 doi:10.20944/preprints201804.0046.v1

Article

An Approximation Method of Fuzzy Numbers Based on Extended Fuzzy Transforms

Ferdinando Di Martino [1,2*], Salvatore, Sessa [1,2]

[1] Università degli Studi di Napoli Federico II, Dipartimento di Architettura, Via Toledo 402, 80134 Napoli, Italy

[2] Università degli Studi di Napoli Federico II, Centro Interdipartimentale di ricerca A. Calza Bini, Via Toledo 402, 80134 Napoli, Italy

* Correspondence: fdimarti@unina.it; Tel.: +39-081-253-8904

Abstract: We propose a new Mamdani fuzzy rule-based system in which the fuzzy sets in the antecedents and consequents are assigned in a discrete set of points and approximated by using the extended inverse fuzzy transforms, whose components are calculated by verifying that the dataset is sufficiently dense with respect to the uniform fuzzy partition. We test our system in the problem of spatial analysis consisting in the evaluation of the liveability of residential housings in all the municipalities of the district of Naples (Italy). Comparisons are done with the results obtained by using trapezoidal fuzzy numbers in the fuzzy rules.

Keywords: Extended fuzzy transform, fuzzy number, rule management system, spatial analysis

1. Introduction

The A fuzzy number (FN) is a fuzzy set with membership function A: Reals →[0,1] defined as

$$A(x) = \begin{cases} 0 & \text{IF } x < a \\ A^{-}(x) & \text{IF } a \le x < c \\ 1 & \text{IF } c \le x \le d \\ A^{+}(x) & \text{IF } d < x \le b \\ 0 & \text{IF } x > b \end{cases} \tag{1}$$

where $a \le c \le d \le b$, A^-: [a,c] →[0,1] is a not decreasing continuous function with $A^-(a) = 0$, $A^-c) = 1$ and A^+: [b,d] →[0,1] is a not increasing continuous function with $A^+(d) = 1$, $A^+(b) = 0$. A^- and A^+ are called *left-side* and *right side* of A, respectively.

Complicated left-side and right-side functions can generate serious computational difficulties when imprecise information is modeled by FNs. In order to overcome this problem, the original FN can be approximated with other easier functions. The simplest FNs used in fuzzy modeling, fuzzy control and fuzzy decision making are the trapezoidal and triangular FNs. In a trapezoidal FN the functions A^- and A^+ are linear: for instance,

$A^-(x) = (x-a)/(c-a)$ and $A^+(x) = (b-x)/(b-d)$ with $a \le b \le c \le d$, $a \ne c$, $b \ne d$. In a triangular FN it is assumed d=c. Other simple FNs widely used is the degenerated left (resp., right) size semi-trapezoidal FNs with $a = c < d < b$ (resp., $a < c < d = b$). In many problems trapezoidal, triangular or semi-trapezoidal approximations of FNs could give a loss of information not negligible and this can significantly affect the reliability of the results.

Furthermore, the membership functions of FNs used in applications are not generally known: for example, when they are obtained as relative frequencies of measured occurrences in in a discrete set of points, or in collaborative applications in which a set of stakeholders evaluate separately the membership degrees of a FN and the function is assigned as an average of these membership degrees. For making understandable this idea, in the example of Fig 1 the membership degree *f*(T) of the fuzzy set "daily temperature T" (measured in °C) for a discrete set of 100 points is the average of the membership degrees evaluated separately by many stakeholders.

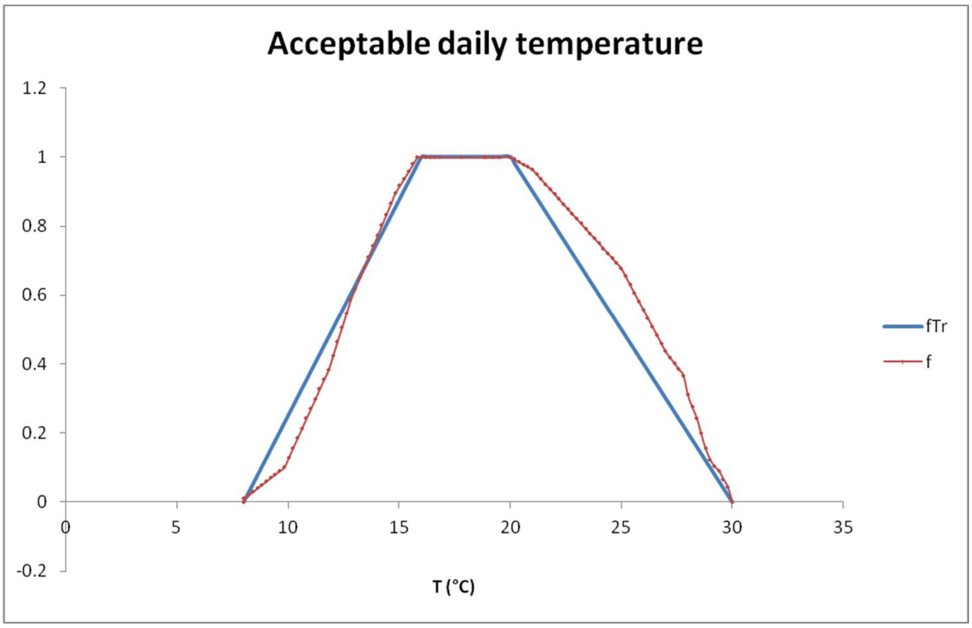


Fig. 1. Example of FN constructed for a discrete set of points and approximated with a trapezoidal membership function

Recently many methods are proposed in order to approximate FNs with easier FNs using a suitable metric (see, e.g. [1, 6, 23, 29, 30, 31]). Some authors investigate approximations by adding some restrictions to preserve properties of a FN as core [2], ambiguity [3, 4, 5], expected interval, translation invariance and scale invariance [19, 20]. As pointed out in [8], by using a trapezoidal FN as approximation function by, only a limited number of characteristics can be preserved since a trapezoidal FN depends only by four parameters, and the best approach to preserve multiple characteristics is to use sequences of FNs. In [7] a new method is proposed based on the inverse fuzzy transform (iF-transform) [24] in order to construct sequence of FNs which converge uniformly to a FN, preserving properties as its support, core, ambiguity, quasi-concavity and expected interval. The F-transform method was already used image analysis (see, e.g. [10, 13, 14, 26, 27]), data analysis applications (see, e.g. [11, 12, 15, 25]). In [28] the bi-dimensional F-transform is used to approximate type-2 FNs. In [7] the extended iF-transform method, proposed in [27], is applied to approximate FNs preserving the support and the quasi-concavity property. The main advantage of this method is to reach the desired approximation with a linear rate of uniform convergence. However, when the membership function is given in a discrete set of point, it is necessary to verify that this dataset is sufficiently dense with respect to the uniform fuzzy partition of the support of the FN. More specifically, the F-transform method divides the interval [a,b] in n sub-intervals of width $h = (n-1)/(b-a)$. The points $x_1= a$, $x_2= a+h$,..., $x_i= a+(i-1)h$,..., $x_n= b$ are called nodes: an uniform fuzzy partition of [a,b] is created by assigning n fuzzy sets with continuous membership functions $A_1,\ldots,A_n : [a,b] \rightarrow [0,1]$, called basic functions, where $A_i(x) = 0$ if $x \notin (x_{i-1},x_{i+1})$, $i = 1,\ldots,n$. When the input data form a dataset of points in [a,b], it is necessary to control that this set is dense with respect to the uniform fuzzy partition, namely we must verify that at least one data point with non-zero membership degree falls within a sub-interval (x_i-1,x_i+1) for i=1,...,n. In Fig. 2 we show an example of dataset not sufficiently dense with respect to the fuzzy partition: no data is included in (x_{i-1},x_{i+1}).

 doi:10.20944/preprints201804.0046.v1

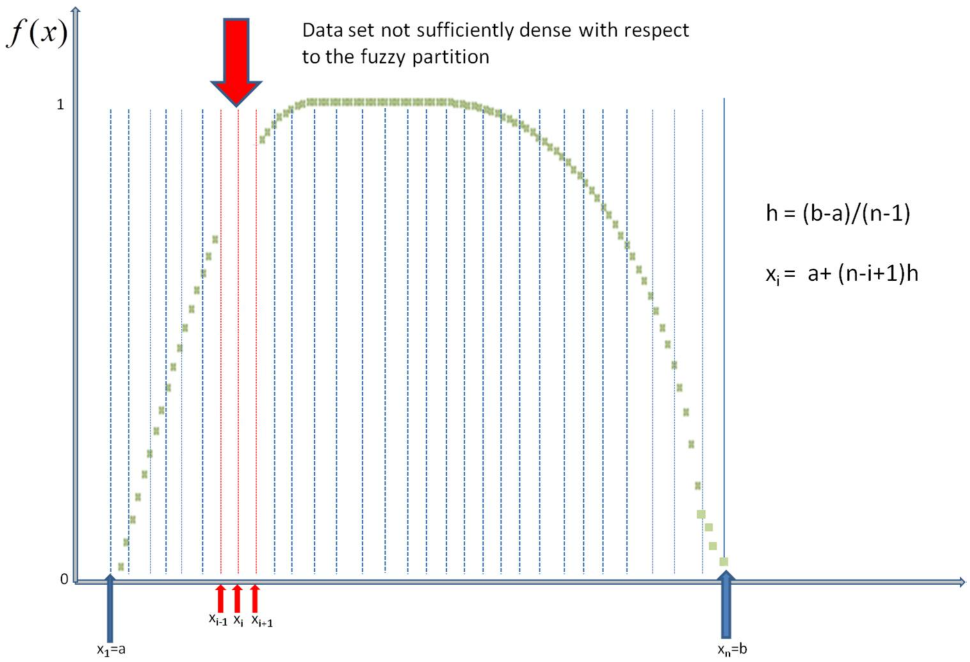


Fig. 2. Example of input dataset non sufficiently dense with respect to the fuzzy partition.

