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Article

Characterization of Caristi Type Mapping through its Absolute Derivative

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Abstract: The purpose of this article is to characterize the Caristi type mapping by the absolute derivative. The equivalences of the Caristi mapping with contraction mapping is discussed too. In addition, it was shown that the contraction mapping can be characterized through its absolute derivative.

Keywords: Fixed point theorem; Differentiation; Metric space.

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1. Introduction and Priliminaries

In 1976 [1] J. Caristi introduced the fixed point theorem in the metric space which was one of the generalizations of the Banach's fixed point theorem. The method performed out is different the generalizations introduced by the other researchers, namely his mapping involves a real-valued function with metric space domains. Today the mappings is called Caristi type mapping. Many mathematicians regard Caristi's theorem is similar to Ekeland's variational principles [5] which do not highlight the existence of a fixed point.

Development the Caristi's fixed point theorem has been carried out by researchers through a variety of different ways such as combines the Banach fixed point theorem to that Caristi's fixed point theorem [7]. In 1996, Kada-Suzuki and Takahashi used the w-distance function to characterize the Caristi type mapping [8]. Further, there exist several results involving set-valued mapping into Caristi type conditions, see [10,11,17]. In this article we develop the Caristi's fixed point theorem which involve by utilizing the derivatives of Caristi type function.

Let (X,d) be a complete metric space and $K \subset X$. Caristi's fixed point theorem states that each mapping $f : K \longrightarrow K$ satisfies the condition: there exists a lower semi-continuous function $\varphi : K \longrightarrow [0,+\infty)$ such that

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x) \tag{1}$$

for each $x \in X$ has a fixed point.

Some of the authors mention that a mapping $f : K \longrightarrow K$ is called the Caristi type mapping if the inequalities (1) is satisfied.

One advantage of the Caristi type mapping can be used to characterize completeness of a metric space.That is if the Caristi type mappings have a fixed point on arbitrary metric spaces , then the metric space is complete see for example Kirk [9]. Not all of the mappings which have the fixed point

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results in the completeness of the metric space. It is well-known that the fixed point property for contraction mappings does not characterize metric completeness (see [16]).

A mapping $f : X \longrightarrow X$ is called a contraction, if there exists a real number $0 \leq k < 1$, such that

$$d(f(x), f(y)) \leq kd(x, y) \tag{2}$$

for all $x, y \in X$.

Theorem 1. *(Banach's Fixed Point Theorem) Let (X, d) be a complete metric space. If $f : X \longrightarrow X$ is a contraction on X, then f has a unique fixed point.*

The relation between the contraction real-valued function and its derivative is described as follows.

Lemma 2. *The function $f : [a, b] \longrightarrow \mathbb{R}$ is a differentiable on (a, b). Then f is a contraction on $[a, b]$ if and only if there exists a real numbers $0 \leq k < 1$ such that $|f'(x)| \leq k$ for all $x \in [a, b]$.*

Some of the generalizations Banach's fixed point theorem were presented in below.

Theorem 3. *(Kannan's Fixed Point Theorem)[**?**] Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a function. If there exists real numbers $0 \leq \alpha < \frac{1}{2}$, such that*

$$d(f(x), f(y)) \leq \alpha[d(x, f(x)) + d(y, f(y))] \tag{3}$$

for all $x, y \in X$, then f has a unique fixed point.

The mapping f that satisfies (3) is called Kannan type mapping.

Theorem 4. *(Reich's Fixed Point Theorem)[14] Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a function. If the real numbers a, b, c are non negative and $a + b + c < 1$ such that*

$$d(f(x), f(y)) \leq ad(x, f(x)) + bd(y, f(y)] + cd(x, y) \tag{4}$$

for all $x, y \in X$, then f has a unique fixed point.

The mapping f that satisfies (4) is called Reich type mapping.

The mapping that satisfies the inequalities (3) or (4) is categorized as contraction type mapping. If f is a contraction mapping with the constant contraction $0 \leq k < 1$, then f is a Caristi type mapping with a function (for example see [3]).

$$\varphi(x) = \frac{1}{1-k} d(x, f(x)).$$

The other result if f is a Reich type mapping, then f is Caristi type mapping with a function

$$\varphi(x) = \frac{1-c}{1-a-b-c} d(x, f(x)).$$

Similarly, Kannan type mapping (3) is included in the Caristi type mapping class. Thus, the class of the Caristi type mapping is very large, including at least the above mentioned types of contraction mappings [13].

