doi:10.20944/preprints201803.0017.v1

INEQUALITIES AND APPROXIMATIONS FOR THE FINITE HILBERT TRANSFORM: A SURVEY OF RECENT RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

ABSTRACT. In this paper we survey some recent results due to the author concerning various inequalities and approximations for the finite Hilbert transform of a function belonging to several classes of functions, such as: Lipschitzian, monotonic, convex or with the derivative of bounded variation or absolutely continuous. More accurate estimates in the case that the higher order derivatives are absolutely continuous, are also provided. Some quadrature rules with error bounds are derived. They can be used in the numerical integration of the finite Hilbert transform and, due to the explicit form of the error bounds, enable the user to predict a priory the accuracy.

1. Introduction

Let $\Omega = (-1,1)$ where $1 \leq p < \infty$, the usual $\mathfrak{L}^p$-space with respect to the Lebesgue measure λ restricted to the open interval Ω will be denoted by $\mathfrak{L}^p(\Omega)$.

We define a linear operator T (see [24]) from the vector space $\mathfrak{L}^1(\Omega)$ into the vector space of all λ-measurable functions on Ω as follows. Let $f \in \mathfrak{L}^1(\Omega)$. The Cauchy principal value

$$\frac{1}{\pi} PV \int_{-1}^{1} \frac{f(\tau)}{\tau - t} d\tau = \lim_{\varepsilon \to 0+} \left[\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1} \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau \tag{1.1}$$

exists for λ-almost every $t \in \Omega$.

We denote the left-hand side of (1.1) by $(Tf)(t)$ for each $t \in \Omega$ for which $(Tf)(t)$ exists. The so-defined function Tf, which we call the *finite Hilbert Transform* of f, is defined λ-almost everywhere on Ω and is λ-measurable; (see for example [2, Theorem 8.1.5]). The resulting linear operator T will be called the *finite Hilbert transform operator* or Cauchy kernel operator.

It is known that $\mathfrak{L}^1(\Omega)$ is not invariant under T, namely, $T(\mathfrak{L}^1(\Omega)) \not\subset \mathfrak{L}^1(\Omega)$ [19, Proof of Theorem 1 (b)].

The following basic results are well known and their proofs may be found in Propositions 8.1.9 and 8.2.1 of [2] respectively.

Theorem 1 (M. Riesz). *Let $1 < p < \infty$. Then $T(\mathfrak{L}^p(\Omega)) \subset \mathfrak{L}^p(\Omega)$ and the linear operator*

$$T_p : f \mapsto Tf,\ f \in \mathfrak{L}^p(\Omega)$$

on $\mathfrak{L}^p(\Omega)$ is continuous.

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Finite Hilbert Transform, Lipschitzian, Monotonic, Convex functions, Midpoint and Trapezoid inequalities, Ostrowski's inequality, Taylor's formula,

 doi:10.20944/preprints201803.0017.v1

Theorem 2 (Parseval). *Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. Then*

$$\int_{-1}^{1} (fTg + gTf)\, d\lambda = 0 \tag{1.2}$$

for every $f \in \mathfrak{L}^p(\Omega)$ and $g \in \mathfrak{L}^q(\Omega)$.

We introduce the following definition.

Definition 1. *A function $f : \Omega \to \mathbb{C}$ is said to be α-Hölder continuous $(0 < \alpha \leq 1)$ in a subinterval Ω_0 of Ω if there exists a constant $c > 0$, dependent upon Ω_0, such that*

$$|f(s) - f(t)| \leq c\,|s-t|^{\alpha}, \quad s,\ t \in \Omega_0. \tag{1.3}$$

A function on Ω is said to be *locally α-Hölder continuous* if it is α-Hölder continuous in every compact subinterval of Ω. We denote by $H^{\alpha}_{loc}(\Omega)$ the space of all locally $\alpha-$Hölder continuous functions on Ω.

The class of Hölder continuous functions on Ω is independent because the finite Hilbert transform of such a function exists everywhere on Ω (see [18, Section 3.2] or [23, Lemma II.1.1]).

This is in contrast to the λ-almost everywhere existence of the finite Hilbert transform of functions in $\mathfrak{L}^1(\Omega)$.

There are continuous functions $f \in \mathfrak{L}^1(\Omega)$ such that $(Tf)(t)$ does not exist at some point $t \in \Omega$. An example is given by the function f defined by (see [24])

$$f(t) = \begin{cases} 0 & \text{if} \quad -1 < t \leq 0, \\ \frac{1}{\ln t - \ln 2} & \text{if} \quad 0 < t < 1. \end{cases}$$

It readily follows that $(Tf)(0)$ does not exist.

In paper [24] it is proved amongst others the following result.

Theorem 3 (Okada-Elliot, 1994). *The space $\mathfrak{L}^p(\Omega) \cap H^{\alpha}_{loc}(\Omega)$ is invariant under the finite Hilbert transform operator T and the restriction of T to that space is continuous whenever $1 < p < \infty$. This, however, is not true when $p = 1$.*

Allover this paper, we consider the finite Hilbert transform on the open interval (a, b) defined by

$$(Tf)(a, b; t) := \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau$$

for $t \in (a, b)$ and for various classes of functions f for which the above Cauchy principal value integral exists.

For several recent papers devoted to inequalities for the finite Hilbert transform (Tf), see [20], [21], [22], [25] and [26].

In this paper we survey some recent results due to the author concerning various inequalities and approximations for the finite Hilbert transform of a function belonging to several classes of functions, such as: Lipschitzian, monotonic, convex or with the derivative of bounded variation or absolutely continuous. More accurate estimates in the case that the higher order derivatives are absolutely continuous, are also provided. Some quadrature rules with error bounds are derived. They can be used in the numerical integration of the finite Hilbert transform and, due to the explicit form of the error bounds, enable the user to predict a priory the accuracy.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

2. Inequalities for Some Classes of Functions

2.1. **Some Estimates for α-Hölder Continuous Mappings.** We say that the function $f : [a, b] \to \mathbb{R}$ is α-H-Hölder continuous on (a, b), if

$$|f(t) - f(s)| \leq H |t - s|^{\alpha} \quad \text{for all } t,\ s \in (a, b), \tag{2.1}$$

where $\alpha \in (0, 1]$, $H > 0$.

The following theorem holds.

Theorem 4 (Dragomir et al., 2001 [3]). *If $f : [a, b] \to \mathbb{R}$ is α-H-Hölder continuous on (a, b), then we have the estimate*

$$\left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln\left(\frac{b - t}{t - a}\right) \right| \leq \frac{H}{\alpha\pi} \left[(t - a)^{\alpha} + (b - t)^{\alpha} \right] \leq \frac{2^{1-\alpha}}{\alpha\pi} H (b - a)^{\alpha}, \tag{2.2}$$

for all $t \in (a, b)$.

Proof. As for the mapping $f : (a, b) \to \mathbb{R}$, $f(t) = 1$, $t \in (a, b)$, we have

$$\begin{aligned}
(Tf)(a, b; t) &= \frac{1}{\pi} PV \int_a^b \frac{1}{\tau - t} d\tau \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{1}{\tau - t} d\tau + \int_{t+\varepsilon}^b \frac{1}{\tau - t} d\tau \right] \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\ln |\tau - t| \big|_a^{t-\varepsilon} + \ln (\tau - t) \big|_{t+\varepsilon}^b \right] \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\ln \varepsilon - \ln (t - a) + \ln (b - t) - \ln \varepsilon \right] \\
&= \frac{1}{\pi} \ln \left(\frac{b - t}{t - a} \right), \quad t \in (a, b).
\end{aligned}$$

Then, obviously

$$\begin{aligned}
(Tf)(a, b; t) &= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau \\
&= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} PV \int_a^b \frac{1}{\tau - t} d\tau
\end{aligned}$$

from where we get the equality

$$(Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b - t}{t - a} \right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau \tag{2.3}$$

for all $t \in (a, b)$.

By (2.3) and by the modulus properties, we have

$$\begin{aligned}
&\left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b - t}{t - a} \right) \right| \\
&= \frac{1}{\pi} \left| PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau \right| \leq \frac{1}{\pi} PV \int_a^b \left| \frac{f(\tau) - f(t)}{\tau - t} \right| d\tau \\
&\leq \frac{1}{\pi} PV \int_a^b \frac{|\tau - t|^{\alpha}}{|\tau - t|} d\tau = \frac{1}{\pi} PV \int_a^b \frac{d\tau}{|\tau - t|^{1-\alpha}}.
\end{aligned} \tag{2.4}$$

However,

$$PV\int_a^b \frac{d\tau}{|\tau-t|^{1-\alpha}} = \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon}\frac{d\tau}{(t-\tau)^{1-\alpha}}+\int_{t+\varepsilon}^b\frac{d\tau}{(\tau-t)^{1-\alpha}}\right]$$
$$= \lim_{\varepsilon\to 0+}\left[\left.\frac{-(t-\tau)^{\alpha}}{\alpha}\right|_a^{t-\varepsilon}+\left.\frac{(\tau-t)^{\alpha}}{\alpha}\right|_{t+\varepsilon}^b\right]$$
$$= \lim_{\varepsilon\to 0+}\left[\frac{(t-a)^{\alpha}-\varepsilon^{\alpha}}{\alpha}+\frac{(b-t)^{\alpha}-\varepsilon^{\alpha}}{\alpha}\right]$$
$$= \frac{(t-a)^{\alpha}+(b-t)^{\alpha}}{\alpha}.$$

Using (2.4), we get the first inequality in (2.2).

Consider the mapping $\phi(t) := (t-a)^{\alpha}+(b-t)^{\alpha}$, $\alpha\in(0,1]$, $t\in[a,b]$. Then, obviously

$$\phi'(t) = \frac{\alpha\left[(b-t)^{1-\alpha}-(t-a)^{1-\alpha}\right]}{[(t-a)(b-t)]^{1-\alpha}}.$$

We observe that $\phi'(t)=0$ iff $t=\frac{a+b}{2}$ and $\phi'(t)>0$ if $t\in\left(a,\frac{a+b}{2}\right)$ and $\phi'(t)<0$ if $t\in\left(\frac{a+b}{2},b\right)$, which shows that

$$\max_{t\in[a,b]}\phi(t)=\phi\left(\frac{a+b}{2}\right)=2\left(\frac{b-a}{2}\right)^{\alpha}=2^{1-\alpha}(b-a)^{\alpha}$$

and the last part of (2.2) is proved. □

The following two corollaries are natural.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be an $L-$Lipschitzian mapping on $[a,b]$, i.e., f satisfies the condition*

$$|f(t)-f(s)|\le L|t-s| \quad \text{for all } t,s\in[a,b],\ (L>0). \tag{2.5}$$

Then we have the inequality

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right|\le\frac{L(b-a)}{\pi} \tag{2.6}$$

for all $t\in(a,b)$.