The paper is organized as follows: Section 2 contains the basic notions of fuzzy number and F-transform, in Section 3 we introduce the extended iF-transform method which in Section 4 is applied to a Fuzzy Rule Based Systems (FRBS). In Section 5 we give the results of our tests and final considerations are reported in Section 6.

1.1 Preliminaries

As already shown in [11, 12], the extended iF-transform method, proposed in [7], approximates a function assigned on a discrete set of points by means of an iterative process. Strictly speaking, we set initially the dimension n of the fuzzy partition to a value n_0, afterwards it is necessary to verify at any step that the dataset is sufficiently dense with respect to the fuzzy partition and that the approximation error is less or equal to a prefixed threshold: in this case the process stops and the direct F-transform components are stored, otherwise n is set to n + 1 and the process is iterated by considering a finer fuzzy partition. Below we schematize the pseudocode of this process.

We propose a new Mamdani FRBS in which we use the extended iF-transform to approximate FNs and we apply the above process for constructing the input fuzzy sets in the antecedent and the output fuzzy sets.

Approximation of a set of data by using the extended inverse F-transform

```
n:=n0
Create the fuzzy partition
Calculate the direct F-transform components
    WHILE the dataset is sufficiently dense with respect to the fuzzy partition
        Calculate the approximation error
        IF (approximation error ≤ threshold) THEN
            Store the direct F-transform components
            RETURN "SUCCESS"
```

 doi:10.20944/preprints201804.0046.v1


```
        END IF
        n:=n+1
        Calculate the extended direct F-transform components
    END WHILE
    RETURN "ERROR: Dataset not sufficiently dense"
    END
```

The extended iF-transform method for approximation of the FNs is used to fuzzify the crisp input data. The min and max operators are applied as AND and OR connectives in the antecedent of the fuzzy rules to calculate the strength of any rule. The defuzzification process of the output fuzzy set is carried out via the discrete Center of Gravity (CoG) method. For example, we consider a system formed by two rules in the form:

$$\begin{cases} r_1 : (x \text{ is } \mathrm{A}_1) \ \text{ OR } \ (y \text{ is } \mathrm{B}_1) \rightarrow z \text{ is } \mathrm{C}_1 \\ r_2 : (x \text{ is } \mathrm{A}_2) \ \text{ AND } \ (y \text{ is } \mathrm{B}_2) \rightarrow z \text{ is } \mathrm{C}_2 \end{cases} \tag{2}$$

where A_1 and A_2 are two FNs for the linguistic input variable x, B_1 and B_2 are two FNs for the input linguistic variable y, C_1 and C_2 are two FNs for the output variable z. Applying the extended iF-transforms to evaluate each fuzzy set, we suppose that $A_1^-(x) = 0.4$, $A_1^+(x) = 0.7$, $B_1^-(x) = 0.7$, $B_2^+(x) = 0.3$. With max (resp., min) operator as connective OR (resp., AND), we obtain the value of the two rules: r_1 = *max*(0.4, 0.7) = 0.7 and r_2 = *min*(0.7, 0,3) = 0.3. In the defuzzification process we reconstruct the output fuzzy set as

$$f_C(z) = \max\left[\min\left(C_1(z), s_1\right), \min\left(C_2(z), s_2\right)\right] \tag{3}$$

The CoG method is useful for obtaining the final crisp value $\hat{z}$ of the output variable as

$$\hat{z} = \sum_{i=1}^{Nc} C(z_i) \cdot z_i \Big/ \sum_{i=1}^{Nc} C(z_i) \tag{4}$$

where Nc is the number of rules and $z_1 < z_2 < \ldots < z_{Nc}$ are points of the support of C. In Fig. 3 we give an example.

2. Fuzzy Numbers and F-transforms

2. 1 Fuzzy Numbers and F-transforms

Given a value $\alpha \in [0,1]$, we denote with A_α, called *α-cut* of a FN A, the crisp set containing the elements $x \in R$ with a membership degree greater or equal to α. We also use the interval

$$[A]_\alpha = [\mathrm{a}_1(\alpha), \mathrm{a}_2(\alpha)] \tag{5}$$

Where

$$a_1(\alpha) = \inf\left\{x \in R : A(x) \geq \alpha\right\} \tag{6}$$

$$a_2(\alpha) = \sup\left\{x \in R : A(x) \geq \alpha\right\} \tag{7}$$

For $\alpha = 1$, $[\mathrm{A}]_1 = [\mathrm{a}_1(1), \mathrm{a}_2(1)]$ is called the *core* of the FN and denoted by *core(A)*.

The support of a fuzzy set is given by the closure of the crisp set:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 April 2018 doi:10.20944/preprints201804.0046.v1

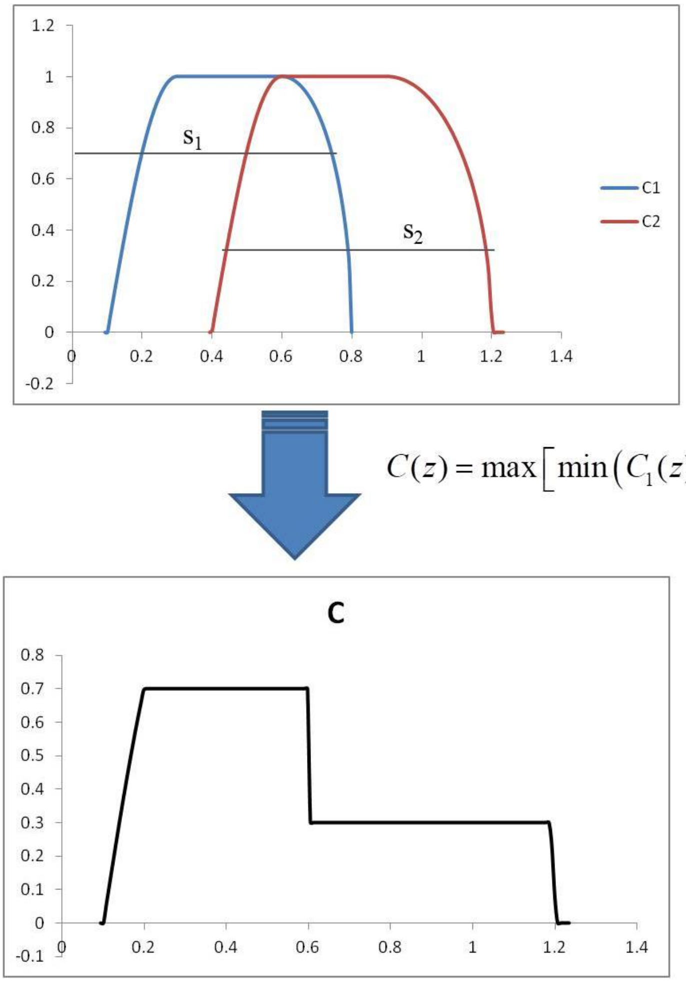
$$\mathrm{supp}(A)=\{x\in R\,|\,A(x)>0\} \tag{8}$$

Given two arbitrary FNs A and B, two metrics are considered in [17, 18]: the *Chebyshev distance*

$$d(A,B)=\sup\{x\in R:|A(x)-B(x)|\} \tag{9}$$

and the extension of the Euclidean metric given by

$$d(A,B)=\sqrt{\int_0^1[a_1(\alpha)-b_1(\alpha)]^2d\alpha+\int_0^1[a_2(\alpha)-b_2(\alpha)]^2\,d\alpha} \tag{10}$$



$$C(z)=\max\left[\min\left(C_1(z),s_1\right),\min\left(C_2(z),s_2\right)\right]$$

Fig. 3. Defuzzification of the output fuzzy set

Two properties of A are given in [9] called Ambiguity and Value defined as

$$Amb_r(A)=\int_0^1 r(\alpha)\cdot[a_2(\alpha)-a_1(\alpha)]d\alpha \tag{11}$$

And

$$Val_r(A) = \int_0^1 r(\alpha)\cdot(a_2(\alpha)+a_1(\alpha))d\alpha \tag{12}$$

respectively, where r: [0,1]→[0,1] is a not decreasing function called reducing function with with r(0) = 0 and r(1) = 1. Another important propriety is the Expected Interval of A, introduced in [16, 21], defined as follows

$$EI(A) = \left[\int_0^1 a_1(\alpha)d\alpha, \int_0^1 a_2(\alpha)d\alpha\right] \tag{13}$$

We have EI(A) = [(a+c)/2, (d+b)/2] for a trapezoidal FN A.

2. 2 Direct and Inverse F-transforms

Following the definitions and notations of [24], let $n \geq 2$ and $\mathbf{P} = \{x_1, x_2, \ldots, x_n\}$ be a set of points of [a,b], called nodes, such that $x_1 = a < x_2 < \ldots < x_n = b$. Let $\{A_1,\ldots,A_n\}$ be an assigned family of fuzzy sets with membership functions $A_1(x),\ldots,A_n(x)$: [a,b] → [0,1], called basic functions. We say that it constitutes a fuzzy partition of [a,b] if the following properties hold:

(1) $A_i(x_i) = 1$ for every $i = 1,2,\ldots,n$;

(2) $A_i(x) = 0$ if $x \notin (x_{i-1},x_{i+1})$ for $i=2,\ldots,n$;

(3) $A_i(x)$ is a continuous function on [a,b];

(4) $A_i(x)$ strictly increases on $[x_{i-1},x_i]$ for $i = 2, \ldots, n$ and strictly decreases on $[x_i,x_i+1]$ for $i = 1,\ldots, n-1$;

(5) $\sum_{i=1}^{n} A_i(x) = 1$ for every $x \in [a,b]$.

