.** *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then $\gamma_v(G) = p$, if G be edgeless, i.e $G = \bar{K_n}$.*

Proof. Since G is edgeless, Hence V is only α-strong dominating set in G and none of arcs are α-strong. so we have $\gamma_v(G) = \min_{D\in S}[\Sigma_{u\in D}\sigma(u)] = \Sigma_{u\in v}\sigma(u) = p$ by Definition (3.4). so we can write $\gamma_v(\bar{K_n}) = p$ by our notations. □

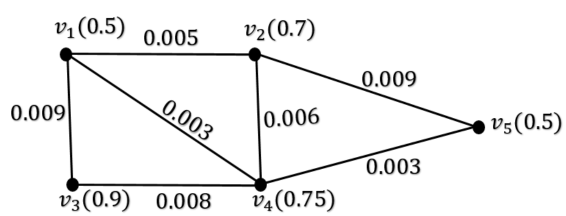

Figure 1. **Vertex domination**

It is interesting to note the converse of Proposition (3.11) that does not hold.

Example 3.12. We show the converse of Proposition (3.11) does not hold. For this purpose, Let $V = \{v_1, v_2, v_3, v_4, v_5\}$. We define σ on V by $\sigma : V \to [0, 1]$ such that

$$\sigma(v_1) = 0.5, \sigma(v_2) = 0.7, \sigma(v_3) = 0.9, \sigma(v_4) = 0.75, \sigma(v_5) = 0.5$$

Now, The function $\mu : V \times V \to [0, 1]$ is defined by

$$\mu(v_1v_2) = 0.005, \mu(v_1v_4) = 0.003, \mu(v_1v_3) = 0.009, \mu(v_2v_4) = 0.006, \mu(v_2v_5) = 0.009,$$

$\mu(v_3v_4) = 0.008, \mu(v_4v_5) = 0.003$ such that $\forall u, v \in V, \mu(u, v) \le \sigma(u) \wedge \sigma(v)$. Finally, Let V, σ, and μ be the vertices, value of vertices and value of edges respectively. In other words, By attention to fuzzy graph In Figure (1), the arcs v_2v_5, v_2v_4, v_3v_4 and v_1v_3 are α-strong and the arcs v_1v_4, v_1v_2 and v_4v_5 are not α-strong. So the set $\{v_2, v_3\}$ is the α-strong dominating set. This set is also vertex dominating set in fuzzy graph G. Hence $\gamma_v(G) = 1.75 + 0.9 + 0.7 = 3.35 = \Sigma_{u \in v}\sigma(u) = p$. So $G \neq \bar{K_5}$ but $\gamma_v(G) = p$.

Proposition 3.13 (Star fuzzy graph)**.** *Let G be a star fuzzy graph. Then $G = K_{1,\sigma}$ and $\gamma_v(K_{1,\sigma}) = \sigma(u) + 1$ where u is center of G.*

Proof. Let G be the star fuzzy graph with $V = \{u, v_1, v_2, \cdots, v_n\}$ such that u and v_i are center and leaves of G, for $1 \le i \le n$ respectively. So $G^* = K^*_{1,n}$ is underlying crisp graph of G. $\{u\}$ is vertex dominating set in G and all arcs are α-strong by Remark (3.9) and due to G is bipartite fuzzy graph. Hence the result follows. □

Proposition 3.14 (Bipartite fuzzy graph)**.** *Let G be the bipartite fuzzy graph which is not star fuzzy graph. Then $G = K_{\sigma_1,\sigma_2}$ and $\gamma_v(K_{\sigma_1,\sigma_2}) = \min_{u \in V_1, v \in V_2}(\sigma(u) + \sigma(v)) + 2$.*

Proof. Let $G \neq K_{1,\sigma}$ be bipartite fuzzy graph. Then both of V_1 and V_2 include more than one vertex. In K_{σ_1,σ_2} , all arcs are α-strong by Remark (3.9). Also each node in V_1 is dominated as α-strong with all nodes in V_2 and conversely. Hence in K_{σ_1,σ_2}, the α-strong dominating sets are V_1 and V_2 and any set containing 2 nodes, one in V_1 and other in V_2. Hence $\gamma_v(K_{\sigma_1,\sigma_2}) = \min_{u \in V_1, v \in V_2}(\sigma(u) + \sigma(v)) + 2$. So the theorem is proved. □

We give an upper bound for the vertex domination number of fuzzy graphs, Proposition (3.15).

