Title:

Mathematical Modeling of Rogue Waves, a Review of Conventional and Emerging Mathematical Methods and Solutions

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Abstract

Anomalous waves and rogue events are closely associated with irregularities and unexpected events occurring at various levels of physics, such as in optics, in oceans and in the atmosphere. Mathematical modeling of rogue waves is a highly actual field of research, which has evolved over the last four decades into a specialized part of mathematical physics. The applications of the mathematical models for rogue events is directly relevant to technology development for prediction of rogue ocean waves, and for signal processing in quantum units. In this manuscript, a comprehensive view of the most recent development in conventional methods for representing rogue waves is carried out, along with discussion of the devised forms and solutions. The standard nonlinear Schrödinger equation, the Hirota equation, the MMT equation and further to other models are discussed, and their properties highlighted. This review shows that the most recent advancement in modeling rogue waves give models which can be used to establish methods for prediction of rogue waves at open seas, which is important for the safety and activity of marine vessels and installations. The study further puts emphasis on the difference between the methods, and how the resulting models form a basis for representing rogue waves.

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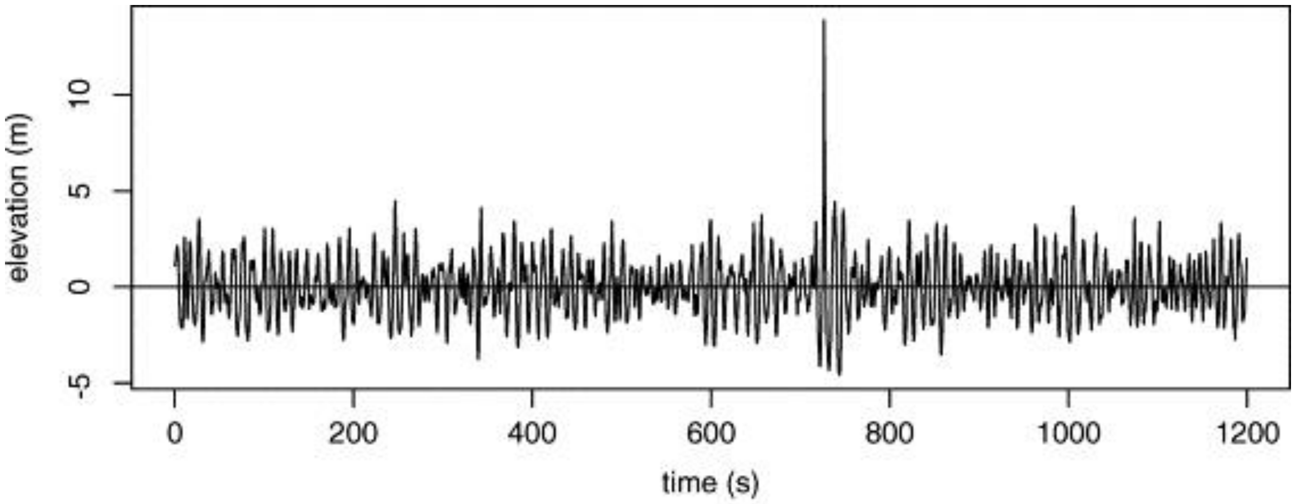
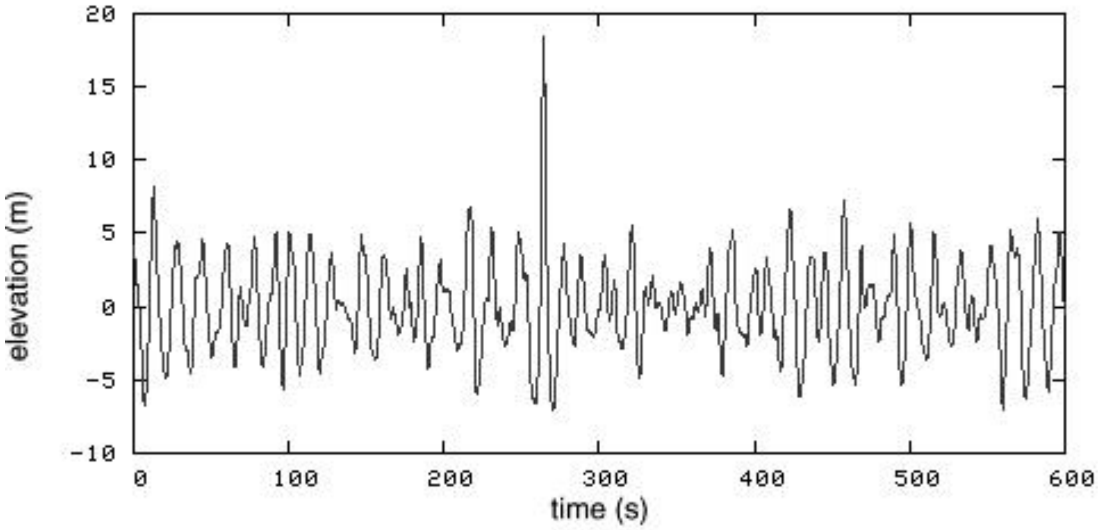
Introduction

Anomalous waves, or "rogue waves", represent a rare phenomenon at sea which occurs approximately 10-15 times pr. year [1,2] and cause yearly millions of dollars of loss of cargo and loss of lives [3]. Rogue waves are abnormally elevated waves, with a 2-3X height of the average wave normal and with unusually steep shapes [4,5]. Rogue waves were recorded for the first time in 1995, when the "New Years Wave" hit the Draupner platform with a wave height of 27 meters, 2.25X the average during that winter storm in December [4]. The laser-installation on the deck, which regularly records the elevation of the platform over the sea bed, registered the soletary giant wave with its 15.4 m elevation above and 11,6 meter below the zero-level [4]. The shape of the wave was symmetrical (Fig 1) with a Gaussian-bell shape and with a particular narrow wavelength. This shape and behavior of anomalous waves is conserved across several observations made in the last 25 years, including the rogue wave that hit the North Alwyn platform in November 1997 [6], the Gorm platform in 1984 [6] and from Storm 172 on the North Alwyn field 100 miles east of Shetland [5]The latter was particularly unusual, with a height 3.19X the average (Fig 1).