Furthermore, we say that the fuzzy sets $\{A_1,\ldots,A_n\}$ form an h-uniform fuzzy partition of [a,b] if

(6) $n \geq 3$ and $x_i = a + h\cdot(i-1)$, where $h = (b-a)/(n-1)$ and $i = 1, 2, \ldots, n$ (that is the nodes are equidistant);

(7) $A_i(x_i - x) = A_i(x_i + x)$ for every $x \in [0,h]$ and $i = 2,\ldots, n-1$;

(8) $A_{i+1}(x) = A_i(x - h)$ for every $x \in [x_i,x_{i+1}]$ and $i = 1,2,\ldots, n-1$.

Let *f*(x) be a continuous function on [a,b]. The following quantity

$$F_i = \int_a^b f(x)A_i(x) \Bigg/ \int_a^b A_i(x) \tag{14}$$

for $i = 1, \ldots, n$, is the i*th* component of the direct F-transform $[F_1, F_2, \ldots, F_n]$ of f with respect to the family of basic functions $\{A_1, A_2, \ldots, A_n\}$. If this fuzzy partition is h-uniform, the components are the following [24]:

$$F_i = \begin{cases} 2h^{-1}\int_{x_1}^{x_2} f(x)A_1(x)dx & if \quad i=1 \\ h^{-1}\int_{x_{i-1}}^{x_i} f(x)A_i(x)dx & if \quad i=2,\dots,n-1 \\ 2h^{-1}\int_{x_{n-1}}^{x_n} f(x)A_n(x)dx & if \quad i=n \end{cases} \tag{15}$$

The following function

$$f_{F,n}(x) = \sum_{i=1}^{n} F_i A_i(x) \tag{16}$$

where $x \in [a,b]$, is defined the iF-transform of f with respect to $\{A_1, A_2, \dots, A_n\}$ and it approximates f in the sense of the following theorem [24]:

Theorem 1. Let f(x) be a continuous function on [a,b]. For every $\varepsilon > 0$, then there exist an integer $n(\varepsilon)$ and a fuzzy partition $\{A_1, A_2, \dots, A_{n(\varepsilon)}\}$ of [a,b] such that $|f(x) - f_{F,n(\varepsilon)}| < \varepsilon$ with respect to the existing fuzzy partition.

In the discrete case we know that the function f assumes assigned values in the points $p_1,\dots,p_m$ of [a,b],. If the set $\{p_1,\dots,p_m\}$ is sufficiently dense with respect to the fixed partition $\{A_1, A_2, \dots, A_n\}$, that is for each $i = 1,\dots,n$ there exists an index $j \in \{1,\dots,m\}$ such that $A_i(p_j) > 0$, we can define the n-tuple $\{F_1, F_2,\dots, F_n\}$ as the discrete direct F-transform of f with respect to $\{A_1, A_2, \dots, A_n\}$, where each F_i is given by

$$F_i = \sum_{j=1}^{m} f(p_j)\cdot A_i(p_j) \Big/ \sum_{j=1}^{m} A_i(p_j) \tag{17}$$

for $i=1,\dots,n$. Similarly we define the discrete iF-transform of f with respect to the $\{A_1, A_2, \dots, A_n\}$ by setting

$$f_{F,n}(p_j) = \sum_{i=1}^{n} F_i A_i(p_j) \tag{18}$$

for every $j \in \{1,\dots,m\}$. We have the following theorem [24]:

Theorem 2. Let $f(x)$ be a function assigned on a set of points $\{p_1,\dots, p_m\} \subseteq [a,b]$. Then for every $\varepsilon > 0$, there exist an integer $n(\varepsilon)$ and a related fuzzy partition $\{A_1, A_2, \dots, A_{n(\varepsilon)}\}$ of [a,b] such that $\{p_1,\dots,p_m\}$ is sufficiently dense with respect to the existing fuzzy partition and for every $p_j \in [a, b]$, $j = 1,\dots,m$, the following inequality

$$|f(p_j) - f_{F,n(\varepsilon)}(p_j)| < \varepsilon \tag{19}$$

remains true.

3. The extended iF-transform and fuzzy numbers

In [27] the extended iF-transform of a continuous function f is introduced in order to preserve the monotonicity as follows. For an h-uniform fuzzy partition $\{A_1, A_2, \dots, A_n\}$, the function f is extended to [a-h,b+h] as follows:

$$\overline{f}(x) = \begin{cases} 2f(a) - f(2a-x) & \text{if } x \in [a-h,a] \\ f(x) & \text{if } x \in [a,b] \\ 2f(b) - f(2b-x) & \text{if } x \in [b,b+h] \end{cases} \tag{20}$$

Then the following basic functions are defined as

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 April 2018 doi:10.20944/preprints201804.0046.v1

$$\begin{aligned} &\overline{A}_1(x) = \begin{cases} A_1(2a-x) & \text{if } x \in [a\text{-}h, a] \\ A_1(x) & \text{if } x \in [a, a+h] \end{cases} \\ &\overline{A}_i(x) = A_i(x) \qquad \text{for } i = 2,\dots,n\text{-}1 \\ &\overline{A}_n(x) = \begin{cases} A_n(x) & \text{if } x \in [b\text{-}h, b] \\ A_n(2b-x) & \text{if } x \in [b, b+h] \end{cases} \end{aligned} \tag{21}$$

Then the i*th* component $\overline{F}_i$ of the extended direct F-transform of f with respect to the family of basic functions $\{A_1, A_2, \dots, A_n\}$ is given by

$$\begin{aligned} &\overline{F}_1 = \frac{1}{h}\int_{a-h}^{a+h} \overline{f}(x)\overline{A_1}(x)dx, \\ &\overline{F}_i(x) = F_i(x) \qquad i = 2,..,n-1 \\ &\overline{F}_n = \frac{1}{h}\int_{b-h}^{b+h} \overline{f}(x)\overline{A_n}(x)dx \end{aligned} \tag{22}$$

Hence the extended iF-transform of *f* is given by

$$\overline{f}_{F,n}(x) = \overline{F_1}\,\overline{A}_1(x) + \sum_{i=2}^{n-1} F_i A_i(x) + \overline{F_n}\,\overline{A_n}(x) \qquad x \in [a\text{-}h,\, b+h] \tag{23}$$

By [8, Lemma 9], we obtain that

$$\begin{aligned} \overline{f_{F,n}}(a) = \overline{F}_1 = f(a) \\ \overline{f_{F,n}}(b) = \overline{F}_n = f(b) \end{aligned} \tag{24}$$

Let S be a fuzzy number with a continuous membership function and supp(S) = [a,b]. We consider an h-uniform fuzzy partition $\{A_1, A_2, \dots, A_n\}$ of [a,b] with $n \geq 3$ and let $\overline{S}_{F,n}(x)$ be the extended iF-transform of S. We obtain that [8, Prop. 11]:

$$\begin{aligned} &\overline{S}_{F,n}(a) = \overline{S}_{F,n}(b) = 0 \\ &\overline{S}_{F,n}(x) > 0 \quad \forall x \in (a,b) \\ &\overline{S}_{F,n}(x) = \sum_{i=2}^{n-1} S_i A_i(x) \end{aligned} \tag{25}$$

where S_i is the i*th* component of the direct F-transform of S (cfr., formulae (15)). Theorem 13 of [8] provides the approximation property of the extended iF-transform as follows.

Theorem 3. Let S be a FN having a continuous membership function and supp(S) = [a,b]. Let a fuzzy partition $\{A_1, A_2, \dots, A_n\}$ of [a,b] be h-uniform with $n \geq 3$ and $\overline{S}_{F,n}(x)$ be the extended iF-transform of S calculated by (23). Then the following inequality holds:

$$\sup_{x \in [a,b]} \left| \overline{S}_{F,n}(x) - S(x) \right| \leq 2\omega(S,h) \tag{26}$$

where $\omega(S,h)$ is the modulus of continuity of S given by

$$2\omega(S,h) = \sup_{x,y \in [a,b]:|a-b| \leq h} \left| S(x) - S(y) \right| \tag{27}$$

Another important theorem [8, Th. 14] proves that the extended iF-transform preserves the shape and the approximation properties of a FN as follows.

Theorem 4. Let S be a FN having a continuous membership function, supp(S) = [a,b] and core(S) = [c,d], a < c < d < b. Let a fuzzy partition {A_1, A_2, …, A_n} of [a,b] be h-uniform with n ≥ 3 and a fuzzy set T such that T(x) = $\overline{S}_{F,n}(x)$ calculated by (23) in [a,b] and T(x)=0 if x $\notin$ [a,b]. If h = (b-a)/(n-1) is such that h ≤ min{(d-c)/4, c-a, b-d}, then T is a FN for which the following hold:

supp(T) = supp(S);
If core(T) = [c′,d′], then c ≤ c′ ≤ d′ ≤ d, |c-c′| ≤ 2h, |d-d′| ≤ 2h;
$\sup_{x\in[a,b]} |T(x)-S(x)| \le 4\omega(S,h)$;
If S^- strictly increases on [a,c], then T strictly increases on [a,c′];
If S^+ strictly decreases on [d,b], then T strictly decreases on [d′,b].