Corollary 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. If $f'\in L_{\infty}[a,b]$, then, for all $t\in(a,b)$, we have*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right|\le\frac{\|f'\|_{\infty}(b-a)}{\pi}, \tag{2.7}$$

where $\|f'\|_{\infty}=\operatorname{essup}_{t\in(a,b)}|f'(t)|<\infty$.

The following result also holds.

Corollary 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$ whose derivative $f'\in L_p[a,b]$, $p\in(1,\infty)$. Then for all $t\in(a,b)$ we have*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right| \tag{2.8}$$
$$\le\frac{p}{(p-1)\pi}\left[(t-a)^{\frac{p-1}{p}}+(b-t)^{\frac{p-1}{p}}\right]\|f'\|_p\le\frac{2^{\frac{1}{p}}p}{(p-1)\pi}(b-a)^{\frac{p-1}{p}}\|f'\|_p,$$

where, as is usually the case, $\|f'\|_p := \left(\int_a^b |f'(t)|^p dt\right)^{\frac{1}{p}} < \infty$.

Proof. As f is absolutely continuous on $[a,b]$, we can state that

$$|f(x) - f(y)| = \left|\int_x^y f'(t)\,dt\right| \le \left|\int_x^y |f'(t)|\,dt\right| \tag{2.9}$$
$$\le \left|\int_x^y dt\right|^{\frac{1}{q}} \left|\int_x^y |f'(t)|^p dt\right|^{\frac{1}{p}} \quad \text{(by Hölder's inequality)}$$
$$\le |y-x|^{\frac{1}{q}} \left(\int_a^b |f'(t)|^p dt\right)^{\frac{1}{p}} = |y-x|^{\frac{1}{q}} \|f'\|_p,$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Thus, f is α-H-Hölder continuous with $\alpha = \frac{1}{q} = \frac{p-1}{p} \in (0,1)$ and $H = \|f'\|_p$. Applying Theorem 4 we get the desired result (2.8). □

The particular case for euclidean norms may be useful.

Corollary 4. *If f is absolutely continuous on $[a,b]$ and $f' \in L_2[a,b]$, then for all $t \in (a,b)$ we have:*

$$\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\right| \tag{2.10}$$
$$\le \frac{2}{\pi}\left(\sqrt{t-a} + \sqrt{b-t}\right)\|f'\|_2 \le \frac{2\sqrt{2}}{\pi}\sqrt{b-a}\,\|f'\|_2.$$

2.2. **Some Results for Monotonic Functions.** The following result holds.

Theorem 5 (Dragomir et al., 2001 [3])**.** *Let $f : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing (nonincreasing) function on $[a,b]$. If the finite Hilbert transform $(Tf)(a,b,\cdot)$ exists in every $t \in (a,b)$, then*

$$(Tf)(a,b;t) \ge (\le) \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a}\right) \tag{2.11}$$

for all $t \in (a,b)$.

Proof. Using the equality (2.3) we have

$$(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) \tag{2.12}$$
$$= \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau$$
$$= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} \frac{f(\tau) - f(t)}{\tau - t} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau) - f(t)}{\tau - t} d\tau\right].$$

If we assume that f is nondecreasing, then for both $\tau \in [a, t-\varepsilon]$ and $\tau \in [t+\varepsilon, b]$ we have

$$\frac{f(\tau) - f(t)}{\tau - t} \ge 0$$

which shows that the right side of (2.12) is positive and hence the inequality (2.11) holds. □

The following result can be useful in practice.

Corollary 5. *Let $f : [a, b] \to \mathbb{R}$ and $e : [a, b] \to \mathbb{R}$, $e(t) = t$ such that $f - me$, $Me - f$ are monotonic nondecreasing, where m, M are given real numbers. If $(Tf)(a, b, \cdot)$ exists in every point $t \in (a, b)$, then we have the inequality*

$$\frac{(b-a)\,m}{\pi} \leq (Tf)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a}\right) \leq \frac{(b-a)\,M}{\pi} \tag{2.13}$$

for all $t \in (a, b)$.

Proof. We simply apply Theorem 5 for the mappings $f - me$ and $Me - f$ which are monotonic on $[a, b]$.

For example, for the first mapping $f - me$ we get

$$T(f-me)(a,b;t) \geq \frac{1}{\pi}\left[f(t) - mt\right] \ln\left(\frac{b-t}{t-a}\right) \tag{2.14}$$

for all $t \in (a, b)$.

However,

$$T(f-me)(a,b;t) = T(f)(a,b;t) - mT(e)(a,b;t)$$

and as

$$\begin{aligned}
T(e)(a,b;t) &= \frac{1}{\pi} \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon} \frac{\tau}{\tau-t}d\tau + \int_{t+\varepsilon}^{b} \frac{\tau}{\tau-t}d\tau\right] \\
&= \frac{1}{\pi} \lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon} \left(1+\frac{t}{\tau-t}\right)d\tau + \int_{t+\varepsilon}^{b} \left(1+\frac{t}{\tau-t}\right)d\tau\right] \\
&= \frac{1}{\pi} \lim_{\varepsilon\to 0+}\left[t-\varepsilon-a+\int_a^{t-\varepsilon} \frac{t}{\tau-t}d\tau + b-t-\varepsilon + \int_{t+\varepsilon}^{b} \frac{t}{\tau-t}d\tau\right] \\
&= \frac{1}{\pi}\left(b-a+tT(1)(a,b;t)\right) \\
&= \frac{b-a+tT(1)(a,b;t)}{\pi} = \frac{1}{\pi}\left[b-a+t\ln\left(\frac{b-t}{t-a}\right)\right]
\end{aligned}$$

then by (2.14) we get

$$T(f)(a,b;t) - \frac{(b-a)\,m}{\pi} - \frac{mt}{\pi}\ln\left(\frac{b-t}{t-a}\right) \geq \frac{1}{\pi} f(t)\ln\left(\frac{b-t}{t-a}\right) - \frac{mt}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$

and the first inequality in (2.13) is obtained.

The second inequality goes likewise by applying Theorem 5 to the mapping $Me - f$. □

Remark 1. *If the mapping is differentiable on (a, b) the condition that $f - me$, $Me - f$ are monotonic nondecreasing is equivalent with the following more practical condition*

$$m \leq f'(t) \leq M \quad \text{for all } t \in (a,b). \tag{2.15}$$

Remark 2. *From (2.13) we may deduce the following approximation result*

$$\left|(Tf)(a,b;t) - \frac{1}{\pi} f(t)\ln\left(\frac{b-t}{t-a}\right) - \frac{M+m}{2\pi}(b-a)\right| \leq \frac{M-m}{2\pi}(b-a). \tag{2.16}$$

for all $t \in (a, b)$.

The above procedure for estimating the finite Hilbert transform can be extended as follows.

Theorem 6 (Dragomir et al., 2001 [3]). *Let $f, g : [a,b] \to \mathbb{R}$ and $\gamma, \Gamma \in \mathbb{R}$ be such that $f - \gamma g$, $\Gamma g - f$ are monotonic on $[a,b]$. If $(Tf)(a,b,\cdot)$ and $(Tg)(a,b,\cdot)$ exist in every point $t \in (a,b)$, then we have the inequality:*

$$\begin{aligned} (2.17) \qquad & \gamma\left[T(g)(a,b;t) - \frac{1}{\pi} g(t) \ln\left(\frac{b-t}{t-a}\right)\right] \\ & \leq (Tf)(a,b;t) - \frac{1}{\pi} f(t) \ln\left(\frac{b-t}{t-a}\right) \\ & \leq \Gamma\left[T(g)(a,b;t) - \frac{1}{\pi} g(t) \ln\left(\frac{b-t}{t-a}\right)\right] \end{aligned}$$

for all $t \in (a,b)$.

Proof. As above, we apply Theorem 5 for the monotonic nondecreasing function $f - \gamma g$ to get

$$T(f - \gamma g)(a,b;t) \geq \frac{1}{\pi}\left[f(t) - \gamma g(t)\right] \ln\left(\frac{b-t}{t-a}\right) \tag{2.18}$$

and as, by the linearity of T, we have

$$T(f - \gamma g)(a,b;t) = T(f)(a,b;t) - \gamma T(g)(a,b;t),$$

then, by (2.18) we obtain the first inequality in (2.17).

The second inequality goes likewise and we omit the details. □

Remark 3. *If we assume that the mappings f, g are differentiable on (a,b), $g'(t) > 0$ on (a,b) and*

$$\gamma = \inf_{t \in (a,b)} \frac{f'(t)}{g'(t)}, \quad \Gamma = \sup_{t \in (a,b)} \frac{f'(t)}{g'(t)},$$

then the inequality (2.17) holds.

2.3. **Some Results for Convex Functions.** Now, if we assume that the mapping $f : (a,b) \to \mathbb{R}$ is convex on (a,b), then it is locally Lipschitzian on (a,b) and then the finite Hilbert transform of f exists in every point $t \in (a,b)$.

The following result holds.

Theorem 7 (Dragomir et al., 2001 [3]). *Let $f : (a,b) \to \mathbb{R}$ be a convex mapping on (a,b). Then we have*

$$\begin{aligned} (2.19) \qquad & \frac{1}{\pi}\left[f(t) \ln\left(\frac{b-t}{t-a}\right) + f(t) - f(a) + l(t)(b-t)\right] \\ & \leq (Tf)(a,b;t) \\ & \leq \frac{1}{\pi}\left[f(t) \ln\left(\frac{b-t}{t-a}\right) + f(b) - f(t) + l(t)(t-a)\right], \end{aligned}$$

where $l(s) \in \left[f'_-(s), f'_+(s)\right]$, $s \in (a,b)$.

Proof. By the convexity of f we can state that for all $a \leq c < d \leq b$ we have

$$\frac{f(d) - f(c)}{d - c} \geq l(c), \tag{2.20}$$

where $l(c) \in \left[f'_-(c), f'_+(c)\right]$.

Using (2.20), we have

$$\int_a^{t-\varepsilon} \frac{f(t)-f(\tau)}{t-\tau} d\tau \geq \int_a^{t-\varepsilon} l(\tau)\, d\tau \tag{2.21}$$

and

$$\int_{t+\varepsilon}^{b} \frac{f(\tau)-f(t)}{\tau-t} d\tau \geq \int_{t+\varepsilon}^{b} l(t)\, d\tau = l(t)(b-t-\varepsilon) \tag{2.22}$$

and then, by adding (2.21) and (2.22), we get

$$\begin{aligned} &\lim_{\varepsilon\to 0+} \left[\int_a^{t-\varepsilon} \frac{f(t)-f(\tau)}{t-\tau} d\tau + \int_{t+\varepsilon}^{b} \frac{f(\tau)-f(t)}{\tau-t} d\tau\right] \\ &\geq \lim_{\varepsilon\to 0+} \left[\int_a^{t-\varepsilon} l(\tau)\, d\tau + l(t)(b-t-\varepsilon)\right] \\ &= \int_a^{t} l(\tau)\, d\tau + l(t)(b-t) = f(t) - f(a) + l(t)(b-t) \end{aligned}$$

Consequently, we have

$$PV\int_a^{b} \frac{f(\tau)-f(t)}{\tau-t} d\tau \geq f(t) - f(a) + l(t)(b-t)$$

and by the identity (2.3), we deduce the first inequality in (2.19).