Rogue waves are known to have sunk over 20 supercarriers since 1970 [6] and carry a force of 16-20 times (100MT/m²) that of a 12 meter wave, and can easily break ship structures which are designed to withstand far lower impact forces (6MT/m²) [3]. Rogue waves are an eminent threat to shipping and naval activities, and increase in prevalence with climate-change weather patterns [7] In this context, the insurance sector has sought for new models for predicting rogue waves and for fortifying naval structures [3], as both off-shore installations, shipping and also cruise-ships have been increasingly exposed to rogue waves in the last decades [3,6]. This development has also sparked the project "Max Wave" [2] which has contributed with new models and algorithms for predicting rogue waves by the use of satellite observation data. Rogue waves occur also in optical systems [8] in the atmosphere [9] and in plasma [10].

Earlier mathematical models and derived algorithms that were used to predict wave patterns were originally developed by using the linear Gaussian random model, and rogue phenomena at sea were largely disregarded as superstition. The linear Gaussian model is essentially a superposition of elementary waves and predicts the occurrence of a rogue event at a very low probability. This low probability has however failed to commute with the high number of cases of rogue events reported at sea and the laser-readings made in the last decades from off-shore installations, and non-linear models are therefore replacing the original linear Gaussian representation of oceanic waves as the principal model for predicting probability of rogue events at sea for the insurance industry.

Non-linear models have been studied by several groups, and include the modified non-linear Schrödinger Equation (NLSE) [6], the Peregrine soliton model [11] the Levi-Civita and Nekrasov models [12,13], the Davey–Stewartson model [14], the fourth order partial differential equation of Kadomtsev–Petviashvili, the one-dimensional Korteweg–de Vries equation for shallow water surfaces, the second-order Zakharov partial differential equation [15], and the fully nonlinear potential equations. Other systems have recently been developed, and are here reviewed in detail given their relevance to rogue wave ocean phenomena, including the inhomogenous non-linear Schrödinger equation [16], the Akhmediev model [17–20], and the recent models developed by Cousins and Sapsis [21–23].

The non-linear Schrödinger equation in prediction of rogue-waves.

Rogue waves occur both in oceans as well as in optical systems [11] and possibly in other wave-systems. A series of new classes of waves have been discovered in non-linear optics I recent years. One of these new types are the self-similar pulses [24–27]. Self-similar pulses are wave-amplitudes measured in fiber amplifiers [28], which experience an optical gain together with a Kerr-nonlinearity (Fig 2). During the induction of the self-similar impulse in the solid, a fluid or any wave-carrying medium, the shape of the resulting rogue wave no longer depends on the shape or duration of the seed pulses, but depends only on the seed pulse energy (chirping). This creates a large effect on the amplitude, which is largely independent on the initial conditions of the wave pattern and has also been observed in ocean wave systems [29]. The exotic nature of this wave-event has attracted various groups to predict the rogue wave pattern [16,24,28,30,31] using the variations of the non-linear Schrödinger equation (NLSE). One group in particular, developed the variable coefficient inhomogenous nonlinear Schrödinger equation (vci-NLSE) for optical signals [16] :

$$i\psi_x + \frac{1}{2}\beta(x)\psi_{tt} + \chi(x)|\psi|^2\psi + \alpha(x)t^2\psi = i\gamma(x)\psi$$

(1)

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which derives from the Zakharov equation [32] Here $\psi(t,x)$ is the complex function for the electrical (wave) field, and x and t are respectively the propagation distance function and retarded time function. The parameter $\alpha(x)$ defines the normalized loss rate and the function $\alpha(x)t^2$ accounts for the chirping effects, which indicate that the initial chirping parameter is the square of the normalized growth rate. The parameter $\beta(x)$ defines the group-velocity dispersion, $\chi(x)$ defines non-linearity parameters, and $\gamma(x)$ defines loss or gain effects of the wave-signal.

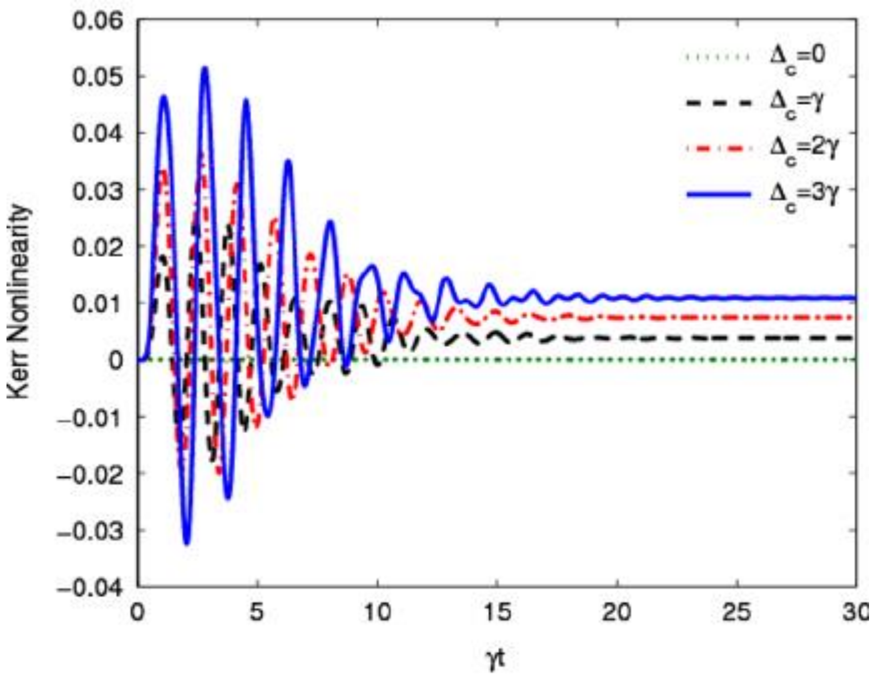

Figure 2. Observed Kerr-Nonlinearity in a a crystal exposed to a magnetic field. [65]

This equation is adaptable both for oceanic waves, as well as for optical non-linear wave guides. Equation 1 is essentially the same as the generalized Gross-Pitaevskii equation with the harmonic oscillator potentials in the Bose-Einstein condensates [33] and can be solved by applying the similarity transformation [34] by replacing ψ(t,x) in equation 1 with:

$$\psi(t,x) = \rho(x)\Psi(T,X)e^{i\phi(t,x)}$$

where ρ(x) is the amplitude, and X and T represent the differential functions describing original propagation distance and the similarity variable while φ(t,x) is the linear variable function of the exponential term, which all must be considered well to avoid singularity of the system ψ(x,t) [16]. The similarity transformation gives:

$$i\Psi_Z + \frac{1}{2}\Psi_{TT} + |\Psi|^2\Psi = 0 \tag{2}$$

which is the standard non-linear Schrödinger equation.