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The advantage of Caristi type mapping which was described above, motivates us to develop the mapping. The main characteristic of the Caristi type mapping lays in the existence of a non-negative real-valued function φ. Therefore, we highlight the existence of the function. In the case, we replaced its function by the absolute derivative function that will be described below.

Suppose that $f : [a,b] \longrightarrow \mathbb{R}$ is real valued function and the point $p \in (a,b)$. We say that the function f is differentiable at $p \in (a,b)$ if there exists $f'(p) \in \mathbb{R}$ such that

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p).$$

The definition is called the classical definition in the context.

In 1971, E. Braude introduced the derivative of the metric valued function with abstract metric domains which is known as "metrically differentiable" (see [12]).

Definition 5. Let (X,d) and (Y,ρ) be two metric spaces and let $p \in X$ be a limit point. The function $f : X \longrightarrow Y$ is said ***metrically differentiable*** at p if a real number $f'(p)$ exists with the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X, x \neq y$ and $0 < d(x,p) < \delta, 0 < d(y,p) < \delta$, then

$$\left| \frac{\rho(f(x), f(y))}{d(x,y)} - f'(p) \right| < \epsilon \tag{5}$$

In 1975, K. Skaland defined it but is weaker than Braude's definition.

Definition 6. [15] Let (X,d) and (Y,ρ) be a metric spaces and let $p \in X$ be a limit point. The function $f : X \longrightarrow Y$ is said ***differentiable*** at p if real number $f'(p)$ exists with the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in N_\delta(p)$ then

$$\left| \frac{\rho(f(x), f(p))}{d(x,p)} - f'(p) \right| < \epsilon. \tag{6}$$

A non-negative real number $f'(p)$ is called the *metrically derivative* or the *quasiderivative* of the function f at the point $p \in X$. Recently, differentiation in metric spaces, as discussed in [2], explain two kinds derivative, namely the *absolute derivative* (such as definition 6) and the *strongly absolute derivative* (such as definition 5).

Throughout this paper, we use the notation f'_{abs} as an absolute derivative of the function f and a function differentiable in the sense of the metric is called ***metrically differentiable.***

Theorem 7. *([2],[15]) Let (X,d) and (Y,ρ) be two metric spaces. If $f : X \longrightarrow Y$ is metrically differentiable at $c \in X$, then f is continuous at c.*

The following a result relate differentiability in the sense of the metric and differentiability in the sense classical on the real line ([2]Proposition 3.1).

Proposition 8. *If $A \subset \mathbb{R}$ and $p \in A$ is a limit point of A, then for any mapping $f : A \longrightarrow \mathbb{R}$, we have the following:*

1. *If f is continuously differentiable in the sense classical at p, then f is metrically differentiable (strongly) at p,and*

$$f'_{abs}(p) = |f'(p)|$$

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2. *If f is differentiable in the sense classical at p, then f is metrically differentiable p,and*

$$f'_{abs}(p) = |f'(p)|$$

Definition 9. Let $f : X \longrightarrow \mathbb{R}$ be a function and let $x_0 \in X$ be a ponit. The function f is called lower semi-continuous at point x_0 if every $\epsilon > 0$ there exists $\delta > 0$ such that for each $x \in B_\epsilon(x_0)$ we have

$$f(x_0) - \epsilon \leq f(x). \tag{7}$$

The function f is upper semi-continuous if the function $(-f)$ is lower semi-continuous. The function f is called continuous on X if it is lower semi-continuous and upper semi-continuous at every points in X.

Proposition 10. *Let $f : X \longrightarrow \mathbb{R}$ be a function and let $\{x_n\}$ be a sequence that converges to $x_0 \in X$. If f lower semi-continuous at x_0, then*

$$\liminf_n f(x_n) \geq f(x_0). \tag{8}$$

If f is upper semi-continuous at x_0, then

$$\limsup_n f(x_n) \leq f(x_0). \tag{9}$$

2. Results

2.1. Modified Caristi's Fixed Point Theorem

In this subsection we modify the Caristi's fixed point theorem. The modification is replacing the function φ in Caristi's fixed point theorem by the absolute derivative function of the function f. Note that, the function f is differentiable in the sense of the metric on metric spaces. Thus the characterization of Caristi type mapping can be characterized by absolute derivative such as Theorem 13 below.