Similarly, by the convexity of f we have for $a \leq c < d \leq b$

$$l(d) \geq \frac{f(d)-f(c)}{d-c}, \tag{2.23}$$

where $l(d) \in \left[f'_-(d), f'_+(d)\right]$.

Using (2.23) we may state

$$\int_a^{t-\varepsilon} \frac{f(t)-f(\tau)}{t-\tau} d\tau \leq \int_a^{t-\varepsilon} l(t)\, d\tau = l(t)(t-\varepsilon-a)$$

and

$$\int_{t+\varepsilon}^{b} \frac{f(\tau)-f(t)}{\tau-t} d\tau \leq \int_{t+\varepsilon}^{b} l(\tau)\, d\tau.$$

Now, in the same manner as that employed above, we may obtain the second part of (2.19). □

Corollary 6. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable convex function on (a,b). Then we have the inequality*

$$\begin{aligned} &\frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right) + f(t) - f(a) + f'(t)(b-t)\right] \\ &\leq (Tf)(a,b;t) \\ &\leq \frac{1}{\pi}\left[f(t)\ln\left(\frac{b-t}{t-a}\right) + f(b) - f(t) + f'(t)(t-a)\right] \end{aligned} \tag{2.24}$$

for all $t \in (a,b)$.

3. Inequalities of Trapezoid Type

3.1. Trapezoid Type Inequalities. The following theorem holds.

Theorem 8 (Dragomir et al., 2002 [4])**.** *Let $f:[a,b]\to\mathbb{R}$ be such that $f':(a,b)\to\mathbb{R}$ is absolutely continuous on (a,b). Then we have the bounds*

$$\left|T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]\right| \tag{3.1}$$

$$\leq\begin{cases}\frac{\|f''\|_\infty}{4\pi}\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right], & \text{if } f''\in L_\infty[a,b];\\ \frac{q\|f''\|_p}{2\pi(q+1)^{\frac{q+1}{q}}}\left[(t-a)^{\frac{q+1}{q}}+(b-t)^{\frac{q+1}{q}}\right], & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2\pi}\|f''\|_1(b-a) & \end{cases}$$

$$\leq\begin{cases}\frac{\|f''\|_\infty(b-a)^2}{8\pi}, & \text{if } f''\in L_\infty[a,b];\\ \frac{q\|f''\|_p(b-a)^{\frac{q+1}{q}}}{2\pi(q+1)^{\frac{q+1}{q}}}, & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2\pi}\|f''\|_1(b-a) & \end{cases}$$

for all $t\in(a,b)$, where $\|\cdot\|_p$ are the usual Lebesgue norms in $L_p[a,b]$ $(1\leq p\leq\infty)$.

Proof. We start with the following elementary identity which can be proved using the integration by parts formula

$$\int_\alpha^\beta g(u)\,du=\frac{g(\alpha)+g(\beta)}{2}(\beta-\alpha)+\int_\alpha^\beta\left(\frac{\alpha+\beta}{2}-u\right)g'(u)\,du, \tag{3.2}$$

provided that g is absolutely continuous on the interval $[\alpha,\beta]$ if $\alpha\leq\beta$ (or $[\beta,\alpha]$ if $\beta\leq\alpha$).

As for the mapping $f:(a,b)\to\mathbb{R}$, $f(t)=1$, $t\in(a,b)$, we have

$$(Tf)(a,b;t)=\frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\ t\in(a,b)$$

then obviously

$$\begin{aligned}(Tf)(a,b;t)&=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau\\ &=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{f(t)}{\pi}PV\int_a^b\frac{d\tau}{\tau-t}\end{aligned}$$

from where we get the equality

$$(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau. \tag{3.3}$$

Using (3.2), we obtain

$$\begin{aligned}
&PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau\\
&=PV\int_a^b \frac{\int_t^\tau f'(u)\,du}{\tau-t}d\tau\\
&=PV\int_a^b \frac{\frac{f'(\tau)+f'(t)}{2}(\tau-t)+\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du}{\tau-t}d\tau\\
&=\frac{1}{2}\int_a^b [f'(\tau)+f'(t)]\,d\tau+PV\int_a^b\frac{\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du}{\tau-t}d\tau\\
&=\frac{1}{2}[f(b)-f(a)+f'(t)(b-a)]+PV\int_a^b\frac{\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du}{\tau-t}d\tau
\end{aligned}$$

and then, by (3.3), we can state the identity

$$\begin{aligned}
&T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}[f(b)-f(a)+f'(t)(b-a)] \qquad (3.4)\\
&=\frac{1}{\pi}PV\int_a^b\frac{\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du}{\tau-t}d\tau.
\end{aligned}$$

Using the property of modulus

$$\begin{aligned}
&\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}[f(b)-f(a)+f'(t)(b-a)]\right| \qquad (3.5)\\
&\leq\frac{1}{\pi}PV\int_a^b\left|\frac{\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du}{\tau-t}\right|d\tau=:A(a,b;t).
\end{aligned}$$

Now, it is obvious that

$$\begin{aligned}
\left|\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du\right| &\leq \operatorname*{essup}_{u\in[a,b]}|f''(u)|\left|\int_t^\tau\left|\frac{t+\tau}{2}-u\right|du\right|\\
&=\|f''\|_\infty\frac{|t-\tau|^2}{4},\quad\text{for all}\quad t,\tau\in(a,b).
\end{aligned}$$

Then

$$\begin{aligned}
A(a,b;t)&\leq\frac{1}{\pi}PV\int_a^b\frac{\|f''\|_\infty|t-\tau|}{4}d\tau\\
&=\frac{\|f''\|_\infty}{4}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}(t-\tau)\,d\tau+\int_{t+\varepsilon}^b(\tau-t)\,d\tau\right]\\
&=\frac{\|f''\|_\infty}{4}\lim_{\varepsilon\to0+}\left[\frac{(t-a)^2}{2}+\frac{\varepsilon^2}{2}+\frac{(b-t)^2}{2}-\frac{\varepsilon^2}{2}\right]\\
&=\frac{\|f''\|_\infty}{4\pi}\cdot\frac{(t-a)^2+(b-t)^2}{2}\\
&=\frac{\|f''\|_\infty}{4\pi}\left[\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right].
\end{aligned}$$

 doi:10.20944/preprints201803.0017.v1

Using Hölder's integral inequality, we can state for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}&\left|\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du\right|\\&\le\left|\int_t^\tau|f''(u)|^p\,du\right|^{\frac{1}{p}}\left|\int_t^\tau\left|\frac{t+\tau}{2}-u\right|^q du\right|^{\frac{1}{q}}\\&\le\left(\int_a^b|f''(u)|^p\,du\right)^{\frac{1}{p}}\left|\int_t^{\frac{t+\tau}{2}}\left(\frac{t+\tau}{2}-u\right)^q du+\int_{\frac{t+\tau}{2}}^\tau\left(u-\frac{t+\tau}{2}\right)^q du\right|^{\frac{1}{q}}\\&=\|f''\|_p\frac{|t-\tau|^{\frac{q+1}{q}}}{2(q+1)^{\frac{1}{q}}}\quad\text{for all } t,\tau\in(a,b).\end{aligned}$$

Then

$$\begin{aligned}A(a,b;t)&\le\frac{1}{\pi}PV\int_a^b\frac{\|f''\|_p|t-\tau|^{q^{-1}}}{2(q+1)^{\frac{1}{q}}}d\tau\\&=\frac{\|f''\|_p}{2\pi(q+1)^{\frac{1}{q}}}PV\int_a^b|t-\tau|^{q^{-1}}\,d\tau\\&=\frac{\|f''\|_p}{2\pi(q+1)^{\frac{1}{q}}}\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}(t-\tau)^{q^{-1}}\,d\tau+\int_{t+\varepsilon}^b(t-\tau)^{q^{-1}}\,d\tau\right]\\&=\frac{\|f''\|_p}{2\pi(q+1)^{\frac{1}{q}}}\lim_{\varepsilon\to0+}\left[\frac{(t-a)^{q^{-1}+1}-\varepsilon^{q^{-1}+1}}{q^{-1}+1}+\frac{(b-t)^{q^{-1}+1}-\varepsilon^{q^{-1}+1}}{q^{-1}+1}\right]\\&=\frac{\|f''\|_p}{2\pi(q+1)^{\frac{1}{q}}}\left[\frac{(t-a)^{q^{-1}+1}+(b-t)^{q^{-1}+1}}{q^{-1}+1}\right]\\&=\frac{q\|f''\|_p\left[(t-a)^{\frac{q+1}{q}}+(b-t)^{\frac{q+1}{q}}\right]}{2\pi(q+1)^{\frac{q+1}{q}}}\end{aligned}$$

and the second bound in (3.1) is proved.

Finally, we observe that

$$\begin{aligned}\left|\int_t^\tau\left(\frac{t+\tau}{2}-u\right)f''(u)\,du\right|&\le\sup_{u\in[t,\tau]}\left|\frac{t+\tau}{2}-u\right|\left|\int_t^\tau|f''(u)|\,du\right|\\&\le\frac{|t-\tau|}{2}\int_a^b|f''(u)|\,du=\frac{|t-\tau|}{2}\|f''\|_1.\end{aligned}$$

Consequently,

$$A(a,b;t)\le\frac{1}{\pi}PV\int_a^b\frac{1}{2}\|f''\|_1\,d\tau=\frac{1}{2\pi}\|f''\|_1(b-a)$$

and the theorem is proved. □

Remark 4. *It is obvious that for small intervals (a,b), the approximation provided by (3.1) is accurate.*

The best inequality we can get from the first and second part of (3.1) is the one for $t=\frac{a+b}{2}$, and thus we can state the following corollary.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Corollary 7. *With the assumptions of Theorem 8, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'\left(\frac{a+b}{2}\right)(b-a)\right]\right| \tag{3.6}$$
$$\leq\begin{cases}\frac{\|f''\|_{\infty}(b-a)^2}{16\pi} & \text{if } f''\in L_{\infty}[a,b];\\ \frac{q\|f''\|_p}{2^{\frac{q+1}{q}}(q+1)^{\frac{q+1}{q}}\pi}(b-a)^{\frac{q+1}{q}}, & \text{if } f''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

The following result also holds.