The transformation and integrability conditions derived by [16] show that the factors of the wave system, such as effective wave propagation, distance, central position amplitude, the width and phase of the pulse are ultimately dependent on the group velocity dispersion and on the non-linearity parameters of the system (α, β, γ, χ). The “self-similar” solution found in the process of the transformation of the variable coefficient inhomogenous nonlinear Schrödinger equation into the standard nonlinear Schrödinger equation can ultimately be controlled under dispersion and non-linearity management [16]. Once transformed from the iNLSE, the solutions to the NLSE are derived by the derivation of polynomial conjugates to the root exponential function. This process is reviewed in detail here from the studies by [18].

The solutions to the NLSE

The NLSE equation has been solved by various groups, including [15,18,34–36]. Following one of the most recent works by [17,18] in particular, the steps for deriving exact solutions to the NLSE are defined by identifying rational solutions [17] for the homogenous nonlinear system (2) by using the Darboux transformation [37]. This method used to derive rational solutions is adaptable to both to specific optical rogue waves as well as ocean rogue waves, which both can be represented by the NLSE. The main definition of a rogue event is that the wave "appears from nowhere and vanish without a trace", which is feature closely related to the behavior of solitons. Solitons are independent waves, which self-propagate and exit a collision unchanged. The origin of solitons arises from the first observation of a single solitary wave in the North Sea, made in 1834 by J. S. Russell, who later reproduced the solitary wave in a tank. Since then, solitons have been mainly studied in optical systems, and are represented as solutions to several types of nonlinear PDEs, including the NLSE, the Korteweg de Vries equation and the Sine-Gordon equation. This type of rogue behavior, is described by the rational solutions derived from the NLSE [18], which describe *an induction of a system instability to the top of a plane wave amplitude*, which is transferred to the highest amplitude and then decays exponentially towards zero [17]. This behaviour is represented by Ma-solitons and as Akhmediev breathers or "Akhmediev solitons" [17,17,38–40]. The difference between these two soliton models lies in the initial conditions, where the Ma-solitons originates from the initial conditions while the Akhmediev solitons arise during evolution of the system given by modulation instability [38,41,42].

When solving the NLSE according to the Akhmediev scheme [18] , their method describes the modeled envelope function (ψ) as a solution ranked into an order of hierarchy, starting from first, and progressing to the second, third or fourth order [18]. The difference between each order is the increasing amplitude of the rogue wave (first order -lowest amplitude, fourth order sharpest peak and highest amplitude). The envelope function ψ is expressed as a ratio of polynomials multiplied to the exponential root function, e^{ix}. The polynomials, which are given by functions of variable x and t, are solved by performing the Darboux transformation on the NLSE system [18].

Akhmediev and colleagues furthermore apply in the method a compatibility-check between the root function e^{ix} and the reference-state for two specified column matrix elements, which define initial conditions for the NLSE. These matrix elements (vectors) are given specifically by Akhmediev and colleagues [18] as two differential equations:

$$r_x = il^2 r + il\psi^* s - i/2|\psi|2r + \frac{1}{2}\psi^* s$$

(3)

$$s_x = il^2 s + il\psi r - \frac{1}{2}\psi tr + \frac{i}{2}|\psi|2s \tag{4}$$

which are split into real and imaginary parts, before being simplified and solved to fit into the modified Darboux scheme [18,37] to the two linear differential forms:

$$r_l(x,t) = \sqrt{2}[t - \frac{1}{2} + ix]e^{(-ix/2)} \tag{5}$$

$$s_l(x,t) = \sqrt{2}[x - i(t + \frac{1}{2})]e^{(-ix/2)} \tag{6}$$

Where the two vectors (5) and (6) are used in the Darboux scheme to find ψ_j, where *j* is the order of hierarchy.

The general solution to the NLSE, derived from this scheme [18] is given by the following general form of the envelope (for any order in the hierarchy):

$$\psi_j(x,t) = [(-1)^j + \frac{G_j(x,t) + ixH_j(x,t)}{D_j(x,t}]e^{ix} \tag{7}$$

where G, H and D are the polynomials of the two variables x and t (mentioned above), and *j* is the order of solution in the hierarchy. The first order-solution [18] has the following polynomials: G = 1, H = 2 and D = $1 + 4t^2 + 4x^2$ which give the following envelope function (Fig 3) :

$$\psi_1 = [1 - 4\frac{1 + 2ix}{1 + 4t^2 + 4x^2}]e^{ix}$$

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(8)

For the second-order solution, [18] identify the vectors r_2 and s_2 by solving the equations (3) and (4) using the form of ψ_1 given (8). This gives the second-order solution with the identified polynomials:

$G_2 = 3/8 - 3t^2 - 2t^4 - 9x^2 - 10x^4 - 12t^2x^2$

$H_2 = 15/4 + 6t^2 - 4t^2 - 2x^2 - 4x^4 + 8t^2x^2$

$D_2 = 1/8[\,3/4 + 9t^2 + 4t^2 + 16/3t^6 + 33x^2 + 36x^4 + 16/3x^6]$

inserted in (7), where *j*=2. The second-order rational solution is shown in figure 4. The third and fourth order rational solutions are furthermore calculated and given in [18].