For the proof of the Theorem 13 we need the following Lemma.

Lemma 11. *Let (X, d) be a metric space and $f : X \longrightarrow X$ be a continuously metrically differentiable on X. For each $x, y \in X$ we define relation "$\preceq_d$" as follows.*

$$x \preceq_d y \Longleftrightarrow d(x, y) \leq f'_{abs}(x) - f'_{abs}(y). \tag{10}$$

Then the relation "$\preceq_d$" is partially ordered on X.

Proof. (i) It clear that $d(x, x) = f'_{abs}(x) - f'_{abs}(x) = 0$ so that $x \preceq_d x$ is reflexive.

(ii) If $x \preceq_d y$ then $d(x, y) \leq f'_{abs}(x) - f'_{abs}(y)$ and if $y \preceq_d x$ then $d(y, x) \leq f'_{abs}(y) - f'_{abs}(x)$. This implies $2\, d(x, y) \leq 0$ and so that $x = y$ (symmetry).

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(iii) If $x \preceq_d y$ then $d(x,y) \leq f'_{abs}(x) - f'_{abs}(y)$ and if $y \preceq_d z$ then $d(y,z) \leq f'_{abs}(y) - f'_{abs}(z)$. By metric we obtain

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &\leq (f'_{abs}(x) - f'_{abs}(y)) + (f'_{abs}(y) - f'_{abs}(z)) \\ &= f'_{abs}(x) - f'_{abs}(z). \end{aligned}$$

This means $x \preceq_d z$ (transitive) $\square$

Lemma 12. *(Zorn's Lemma) Let X be non-empty partially ordered. If every totally ordered subset M of X has an upper bound in X, then X has at least one maximal element.*

The following is a modification of the Caristi's fixed point theorem by absolute derivative.

Theorem 13. *Let (X,d) be a complete metric space and let $f : X \longrightarrow X$ be a continuously metrically differentiable on X such that*

$$d(x, f(x)) + f'_{abs}(f(x)) \leq f'_{abs}(x) \tag{11}$$

for each $x \in X$. Then f has a fixed point in X.

Proof. For each $x, y \in X$ defined relation "$\preceq_d$" in X as follows

$$x \preceq_d y \Longleftrightarrow d(x,y) \leq f'_{abs}(x) - f'_{abs}(y). \tag{12}$$

According to Lemma 11, the pairs $(X, \preceq_d)$ is a partially ordered. Let $x_0 \in X$ fixed. By Zorn's Lemma, totally ordered subset M of X containing x_0.
Let $M = \{x_\alpha\}_{\alpha\in\Gamma} \subset X$, where Γ is a totally ordered set. This means there is an x_β such that $x_\alpha \preceq_d x_\beta$ for all $\alpha \in \Gamma$. Now we define

$$x_\alpha \preceq_d x_\beta \Longleftrightarrow \alpha \preceq_d \beta \tag{13}$$

for all $\alpha, \beta \in \Gamma$.

From (12), the sequence of real number $\{f'_{abs}(x_\alpha)\}$ is decreasing in $[0, \infty]$ hence there exists a real number $r \geq 0$ such that $f'_{abs}(x_\alpha)$ converges to r when α increases.

Let be given $\epsilon > 0$ arbitrary then there exists $\alpha_0 \in \Gamma$ such that for $\alpha \succeq_d \alpha_0$ this holds

$$r \leq f'_{abs}(x_\alpha) \leq f'_{abs}(x_{\alpha_0}) \leq r + \epsilon. \tag{14}$$