Theorem 9 (Dragomir et al., 2002 [4]). *Let $f:[a,b]\to\mathbb{R}$ be such that $f'':(a,b)\to\mathbb{R}$ is absolutely continuous on (a,b). Then we have the bounds*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]\right| \tag{3.7}$$
$$\leq\begin{cases}\frac{\|f'''\|_{\infty}(b-a)}{12\pi}\left[\frac{(b-a)^2}{12}+\left(t-\frac{a+b}{2}\right)^2\right], & \text{if } f'''\in L_{\infty}[a,b];\\ \frac{q\|f'''\|_p[B(q+1,q+1)]^{\frac{1}{q}}}{2\pi(2q+1)}\left[(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}\right], & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\|f'''\|_1}{8\pi}\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right]\end{cases}$$

$$\leq\begin{cases}\frac{\|f'''\|_{\infty}(b-a)^3}{36\pi}, & \text{if } f'''\in L_{\infty}[a,b];\\ \frac{q\|f'''\|_p[B(q+1,q+1)]^{\frac{1}{q}}}{2\pi(2q+1)}(b-a)^{2+\frac{1}{q}}, & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{\|f'''\|_1}{16\pi}(b-a)^2\end{cases}$$

for all $t\in(a,b)$, where $\|\cdot\|_p$ are the usual $p-$norms and $B(\cdot,\cdot)$ is Euler's Beta mapping

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\quad \alpha,\beta>0. \tag{3.8}$$

Proof. Using the integration by parts formula, we obtain the equality:

$$\int_{\alpha}^{\beta}g(u)\,du=\frac{g(\alpha)+g(\beta)}{2}(\beta-\alpha)-\frac{1}{2}\int_{\alpha}^{\beta}(u-\alpha)(\beta-u)g''(u)\,du, \tag{3.9}$$

where g is such that g' is absolutely continuous on $[\alpha,\beta]$ (if $\alpha<\beta$), or on $[\beta,\alpha]$ (if $\beta<\alpha$).

By a similar procedure to that in Theorem 8, we have

$$(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right] \tag{3.10}$$
$$=-\frac{1}{2\pi}PV\int_a^b\frac{\int_t^{\tau}(u-t)(\tau-u)f'''(u)\,du}{\tau-t}d\tau.$$

Using the property of modulus, we have

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]\right|$$
$$\leq\frac{1}{2\pi}PV\int_a^b\left|\frac{\int_t^\tau(u-t)(\tau-u)f'''(u)\,du}{\tau-t}\right|d\tau=:B(a,b;t).$$

Firstly, let us observe that

$$\left|\int_t^\tau(u-t)(\tau-u)f'''(u)\,du\right|\leq\sup_{u\in[t,\tau]}\left|f'''(u)\right|\left|\int_t^\tau|u-t|\,|\tau-u|\,du\right|$$
$$\leq\|f'''\|_\infty\frac{|t-\tau|^3}{6}$$

and then

$$B(a,b;t)\leq\frac{\|f'''\|_\infty}{12\pi}PV\int_a^b|t-\tau|^2\,d\tau$$
$$=\frac{\|f'''\|_\infty}{12\pi}\cdot\frac{(b-t)^3+(t-a)^3}{3}$$
$$=\frac{\|f'''\|_\infty(b-a)}{12\pi}\left[\frac{(b-a)^2}{12}+\left(t-\frac{a+b}{2}\right)^2\right],$$

which proves the first part of (3.7).

For the second part, we apply Hölder's integral inequality to obtain

$$\left|\int_t^\tau(u-t)(\tau-u)f'''(u)\,du\right|\leq\left|\int_t^\tau|u-t|^q|\tau-u|^q\,du\right|^{\frac{1}{q}}\left|\int_t^\tau|f'''(u)|^p\,du\right|^{\frac{1}{p}}$$
$$\leq\|f'''\|_p|t-\tau|^{2+\frac{1}{q}}\left[B(q+1,q+1)\right]^{\frac{1}{q}}$$

for all $t,\tau\in(a,b)$.

Indeed, if we assume that $\tau>t$, then, by $u=(1-s)t+s\tau$, $s\in[0,1]$, we get

$$\int_t^\tau|u-t|^q|\tau-u|^q\,du=\int_t^\tau(u-t)^q(\tau-u)^q\,du$$
$$=(\tau-t)^{2q+1}\int_0^1 s^q(1-s)^q\,ds$$
$$=(\tau-t)^{2q+1}B(q+1,q+1).$$

Consequently, we have

$$B(a,b;t)\leq\frac{\|f'''\|_p\left[B(q+1,q+1)\right]^{\frac{1}{q}}}{2\pi}PV\int_a^b|t-\tau|^{1+\frac{1}{q}}\,d\tau$$
$$=\frac{\|f'''\|_p\left[B(q+1,q+1)\right]^{\frac{1}{q}}}{2\pi}\cdot\frac{(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}}{1+\frac{1}{q}}$$
$$=\frac{q\|f'''\|_p\left[B(q+1,q+1)\right]^{\frac{1}{q}}}{2\pi(2q+1)}\left[(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}\right]$$

and the second part of (3.7) holds.

For the last part, we observe that

$$\left|\int_t^\tau (u-t)(\tau-u) f'''(u)\,du\right| \le \max_u |(u-t)(\tau-u)| \int_t^\tau |f'''(u)|\,du \le \|f'''\|_1 \frac{(t-\tau)^2}{4}$$

since a simple calculation shows that

$$\max_{\substack{u\in[t,\tau]\\(u\in[\tau,t])}} |(u-t)(\tau-u)| = \frac{(t-\tau)^2}{4}.$$

Thus, we can write the following inequality

$$\begin{aligned} B(a,b;t) &\le \frac{\|f'''\|_1}{8\pi} PV \int_a^b |t-\tau|\,d\tau \\ &= \frac{\|f'''\|_1}{8\pi}\cdot\frac{(b-t)^2+(t-a)^2}{2} \\ &= \frac{\|f'''\|_1}{8\pi}\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right] \end{aligned}$$

and the theorem is proved. □

Remark 5. *It is obvious that if $(b-a)\to 0$, then (3.7) provides an accurate approximation for the finite Hilbert transform.*

Taking into account the fact that all the mappings depending on t from the right hand side of (3.7) are convex on the interval (a,b), it is obvious that the best inequality from (3.7) is that one for which $t=\frac{a+b}{2}$.

Corollary 8. *Let f be as in Theorem 9. Then we have the inequality*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'\left(\frac{a+b}{2}\right)(b-a)\right]\right| \tag{3.11}$$
$$\le \begin{cases} \frac{\|f'''\|_\infty (b-a)^3}{144\pi}, & \text{if } f'''\in L_\infty[a,b]; \\ \frac{q\|f'''\|_p [B(q+1,q+1)]^{\frac{1}{q}}}{2^{2+\frac{1}{q}}\pi(2q+1)}(b-a)^{2+\frac{1}{q}}, & \text{if } f'''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'''\|_1 (b-a)^2}{32\pi}. \end{cases}$$

3.2. **Agarwal-Dragomir Type Inequalities.** In [1], R. P. Agarwal and S. S. Dragomir proved the following trapezoid type inequality

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b g(x)\,dx-\frac{g(a)+g(b)}{2}\right| \\ &\le \frac{[g(b)-g(a)-m(b-a)][M(b-a)-g(b)+g(a)]}{2(M-m)(b-a)} \\ &\le \frac{(M-m)(b-a)}{8}, \end{aligned} \tag{3.12}$$

 doi:10.20944/preprints201803.0017.v1

provided that $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $M=\sup_{x\in(a,b)} g'(x)<\infty$, $m=\inf_{x\in(a,b)} g'(x)>-\infty$ and $M>m$.

Using the above inequality, we can state and prove the following theorem.

Theorem 10 (Dragomir et al., 2002 [4])**.** *Let $f:[a,b]\to\mathbb{R}$ be such that $f':(a,b)\to\mathbb{R}$ is absolutely continuous on (a,b) and*

$$\Gamma=\sup_{t\in(a,b)} f''(t)<\infty,\ \ \gamma=\inf_{t\in(a,b)} f''(t)>-\infty,\ \Gamma>\gamma. \tag{3.13}$$

Then we have the bound:

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]\right| \tag{3.14}$$
$$\leq\frac{(\Gamma-\gamma)}{8\pi}\left[\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right]\leq\frac{(\Gamma-\gamma)(b-a)^2}{16\pi},$$

for all $t\in(a,b)$.

Proof. Applying the inequality (3.12) written for $f'(\cdot)$ in the following version:

$$\left|\frac{1}{\tau-t}\int_t^\tau f'(u)\,du-\frac{f'(\tau)+f'(t)}{2}\right|\leq\frac{(\Gamma-\gamma)|t-\tau|}{8},$$

we can state the inequality

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{f'(\tau)+f'(t)}{2}\right|\leq\frac{(\Gamma-\gamma)|t-\tau|}{8} \tag{3.15}$$

for all $t,\tau\in[a,b]$, $t\neq\tau$.

The following property of the Cauchy-Principal Value follows by the properties of integral, modulus and limit,

$$\left|PV\int_a^b A(t,s)\,ds\right|\leq PV\int_a^b |A(t,s)|\,ds, \tag{3.16}$$

holds, assuming that the PV involved exist for all $t\in(a,b)$.

Using (3.4) and (3.5), we may write:

$$\left|PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-PV\int_a^b\frac{f'(\tau)+f'(t)}{2}d\tau\right| \tag{3.17}$$
$$\leq PV\int_a^b\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{f'(\tau)+f'(t)}{2}\right|d\tau$$
$$\leq PV\int_a^b\frac{(\Gamma-\gamma)|t-\tau|}{8}d\tau,$$

and as

$$\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau=(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right),$$
$$\frac{1}{\pi}PV\int_a^b\frac{f'(\tau)+f'(t)}{2}d\tau=\frac{1}{2\pi}\left[f(b)-f(a)+f'(t)(b-a)\right]$$

and

$$\frac{1}{\pi}PV\int_a^b |t-\tau|\,d\tau = \frac{(t-a)^2+(b-t)^2}{2\pi} = \frac{1}{\pi}\left[\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right]$$

then by (3.17) we deduce the desired inequality (3.14). □

It is obvious that the best inequality we can get from (3.14) is that one for which we have $t=\frac{a+b}{2}$, obtaining the following result.

Corollary 9. *Under the assumptions of Theorem 10, we have the inequality:*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{1}{2\pi}\left[f(b)-f(a)+f'\left(\frac{a+b}{2}\right)(b-a)\right]\right| \le \frac{(\Gamma-\gamma)(b-a)^2}{32\pi}. \tag{3.18}$$

3.3. **Compounding Trapezoid Type Inequalities.** The following inequality concerning the trapezoid inequality for absolutely continuous functions holds (see for example [14, p. 32]).