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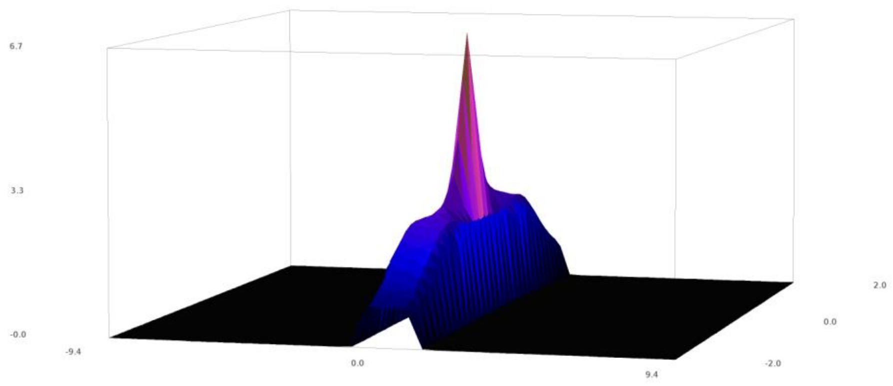

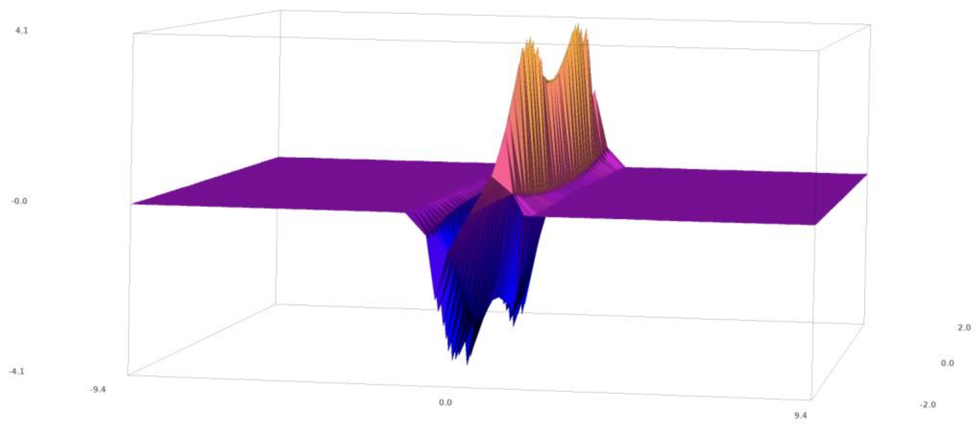

Figure 4. The plot of ψ_2, the second-order solution to the standard NLSE. Top: The real part; Bottom: The imaginary part. [18] Plotted with SageMATH [66,67]

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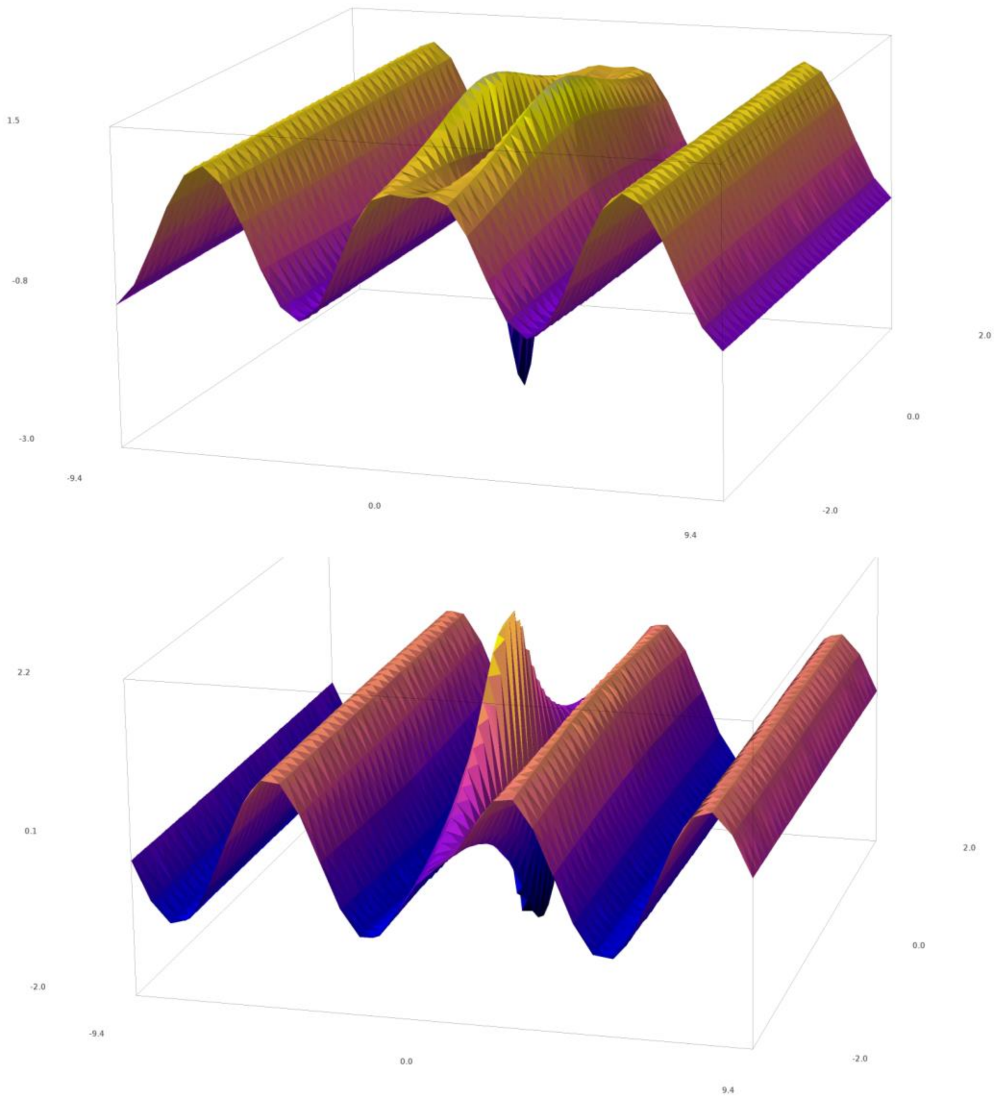

Figure 3. The plot of ψ₁, the first-order solution to the standard NLSE. Top: The real part; Bottom: The imaginary part. [18]. Plotted with SageMATH [66,67].