If $\beta \succeq_d \alpha \succeq_d \alpha_0$, then according to (12), (13) and (14) then we obtain

$$d(x_\alpha, x_\beta) \leq f'_{abs}(x_\alpha) - f'_{abs}(x_\beta) \leq r + \epsilon - r = \epsilon. \tag{15}$$

which implies that $\{x_\alpha\}$ is Cauchy net in a complete metric space X so that there exists $x \in X$ such that $x_\alpha \to x$ (as α increases). Since the real function f'_{abs} is continuous, certainly it a lower semi-continuous so that $f'_{abs}(x_\alpha) \leq r$.
If $\beta \succeq_d \alpha$, then $x_\beta \succeq_d x_\alpha$ then

$$d(x_\alpha, x_\beta) \leq f'_{abs}(x_\alpha) - f'_{abs}(x_\beta)$$

by inequality (12). If β is increasing then we obtain

$$d(x_\alpha, x) \leq f'_{abs}(x_\alpha) - f'_{abs}(x).$$

In this case implies that $x_\alpha \preceq_d x$ for all $\alpha \in \Gamma$. In particular $x_0 \preceq_d x$. Since M is maximal, of course $x \in M$. Moreover, if we let $y = f(x)$ the condition (11) implies that

$$x_\alpha \preceq_d x \preceq_d y = f(x) \tag{16}$$

for all $\alpha \in \Gamma$. Again by maximality, $f(x) \in M$. Since $x \in M$ we have

$$y = f(x) \preceq_d x. \tag{17}$$

Based on the inequality (16) , (17) and (11) yields 2 $d(x, f(x)) = 0$. Hence $f(x) = x$ or the function f has a fixed point $x \in X$. □

2.2. *Absolute derivative test*

In this subsection, we will investigate the relation of the contraction mapping and its absolute derivative.

Definition 14. Let (X, d) be a metric space. A set $K \subset X$ is said to be *d-convex* (metrically convex) if for each $x, y \in K$ there is an "interval" $[x, y]$ in K.

An interval $[x, y]$ in K is image of an arc or path (homeomorphism) $\gamma : [0, 1] \longrightarrow K$ such that $\gamma(0) = x, \gamma(1) = y$ and for $0 \le p < q < r \le 1$, we have $d(\gamma(p), \gamma(r)) = d(\gamma(p), \gamma(q)) + d(\gamma(q), \gamma(r))$. So we can say that K is d-convex if for every $x, y \in K$ there exists $z \in K$ such that

$$d(x, y) = d(x, z) + d(z, y). \tag{18}$$

The metric space (X, d) is said to be *locally* d-convex if every point $x \in X$ has a d-convex neighborhood $N_r(x)$ for some $r > 0$.

In 1982, Gerald Jungck states that a function which locally Lipschitzian on a d-convex subset K of metric space is globally Lipschitzian on K with the same Lipschitzian constant [4]. Precisely as follows.

Theorem 15. *([4]) Let K be a d-convex subset of metric space (X, d), let $f : K \longrightarrow X$ and suppose that $L \in (0, \infty)$. If for each $a \in K$ there exists $\delta_a > 0$ such that $d(f(a), f(x)) \le Ld(a, x)$ for all $x \in N_{\delta_a}(a) \cap K$, then $d(f(x), f(y)) \le Ld(x, y)$ for all $x, y \in K$.*

The following is the other main result a kind of Mean Value Theorem in the metric space version.

Theorem 16. *Let K be a d-convex subset of metric space (X, d) and let $f : K \longrightarrow X$ be a metrically differentiable on K with $f'_{abs}(x) \ne 0$ for all $x \in K$. If for each $x \in K$ there exists $\delta_x > 0$ and $c_x \in N_{\delta_x}(x) \cap K$ such that $d(f(x), f(z)) \le f'_{abs}(c_x) d(x, z)$ for all $z \in N_{\delta_x}(x) \cap K$, then*

$$d(f(x), f(y)) \le f'_{abs}(c) d(x, y) \tag{19}$$

for all $x, y \in K$ and for some $c \in K$.