The preservation of the properties "Ambiguity" and "Value" of a FN and its approximation with an extended iF-transform is given by the following theorem in [8,Theorem 27]:

Theorem 5. Let S be a FN having a continuous membership function with supp(S) = [a,b] and core(S) = [c,d], a < c < d < b. Let a fuzzy partition {A_1, A_2, …, A_n} of [a,b] be h-uniform with n ≥ 3 and a fuzzy set T such that T(x) = $\overline{S}_{F,n}(x)$ given by (23) in [a,b] and T(x)=0 if x $\notin$ [a,b]. Let core(T) = [c′,d′] with c′ ≤ d′ . By putting $\delta_h = 2\omega(f,h)$, we obtain that

$$\left|Amb_r(S) - Amb_r(T)\right| \le \left(\tilde{K}_{h,1}(S) + \tilde{K}_{h,2}(S)\right)\delta_h \tag{28}$$

$$\left|Val_r(S) - Val_r(T)\right| \le \left(\tilde{K}_{h,1}(S) + \tilde{K}_{h,2}(S)\right)\delta_h \tag{29}$$

where $\tilde{K}_{h,1}(S)=c-a+|c|+4h$ and $\tilde{K}_{h,2}(S)=b-d+|b|+4h$.

In order to apply the extended iF-transform to approximate a FN S with one-element core, in [8] the concept of regular h-uniform partition of [a,b] is introduced as an h-uniform partition of [a,b] such that A_1 is differentiable in [a,x_2], A_i is differentiable in
[x_{i-1},x_{i+1}] for i = 2,…,n-1 and A_n is differentiable in [x_{n-1},b]. Thus we can define the normalized extended iF-transform given as

$$\overline{\overline{S}}_{F,n}(x) = \frac{\overline{S}_{F,n}(x)}{\max\limits_{x\in[a,b]}\left(\overline{S}_{F,n}(x)\right)} \qquad x \in [a-h, b+h] \tag{30}$$

A theorem similar to Theorem 5 is given in [8, Theorem 29] as follows.

Theorem 6. Let S be a FN having a continuous membership function with supp(S) = [a,b] and core(S) = {c}, a < c < b. Let be a regular h-uniform partition {A_1, A_2, …, A_n} of [a,b] and a fuzzy set T(x) = $\overline{\overline{S}}_{F,n}(x)$ given by (30) in [a,b] and T(x)=0 if x $\notin$ [a,b]. Let core(T) = [c′,d′] with c′ ≤ d′ and $\delta_h = \frac{8}{1-4\omega(f,h)}\omega(f,h)$. Then the following properties hold:

$$\left|Amb_r(S) - Amb_r(T)\right| \le \left(\tilde{K}_1(S) + \tilde{K}_2(S)\right)\delta_h \tag{31}$$

$$\left|Val_r(S) - Val_r(T)\right| \le \left(\tilde{K}_1(S) + \tilde{K}_2(S)\right)\delta_h \tag{32}$$

where $\tilde{K}_1(S)=c-a+3|c|+2\max(|a|,|b|)$ and $\tilde{K}_2(S)=b-c+3|c|+2\max(|a|,|b|)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 April 2018 doi:10.20944/preprints201804.0046.v1

Now we suppose that the membership values of a FN S in the form (1) are assigned on a discrete set of m points $a = p_1 < p_2 < \ldots < p_{m-1} < p_m = b$. We consider an h-uniform fuzzy partition $\{A_1, A_2, \ldots, A_n\}$ of [a,b]. If the set of points are sufficiently dense with respect to the fuzzy partition, i.e. if

$$\sum_{j=1}^{m} A_i(p_j) > 0 \quad i = 2, \ldots, n-1 \tag{33}$$

then the extended iF-transform of *S* is defined for any $x \in [a,b]$ as [8]:

$$\overline{S}_{F,n}(x) = \overline{S}_1 A_1(x) + \sum_{i=2}^{n-1} S_i A_i(x) + \overline{S}_n A_n(x) \tag{34}$$

where $\overline{S}_1 = S(a), \overline{S}_n = S(b)$ and S_i is the i*th* component of the direct F-transform of S in [a,b] for i=1,…,n. Similarly it can be proved that all the above properties of the extended iF-transform of a FN with continuous membership function apply in the discrete case as well.

4. Extended iF-transform and Fuzzy Rule Based System

Let the expert knowledge be formed by a set of fuzzy rules in a linguistic fuzzy model:

$$R_k: \text{IF } (x_1 = X_{1k}) \,\Delta\, (x_2 = X_{2k}) \,\Delta \ldots \Delta\, (x_n = X_{nk}) \text{ THEN } (y = Y_i) \tag{35}$$

where $x_1, x_2, \ldots, x_n$ are input variables, y is the output variable, $X_{1i}, X_{2i}, \ldots, X_{ni}, Y_i$ are fuzzy sets and the operator Δ is an AND or an OR operator. We construct a fuzzy rule set considering only AND connectives, splitting rules in which there are OR connectives in the antecedent.

We propose a FRBS in which the FNs of the fuzzy rule set are approximated by using extended iF-transforms. We suppose that the fuzzy sets in the antecedent and consequent of each rule are given by FNs whose membership functions are assigned in a discrete set of points $p_1 = a < p_2 < \ldots < p_{m-1} < p_m = b$. An example of this case occurs when, in a collaborative project, the membership values of a fuzzy set are given over a discrete set of points by means of averages of membership values assigned by different stakeholders.

Let [a,b] be the core and [c,d] be the support of this FN. We approximate the membership function of it by the extended iF-transform calculated with (34). As already said above in Section 3, we find a fuzzy partition such that the set of points is sufficiently dense with respect to it and we apply the iterative process given in Section 1.2. For each FN in the antecedents and in the consequents of the fuzzy rules, we calculate the discrete extended direct F-transform storing them in the fuzzy rule set. The crisp input data are fuzzified via (34) by using the stored direct F-transform components of the FNs. The inference engine applies to the max-min Mamdani inference model to calculate the strength of each rule and to obtain the final fuzzy set aggregating the output fuzzy sets. The crisp output value is obtained by applying the CoG method. The FRBS is schematized in Fig. 4.

The extended iF-transform approximation function approximates each fuzzy number by considering the set of points in which is assigned its membership function. This function creates an h-uniform fuzzy partition of the support of the fuzzy set and verify that the set of points is sufficiently dense with respect to the fuzzy partition. Initially n is set to a value n_0 (for example, n_0 = 3). If the set of points is not sufficiently dense with respect to the fuzzy partition, the F-transform approximation method cannot be applied, otherwise the extended direct F-transform components and the approximation error are calculated.

If this error is less than a defined threshold, the process stops and the extended direct F-transform components are stored, otherwise n is increased by 1 and the process is iterated.

If the set of points is not sufficiently dense with respect to the fuzzy partition, the process stops with an error and the previous extended direct F-transform components are stored.

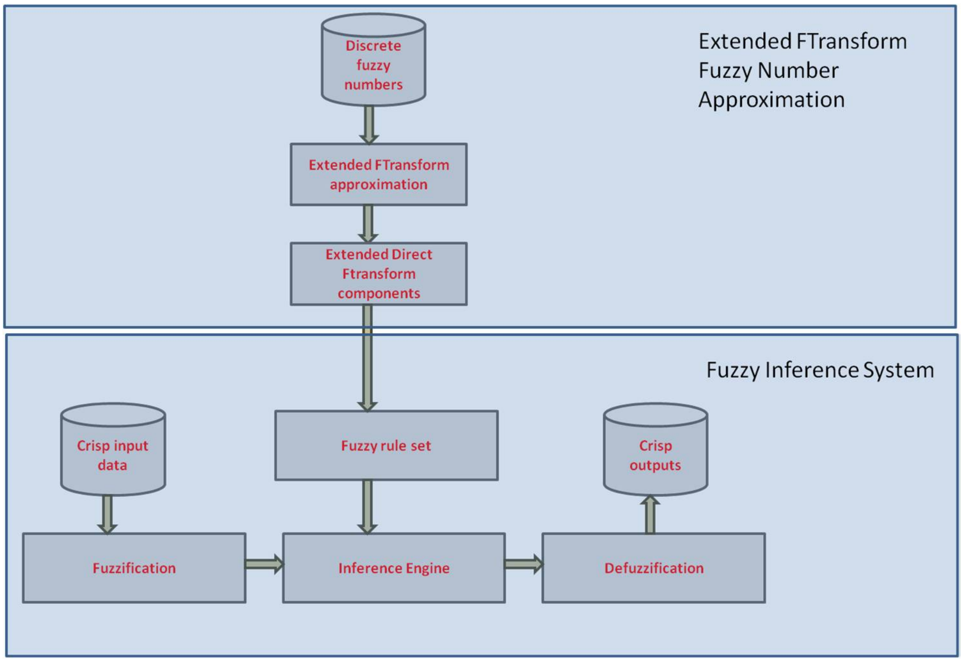


Fig. 4. Schema of the proposed FRBS

In this last case, the best possible approximation of the FN is obtained, even if the approximation error is higher than the threshold. In order to create an h-uniform fuzzy partition of [a,b], the following basic functions are used:

$$A_1(x)=\begin{cases}0.5(\cos\frac{\pi}{h}(x-a)+1) & \text{if } x\in[a,x_2]\\ 0 & \text{otherwise}\end{cases}$$
$$A_i(x)=\begin{cases}0.5(\cos\frac{\pi}{h}(x-x_i)+1) & \text{if } i\in[x_{i-1},x_{i+1}]\\ 0 & \text{otherwise}\end{cases}\quad i=2,\dots,n-1 \tag{36}$$
$$A_n(x)=\begin{cases}0.5(\cos\frac{\pi}{h}(x-x_{n-1})+1) & \text{if } i\in[x_{n-1},b]\\ 0 & \text{otherwise}\end{cases}$$

The approximation error is given by the Root Mean Square Error (RMSE) defined as

$$RMSE=\sqrt{\sum_{j=1}^{n}(\bar{S}_{F,n}(p_j)-S(p_j))^2\Big/ n} \tag{37}$$

The threshold for the RMSE is set as a positive value much smaller than 1. The extended iF-transform method is schematized in the following pseudocode.