The same hierarchy-dependency for the solutions is given in the approach by [16], for the transformed vci-NLSE, who define the general solutions for the NLSE in the n-th order given by:

$$\psi_n = \frac{1}{W}\sqrt{\frac{\beta}{\chi}}[(-1)^{-1} + \frac{G_n + i(Z - Z_0)H_n}{F_n} \times e^{i[(1-\frac{v^2}{2})(Z-Z_0)+vT+\phi]} \tag{9}$$

where each factor is defined precisely for the first and second order rational solutions [16]. Similarly to the hierarchy solutions of [18], the increasing order gives higher and higher rogue waves, compared to their surrounding waves. The first and second order rational solutions given in

[16] reflect respectively a 3X and 5X rogue wave height, compared to the surrounding waves. For plots of these, refer to [16].

The similarity between (9) and (7) is striking, and both retain the basic form of a complex polynomial multiplied by a complex exponential root function giving soliton solutions. The root functions of (7) and (9) are shown in their generic form in Fig 5, which depicts the distinction between the seed pulse for the the regular NLSE and the vci-NLSE, as studied respectively by [16,18] for the rogue wave problem. The root function for the vci-NLSE (Fig 5) shows its specific pattern of wave accumulation, which is similar to the formation of wave-packets from a wave impulse. This pattern is conserved with the physical behavior of rogue wave formation, where the rogue wave forms during a focusing phase [43].

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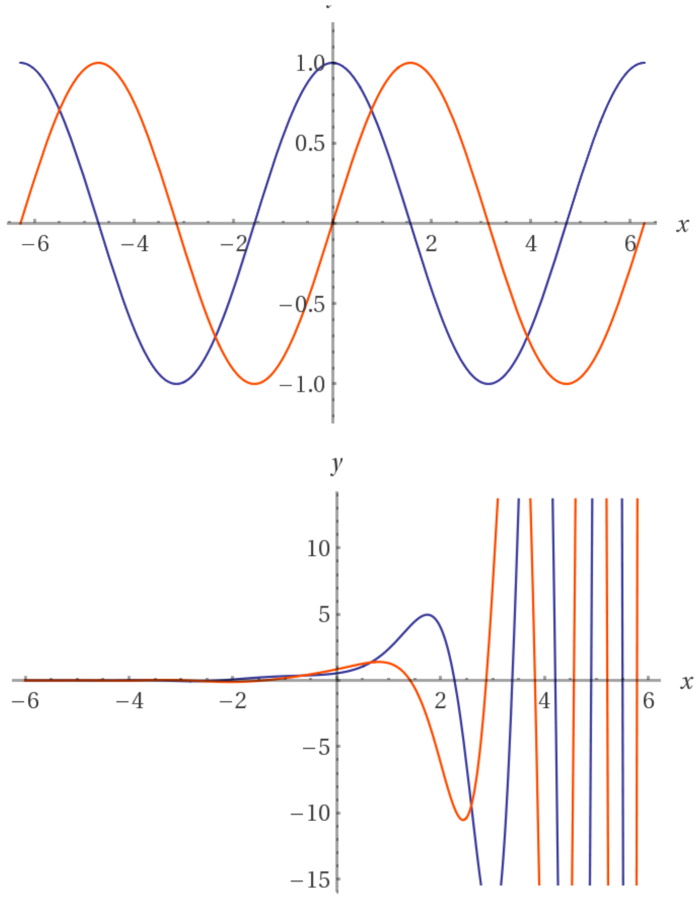

Figure 5. The root functions for the standard NLSE and the inhomogenous variable coefficient NLSE. Top: The seed impulse used in the solutions to the standard NLSE ($f(x)=e^{ix}$) [18] Bottom: A generic form of the seed impulse used in the solutions of the inhomogenous variable coefficient NLSE [16] ($f(x) = \exp(i(1-x^2/2)+x)$. Real part (Blue) and imaginary part (Red).

Other approaches to solve the NLSE have been given by [31], who used the inverse scattering method of transformation, which is a generalization of the Fourier analysis. Their solutions differ from the methods discussed above, and are periodic and are ascribed by a complex envelope function for the deep water train with added higher-order terms from the perturbation procedure [31]. One of the solutions are shown in figure 6, which shows the following variant of the Osborne models:

$$\psi = \frac{cos(\sqrt{2}x)sech(\sqrt{2}t) + i\sqrt{2}tanh(2t)}{\sqrt{2} - cos(\sqrt{2}x)sech(\sqrt{2}t)} e^{2it}$$

which is a periodic function in space, derived from the general form given in [31]. The disadvantage of this system, compared to single peaks derived from [18] lies in their periodicity and multiple peaks, while the rational solutions behind the single peak models of [17,18] are the first to serve as prototypes for rogue waves.