Proof. Suppose the points $x \ne y \in K$. Since K is d-convex, there is a path $\gamma : [0, 1] \longrightarrow K$ such that $\gamma(0) = x, \gamma(1) = y$ and the image $\gamma([0, 1]) = [x, y]$. The hypothesis concerning of f implies that for each $t \in [0, 1]$ there exists $\delta_t > 0$ and $c_t \in [0, 1]$ such that

$$d(f(\gamma(t)), f(z)) \le f'_{abs}(\gamma(c_t)) d(\gamma(t), z) \tag{20}$$

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when $z, \gamma(c_t) \in N_{\delta_t}(\gamma(t)) \cap K$.
Since γ is continuous, for each $t \in [0,1]$ we can choose $r_t > 0$ such that $I_t = (t - r_t, t + r_t) \subset [0,1]$ and

$$d(\gamma(t)), \gamma(t')) < \delta_t \tag{21}$$

for all $t' \in I_t = (t - r_t, t + r_t)$. In particular we choose $r_0, r_1 > 1$ such that $I_0 = [0, r_0) \subset [0,1]$ and $I_1 = (1 - r_1, 1] \subset [0,1]$.
Let $\{I_t \mid t \in [0,1]\}$ be an open cover of the connected set $[0,1]$. Since $[0,1]$ compact there is finite open cover $I_{t_0}, I_{t_1}, \cdots I_{t_n}$ such that $[0,1] \subset \bigcup_{i=0}^{n} I_{t_i}$ and $I_{t_i} \cap I_{t_j} \neq \emptyset$ for $i \neq j$. In this case $t_0 = 0$ and $t_1 = 1$. Moreover, $t_i \in I_{t_i}$ for $1 < i < n$ and $t_{i-1} < t_1$. Now we can choose the point $c_i \in I_{t_{i-1}} \cap I_{t_i}$ so that $t_{i-1} < c_i < t_i$ for $i = 1, 2, \ldots n$. From (21) we have

$$d(\gamma(t_{i-1})), \gamma(c_i)) < \delta_{t_{i-1}} \quad and \quad d(\gamma(c_i)), \gamma(t_i)) < \delta_{t_i} \tag{22}$$

so that from (20) we obtain

$$\begin{aligned} d(f(\gamma(t_{i-1}), f(\gamma(t_i)) &\leq d(f(\gamma(t_{i-1})), f(\gamma(c_i))) + d(f(\gamma(c_i)), f(\gamma(t_i))) \\ &\leq f'_{abs}(\gamma(c_i))d(\gamma(t_{i-1}), \gamma(c_i)) + f'_{abs}(\gamma(c_i)d(\gamma(c_i), \gamma(t_i)) \\ &= f'_{abs}(\gamma(c_i))d(\gamma(t_{i-1}), \gamma(t_i)) \end{aligned} \tag{23}$$

since $t_{i-1} < c_i < t_i$ and convexity of K. Consequently,

$$\begin{aligned} d(f(x), f(y)) &= d(f(\gamma(t_0)), f(\gamma(t_n)) \\ &\leq \sum_{i=1}^{n} d(f(\gamma(t_{i-1})), f(\gamma(t_i))) \leq \sum_{i=1}^{n} f'_{abs}(\gamma(c_i))d(\gamma(t_{i-1}), \gamma(t_i)) \\ &= \sum_{i=1}^{n} f'_{abs}(\gamma(c_i))d(\gamma(t_0), \gamma(t_n)) = \sum_{i=1}^{n} f'_{abs}(\gamma(c_i))d(\gamma(0), \gamma(1)) \\ &= \sum_{i=1}^{n} f'_{abs}(\gamma(c_i))d(x, y), \end{aligned}$$

again using the fact that K is convex. So for each $x, y \in K$ there is $c \in K$ such that

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(c)) + d(f(c), f(y)) \leq f'_{abs}(c)d(x, c) + f'_{abs}(c)d(c, y) \\ &= f'_{abs}(c)[d(x, c) + d(c, y)] \\ &= f'_{abs}(c)d(x, y). \end{aligned}$$

In this case that $f'_{abs}(c) = \sum_{i=1}^{n} f'_{abs}(\gamma(c_i))$. □

The relation between a contraction mapping and its absolute derivative as in the real-valued function (Lemma 2) is presented in the next.