Algorithm: Extended F-transform approximation	
Description:	*Approximate a fuzzy number with an extended iF-transform*
Input:	*Initial fuzzy partition size* n_0
	Threshold parameter
	A set of m points and their membership function value (p_1, $f(p_1)$),..., (p_n, $f(p_n)$)
Output:	*RMSE error*
	Extended Direct F-transform components

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 April 2018 doi:10.20944/preprints201804.0046.v1


```
n:= n0
Read the dataset of points
Create a h-uniform fuzzy partition by using the basic functions (36)
Calculate the extended direct F-transform components
WHILE the dataset is sufficiently dense with respect to the fuzzy partition
Calculate the RMSE approximation error   (37)
IF (RMSE approximation error ≤ threshold) THEN
        Store the extended direct F-transform components and the RSME error
        RETURN "Success"
  END IF
  n:=n+1
  Create a h-uniform fuzzy partition by using the basic functions (36)
  Calculate the extended direct F-transform components
END WHILE
Store the extended direct previous F-transform components (n = n-1) and the RMSE
     error
RETURN "ERROR: Dataset non sufficiently dense
END
```

The fuzzification reads the input data and calculates the membership degree of each fuzzy set related to the input variable using (34). The strength of each rule is obtained via the min connective. If $f'_{X_{hk}}(x_k)$ is the approximated membership degree of the input variable x_k, the strength of the k*th* rule is the following:

$$f_B(y) = \max\left\{f'_{X_{h1}}(x_1), f'_{X_{h2}}(x_2), \dots, f'_{X_{hk}}(x_k)\right\} \tag{38}$$

The output fuzzy set is constructed as follows:

$$f_B(y) = \max\left\{\min\left(f'_{Y_1}(y), s_1\right), \min\left(f'_{Y_2}(y), s_2\right), \dots, \min\left(f'_{Y_r}(y), s_r\right)\right\} \tag{39}$$

where $f'_B(y)$ is the approximated membership function of the output variable to the fuzzy set in the consequent of the k*th* rule. The defuzzification function implements the CoG algorithm for converting the fuzzy output in a crisp number. We partition the support of the output fuzzy set in N_B intervals with equal width. Let y_i be the value of the midpoint of the i*th* interval. The output crisp value $\hat{y}$ is as follows:

$$\hat{y} = \sum_{i=1}^{NB} f_B(y_i) \cdot y_i \Big/ \sum_{i=1}^{N_B} f_B(y_i) \tag{40}$$

We test our FRBS to a spatial decision problem in Section 5.

5. Experimental results: the liveability in residential housings

We apply the extended F-transform in a FRBS based on a set of census data of the 92 municipalities of the district of Naples (Italy), related to the residential housing. Our aim is to evaluate their liveability whose crisp output variable is evaluated in percentage on the basis of a set of fuzzy rules extracted by experts in which the following six linguistic input variables are considered: x1 = average surface of the housings in m2, x2 = percentage of housings with six or more rooms, x3 = percentage of residential buildings built since 2000, x4 = percentage of housings with centralized or autonomous heating system, x5 = percentage of housings with two or more showers or bathtubs, x6 = percentage of housings with two or more restrooms. The crisp input data are extracted from the ISTAT dataset. The crisp value of the variable x1 is given by the total surface of

 doi:10.20944/preprints201804.0046.v1

the housings in the municipality dividing by the number of housings. The crisp values of the variables x2, …, x6 are obtained dividing the corresponding absolute value recorded in the dataset by the total number of housings in the municipality. The domain of any variable is partitioned in 5 fuzzy sets labeled as "Low", Mean Low", "Mean", "Mean High", "High". The fuzzy rule set contains the following 62 fuzzy rules constructed by a set of twenty experts.

Table 1. The fuzzy rule set used for evaluating the liveability in residential housings

ID	Rule
r1	IF (x1 = High) AND (x2 = High) AND (x3 = High) THEN y = High
r2	IF (x1 = High) AND (x2 = Mean High) AND (x4 = Mean High) THEN y = Mean High
r3	IF (x1 = High) AND (x3 = High) THEN y = High
r4	IF (x1 = High) AND (x4 = High) THEN y = High
r5	IF (x1 = High) AND (x3 = Mean High) AND (x5 = High) THEN y = High
r6	IF (x1 = High) AND (x3 = Mean High) AND (x6 = High) THEN y = High
r7	IF (x1 = High) AND (x3 = Mean High) AND (x5 = Mean High) THEN y = Mean High
r8	IF (x1 = High) AND (x3 = Mean High) AND (x6 = Mean High) THEN y = Mean High
r9	IF (x1 = High) AND (x4 = Mean High) AND (x5 = High) THEN y = High
r10	IF (x1 = High) AND (x4 = Mean High) AND (x6 = High) THEN y = High
r11	IF (x1 = High) AND (x4 = Mean High) AND (x5 = Mean High) THEN y = Mean High
r12	IF (x1 = High) AND (x4 = Mean High) AND (x6 = Mean High) THEN y = Mean High
r13	IF (x2 = High) AND (x3 = High) THEN y = High
r14	IF (x2 = High) AND (x4 = High) THEN y = High
r15	IF (x3 = High) AND (x4 = High) THEN y = High
r16	IF (x3 = High) AND (x4 = Mean High) AND (x5 = High) THEN y = High
r17	IF (x3 = High) AND (x4 = Mean High) AND (x5 = Mean High) THEN y = Mean High
r18	IF (x3 = High) AND (x4 = Mean High) AND (x5 = Mean) THEN y = Mean High
r19	IF (x3 = High) AND (x4 = Mean High) AND (x6 = High) THEN y = High
r20	IF (x3 = High) AND (x4 = Mean High) AND (x6 = Mean High) THEN y = Mean High
r21	IF (x4 = High) AND (x5 = High) THEN y = High
r22	IF (x1 = Mean High) AND (x3 = Mean High) THEN y = Mean High
r23	IF (x1 = Mean High) AND (x3 = Mean) THEN y = Mean High
r24	IF (x1 = Mean High) AND (x4 = Mean High) THEN y = Mean High
r25	IF (x1 = Mean High) AND (x4 = Mean) THEN y = Mean High
r26	IF (x2 = Mean High) AND (x3 = High) THEN y = Mean High
r27	IF (x2 = Mean High) AND (x3 = Mean High) THEN y = Mean High
r28	IF (x2 = Mean High) AND (x4 = High) THEN y = Mean High
r29	IF (x2 = Mean High) AND (x4 = Mean High) THEN y = Mean High
r30	IF (x1 = Mean) AND (x3 = Mean High) THEN y = Mean
r31	IF (x1 = Mean) AND (x3 = Mean) THEN y = Mean

 doi:10.20944/preprints201804.0046.v1

r32	IF (x1 = Mean) AND (x4 = Mean High) THEN y = Mean
r33	IF (x1 = Mean) AND (x4 = Mean) THEN y = Mean
r36	IF (x2 = Mean) AND (x3 = Mean) THEN y = Mean
r37	IF (x2 = Mean) AND (x4 = Mean) THEN y = Mean
r38	IF (x3 = Mean) AND (x5 = Mean) THEN y = Mean
r39	IF (x3 = Mean) AND (x6 = Mean) THEN y = Mean
r40	IF (x4 = Mean) AND (x5 = Mean) THEN y = Mean
r41	IF (x4 = Mean) AND (x6 = Mean) THEN y = Mean
r42	IF (x1 = Mean) AND (x3 = Mean Low) THEN y = Mean Low
r43	IF (x1 = Mean) AND (x4 = Mean Low) THEN y = Mean Low
r44	IF (x1 = Mean Low) AND (x3 = Mean) THEN y = Mean Low
r45	IF (x1 = Mean Low) AND (x4 = Mean) THEN y = Mean Low
r46	IF (x2 = Mean Low) AND (x3 = Mean) THEN y = Mean Low
r47	IF (x2 = Mean Low) AND (x4 = Mean) THEN y = Mean Low
r48	IF (x3 = Mean Low) AND (x5 = Mean Low) THEN y = Mean Low
r49	IF (x3 = Mean Low) AND (x6 = Mean Low) THEN y = Mean Low
r50	IF (x4 = Mean Low) AND (x5 = Mean Low) THEN y = Mean Low
r51	IF (x4 = Mean Low) AND (x6 = Mean Low) THEN y = Mean Low
r52	IF (x3 = Mean Low) AND (x5 = Mean Low) THEN y = Mean Low
r53	IF (x1 = Low) AND (x4 = Mean Low) THEN y = Low
r54	IF (x1 = Low) AND (x4 = Low) THEN y = Low
r55	IF (x2 = Low) AND (x4 = Mean Low) THEN y = Low
r56	IF (x2 = Low) AND (x4 = Low) THEN y = Low
r57	IF (x2 = Low) AND (x5 = Low) THEN y = Low
r58	IF (x2 = Low) AND (x6 = Low) THEN y = Low
r59	IF (x3 = Low) AND (x5= Low) THEN y = Low
r60	IF (x3 = Low) AND (x6= Low) THEN y = Low
r61	IF (x4 = Low) AND (x5= Low) THEN y = Low
r62	IF (x4 = Low) AND (x6= Low) THEN y = Low

In the pre-processing phase we apply the extended F-transform based algorithm to approximate the five FNs associated to each variable. Each FN is obtained as average of the membership values assigned by the experts in 200 points.

In Fig. 5 we show some FNs and their approximations obtained by applying the extended F-transform. We set the threshold to 0.01, so having a RMSE less than 0.01 for every FN.