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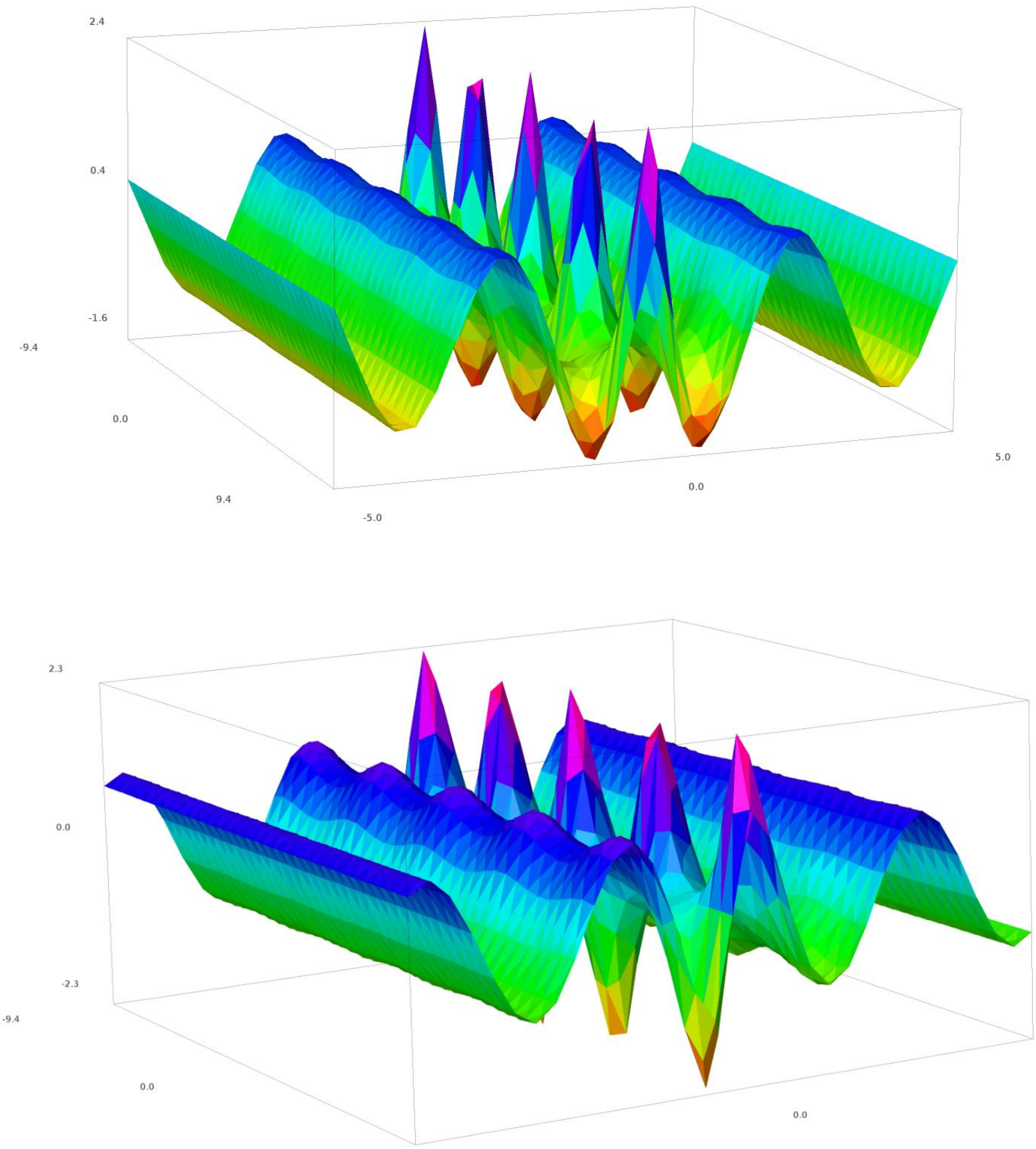

Figure 6. The selected wavefunction from the Osborne models [31]. Top: Real part; Bottom: Imaginary part. Plotted with SAGEMATH [66,67].

The Korteweg de Vries equation

Wave systems defined by higher order nonlinear PDEs, such as (2), can be solved also by the bilinearization technique [44]. This technique involves the step of transforming the differential equation into a more tractable form by replacing the unknown time- and position-dependent envelope function with a new form [44]. After this replacement has been performed, the bilinearization technique applies Hirota bilinear operators for a modified Bäcklund transformation technique [45], which assists in rewriting the original PDE into a simplified PDE composed of bilinear operators, from where exact soliton solutions can be identified. The most suitable example [44] for the application of the bilinearization technique is on the Korteweg de Vries (KdV) equation:

$$\Psi_t + 6\Psi\Psi_x + \Psi_{xxx} = 0 \tag{10}$$

where the boundary conditions are that $\psi \rightarrow 0$ as $|x| \rightarrow \infty$. The real wavefunction is differentiated according to the spatial and temporal dimensions as denoted. In the bilinearization technique, a transformation of the wavefunction to another form is the first step, where an ideal steady-state form is proposed [44] to:

$$\psi(x,t) = (p^2/2)sech^2(\eta/2) \tag{11}$$

where

$$\eta = px - p^3t + \eta_0$$

and η_0 and p are arbitrary constants. By the bilinearization technique [44], one can rewrite (11) to the form:

$$\psi(x,t) = 2p^2(e^{\eta/2} + e^{-\eta/2}) \tag{12}$$

which is converted to its functional form:

$$\psi(x,t) = 2\frac{\partial^2 ln(f(x,t)}{\partial x^2} \tag{13}$$

with $f(x) = 1+e^{\eta}$

By the bilinearization technique [44] substitutes (13) into the original KdG equation (10) and integrates it with respect to x:

$$f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3(f_{xx})^2 = 0 \tag{14}$$

which is the original version of the bilinearized variant of the Korteweg de Vries equation (10) as derived by [45] The solution to (14), $f(x) = 1+e^{\eta}$, is defined as a more fundamental quantity than ψ (12) for the structure of the original nonlinear PDE (10).

In the method of bilinearization, the Hirota bilinear operators are introduced. These are defined by the following definition [45]

$$D_t^n D_x^m a \cdot b = (\partial/\partial t - \partial/\partial t')^n (\partial/\partial t - \partial/\partial t')^m a(x,t) b(x',t')|_{x=x',t'=t} \tag{15}$$

with m and n being arbitrary positive integers. At this stage, the converted form of the KdV equation (14) is rewritten as a PDE composed of Hitora operators:

$$D_x(D_t + D_x^3) f \cdot f = 0 \tag{16}$$

which is a simplified form for the identification of exact solutions using the Bäcklund transformation for the original nonlinear PDE (10).