Proposition 17. *Let K be a d-convex subset of metric space (X, d) and let $f : K \longrightarrow K$ be a continuously metrically differentiable on K. Then f is a contraction if and only if there exists a number $0 \leq k < 1$ such that $f'_{abs}(x) \leq k < 1$ for all $x \in K$.*

Proof. If f is contraction on K, then there is $0 \leq k < 1$ such that

$$d(f(x), f(y)) \leq kd(x, y) \tag{24}$$

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for each $x, y \in K$.
According to the hypothesis, the function f is metrically differentiable on K. It implies for each $p \in K$ the limit

$$\lim_{d(x,y)\to d(p,p)} \frac{d(f(x), f(y))}{d(x,y)}$$

does exists and equals $f'_{abs}(p)$ (see Definition 5). From (24) we obtain

$$f'_{abs}(p) = \lim_{d(x,y)\to d(p,p)} \frac{d(f(x), f(y))}{d(x,y)} \leq k < 1,$$

for all $p \in K$. In the other word $f'_{abs}(x) \leq k < 1$ for all $x \in K$.

Conversely, if f is continuously metrically differentiable on d-convex K, then for each $x, y \in K$ there exists $c \in K$ such that

$$d(f(x), f(y)) \leq f'_{abs}(c)d(x,y). \tag{25}$$

by Theorem 16. Since $f'_{abs}(x) \leq k < 1$ for all $x \in K$, it allow that we obtain

$$d(f(x), f(y)) \leq f'_{abs}(c)d(x,y) \leq kd(x,y).$$

for all $x, y \in K$. It proved the function f contraction on K. □

Corollary 18. *Let (X, d) be a complete metric space and d-convex and let $f : X \longrightarrow X$ be a continuously metrically differentiable on X. If there exists a number $0 \leq k < 1$ such that $f'_{abs}(x) \leq k$ for all $x \in K$, then f has a unique fixed point.*

The same as result before, if f is contraction maps with constant Lipschitz $(0 \leq k < 1)$ and f is continuously metrically differentiable with $f'_{abs}(x) = \dfrac{1}{1-k} d(x, f(x))$ then f is the Caristi type mapping.

When do Caristi type mapping to be contraction mapping? Here is a statement.

Proposition 19. *Let K be a subset of metric space (X, d). Suppose $f : K \longrightarrow K$ is continuously metrically differentiable on K and satisfies property as follows*

(a) $f'_{abs}(x) = d(x, f(y))$ for all $x \neq y \in K$
(b) For each $x, y \in K$ there exist $0 \leq k < 1$ such that $d(x, f(x)) = d(y, f(y)) - (k-1)d(x,y)$.

If f is Caristi type mapping, then f is contraction mapping.

Proof. Since f is Caristi type mapping, we have

$$f'_{abs}(f(x)) \leq f'_{abs}(x) - d(x, f(x)) \tag{26}$$

for each $x \in K$. From the properties (a) $f'_{abs}(x) = d(x, f(y))$ for all $x \neq y \in K$ so that inequalities(26) become

$$d(f(x), f(y) \leq d(x, f(y)) - d(x, f(x))$$

for all $x \neq y \in K$. According to the properties (b) there is $0 \leq k < 1$. Hence holds

$$\begin{aligned} d(f(x), f(y) &\leq d(x, f(y)) - d(x, f(x)) \\ &\leq d(x,y) + d(y, f(y)) - d(x, f(x)) \\ &= d(x,y) + (k-1)d(x,y) \\ &= kd(x,y) \end{aligned}$$

for all $x \neq y \in K$. □

Corollary 20. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a continuously metrically differentiable on X such that satisfies (a) and (b). If f is Caristi type mapping, then f has a unique fixed point.*

3. Discussion

A beginning of the question is when do a derivative function has a fixed point in metric spaces? This problem is not the same as the fixed point property for the function since the existence of the derivative depends on the given function. Therefore, the authors agreed in the discussion to study the differentiation in the metric space. These results can simplify previous studies of Caristi type mapping such as presented in this manuscript. Future research direction in particularly in the context of real-valued functions may highlight the existence of fixed point for its absolute derivative. In addition, the value of the derivative can be extended to the metric space.

4. Conclusions

The results of this paper can be used as an alternative to the investigation of the existence of fixed points for a mapping. The facts in this paper show that an absolute derivative plays an important role in the investigation the fixed point of a mapping. In addition, it is a significant result to characterize the contraction mapping and it is also requirements of equivalence between Caristi type mapping and ordinary contraction mapping.

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