Proposition 3.15. *For any fuzzy graph $G = (V, \sigma, \mu)$, We have $\gamma_v \le p$.*

Proof. $\gamma_v(\bar{K_n}) = p$ by Theorem (3.11). So the result follows. □

For the vertex domination number γ_v the following theorem gives a Nordhaus-Gaddum type result.

For any fuzzy graph the Nordhaus-Gaddum(NG)'result holds, Theorem (3.16).

Theorem 3.16. *For any fuzzy graph $G = (V, \sigma, \mu)$, The Nordhaus-Gaddum result holds. In other words, we have $\gamma_v + \bar{\gamma_v} \le 2p$.*

Proof. G is fuzzy graph. So $\bar{G}$ is also fuzzy graph. We implement Theorem (3.15) on G and $\bar{G}$. Then $\gamma_v \le p$ and $\bar{\gamma_v} \le p$. Hence $\gamma_v + \bar{\gamma_v} \le 2p$. So the theorem is proved. □

The following theorems on dominating sets in graphs are the first results about domination and were presented by Ore in his book Theory of Graphs [69].

Definition 3.17 ([47])**.** A α-strong dominating set D is called a minimal α-strong dominating set if no proper subset of D is a α-strong dominating set.

Theorem 3.18 ([47])**.** *Let G be a fuzzy graph without isolated nodes. If D is a minimal α-strong dominating set then $V - D$ is a α-strong dominating set.*

Finding a domatic partition of size 1 is trivial and finding a domatic partition of size 2 (or establishing that none exists) is easy but finding a maximum-size domatic partition (i.e., the domatic number), is computationally hard. Finding domatic partition of size two in fuzzy graph G of order $n \ge 2$ is easy by the following.

Theorem 3.19 ([47])**.** *Every connected graph G of order $n \ge 2$ has a α-strong dominating set D whose complement $V - D$ is also a α-strong dominating set.*

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We improve upper bound for the vertex domination number of fuzzy graphs without isolated nodes, Theorem (3.20).

Theorem 3.20. *For any fuzzy graph $G = (V, \sigma, \mu)$ without isolated nodes, We have $\gamma_v \leq \frac{p}{2}$.*

Proof. Let D be a minimal dominating set of G. Then by Theorem (3.19), V-D is a α-strong dominating set of G. Then $\gamma_v(G) \leq w_v(D)$ and $\gamma_v(G) \leq w_v(V - D)$.

Therefore $2\gamma_v(G) \leq w_v(D) + w_v(V - D) \leq p$ which implies $\gamma_v \leq \frac{p}{2}$. Hence the proof is completed. □

We also improve Nordhaus-Gaddum(NG)'result for fuzzy graphs without isolated nodes, Corollary (3.21).

Corollary 3.21. *Let G be a fuzzy graph such that both G and $\bar{G}$ have no isolated nodes. Then $\gamma_v + \bar{\gamma_v} \leq p$,where $\bar{\gamma_v}$ is the vertex domination number of $\bar{G}$. Moreover, equality holds if and only if $\gamma_v = \bar{\gamma_v} = \frac{p}{2}$.*

Proof. By the Implement of Theorem (3.20) on G and $\bar{G}$, we have $\gamma_v(G) = \gamma_v \leq \frac{p}{2}$, and $\gamma_v(\bar{G}) = \bar{\gamma_v}(G) = \bar{\gamma_v} \leq \frac{p}{2}$. So $\gamma_v + \bar{\gamma_v} \leq \frac{p}{2} + \frac{p}{2} = p$. Hence $\gamma_v + \bar{\gamma_v} \leq p$.