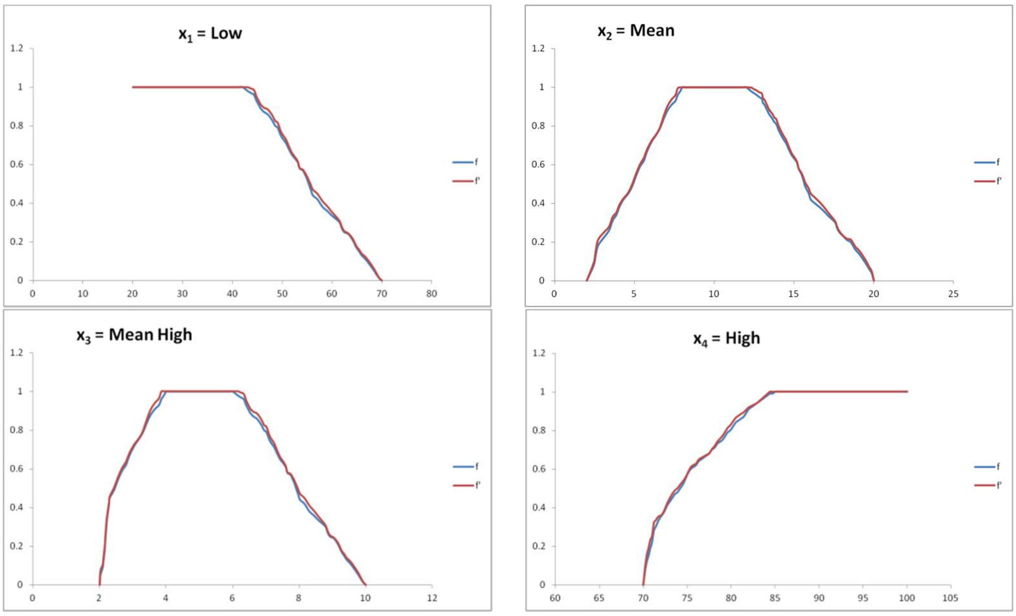


Fig. 5. Fuzzy numbers x_1 = Low, x_2 = Mean, x_3 = Mean High and x_4 = High (in blue) and their extended iF-transform approximations (in red).

The FNs (x_1 = Low) and (x_4 = High) have a degenerated side. In Table 2.i we show the parameters a, c, d, b of each FN x_i (i = 1, 2, 3, 4, 5, 6 and the RMSE, respectively.

Table 2.1. Parameters and RMSE of the approximation for fuzzy sets of x_1

		x_1 (m²)			×
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	20	20	45	70	9.11×10^{-3}
Mean Low	45	70	75	90	9.91×10^{-3}
Mean	75	90	95	100	9.17×10^{-3}
Mean High	95	100	115	125	9.76×10^{-3}
High	110	120	150	150	9.34×10^{-3}

Table 2.2. Parameters and RMSE of the approximation for fuzzy sets of x_2

		x_2			
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	0	0	1	4	9.18×10^{-3}
Mean Low	0.5	3	6	8	9.43×10^{-3}
Mean	2	7	12	20	9.19×10^{-3}
Mean High	8	12	15	25	9.57×10^{-3}
High	15	25	50	50	9.15×10^{-3}

Table 2.3. Parameters and RMSE of the approximation for fuzzy sets of x_3

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 April 2018 doi:10.20944/preprints201804.0046.v1

x_3					
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	0	0	0.5	1	9.21×10^{-3}
Mean Low	0.4	0.6	1	1.5	9.35×10^{-3}
Mean	1	2	4	6	9.33×10^{-3}
Mean High	2	4	7	10	9.02×10^{-3}
High	6	10	30	30	9.26×10^{-3}

Table 2.4. Parameters and RMSE of the approximation for fuzzy sets of x_4

x_4					
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	0	0	30	40	9.24×10^{-3}
Mean Low	30	50	60	70	9.29×10^{-3}
Mean	60	65	70	80	9.49×10^{-3}
Mean High	75	80	85	90	9.35×10^{-3}
High	85	95	100	100	9.08×10^{-3}

Table 2.5. Parameters and RMSE of the approximation for fuzzy sets of x_5

x_5					
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	0	0	10	15	9.30×10^{-3}
Mean Low	7	15	20	25	9.52×10^{-3}
Mean	20	25	30	35	9.25×10^{-3}
Mean High	30	35	40	50	9.31×10^{-3}
High	40	50	100	100	9.37×10^{-3}

Table 2.6. Parameters and RMSE of the approximation for fuzzy sets of x_6

x_6					
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	0	0	10	15	9.32×10^{-3}
Mean Low	7	15	25	30	9.19×10^{-3}
Mean	22	28	32	35	9.24×10^{-3}
Mean High	30	40	45	55	9.48×10^{-3}
High	50	60	100	100	9.28×10^{-3}

In Table 3 we show the parameter a, c, d, b of the FNs used for the output variable y and the RMSE obtained applying the extended F-transform.

doi:10.20944/preprints201804.0046.v1

Table 3. Parameters and RMSE of the approximation for fuzzy sets of output y

y					
Fuzzy number	**a**	**c**	**d**	**b**	**RMSE**
Low	0	0	10	20	9.67×10^{-3}
Mean Low	10	20	30	40	9.32×10^{-3}
Mean	30	40	60	70	9.46×10^{-3}
Mean High	50	70	80	85	9.78×10^{-3}
High	80	90	100	100	9.31×10^{-3}

At the end of the preprocessing phase, the fuzzification of the input data is performed as well. In Figures 6i we show the thematic maps (in a Geographic Information System environment) of the six input variables x_i (i = 1,2, 3, 4, 5, 6), respectively, in the municipalities of the district of Naples. In each map the municipality is classified with the linguistic label of the fuzzy set with the highest approximated membership value.