Lemma 1. *Let $u:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then one has the inequalities:*

$$\left|\int_a^b u(s)\,ds-\frac{u(a)+u(b)}{2}(b-a)\right| \le \begin{cases} \frac{(b-a)^2}{4}\|u'\|_{[a,b],\infty} & \text{if } u'\in L_\infty[a,b];\\[2mm] \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|u'\|_{[a,b],p} & \text{if } u'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{(b-a)}{2}\|u'\|_{[a,b],1}. \end{cases} \tag{3.19}$$

A simple proof of this fact can be done by using the identity

$$\int_a^b u(s)\,ds-\frac{u(a)+u(b)}{2}(b-a) = -\int_a^b\left(s-\frac{a+b}{2}\right)u'(s)\,ds, \tag{3.20}$$

and we omit the details.

The following lemma holds.

 doi:10.20944/preprints201803.0017.v1

Lemma 2. *Let $u:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $t,\tau\in[a,b]$, $t\neq\tau$ and $n\in\mathbb{N}$, $n\geq 1$, we have the inequality:*

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\,ds-\frac{1}{2n}\sum_{i=0}^{n-1}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right| \tag{3.21}$$

$$\leq\begin{cases}\dfrac{|\tau-t|}{4n}\|u'\|_{[t,\tau],\infty} & \text{if}\quad u'\in L_\infty[a,b];\\[2ex] \dfrac{|\tau-t|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}n}\|u'\|_{[t,\tau],p} & \text{if}\quad u'\in L_p[a,b],\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{1}{2n}\|u'\|_{[t,\tau],1};\end{cases}$$

where

$$\|u'\|_{[t,\tau],\infty}:=\underset{\substack{s\in[t,\tau]\\(s\in[\tau,t])}}{\operatorname{essup}}|u'(s)|$$

and

$$\|u'\|_{[t,\tau],p}:=\left|\int_t^\tau|u'(s)|^p\,ds\right|^{\frac{1}{p}}$$

for $p\geq 1$.

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$E_n: x_i=t+i\cdot\frac{\tau-t}{n},\quad i=\overline{0,n}.$$

If we apply the inequality (3.19) on the interval $[x_i,x_{i+1}]$, we may write that:

$$\left|\int_{x_i}^{x_{i+1}}u(s)\,ds-\frac{u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)}{2}\cdot\frac{\tau-t}{n}\right|$$

$$\leq\begin{cases}\dfrac{(\tau-t)^2}{4n^2}\|u'\|_{[x_i,x_{i+1}],\infty} & \text{if}\quad u'\in L_\infty[a,b];\\[2ex] \dfrac{|\tau-t|^{1+\frac{1}{q}}}{2n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\|u'\|_{[x_i,x_{i+1}],p} & \text{if}\quad u'\in L_p[a,b],\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{|\tau-t|}{2n}\|u'\|_{[x_i,x_{i+1}],1};\end{cases}$$

from where we get

$$\left|\frac{1}{\tau-t}\int_{x_i}^{x_{i+1}}u(s)\,ds-\frac{1}{2n}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|}{4n^2}\|u'\|_{[x_i,x_{i+1}],\infty} \\ \\ \dfrac{|\tau-t|^{\frac{1}{q}}}{2n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\|u'\|_{[x_i,x_{i+1}],p} \\ \\ \dfrac{1}{2n}\|u'\|_{[x_i,x_{i+1}],1}\,. \end{cases}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we may write

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\,ds-\frac{1}{2n}\sum_{i=0}^{n-1}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \sum_{i=0}^{n-1}\left|\frac{1}{\tau-t}\int_{x_i}^{x_{i+1}} u(s)\,ds-\frac{1}{2n}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|}{4n^2}\displaystyle\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],\infty} \\ \\ \dfrac{|\tau-t|^{\frac{1}{q}}}{2n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}}\displaystyle\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],p} \\ \\ \dfrac{1}{2n}\displaystyle\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],1}\,. \end{cases}$$

However,

$$\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],\infty}\leq n\|u'\|_{[t,\tau],\infty}\,,$$

$$\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],p}=\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}|u'(s)|^p\,ds\right|^{\frac{1}{p}}$$

$$\leq n^{\frac{1}{q}}\left[\left(\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}|u'(s)|^p\,ds\right|^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}$$

$$=n^{\frac{1}{q}}\left|\int_t^\tau |u'(s)|^p\,ds\right|^{\frac{1}{p}}=n^{\frac{1}{q}}\|u'\|_{[t,\tau],p}$$

and

$$\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],1}\le\left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|u'(s)|\,ds\right|=\left|\int_t^{\tau}|u'(s)|\,ds\right|=\|u'\|_{[t,\tau],1}$$

and the lemma is proved. □

The following theorem in approximating the Hilbert transform of a differentiable function whose derivative is absolutely continuous holds.

Theorem 11 (Dragomir et al., 2003, [6]). *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a,b]$. If*

$$T_n(f;t)=\frac{f(b)-f(a)+f'(t)(b-a)}{2n\pi}+\frac{b-a}{n\pi}\sum_{i=1}^{n-1}\left[f;t-\frac{t-a}{n}\cdot i,t+\frac{b-t}{n}\cdot i\right], \tag{3.22}$$

then we have the estimate

(3.23)

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-T_n(f;t)\right|$$

$$\le\begin{cases}\frac{1}{4\pi n}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b];\\ \frac{q}{2\pi n(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2\pi n}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}\end{cases}$$

$$\le\begin{cases}\frac{1}{8\pi n}(b-a)^2\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b];\\ \frac{q}{2\pi n(q+1)^{1+\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2\pi n}(b-a)\|f''\|_{[a,b],1}\end{cases}$$

for all $t\in(a,b)$, where $[f;c,d]$ denotes the divided difference

$$[f;c,d]:=\frac{f(c)-f(d)}{c-d}.$$

Proof. Applying Lemma 2 for the function f', we may write that

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{2n}\left[f'(t)+\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right.\right.$$
$$\left.\left.+\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)+f'(\tau)\right]\right|$$
$$\leq\begin{cases}\dfrac{(\tau-t)}{4n}\|f''\|_{[t,\tau],\infty} & \text{if } f''\in L_\infty[a,b];\\[2ex] \dfrac{|\tau-t|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}n}\|f''\|_{[t,\tau],p} & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{1}{2n}\|f''\|_{[t,\tau],1}. \end{cases} \tag{3.24}$$

However,

$$\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)=\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)$$

and then, by (3.24), we may write:

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\left[\frac{f'(t)+f'(\tau)}{2n}+\frac{1}{n}\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right|$$
$$\leq\begin{cases}\dfrac{|\tau-t|}{4n}\|f''\|_{[t,\tau],\infty}\\[2ex] \dfrac{|\tau-t|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}n}\|f''\|_{[t,\tau],p}\\[2ex] \dfrac{1}{2n}\|f''\|_{[t,\tau],1}\end{cases} \tag{3.25}$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

Consequently, we have

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi}PV\int_a^b\left[\frac{f'(t)+f'(\tau)}{2n}\right.\right.$$
$$\left.\left.+\frac{1}{n}\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau\right| \tag{3.26}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

$$\leq \begin{cases} \frac{1}{4\pi n} PV \int_a^b |\tau - t| \, \|f''\|_{[t,\tau],\infty} \, d\tau, \\ \\ \frac{1}{2\pi (q+1)^{\frac{1}{q}} n} PV \int_a^b |\tau - t|^{\frac{1}{q}} \, \|f''\|_{[t,\tau],p} \, d\tau, \\ \\ \frac{1}{2\pi n} PV \int_a^b \|f''\|_{[t,\tau],1} \, d\tau. \end{cases}$$

Since

$$PV \int_a^b \left[\frac{f'(t) + f'(\tau)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} f'\left(t + i \cdot \frac{\tau - t}{n} \right) \right] d\tau$$

$$= \lim_{\varepsilon \to 0+} \left(\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right) \left(\frac{f'(t) + f'(\tau)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} f'\left(t + i \cdot \frac{\tau - t}{n} \right) \right) d\tau$$

$$= \frac{f'(t)(b-a) + f(b) - f(a)}{2n}$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} \left[\lim_{\varepsilon \to 0+} \left(\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right) \left(f'\left(t + i \cdot \frac{\tau - t}{n} \right) \right) d\tau \right]$$

$$= \frac{f'(t)(b-a) + f(b) - f(a)}{2n}$$

$$+ \frac{1}{n} \sum_{i=1}^{n-1} \left[\lim_{\varepsilon \to 0+} \left[\frac{n}{i} \cdot f\left(t + i \cdot \frac{\tau - t}{n} \right) \Big|_a^{t-\varepsilon} + \frac{n}{i} \cdot f\left(t + i \cdot \frac{\tau - t}{n} \right) \Big|_{t+\varepsilon}^b \right] \right]$$

$$= \frac{f'(t)(b-a) + f(b) - f(a)}{2n} + \frac{1}{n} \sum_{i=1}^{n-1} \frac{n}{i} \left[f\left(t + i \cdot \frac{b-t}{n} \right) - f\left(t + i \cdot \frac{a-t}{n} \right) \right]$$

$$= \frac{f'(t)(b-a) + f(b) - f(a)}{2n} + \frac{b-a}{n} \sum_{i=1}^{n-1} \left[f; t + i \cdot \frac{b-t}{n}, t + i \cdot \frac{a-t}{n} \right],$$

and

$$\begin{aligned} PV \int_a^b |\tau - t| \, \|f''\|_{[t,\tau],\infty} \, d\tau &\leq \|f''\|_{[a,b],\infty} PV \int_a^b |\tau - t| \, d\tau \\ &= \|f''\|_{[a,b],\infty} \frac{(t-a)^2 + (b-t)^2}{2} \\ &= \|f''\|_{[a,b],\infty} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right], \end{aligned}$$

$$
\begin{aligned}
PV\int_a^b |\tau-t|^{\frac{1}{q}} \|f''\|_{[t,\tau],p}\, d\tau &\le \|f''\|_{[a,b],p}\, PV\int_a^b |\tau-t|^{\frac{1}{q}}\, d\tau \\
&= \|f''\|_{[a,b],p}\left[\frac{(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}}{1+\frac{1}{q}}\right] \\
&= \frac{q\|f''\|_{[a,b],p}}{q+1}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
PV\int_a^b \|f''\|_{[t,\tau],1}\, d\tau &= PV\left[\int_a^t \|f''\|_{[\tau,t],1}\, d\tau + \int_\tau^b \|f''\|_{[t,\tau],1}\, d\tau\right] \\
&\le \|f''\|_{[a,t],1}(t-a) + \|f''\|_{[t,b],1}(b-t) \\
&\le \max(t-a, b-t)\left[\|f''\|_{[a,t],1} + \|f''\|_{[t,b],1}\right] \\
&= \left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}
\end{aligned}
$$

then, by (3.26) we get

$$
\begin{aligned}
&\left|\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau - \frac{f'(t)(b-a)+f(b)-f(a)}{2n\pi}\right. \\
&\qquad\left. - \frac{b-a}{n\pi}\sum_{i=1}^{n-1}\left[f;t-\frac{t-a}{n}\cdot i, t+\frac{b-t}{n}\cdot i\right]\right| \\
&\le \begin{cases}
\frac{\|f''\|_{[a,b],\infty}}{4\pi n}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right] & \text{if } f''\in L_\infty[a,b]; \\
\frac{q\|f''\|_{[a,b],p}}{2\pi(q+1)^{1+\frac{1}{q}}n}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right] & \text{if } f''\in L_p[a,b], \\
& p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{\|f''\|_{[a,b],1}}{2\pi n}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right].
\end{cases}
\end{aligned} \tag{3.27}
$$