The exact solution structure for the type of Hirota-operator based PDE form (16) of the KdV equation (10) is given as:

$$\Psi = 1 + \epsilon(e^{\eta_1} + e^{\eta_2}) - \epsilon^2 \frac{F(\Omega_1 - \Omega_2, p_1 - p_2)}{F(\Omega_1 + \Omega_2, p_1 + p_2)} e^{\eta_1 + \eta_2} \tag{17}$$

which represents the two-solition solution to the original KdV equation (10). η_1 and η_2 are the functions with the independent variables x and t as given in (11) for each of the solitons, and $\Omega_{1,2}$ = $-p_1^3$ and $-p_2^3$, following the same definition for (11) for each soliton. ε represent the perturbation [44]. The KdV equation (10) has also been solved by Matveev by identifying positon solutions [46], which exert the same behavior as solitons, such as conserved shape after collision, and elastic collision behavior. The positon differs from the soliton in that it has an infinite energy, and is therefore not a strong model for oceanic or optical rogue waves. Positons have however a tendency

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to represent smoother solutions than solitons to the KdV equation, and can have very high peaks compared to the wave normal. The KdW equation has also been solved by a nonlinear Fourier method [47,48] which is represented by a superposition of nonlinear oscillatory modes of the wave-spectrum. The model by Osborne has a capacity to include a large number of non-linear oscillatory patterns, also known as multi-quasi-cnoidal waves, which are used to form the rogue wave by superposition in constructive phases. These solutions to the original KdV equation (10) include several solitons, depending on the number of degrees of freedom selected for the numerical simulation of the KdV equation and result in a 3D wave complex composed of solitons and radiation components in the simulated wavetrain [47]

The extended Dysthe equation

In 1979, Dysthe [49] developed a modification of the perturbation-based NLSE by adding an additional term to the third-order perturbation variant originally developed by Higgins [50]. Dysthe's method gave an NLSE variant, known as the extended Dysthe equation, which gave better agreement with the mean flow response to non-uniformities in deep-water waves. The extended Dysthe equation is given as:

$$\frac{i}{k}\psi_{xyy} + \psi_{yy} + 2ik\psi_x + 2\psi_z = o(\epsilon^4) \tag{18}$$

where the inhomogenous component is the fourth-order perturbation defined by Dysthe [49]. Dysthe transformed this equation to standard NLSE by using dimensionless variables, and added the following perturbation to the general solution:

$$\psi = c_0(1+\alpha)e^{i(\theta' - \frac{1}{2}ic_0^2 t)} \tag{19}$$

where α and θ are small real perturbations of the amplitude and phase respectively. After insertion of (19) in the dimensionless form of (18) and linearizing, Dysthe obtained a simplified system of two PDEs, which the respective plane-wave solutions are in the form:

$$\begin{pmatrix} \alpha' \\ \theta' \end{pmatrix} \propto e^{i(\lambda x + \mu y - \Omega t)} \tag{20}$$

and

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$$\overline{\phi} \propto e^{[Kz+i(\lambda x+\mu y-\Omega t)]} \tag{21}$$

where $K = (\lambda^2+\mu^2)^{1/2}$ and λ, μ and Ω are selected parameters which satisfy a set of dispersion relations set by Dysthe [49].

The stability of the solutions derived by Dysthe show that the Dysthe equation represents a more realistic model than the NLSE, given that it does not predicts a maximum growth rate for all wavevectors, but only for some wavevectors only. This displays that the fourth-order perturbation term added to the NLSE gives a considerable improvement to the results relating to the stability of the finite amplitude wave. It is particularly the first derivative to the transformed variable in the x and z dimension in eqn. 18 which contributes to the results of Dysthe. Dysthe and Trulsen [51,52]further developed this equation by including up to the fifth-order of the derivative of the wave amplitude describing the linear dispersive terms, and simulated successfully [53] the New Year's wave [4] using the extended Dysthe equation [49,53]

The MMT model

The MMT equation is a one-dimensional nonlinear dispersion equation which was originally proposed by Majda, McLaughlin and Tabak [54] The MMT equation gives soliton-like solutions which have been analyzed in detail by Zhakarov [55–57] and gives four-wave resonant interaction between waves, which, when coupled with large scale forces and small-scale damping, yields a family of solutions which exhibit direct and inverse cascades [21]. The MMT equation is given by:

$$i\psi_t = |\partial_x|^{\alpha} + \lambda|\partial_x|^{\beta/4}(||\partial_x|^{\beta/4}\psi|^2|\partial_x|^{\beta/4}\psi) + iD\psi \tag{22}$$

where ψ is a complex scalar and $|\partial_x|^{\alpha}$ is the pseudodifferential operator defined on the real axis through the Fourier transform:

$$|\partial_x|^{\alpha}\psi(k) = |k|^{\alpha}\widehat{\psi}(k) \tag{23}$$

The last term in (22) is the dissipation term, which is tuned to fit ocean waves through the Laplacian operator, $D\psi$, defined in the Fourier space:

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$$\widehat{D\psi}(k) = \begin{cases} -(|k| - k^*)^2 \widehat{\psi}(k) & |k| > k^* \\ 0 & |k| \leqslant k^* \end{cases} \tag{24}$$

This dissipation term, used by [21] is similar to other dissipation models used by Komen and colleagues [58], who have developed concrete models for simulating large wave groups with focusing and defocusing effects. λ is the nonlinearity coefficient and corresponds to the focusing phase when < 0, and to the defocusing phase when > 0. The MMT equation (22) differs from the standard NLSE by that its family of solutions develop in a more exponential pattern, rather then the Gaussian-bell shaped pattern observed for the solutions for the NLSE [21]. The interesting aspect of this pattern of the spectrum of solutions of the MMT equation is in the mode of formation of the rogue wave, where there energy is transferred from and to the surrounding waves. The solutions are in other words induced by the intermittent formation from the localized rogue event arising out from the regular Gaussian background and collapsing into the surrounding waves. The energy of the rogue wave is transferred to the surroundings and experiences a complete zero-point state, merging completely in the background [21].