Fig. 6.1. Thematic map for the input variable x_1

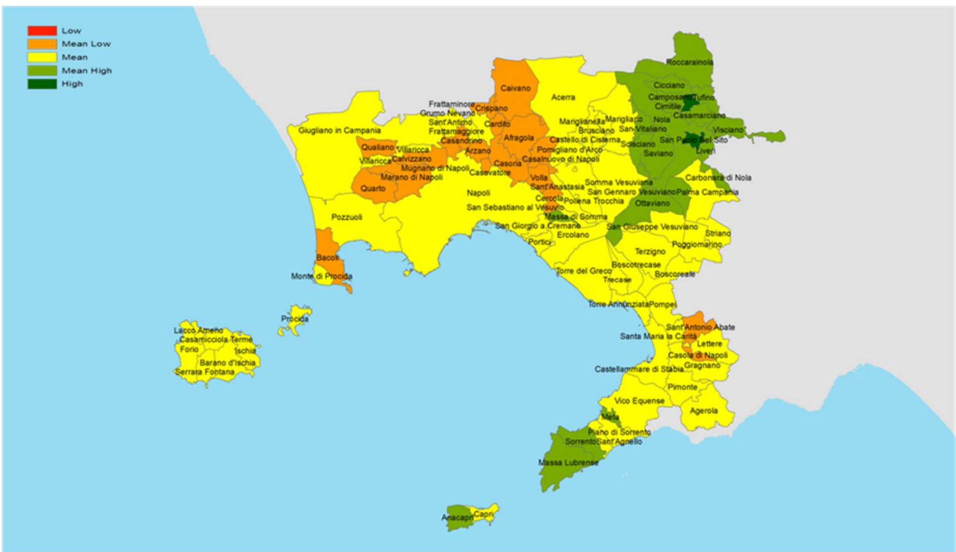


Fig. 6.2. Thematic map for input variable x_2

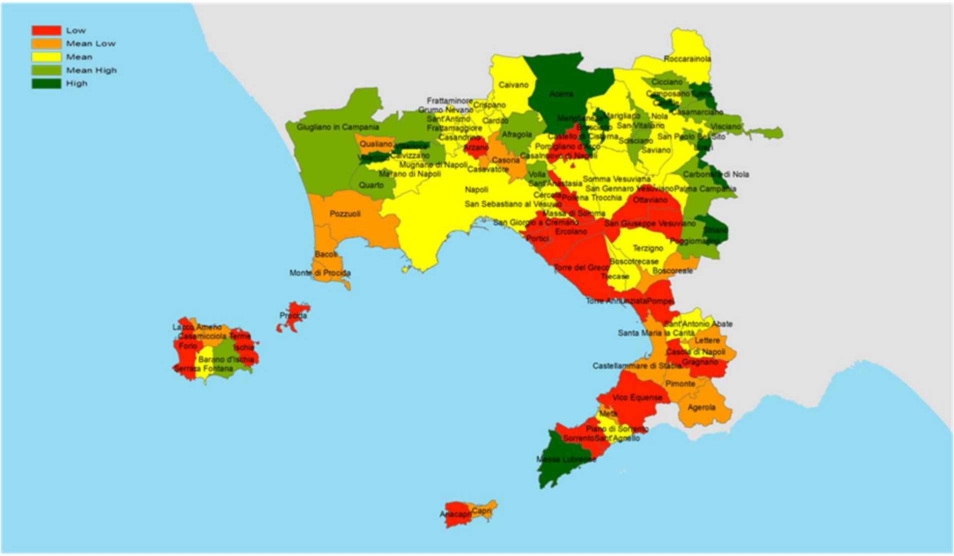


Fig. 6.3. Thematic map for input variable x_3

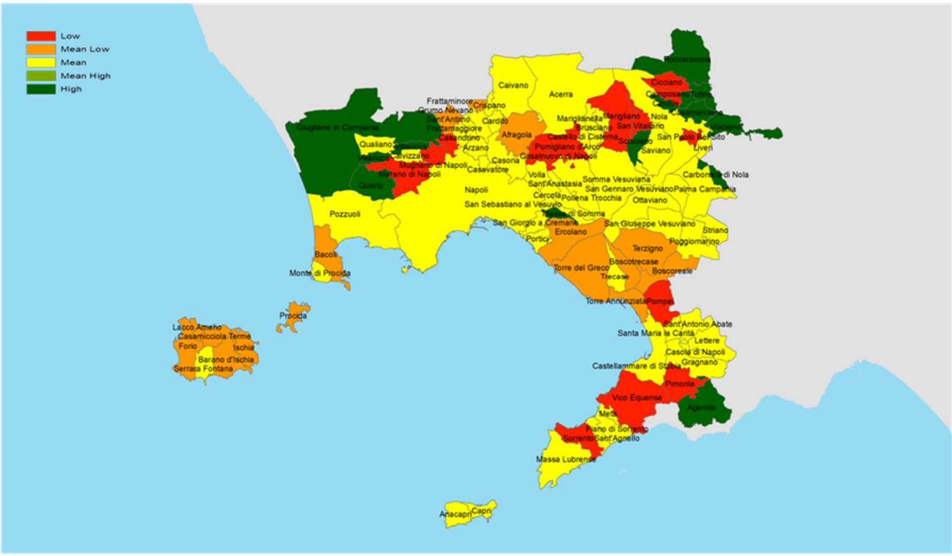


Fig. 6.4. Thematic map for input variable x_4

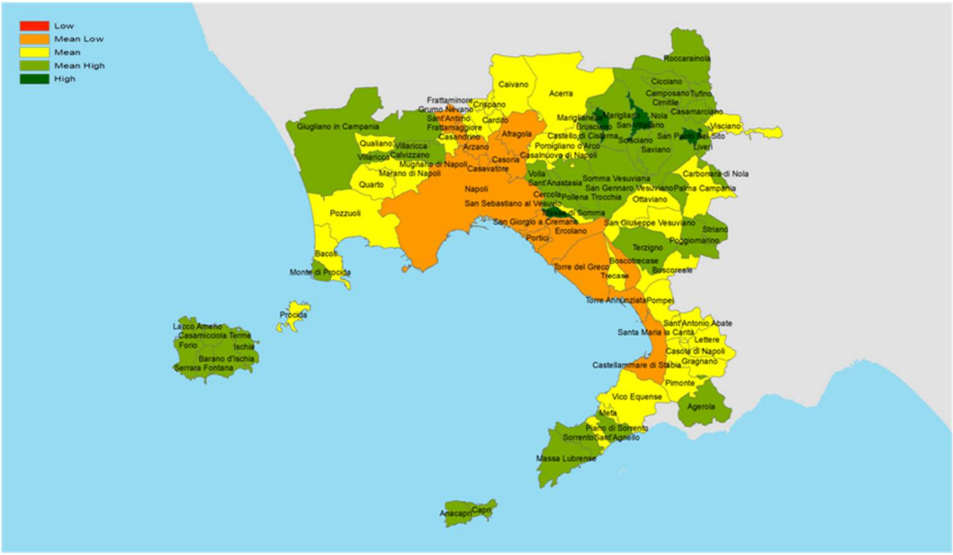


Fig. 6.5. Thematic map for input variable x5

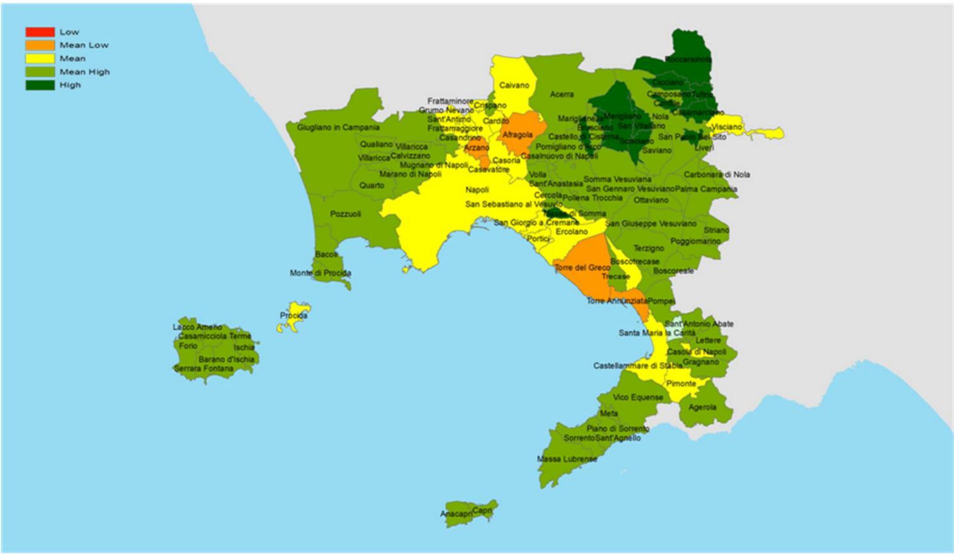


Fig. 6.6. Thematic map for input variable x6

The defuzzified final values of liveability in the residential housings (calculated in percentage) for every municipality are in Table 4.

Table 4. Defuzzified values obtained for liveability of residential housings

Municipality	ŷ	Municipality	ŷ	Municipality	ŷ
Acerra	55.18	Forio	40.83	Procida	20.68
Afragola	27.12	Frattamaggiore	55.02	Qualiano	18.36
Agerola	59.21	Frattaminore	33.8	Quarto	65.52
Anacapri	60.29	Giugliano in	81.75	Roccarainola	82.01

		Campania			
Arzano	23.34	Gragnano	29.64	SanGennaro Vesuviano	76.18
Bacoli	24.65	Grumo Nevano	55	SanGiorgio a Cremano	23.14
Barano d'Ischia	58.36	Ischia	27.13	SanGiuseppe Vesuviano	51.03
Boscoreale	32.23	Lacco Ameno	26.69	San Paolo BelSito	63.46
Boscotrecase	48.7	Lettere	42.57	San Sebastiano al Vesuvio	82.37
Brusciano	63.37	Liveri	81.39	San Vitaliano	66.52
Caivano	44.85	Marano di Napoli	24.93	Santa Maria la Carità	56.94
Calvizzano	52.06	Mariglianella	82.39	Sant'Agnello	33.85
Camposano	50.84	Marigliano	53.68	Sant'Anastasia	64.19
Capri	47.32	Massa di Somma	36.15	Sant'Antimo	47.82
Carbonara di Nola	73.29	MassaLubrense	71.5	Sant'Antonio Abate	58.19
Cardito	47.68	Melito di Napoli	26.87	Saviano	76.84
Casalnuovo di Napoli	23.45	Meta	56.38	Scisciano	88.93
Casamarciano	92.74	Monte di Procida	32.69	Serrara Fontana	54.08
Casamicciola Terme	34.61	Mugnano di Napoli	29.14	Somma Vesuviana	52.11
Casandrino	39.26	Napoli	53.82	Sorrento	20.18
Casavatore	33.15	Nola	75.35	Striano	73.69
Casola di Napoli	38.77	Ottaviano	52.94	Terzigno	52.01
Casoria	34.02	Palma Campania	62.9	Torre Annunziata	25.12
Castellammare di Stabia	44.26	Piano di Sorrento	60.67	Torre del Greco	26.36
Castello di Cisterna	73.89	Pimonte	27	Trecase	55.8
Cercola	49.67	Poggiomarino	75	Tufino	96.44
Cicciano	67.16	Pollena Trocchia	25.13	Vico Equense	20.37
Cimitile	87.38	Pomigliano d'Arco	19.75	Villaricca	78.36
Comiziano	85.46	Pompei	17.89	Visciano	78.45

Crispano	39.07	Portici	22.51	Volla	55.83
Ercolano	28.14	Pozzuoli	39.43		

In Fig. 7 we show a thematic map of the index of liveability in the residential housings: the label of output variable fuzzy set with the greatest membership degree is assigned for every municipality.

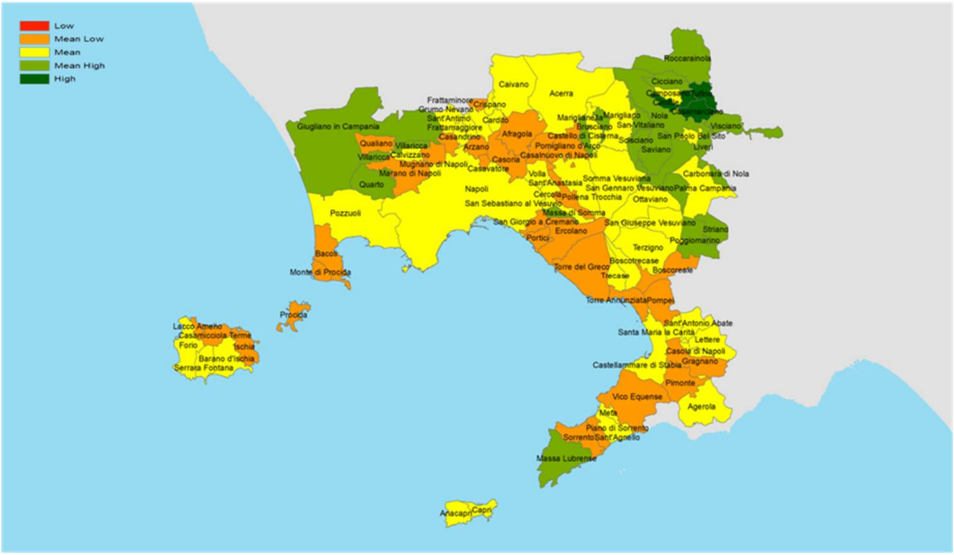


Fig. 7. Thematic map of index of liveability in residential housings

We compare these results with the ones obtained by approximating the input and output variables fuzzy sets with trapezoidal FNs, by using the approximation method of [20]. We apply the inference system to the residential housing dataset again, by using the approximated trapezoidal FNs as fuzzy sets in the antecedents and consequents of the rule set. Then we calculate the RMSE and we calculate the number and the percentage of municipalities classified with a liveability linguistic label different by the one contained in Fig.7.