On the other hand, as for the function $f_0:(a,b)\to\mathbb{R}$, $f_0(t)=1$, we have

$$
(Tf_0)(a,b;t) = \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\ t\in(a,b)
$$

then obviously

$$
\begin{aligned}
(Tf)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau \\
&= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau + \frac{f(t)}{\pi}PV\int_a^b \frac{d\tau}{\tau-t},
\end{aligned}
$$

from where we get the equality:

$$
(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau. \tag{3.28}
$$

Finally, using (3.27) and (3.28), we deduce (3.23). □

Before we proceed with another estimate of the remainder in approximating the Hilbert Transform for functions whose second derivatives are absolutely continuous, we need the following lemma (see for example [14, p. 39]).

Lemma 3. *Let $u : [a,b] \to \mathbb{R}$ be a function such that its derivative is absolutely continuous on $[a,b]$. Then one has the inequalities*

$$\left|\int_a^b u(s)\,ds - \frac{u(a)+u(b)}{2}(b-a)\right| \tag{3.29}$$

$$\leq \begin{cases} \dfrac{(b-a)^3}{12}\|u''\|_{[a,b],\infty} & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{(b-a)^{2+\frac{1}{q}}}{2}[B(q+1,q+1)]^{\frac{1}{q}}\|u''\|_{[a,b],p} & \text{if } u'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^2}{8}\|u''\|_{[a,b],1}. \end{cases}$$

where $B(\cdot,\cdot)$ is the Beta function

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,\quad \alpha,\beta>0.$$

A simple proof of the fact can be done by the use of the following identity:

$$\int_a^b u(s)\,ds - \frac{u(a)+u(b)}{2}(b-a) = -\frac{1}{2}\int_a^b (b-s)(s-a)u''(s)\,ds, \tag{3.30}$$

and we omit the details.

The following lemma also holds.

Lemma 4. *Let $u : [a,b] \to \mathbb{R}$ be a differentiable function such that $u' : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then for any $t,\tau \in [a,b]$, $t \neq \tau$ and $n \in \mathbb{N}$, $n \geq 1$, we have the inequality:*

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\,ds - \frac{1}{2n}\sum_{i=0}^{n-1}\left[u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\right| \tag{3.31}$$

$$\leq \begin{cases} \dfrac{|\tau-t|^2}{12n^2}\|u''\|_{[t,\tau],\infty} & \text{if } u'' \in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{1+\frac{1}{q}}}{2n^2}[B(q+1,q+1)]^{\frac{1}{q}}\|u''\|_{[t,\tau],p} & \text{if } u'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{|\tau-t|}{8n^2}\|u''\|_{[t,\tau],1}, \end{cases}$$

where $B(\cdot,\cdot)$ is the Beta function.

Proof. Consider the equidistant division of $[t,\tau]$ (or $[\tau,t]$)

$$E_n : x_i = t + i\cdot\frac{\tau-t}{n},\quad i=\overline{0,n}.$$

 doi:10.20944/preprints201803.0017.v1

If we apply the inequality (3.29), we may state that

$$\left|\int_{x_i}^{x_{i+1}} u(s)\,ds - \frac{u\left(t+i\cdot\frac{\tau-t}{n}\right)+u\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)}{2}\cdot\frac{\tau-t}{n}\right|$$

$$\leq \begin{cases} \dfrac{|\tau-t|^3}{12n^3}\|u''\|_{[x_i,x_{i+1}],\infty} & \text{if} \quad u''\in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{2+\frac{1}{q}}}{2n^{2+\frac{1}{q}}}\left[B(q+1,q+1)\right]^{\frac{1}{q}}\|u''\|_{[x_i,x_{i+1}],p} & \text{if} \quad u''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{|\tau-t|^2}{8n^2}\|u''\|_{[x_i,x_{i+1}],1}. \end{cases}$$

Dividing by $|\tau-t|>0$ and using a similar argument to the one in Lemma 2, we conclude that the desired inequality (3.31) holds. □

The following theorem in approximating the Hilbert transform of a twice differentiable function whose second derivative f'' is absolutely continuous also holds.

Theorem 12 (Dragomir et al., 2003, [6]). *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that the second derivative f'' is absolutely continuous on $[a,b]$. Then*

$$(3.32)\qquad \left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-T_n(f;t)\right|$$

$$\leq \begin{cases} \dfrac{1}{12n^2\pi}\left[\dfrac{(b-a)^2}{12}+\left(t-\dfrac{a+b}{2}\right)^2\right](b-a)\|f'''\|_{[a,b],\infty} \\ \qquad \text{if} \quad f'''\in L_\infty[a,b]; \\ \dfrac{q\left[B(q+1,q+1)\right]^{\frac{1}{q}}}{2(2q+1)n^2\pi}\left[(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}\right]\|f'''\|_{[a,b],p} \\ \qquad \text{if} \quad f'''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{8\pi n^2}\left[\dfrac{(b-a)^2}{4}+\left(t-\dfrac{a+b}{2}\right)^2\right]\|f'''\|_{[a,b],1} \end{cases}$$

$$\leq \begin{cases} \dfrac{(b-a)^3}{36\pi n^2}\|f'''\|_{[a,b],\infty} & \text{if} \quad f'''\in L_\infty[a,b]; \\ \dfrac{q\left[B(q+1,q+1)\right]^{\frac{1}{q}}(b-a)^{2+\frac{1}{q}}}{2\pi(2q+1)n^2}\|f'''\|_{[a,b],p} & \text{if} \quad f'''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\|f'''\|_{[a,b],1}}{16\pi n^2}(b-a) \end{cases}$$

for all $t\in(a,b)$, where $T_n(f;t)$ is defined by (3.21).

 doi:10.20944/preprints201803.0017.v1

Proof. Applying Lemma 4 for the function f'', we may write that (see also Theorem 11)

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\left[\frac{f'(t)+f'(\tau)}{2n}+\frac{1}{n}\sum_{i=0}^{n-2}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]\right| \tag{3.33}$$

$$\leq\begin{cases}\dfrac{|\tau-t|^2}{12n^2}\|f'''\|_{[t,\tau],\infty} & \text{if } f'''\in L_\infty[a,b];\\[2mm] \dfrac{|\tau-t|^{1+\frac{1}{q}}}{2n^2}[B(q+1,q+1)]^{\frac{1}{q}}\|f'''\|_{[t,\tau],p} & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \dfrac{|\tau-t|}{8n^2}\|f'''\|_{[t,\tau],1}. & \end{cases}$$

for all $t,\tau\in[a,b]$, $t\neq\tau$.

Consequently, we may write:

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi}PV\int_a^b\left[\frac{f'(t)+f'(\tau)}{2n}\right.\right. \tag{3.34}$$

$$\left.\left.+\frac{1}{n}\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau\right|$$

$$\leq\begin{cases}\dfrac{1}{12n^2\pi}PV\displaystyle\int_a^b|\tau-t|^2\|f'''\|_{[t,\tau],\infty}d\tau,\\[2mm] \dfrac{[B(q+1,q+1)]^{\frac{1}{q}}}{2n^2}PV\displaystyle\int_a^b|\tau-t|^{1+\frac{1}{q}}\|f'''\|_{[t,\tau],p}d\tau,\\[2mm] \dfrac{1}{8n^2}PV\displaystyle\int_a^b|\tau-t|\|f'''\|_{[t,\tau],1}d\tau.\end{cases}$$

Since

$$\begin{aligned}&PV\int_a^b|\tau-t|^2\|f'''\|_{[t,\tau],\infty}d\tau\\&\leq\|f'''\|_{[a,b],\infty}PV\int_a^b|\tau-t|^2d\tau=\|f'''\|_{[a,b],\infty}\left[\frac{(t-a)^3+(b-t)^3}{3}\right]\\&=\|f'''\|_{[a,b],\infty}\left[\frac{(b-a)^2}{12}+\left(t-\frac{a+b}{2}\right)^2\right](b-a),\end{aligned}$$

$$\begin{aligned} PV\int_a^b |\tau-t|^{1+\frac{1}{q}} \|f'''\|_{[t,\tau],p}\,d\tau &\le \|f'''\|_{[a,b],p}\, PV\int_a^b |\tau-t|^{1+\frac{1}{q}}\,d\tau \\ &= \|f'''\|_{[a,b],p} \frac{(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}}{2+\frac{1}{q}} \\ &= \frac{q\|f'''\|_{[a,b],p}}{2q+1}\left[(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}\right] \end{aligned}$$

and

$$\begin{aligned} & PV\int_a^b |\tau-t|\,\|f'''\|_{[t,\tau],1}\,d\tau \\ & \le \|f'''\|_{[a,b],1}\, PV\int_a^b |\tau-t|\,d\tau = \frac{(t-a)^2+(b-t)^2}{2}\|f'''\|_{[a,b],1} \\ & = \left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right]\|f'''\|_{[a,b],1}. \end{aligned}$$

Then by (3.34), we deduce the first part of (3.32). □

4. Inequalities of Midpoint Type

4.1. Midpoint Type Inequalities. The following result holds.

Theorem 13 (Dragomir et al., 2002, [5])**.** *Assume that the function $f:[a,b]\to\mathbb{R}$ is such that $f':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\left|T(f)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]\right| \tag{4.1}$$

$$\le\begin{cases} \frac{\|f''\|_\infty}{4\pi}\left[\left(t-\frac{a+b}{2}\right)^2+\frac{(b-a)^2}{4}\right] & \text{if } f''\in L_\infty[a,b]; \\ \frac{q\|f''\|_p}{2\pi(q+1)^{1+\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right] & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f''\|_1}{2\pi}(b-a), \end{cases}$$

$$\le\begin{cases} \frac{\|f''\|_\infty}{8\pi}(b-a)^2 & \text{if } f''\in L_\infty[a,b]; \\ \frac{q\|f''\|_p}{\pi(q+1)^{1+\frac{1}{q}}}(b-a)^{1+\frac{1}{q}} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2\pi}\|f''\|_1(b-a), \end{cases}$$

for any $t\in(a,b)$. The $\|\cdot\|_p$, $p\in[1,\infty]$ denote the usual norms, i.e.,

$$\|g\|_\infty := \operatorname*{essup}_{t\in[a,b]}|g(t)| \quad \text{if } g\in L_\infty[a,b]$$

and

$$\|g\|_p := \left(\int_a^b |g(t)|^p\, dt\right)^{\frac{1}{p}} \quad \text{if} \quad g \in L_p[a,b],\ p \geq 1.$$