The MMT model shows also the formation of quasisolitons which appear in triple-wave packets, as modelled by Zakharov and Pushkarev [55] and differ from regular solitons in that they radiate the energy backwards towards the preceding amplitudes. This behavior of the solutions may be particularly compatible with the simulation of rogue wave events occurring in regions with strong counter-wind currents, such as in the Aghulas-current [59] or in the regions of the Irish sea [2], which are heavily populated by rogue events, on the passage of the Great conveyor belt. The quasibreathers or quasisolitons [55], have the root function similar to the Dysthe-type solutions given in (19). [55] approach the solutions in the form:

$$\psi(t) = e^{i(\Omega - kV)t}\phi_k \tag{25}$$

where Ω and V are constants (Ω < 0 and V>0), and k is the wavenumber, which is an approximate solution to the soliton-like solution for the MMT model. In this approximation, [55] give φ_k the following form:

$$\phi_k = \lambda \frac{\int T_{1234}\phi_1^*\phi_2\phi_3\delta(k + k_1 - k_2 - k_3)dk_1 dk_2 dk_3}{-\Omega + kV - |k|^\alpha} \tag{26}$$

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which represent a form which gives quasi-soliton solutions [60] to the MMT equation. This form of the solutions to the MMT equation radiate energy backwards to the proceeding amplitudes, and represent therefore a energy focusing quite non-similar from the focusing effects modeled by others for rogue patterns (*vide supra*). It is interesting to note that backward radiation plays also a central role for the dynamics of the quasi-solitons, and not only for their energy-accumulation profile. Using the MMT model, [55] developed also a model for collapses of the rogue event, by using self-similar solutions, and model the formation of the wave wedge in the appearing and vanishing state, given by a Fourier-space distribution of the wave-function. [55] have also used the MMT model to develop turbulence-based solution for the localized rogue event, using the initial condition in the form of a NLSE soliton:

$$\psi(x,0) = \frac{q}{2k_m^{9/4}} \frac{e^{ik_m x}}{cosh(qx)} \tag{27}$$

which shows a conserved action and momentum, and an "inner turbulence" localized both in the real and Fourier spaces of the solutions to the modeled envelope function. This "intrinsic turbulence" is described by the authors in affecting the the form of its wave-spectra, which is irregular and with a stochastic behavior. This model of the rogue wave shows quasi-periodic oscillations with slowly diminishing amplitudes over time – caused by the destruction of rogue wave by the surrounding interference, which the authors denoted as a quasi-breather.

The Hirota equation

Multisolitons and breathers for rogue waves have been also successfully modeled [61] by applying the Darboux transformation on the Hirota equation [45]. In their approach, Tao and He [61] developed the Lax pair on the Hirota equation, by using the AKNS [62] procedure to get the Lax pair with the spectral parameters of the Hirota equation given below:

$$iq_t + \alpha(2|q|^2 q + q_{xx}) + i\beta(q_{xxx} + 6|q|^2 q_x) = 0 \tag{28}$$

The Lax pairs were expressed as:

$$\phi_x = M\phi, \phi_t = N\phi \tag{29}$$

giving rise to the extended matrix representation of the operators in the Hirota equation as given by Tao and He [61]. Tao and He further applied the Darboux transformation [37] on the Lax-represented system by using the simple gauge transformation for spectral problems,

$$\phi^{[1]} = T\phi \tag{30}$$

where T is the polynomial applied on the parameter λ given in the Lax pair, and φ is the seed function. Tao and He [61] argue however that regular seed solution $\varphi=e^{ix}$ as described above in the previous sections is too special, and makes the rogue wave model not universal enough. Tao and He develop therefore a different seed function compared to for instance Akhmediev and colleagues [18,38] and develop a more extended form of the seed function by starting from a zero seed solution and a periodic seed solution to construct the complete solutions for the breathers and solitons. At zero seed and with the parameter λ from the Lax pair they set the following seed functions

$$\begin{aligned} f_1 &= e^{-i(\xi+i\eta)x-(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t} \\ f_2 &= e^{i(\xi+i\eta)x+(4\beta i(\xi+i\eta)^3+2\alpha i(\xi+i\eta)^2)t} \end{aligned} \tag{31}$$

back in the Darboux Transformation to get the 1-soliton solution:

$$q^{[1]}_{soliton} = 2\eta e^{2i(-i\xi x-4\beta\xi^3 t-2\alpha\xi^2 t+12\beta\xi\eta^2 t+2\alpha\eta^2 t)} \times sech(-2\eta x - 24\beta\eta\xi^2 t + 8\beta\eta^3 t - 8\alpha\eta\xi t) \tag{32}$$

Tao and He further also report the model for the 2-soliton solution, and finally give the form of the breather solution:

$$q^{[1]}_{breather} = e^{i\rho}\left[c - \frac{2\eta[\eta cosh(2d_2) - i\sigma sinh(2d_2) - ccos(2d_1)]}{ccosh(2d_2) - \eta cos(2d_1)}\right] \tag{33}$$

where d_1, d_2, σ are given [61]. Tao and He finally construct the Rogue wave solutions to the original Hirota equation (eqn . 28) by Taylor expansion on the breather solutions (eqn. 33). The Taylor expansion is carried out at the η variable of the breather solution (eqn. 33) which is given in [61], and forms the general form of the first order rogue wave of the Hirota equation:

$$q_{roguewave} = ke^{i(-2\xi x+\beta t)}\left(1 - \frac{2k_1 + 2k_2 + ik_3 t}{k_1 - k_2}\right)$$

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(34)

where the polynomials k_1, k_2, k_3 are given by Tao and He [61]. The rogue wave model resulting from this form is more general than the model given by Akhmediev and collegues [63] on the Hirota equation. This difference is caused by the appearance of several parameters related to the eigenvalues of the Lax pairs, and gives however a possibility to tune more finely the model to experiments on rogue waves. This advantage of the model by Tao and He increases the ability to modulate the precision of reproducing a rogue wave model by calculations. Tao and He's method grants also the possibility in calculating higher order rogue wave solutions to the Hirota equation by determinant representation of the Darboux Transform, and was carried out in a subsequent work [64].

Conclusions

A review of various mathematical models for representing rogue waves has here been carried out. The methods used to solve the PDEs are directly associated with the efficacy in solving simulations of rogue waves, either on a small scale, such as in acoustics, or at a large scale, such as in anomalous ocean waves or even rogue events in the atmosphere [9]. The usefulness of this study is in collecting, and simplifying the view of the models used to represent rogue events, into a set of consistent equations applicable for computer simulations.

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