Suppose $\gamma_v = \bar{\gamma_v} = \frac{p}{2}$, then obviously $\gamma_v + \bar{\gamma_v} = p$. Conversely, suppose $\gamma_v + \bar{\gamma_v} \leq p$. Then we have $\gamma_v \leq \frac{p}{2}$ and $\bar{\gamma_v} \leq \frac{p}{2}$. If either $\gamma_v < \frac{p}{2}$ or $\bar{\gamma_v} < \frac{p}{2}$, then $\gamma_v + \bar{\gamma_v} < p$, which is a contradiction. Hence the only possibility case is $\gamma_v = \bar{\gamma_v} = \frac{p}{2}$. □

Remark 3.22. Note that when we use the definition of domination number in [13,14,15], Theorem (3.20) and Corollary (3.21) are hold.

Proposition 3.23. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. If all arcs have equal value, the G has no α-strong edge.*

Proof. Obviously the result is hold by using Definition (3.1). □

We give the relationship between M-strong arcs and α-strong arcs, Corollary (3.24).

Corollary 3.24. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph. If all arcs are M-strong, the G has no α-strong edge.*

Proof. Obviously the result is hold by using Proposition (3.23). □

The following example illustrates this concept.

Example 3.25.
In Figure (2) , all arcs are M-strong but there is no α-strong arcs in this fuzzy graph. Obviously this result is hold by using Definition (3.3).

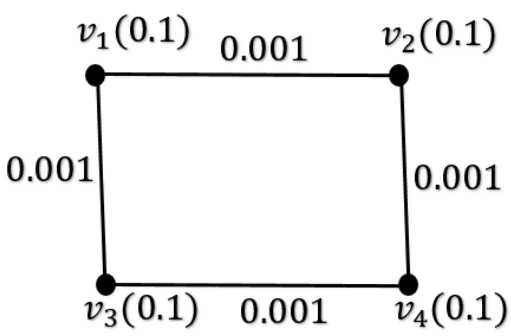

Figure 2. **M-strong arcs and α-strong arcs**

We give a necessary and sufficient condition for vertex domination which is half of order, In fact fuzzy graphs with vertex domination which is half of order is characterized in the special case, Theorem (3.26).

Theorem 3.26. *In any fuzzy graph $G = (V, \sigma, \mu)$ such that values of nodes are equal and all arcs have same value, i.e. for $\forall u_i, u_j \in V$ and $\forall e_i, e_j \in E$, we have $\sigma(u_i) = \sigma(u_j)$ and $\mu(e_i) = \mu(e_j)$. $\gamma_v = \frac{p}{2}$ if and only if For any vertex dominating set D in G, we have $|D| = \frac{n}{2}$.*

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Proof. Suppose D has the conditions. $d_s(D) = 0$ by Proposition (3.23). So $\gamma_v(G) = \Sigma_{u\in D}\sigma(u)$ by using Definition (3.4). Since values of nodes are equal and $|D| = \frac{n}{2}$, we have $\gamma_v(G) = \Sigma_{u\in D}\sigma(u) = \frac{n}{2}\sigma(u) = \frac{1}{2}(n\sigma(u)) = \frac{1}{2}(\Sigma_{u\in V}\sigma(u)) = \frac{1}{2}(p) = \frac{p}{2}$. Hence the result is hold in this case.

Conversely, Suppose $\gamma_v = \frac{p}{2}$. Let $D = \{u_1, u_2, \cdots, u_n\}$ be a vertex dominating set. $d_s(D) = 0$ by Proposition (3.23). So $\gamma_v(G) = \Sigma_{u\in D}\sigma(u)$ by using Definition (3.4). Since $\gamma_v(G) = W_v(D)$, we have $\gamma_v = \frac{p}{2} = \frac{1}{2}(\Sigma_{u\in V}\sigma(u)) = \Sigma_{u\in D}\sigma(u)$. Suppose $n' \neq \frac{n}{2}$. so $\Sigma_{i=1}^{n''}\sigma(v_i)) = 0$ which is a contradiction with $\forall u_i \in V, \sigma(u_i) > 0$. Hence $n' = \frac{n}{2}$, i.e. $|D| = n' = \frac{n}{2}$. So the result is hold in this case. □

The vertex domination of union of two fuzzy graphs is studied, Proposition (3.27).