Proof. As for the mapping $f_0 : (a,b) \to \mathbb{R}$, $f_0(t) = 1$, $t \in (a,b)$, we have

$$T(f_0)(a,b;t) = \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\ t \in (a,b),$$

then, obviously

$$\begin{aligned}(Tf)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau \\ &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau + \frac{f(t)}{\pi}PV\int_a^b \frac{d\tau}{\tau-t},\end{aligned}$$

from where we get the identity

$$(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau. \tag{4.2}$$

If we use the known identity, which can easily be proved using the integration by parts formula,

$$\int_\alpha^\beta g(u)\,du = g\left(\frac{\alpha+\beta}{2}\right)(\beta-\alpha) + \int_\alpha^\beta K(u)\,g'(u)\,du, \tag{4.3}$$

where

$$K(u) := \begin{cases} u-\alpha & \text{if} \quad u \in \left[\alpha, \frac{\alpha+\beta}{2}\right] \\ u-\beta & \text{if} \quad u \in \left(\frac{\alpha+\beta}{2}, \beta\right] \end{cases}$$

and g is absolutely continuous on $[a,b]$, we may write

$$\begin{aligned}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau &= PV\int_a^b \frac{\int_t^\tau f'(u)\,du}{\tau-t}d\tau \\ &= PV\int_a^b \left[\frac{f'\left(\frac{\tau+t}{2}\right)(\tau-t) + \int_t^\tau K(u)\,f''(u)\,du}{\tau-t}\right]d\tau \\ &= PV\int_a^b f'\left(\frac{\tau+t}{2}\right)d\tau + PV\int_a^b \left(\frac{1}{\tau-t}\int_t^\tau K(u)\,f''(u)\,du\right)d\tau \\ &= 2\left[f\left(\frac{b+t}{2}\right) - f\left(\frac{a+t}{2}\right)\right] + PV\int_a^b \left(\frac{1}{\tau-t}\int_t^\tau K(u)\,f''(u)\,du\right)d\tau.\end{aligned}$$

Consequently, by (4.2), we obtain the identity

$$\begin{aligned}&(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right) - f\left(\frac{t+a}{2}\right)\right] \\ &= \frac{1}{\pi}PV\int_a^b \left(\frac{1}{\tau-t}\int_t^\tau K(u)\,f''(u)\,du\right)d\tau,\end{aligned} \tag{4.4}$$

where

$$K(u) = \begin{cases} u-t & \text{if} \quad u \in \left[t, \frac{\tau+t}{2}\right] \\ u-\tau & \text{if} \quad u \in \left(\frac{\tau+t}{2}, \tau\right] \end{cases}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Using the properties of modulus, we get, by (4.4), that

$$\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right) - f\left(\frac{t+a}{2}\right)\right]\right| \tag{4.5}$$
$$\leq \frac{1}{\pi} PV\int_a^b \left|\frac{1}{\tau-t}\int_t^\tau K(u) f''(u)\,du\right| d\tau =: D(a,b;t).$$

Now, it is obvious that

$$\left|\int_t^\tau K(u) f''(u)\,du\right| \leq \sup_{u\in[a,b]} |f''(u)| \left|\int_t^\tau K(u)\,du\right|$$
$$= \|f''\|_\infty \left|\int_t^{\frac{\tau+t}{2}} (u-t)\,du + \int_{\frac{\tau+t}{2}}^\tau (t-u)\,du\right|$$
$$= \|f''\|_\infty \frac{(t-\tau)^2}{4}.$$

Then

$$D(a,b;t) \leq \frac{1}{4\pi}\|f''\|_\infty PV\int_a^b |t-\tau|\,d\tau$$
$$= \frac{\|f''\|_\infty}{4\pi}\cdot\frac{(t-a)^2+(b-t)^2}{2}$$
$$= \frac{\|f''\|_\infty}{4\pi}\left[\left(t-\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{4}\right].$$

Using Hölder's integral equality, we have

$$\left|\int_t^\tau K(u) f''(u)\,du\right| \leq \left|\int_t^\tau |f''(u)|^p\,du\right|^{\frac{1}{p}} \left|\int_t^\tau |K(u)|^q\,du\right|^{\frac{1}{q}}$$
$$\leq \|f''\|_p \left|\int_t^\tau |K(u)|^q\,du\right|^{\frac{1}{q}}$$
$$= \|f''\|_p \left|\int_t^{\frac{\tau+t}{2}} (u-t)^q\,du + \int_{\frac{\tau+t}{2}}^\tau (t-u)^q\,du\right|^{\frac{1}{q}}$$
$$= \|f''\|_p \left[\frac{|\tau-t|^{q+1}}{2^q(q+1)}\right]^{\frac{1}{q}} = \frac{\|f''\|_p |t-\tau|^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}$$

for all $t,\tau\in(a,b)$.

Then

$$D(a,b;t) \leq \frac{1}{\pi}\|f''\|_p PV\int_a^b \frac{|t-\tau|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\,d\tau$$
$$= \frac{q\|f''\|_p\left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}}\right]}{2\pi(q+1)^{1+\frac{1}{q}}}$$

proving the second part of the first inequality in (4.1).

Finally, we observe that

$$\left|\int_t^\tau K(u) f''(u)\,du\right| \le \sup_{u\in[t,\tau]} |K(u)| \left|\int_t^\tau |f''(u)|\,du\right| = \frac{\|f''\|_1}{2\pi}|t-\tau|$$

and then

$$D(a,b;t) \le \frac{1}{2\pi}\|f''\|_1 PV\int_a^b d\tau = \frac{1}{2\pi}\|f''\|_1 (b-a),$$

proving the last part of the second inequality in (4.1).

The last part of (4.1) is obvious. □

The best inequality we can get from (4.1) is embodied in the following corollary.

Corollary 10. *With the assumptions in Theorem 13, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right) - \frac{2}{\pi}\left[f\left(\frac{a+3b}{4}\right) - f\left(\frac{3a+b}{4}\right)\right]\right| \tag{4.6}$$

$$\le \begin{cases} \dfrac{1}{16\pi}\|f''\|_\infty (b-a)^2 & \text{if } f'' \in L_\infty[a,b]; \\ \dfrac{q}{2^{1+\frac{1}{q}}\pi(q+1)^{1+\frac{1}{q}}}\|f''\|_p (b-a)^{1+\frac{1}{q}} & \text{if } f'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \end{cases}$$

Remark 6. *It is also obvious that if $b-a \to 0$, then both the inequalities (4.1) and (4.6) provide accurate approximations.*

4.2. **Other Midpoint Type Inequalities.** The following result holds.

Theorem 14 (Dragomir et al., 2002, [5])**.** *Assume that the function $f:[a,b]\to\mathbb{R}$ is such that $f'':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequalities:*

$$\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - \frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right) - f\left(\frac{t+a}{2}\right)\right]\right| \tag{4.7}$$

$$\le \begin{cases} \frac{\|f'''\|_\infty}{24\pi}(b-a)\left[\frac{(b-a)^2}{12} + \left(t-\frac{a+b}{2}\right)^2\right] & \text{if } f''' \in L_\infty[a,b]; \\ \frac{q\|f'''\|_p}{8\pi(2q+1)^{1+\frac{1}{q}}}\left[(b-t)^{2+\frac{1}{q}} + (t-a)^{2+\frac{1}{q}}\right] & \text{if } f''' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'''\|_1}{8\pi}\left[\frac{(b-a)^2}{4} + \left(t-\frac{a+b}{2}\right)^2\right], & \end{cases}$$

$$\le \begin{cases} \frac{\|f'''\|_\infty (b-a)^3}{72\pi} & \text{if } f''' \in L_\infty[a,b]; \\ \frac{q\|f'''\|_p (b-a)^{2+\frac{1}{q}}}{8\pi(2q+1)^{1+\frac{1}{q}}} & \text{if } f''' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'''\|_1 (b-a)^2}{16\pi}. & \end{cases}$$

 doi:10.20944/preprints201803.0017.v1

Proof. If we use the identity (4.2) and the following identity, which can be proved by applying the integration by parts formula twice,

$$\int_\alpha^\beta g(u)\,du = g\left(\frac{\alpha+\beta}{2}\right)(\beta-\alpha)+\frac{1}{2}\int_\alpha^\beta L(u)\,g''(u)\,du,$$

where

$$L(u) := \begin{cases} (u-\alpha)^2 & \text{if} \quad u\in\left[\alpha,\frac{\alpha+\beta}{2}\right] \\ (u-\beta)^2 & \text{if} \quad u\in\left(\frac{\alpha+\beta}{2},\beta\right] \end{cases}$$

and g is such that g' is absolutely continuous on $[\alpha,\beta]$, then we get

$$\begin{aligned}&(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\ &=\frac{1}{\pi}PV\int_a^b\left[\frac{f'\left(\frac{\tau+t}{2}\right)(\tau-t)\,\frac{1}{2}\int_t^\tau L(u)\,f'''(u)\,du}{\tau-t}\right]d\tau\\ &=\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]+\frac{1}{2\pi}PV\int_a^b\left[\frac{1}{\tau-t}\int_t^\tau L(u)\,f'''(u)\,du\right]d\tau.\end{aligned}$$

Consequently, we have the identity:

$$\begin{aligned}&(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]\\ &=\frac{1}{2\pi}PV\int_a^b\left[\frac{1}{\tau-t}\int_t^\tau L(u)\,f'''(u)\,du\right]d\tau,\end{aligned}\tag{4.8}$$

where

$$L(u) = \begin{cases} (u-t)^2 & \text{if} \quad u\in\left[t,\frac{\tau+t}{2}\right] \\ (u-\tau)^2 & \text{if} \quad u\in\left(\frac{\tau+t}{2},\tau\right] \end{cases}.$$

Using the modulus properties, we may write, by (4.8), that

$$\begin{aligned}&\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{2}{\pi}\left[f\left(\frac{b+t}{2}\right)-f\left(\frac{t+a}{2}\right)\right]\right|\\ &\le\frac{1}{2\pi}PV\int_a^b\left|\frac{1}{\tau-t}\int_t^\tau |L(u)|\,|f'''(u)|\,du\right|d\tau =: E(a,b;t).\end{aligned}\tag{4.9}$$