Table 5. Comparisons obtained approximating input and output FN with trapezoidal FN

Comparison parameter	Value
Mean RMSE index for the fuzzy sets approximation with trapezoidal FNs	6.3×10^{-2}
Mean difference of the final crisp liveability values compared with the ones obtained by using the extended iF-transform method	5.58%
Number of municipalities classified with different linguistic labels	7
Percentage of municipalities classified with different linguistic labels	7.61%

The mean RMSE index obtained by using the trapezoidal FN is 6.3×10^{-2}: this value is greater than the threshold 1×10^{-2} set by applying the extended F-transform. The mean difference in absolute value between the crisp liveability obtained by using the trapezoidal approximation of the input and output FNs with respect to the ones obtained by using the extended IF-transform approximation overcomes 5%: this difference is generated by the greater error obtained by the approximation with trapezoidal FNs. The percentage 7.61% of the municipalities are classified differently in the final map of liveability underlines the effective improvement of the final results obtained with the extended F-transform method. The seven municipalities with different liveability class are given in Table 6.

Table 6. Municipalities with different liveability class

Municipality	Extended IFtr liveability class	Trapezoidal liveability class
Casola di Napoli	Mean	Mean Low
Casoria	Mean Low	Mean
Crispano	Mean	Mean Low
Massa di Somma	Mean	Mean Low
Pozzuoli	Mean	Mean Low
Sant'Agnello	Mean Low	Mean
Scisciano	Mean High	High

We can appropriately select the RMSE threshold in order to increase the reliability of the final results, however we point out that that the choice of a very small threshold can lead to a fuzzy uniform partition too finer for which the dataset of the corresponding values is not sufficiently dense.

6. Conclusions

We present a new method based on the extended F-transform to approximate FNs. We apply this method in a fuzzy rule-based system related to a spatial analysis problem consisting in the evaluation of the liveability of residential housings in the municipality of the district of Naples. In many spatial analysis problems, decision-making systems based on expert rules are used in order to extract thematic maps of a final index. A finer approximation of the membership functions of the fuzzy sets in the antecedents and in the consequents of the fuzzy rules is necessary to guarantee a good reliability of the final thematic maps. In many cases, for example in participatory contexts in which knowledge is provided by different experts, these FNs are assigned on a discrete set of points. In future we propose to apply the extended F-transform method to the approximation of FNs in multi-criteria fuzzy decision making problems.

Acknowledgments: This work is performed under the auspices of INDAM-GNCS (Gruppo Nazionale per il Calcolo Scientifico), Italy.

Conflicts of Interest: The authors declare no conflict of interest with this research.

References

1. S. Abbasbandy, B. Asady, *The nearest trapezoidal fuzzy number to a fuzzy quantity*, Applied Mathematics and Computation 156 (2) (2004) 381–386. Doi: 10.1016/j.amc.2003.07.025
2. S. Abbasbandy, T. Hjjari, *Weighted trapezoidal approximation–preserving cores of a fuzzy number*, Computer and Mathematics with Applications 59 (2010) 3066–3077. Doi: 10.1016/j.camwa.2010.02.026
3. A.I. Ban, L. Coroianu, *Nearest interval, triangular and trapezoidal approximation of fuzzy number preserving ambiguity*, International journal of Approximate Reasoning 53 (2012) 805–836. Doi:10.1016/j.ijar.2012.02.001
4. A.I. Ban, *Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval*, Fuzzy Sets and Systems 159 (2008) 1327–1344. Doi:10.1016/j.fss.2007.09.008
5. L. Coroianu, *Lipschitz functions and fuzzy number approximations*, Fuzzy Sets and Systems 200 (2012) 113–135. Doi: 10.1016/j.fss.2012.01.001
6. L. Coroianu, M. Gagolewski, P. Grzegorzewski, *Nearest piecewise linear* approximation of fuzzy numbers, Fuzzy Sets and Systems 233 (2013) 26–51. Doi:10.1016/j.fss.2013.02.005
7. L. Coroianu, S. G. Gal, B. Bede, *Approximation of fuzzy numbers by Bernstein operators of max–min product kind*, Fuzzy Sets and Systems 257 (2014) 41–66. Doi: 10.1016/j.fss.2013.04.010
8. L. Coroianu, L. Stefanini, *General approximation of fuzzy numbers by F–transform*, Fuzzy Sets and Systems 288 (2016) 46–74. Doi: 10.1016/j.fss.2015.03.015
9. M. Delgado, M.A.Vila, W.Voxman, *On a Canonical Representation of a Fuzzy number*, Fuzzy Sets and Systems 93 (1998) 125–135.Doi: 10.1016/S01650114(96)001443
10. F. Di Martino, V. Loia, I. Perfilieva, S. Sessa, *An image coding/decoding method based on direct and inverse fuzzy transforms*, International Journal of Approximate Reasoning 48 (1) (2008) 110–131. Doi:10.1016/j.ijar.2007.06.008
11. F. Di Martino, V. Loia, S. Sessa, *Fuzzy transforms method and attribute dependency in data analysis*, Information Sciences 180 (4) (2010) 493–505. Doi: 10.1016/j.ins.2009.10.012
12. F. Di Martino, V. Loia, S. Sessa, *Fuzzy transforms method in prediction data analysis*, Fuzzy Sets and Systems 180 (1) (2011) 146–163. Doi:10.1016/j.fss.2010.11.009
13. F. Di Martino, P. Hurtik, I. Perfilieva, S. Sessa, *A color image reduction based on fuzzy transforms*, Information Sciences 266 (4) (2014) 101–111. Doi:10.1016/j.ins.2014.01.014
14. F. Di Martino, S. Sessa, *Complete image fusion method based on fuzzy transforms*, Soft Computing (2017) 1–11. Doi: 10.1007/s00500.
15. F. Di Martino, S. Sessa, *Fuzzy transforms prediction in spatial analysis and its application to demographic balance data*, Soft Computing 21 (13) (2017) 3537–3550. Doi:10.1007/s0050001726218
16. D. Dubois, H. Prade, Operations on fuzzy numbers, Ins. J. Systems Sci. 9 (1978) 613–626. Doi: 10.1080/00207727808941724
17. P.Grzegorzewski, *Metrics and Orders in Space of Fuzzy Numbers*, Fuzzy Sets and Systems 97 (1998) 83–94. Doi: 10.1016/S01650114(96)003223
18. P. Grzegorzewski, *Nearest interval approximation of a fuzzy number*, Fuzzy Sets and Systems 130 (2002) 321–330. Doi:10.1016/S01650114(02)000982
19. P. Grzegorzewski, E. Mrówka, *Trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems 153 (2005) 115–135. Doi: 10.1016/j.fss.2004.02.015
20. P. Grzegorzewski, *Trapezoidal approximations of fuzzy numbers preserving the expected interval — Algorithms and properties*, Fuzzy Sets and Systems 159 (2008) 1354–1364. Doi: 10.1016/j.fss.2007.12.001
21. S. Heilpern, *The expected value of a fuzzy number, Fuzzy Sets and Systems* 47 (1992) 81–86. Doi: 10.1016/01650114(92)900629
22. E. H. Mamdani, *Application of fuzzy logic to approximate reasoning using linguistic synthesis*. IEEE Transactions on Computers 26 (12) (1977) 1182–1191.Doi: 10.1109/TC.1977.1674779
23. E. N. Nasibov, S. Peker, *On the nearest parametric approximation of a fuzzy number*,Fuzzy Sets and Systems 159 (11) (2008) 1365–1375. Doi:10.1016/j.fss.2007.08.005
24. I. Perfilieva, *Fuzzy transforms: theory and applications*, Fuzzy Sets and Systems 157 (2006) 993–1023. Doi: 10.1016/j.fss.2005.11.012
25. I. Perfilieva, V. Novak, A. Dvorak, *Fuzzy transforms in the analysis of data*, International Journal of Approximate Reasoning 40 (2008) 26–46. Doi:10.1016/j.ijar.2007.06.003

474 26. I. . Perfilieva, P. Hurtík, F. Di Martino, S. Sessa, *Image reduction method based on the F–transform*, Soft
475 Computing 21(7) (2017) 1847–1861.Doi: 10.1007/s0050
476 27. I. Perfilieva, B. De Baets, *Fuzzy transforms of monotone functions with application to image compression*,
477 Information Sciences 180 (17) (2010) 3304–3315. Doi: 10.1016/j.ins.2010.04.029
478 28. L. Stefanini, L. Sorini, *Type–2 fuzzy numbers and operations by F–transform*, IFSA World Congress and
479 NAFIPS Annual Meeting (IFSA/NAFIPS), 24–28/06/2013, pp. 1050–1055. Doi:
480 10.1109/IFSA–NAFIPS.2013.6608545
481 29. C-T. Yeh, *Weighted semi–trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems, 165 (2011)
482 61–80. Doi: 10.1016/j.fss.2010.11.001
483 30. C-T. Yeh, *Existence of interval, triangular, and trapezoidal approximations of fuzzy numbers under a general*
484 *condition*, Fuzzy Sets and Systems 310 (2017) 1–13. Doi: 10.1016/j.fss.2016.03.013
485 31. W. Zeng, H. Li, *Weighted triangular approximation of fuzzy numbers*, International Journal of Approximate
486 Reasoning 46 (2007) 137–150. Doi: 10.1016/j.ijar.2006.11.001

487