Proposition 3.27. *Let G_1 and G_2 be fuzzy graphs. The vertex dominating set of $G_1 \cup G_2$ is $D = D_1 \cup D_2$ such that D_1 and D_2 are the vertex dominating set of G_1 and G_2 respectively. Moreover, $\gamma_v(G_1 \cup G_2) = \gamma_v(G_1) + \gamma_v(G_2)$.*

Proof. Obviously the result is hold by using Definition of union of two fuzzy graphs. □

Also the vertex domination of union of fuzzy graphs Family is discussed, Corollary (3.28).

Corollary 3.28. *Let $G_1, G_2, \cdots, G_n$ be fuzzy graphs. The vertex dominating set of $\cup_{i=1}^{n} G_i$ is $D = \cup_{i=1}^{n} D_i$ such that D_i is the vertex dominating set of G_i. Moreover, $\gamma_v(\cup_{i=1}^{n} G_i) = \Sigma_{i=1}^{n}\gamma_v(G_i)$.*

Proof. Obviously the result is hold by using proposition (3.27). □

The concepts of both monotone increasing fuzzy graph property, Definition (3.29), and monotone decreasing fuzzy graph property, Definition (3.31), are introduced.

Definition 3.29. We call a fuzzy graph property P monotone increasing if $G \in P$ implies $G + e \in P$, i.e., adding an edge e to a fuzzy graph G does not destroy the property.

Example 3.30. Connectivity and Hamiltonicity are monotone increasing properties. A monotone increasing property is nontrivial if the empty graph $\bar{K}_n \notin P$ and the complete graph $K_n \in P$.

Definition 3.31. A fuzzy graph property is monotone decreasing if $G \in P$ implies $G - e \in P$, i.e., removing an edge from a graph does not destroy the property.

Example 3.32. Properties of a fuzzy graph not being connected or being planar are examples of monotone decreasing fuzzy graph properties.

Remark 3.33. Obviously, a fuzzy graph property P is monotone increasing if and only if its complement is monotone decreasing. Clearly not all fuzzy graph properties are monotone. For example having at least half of the vertices having a given fixed degree d is not monotone.

Let $\gamma(G)$ denote the domination number of a simple graph G. Then Vizing (1963) [17]conjectured that $\gamma(G)\gamma(H) \leq \gamma(G \times H)$, where $G \times H$ is the graph product. While the full conjecture remains open, Clark and Suen (2000) [23] have proved the looser result $\gamma(G)\gamma(H) \leq 2\gamma(G \times H)$.

Vizing stated the still open conjecture:

Conjecture (Vizing [17]). For all graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G \times H)$. The result in relation with vizing's conjecture by using α-strong arc and monotone decreasing fuzzy graph property is determined, Theorem (3.34).

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Theorem 3.34. *The vizing's conjecture is monotone decreasing property in fuzzy graph G, if the edge e be α-strong and $\gamma_v(G-e)=\gamma_v(G)$.*

Proof. The fuzzy graph $(G-e)\times H$ is the spanning fuzzy subgraph of $G\times H$, for all fuzzy graph H. So $\gamma_v((G-e)\times H)\geq\gamma_v(G\times H)\geq\gamma_v(G)\gamma_v(H)=\gamma_v(G-e)\gamma_v(H)$. Hence vizing's conjecture is also hold for $G-e$. Then the result follows. □

Some results in relation with vizing's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.35).

Corollary 3.35. *Suppose the vizing's conjecture is hold for G. Let K be the spanning fuzzy subgraph of G such that $\gamma_v(K)=\gamma_v(G)$. Then the vizing's conjecture is hold for K.*

Proof. The fuzzy graph $K\times H$ is the spanning fuzzy subgraph of $G\times H$, for all fuzzy graph H. So $\gamma_v(K\times H)\geq\gamma_v(G\times H)\geq\gamma_v(G)\gamma_v(H)=\gamma_v(K)\gamma_v(H)$. Hence the vizing's conjecture is also hold for K. So the result follows. □

The vertex domination of join of two fuzzy graphs is studied, Proposition (3.36).

Proposition 3.36. *Let G_1 and G_2 be fuzzy graphs. The vertex dominating set of $G_1\otimes G_2$ is $D=D_1\cup D_2$ such that D_1 and D_2 are the vertex dominating set of G_1 and G_2 respectively. Moreover, $\gamma_v(G_1\otimes G_2)=\gamma_v(G_1)+\gamma_v(G_2)$.*

Proof. Obviously the result is hold by using Definition of join of two fuzzy graphs and Corollary (3.24) which state in this case, M-strong arcs between two fuzzy graphs is not α-strong which is weak arc changing strength of connectedness of G. □

Also the vertex domination of join of fuzzy graphs Family is discussed, Corollary (3.37).

Corollary 3.37. *Let $G_1,G_2,\cdots,G_n$ be fuzzy graphs. The vertex dominating set of $\otimes_{i=1}^{n}G_i$ is $D=\otimes_{i=1}^{n}D_i$ such that D_i is the vertex dominating set of G_i. Moreover, $\gamma_v(\otimes_{i=1}^{n}G_i)=\Sigma_{i=1}^{n}\gamma_v(G_i)$.*

Proof. Obviously the result is hold by using proposition (3.36). □

Gravier and Khelladi [22] conjecture a Vizing-like inequality for the domination number of the cross product of graphs.

Gravier and Khelladi stated the still open conjecture:

Conjecture (Gravier and Khelladi [22]). For all graphs G and H,

$$\gamma(G)\gamma(H)\leq 2\gamma(G\otimes H).$$

The result in relation with Gravier and Khelladi's conjecture by using α-strong arc and monotone decreasing fuzzy graph property is determined, Theorem (3.38).

Theorem 3.38. *The Gravier and Khelladi's conjecture is monotone decreasing property in fuzzy graph G, if the edge e be α-strong and $\gamma_v(G-e)=\gamma_v(G)$.*

Proof. The fuzzy graph $(G-e)\times H$ is the spanning fuzzy subgraph of $G\times H$, for all fuzzy graph H. So $\gamma_v((G-e)\times H)\geq\gamma_v(G\times H)\geq\gamma_v(G)\gamma_v(H)=\gamma_v(G-e)\gamma_v(H)$. Hence Gravier and Khelladi's conjecture is also hold for $G-e$. Then the result follows. □

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We conclude this section with Some result in relation with Gravier and Khelladi's conjecture by using α-strong arc and spanning fuzzy subgraph is studied, Corollary (3.39).

Corollary 3.39. *Suppose the Gravier and Khelladi's conjecture is hold for G. Let K be the spanning fuzzy subgraph of G such that $\gamma_v(K) = \gamma_v(G)$. Then the Gravier and Khelladi's conjecture is hold for K.*

Proof. The fuzzy graph $K \times H$ is the spanning fuzzy subgraph of $G \times H$, for all fuzzy graph H. So $\gamma_v(K \otimes H) \geq \gamma_v(G \otimes H) \geq \gamma_v(G)\gamma_v(H) = \gamma_v(K)\gamma_v(H)$. Hence the Gravier and Khelladi's conjecture is also hold for K. So the result follows. □

4 Conclusion

Graph theory is one of the branches of modern mathematics having experienced a most impressive development in recent years. One of the most interesting problems in graph theory is that of Domination Theory. Nowadays domination theory ranks top among the most prominent areas of research in graph theory and combinatorics. The theory of domination has been the nucleus of research activity in graph theory in recent times. The fastest growing area within graph theory is a study of domination and related subset problems such independence, covering, matching, decomposition and labelling. Domination boasts a host of applications to social network theory, land surveying, game theory, interconnection network, parallel computing and image processing and so on. Today, this theory gained popularity and remains as a major area of research. At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad hoc networks, biological networks, distributed computing, social networks and web graphs partly explain the increased interest. More than 1200 papers already published on domination in graphs. Without a doubt, the literature on this subject is growing rapidly, and a considerable amount of work has been dedicated to find different bounds for the domination numbers of graphs. However, from practical point of view, it was necessary to define other types of dominations. Most of these new variations required the dominating set to have additional properties.

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