Now, observe that

$$\begin{aligned}\left|\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| &\le \|f'''\|_\infty\left|\int_t^{\frac{\tau+t}{2}}(u-t)^2\,du+\int_{\frac{\tau+t}{2}}^\tau (t-u)^2\,du\right|\\ &=\frac{\|f'''\|_\infty}{12}|t-\tau|^3\end{aligned}$$

and then

$$\begin{aligned}E(a,b;t) &\le \frac{\|f'''\|_\infty}{24\pi}\int_a^b(t-\tau)^2\,d\tau=\frac{\|f'''\|_\infty}{24\pi}\cdot\frac{(b-t)^3+(t-a)^3}{3}\\ &=\frac{\|f'''\|_\infty}{24\pi}\left[\frac{(b-a)^2}{12}+\left(t-\frac{a+b}{2}\right)^2\right](b-a),\end{aligned}$$

giving the first part of the first inequality in (4.7).

 doi:10.20944/preprints201803.0017.v1

Using Hölder's inequality, we may write that

$$\begin{aligned}\left|\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| &\le \|f'''\|_p \left|\int_t^{\frac{\tau+t}{2}} |u-t|^{2q}\,du + \int_{\frac{\tau+t}{2}}^{\tau} |t-u|^{2q}\,du\right|^{\frac{1}{q}} \\ &= \|f'''\|_p \left[\frac{2\cdot\left|\frac{t-\tau}{2}\right|^{2q+1}}{2q+1}\right]^{\frac{1}{q}} = \frac{1}{4(2q+1)^{\frac{1}{q}}}\|f'''\|_p\,|t-\tau|^{2+\frac{1}{q}}\end{aligned}$$

and then

$$\begin{aligned}E(a,b;t) &\le \frac{\|f'''\|_p}{8\pi(2q+1)^{\frac{1}{q}}} PV\int_a^b |t-\tau|^{1+\frac{1}{q}}\,d\tau \\ &= \frac{\|f'''\|_p}{8\pi(2q+1)^{\frac{1}{q}}}\cdot\frac{(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}}{\frac{2q+1}{q}} \\ &= \frac{q\,\|f'''\|_p}{8\pi(2q+1)^{\frac{1}{q}}}\left[(b-t)^{2+\frac{1}{q}}+(t-a)^{2+\frac{1}{q}}\right],\end{aligned}$$

which proves the second part of the first inequality in (4.7).

Finally,

$$\left|\int_t^\tau |L(u)|\,|f'''(u)|\,du\right| \le \sup_{u\in[t,\tau]}|L(u)|\,\|f'''\|_1 = \frac{|t-\tau|}{4}\|f'''\|_1,$$

giving

$$\begin{aligned}E(a,b;t) &\le \frac{\|f'''\|_1}{8\pi}PV\int_a^b |t-\tau|\,d\tau = \frac{\|f'''\|_1}{8\pi}\cdot\frac{(b-t)^2+(t-a)^2}{2} \\ &= \frac{\|f'''\|_1}{8\pi}\left[\frac{(b-a)^2}{4}+\left(t-\frac{a+b}{2}\right)^2\right],\end{aligned}$$

which proves the last part of the first inequality in (4.7). □

The best inequality we may obtain from (4.7) is embodied in the following corollary.

Corollary 11. *With the assumptions of Theorem 14, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right)-\frac{2}{\pi}\left[f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{4}\right)\right]\right| \tag{4.10}$$

$$\le\begin{cases}\dfrac{\|f'''\|_\infty(b-a)^3}{288\pi} & \text{if } f'''\in L_\infty[a,b]; \\[2ex] \dfrac{q\,\|f'''\|_p(b-a)^{2+\frac{1}{q}}}{16\cdot 2^{\frac{1}{q}}\pi(2q+1)^{\frac{1}{q}}} & \text{if } f'''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{\|f'''\|_1(b-a)^2}{32\pi}. \end{cases}$$

4.3. **Compounding Midpoint Type Inequalities.** Before we point out the quadrature formula for the finite Hilbert transform, we need the following two technical lemmas:

Lemma 5. *Let $u:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have the midpoint inequalities:*

$$\left|\int_a^b u(s)\,ds-u\left(\frac{a+b}{2}\right)(b-a)\right| \tag{4.11}$$

$$\leq\begin{cases}\dfrac{(b-a)^2}{4}\|u'\|_{[a,b],\infty} & \text{if } u'\in L_\infty[a,b];\\[2ex] \dfrac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|u'\|_{[a,b],p} & \text{if } u'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{(b-a)}{2}\|u'\|_{[a,b],1}. \end{cases}$$

A simple proof of this fact can be done by using the identity (see for example [14, p. 34]):

$$\begin{aligned}&\int_a^b u(s)\,ds-u\left(\frac{a+b}{2}\right)(b-a) \\ &=-\int_a^{\frac{a+b}{2}}(s-a)f'(s)\,ds+\int_{\frac{a+b}{2}}^b(s-b)f'(s)\,ds.\end{aligned} \tag{4.12}$$

We omit the details.

The following lemma also holds.

Lemma 6. *Let $u:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then for any $t,\tau\in[a,b]$, $t\neq\tau$ and $n\in\mathbb{N}$, $n\geq1$, we have the inequality:*

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\,ds-\frac{1}{n}\sum_{i=0}^{n-1}u\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)\right| \tag{4.13}$$

$$\leq\begin{cases}\dfrac{|\tau-t|}{4n}\|u'\|_{[t,\tau],\infty} & \text{if } u'\in L_\infty[a,b];\\[2ex] \dfrac{|\tau-t|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}n}\|u'\|_{[t,\tau],p} & \text{if } u'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{1}{2n}\|u'\|_{[t,\tau],1}, \end{cases}$$

where

$$\|u'\|_{[t,\tau],\infty}:=\underset{\substack{t\in[t,\tau]\\(t\in[\tau,t])}}{\operatorname{essup}}|u'(t)|$$

 doi:10.20944/preprints201803.0017.v1

and

$$\|u'\|_{[t,\tau],p} := \left| \int_t^\tau |u'(s)|^p \, ds \right|^{\frac{1}{p}}$$

for $p \geq 1$.

Proof. Consider the equidistant division of $[t,\tau]$ (if $t < \tau$) or $[\tau, t]$ (if $\tau < t$) given by

$$E_n : x_i = t + i \cdot \frac{\tau - t}{n}, \quad i = \overline{0, n}.$$

If we apply the inequality (4.11) on the interval $[x_i, x_{i+1}]$, we may write that

$$\left| \int_{x_i}^{x_{i+1}} u(s) \, ds - \frac{\tau - t}{n} \cdot u\left(t + \left(i + \frac{1}{2}\right) \frac{\tau - t}{n}\right) \right|$$

$$\leq \begin{cases} \dfrac{(\tau - t)^2}{4n^2} \|u'\|_{[x_i, x_{i+1}],\infty} & \text{if} \quad u' \in L_\infty[a,b]; \\ \dfrac{|\tau - t|^{1+\frac{1}{q}}}{2n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \|u'\|_{[x_i,x_{i+1}],p} & \text{if} \quad u' \in L_p[a,b], \\ & \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \dfrac{|\tau - t|}{2n} \|u'\|_{[x_i,x_{i+1}],1}, \end{cases}$$

from where we get

$$\left| \frac{1}{\tau - t} \int_{x_i}^{x_{i+1}} u(s) \, ds - \frac{1}{n} \cdot u\left(t + \left(i + \frac{1}{2}\right) \frac{\tau - t}{n}\right) \right|$$

$$\leq \begin{cases} \dfrac{|\tau - t|}{4n^2} \|u'\|_{[x_i, x_{i+1}],\infty}; \\ \dfrac{|\tau - t|^{\frac{1}{q}}}{2n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \|u'\|_{[x_i,x_{i+1}],p}; \\ \dfrac{1}{2n} \|u'\|_{[x_i,x_{i+1}],1}, \end{cases}$$

for all $i = \overline{0, n-1}$.

Summing over i from 0 to $n - 1$ and using the generalized triangle inequality, we may write

$$\left| \frac{1}{\tau - t} \int_t^\tau u(s) \, ds - \frac{1}{n} \sum_{i=0}^{n-1} u\left(t + \left(i + \frac{1}{2}\right) \frac{\tau - t}{n}\right) \right|$$

$$\leq \sum_{i=0}^{n-1} \left| \frac{1}{\tau - t} \int_{x_i}^{x_{i+1}} u(s) \, ds - \frac{1}{n} \cdot u\left(t + \left(i + \frac{1}{2}\right) \frac{\tau - t}{n}\right) \right|$$

$$\leq \begin{cases} \dfrac{|\tau - t|}{4n^2} \sum_{i=0}^{n-1} \|u'\|_{[x_i,x_{i+1}],\infty}\,; \\[2ex] \dfrac{|\tau - t|^{\frac{1}{q}}}{2n^{1+\frac{1}{q}}(q+1)^{\frac{1}{q}}} \sum_{i=0}^{n-1} \|u'\|_{[x_i,x_{i+1}],p}\,; \\[2ex] \dfrac{1}{2n} \sum_{i=0}^{n-1} \|u'\|_{[x_i,x_{i+1}],1}\,. \end{cases}$$

However,

$$\sum_{i=0}^{n-1} \|u'\|_{[x_i,x_{i+1}],\infty} \leq n \|u'\|_{[t,\tau],\infty}\,,$$

$$\begin{aligned} \sum_{i=0}^{n-1} \|u'\|_{[x_i,x_{i+1}],p} &\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u'(s)|^p\, ds \right|^{\frac{1}{p}} \\ &\leq n^{\frac{1}{q}} \left[\left(\sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u'(s)|^p\, ds \right|^{\frac{1}{p}} \right)^p \right]^{\frac{1}{p}} \\ &= n^{\frac{1}{q}} \left| \int_t^{\tau} |u'(s)|^p\, ds \right|^{\frac{1}{p}} = n^{\frac{1}{q}} \|u'\|_{[t,\tau],p} \end{aligned}$$

and

$$\sum_{i=0}^{n-1} \|u'\|_{[x_i,x_{i+1}],1} = \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} |u'(s)|\, ds \right| = \left| \int_t^{\tau} |u'(s)|\, ds \right| = \|u'\|_{[t,\tau],1}$$

and the lemma is proved. □

The following theorem in approximating the Hilbert transform of a differentiable function whose derivative f' is absolutely continuous holds.

Theorem 15 (Dragomir et al., 2004, [7]). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a,b]$. If*

$$M_n(f;t) = \frac{1}{n\pi}(b-a) \sum_{i=0}^{n-1} \left[f; t + \left(i + \frac{1}{2}\right) \frac{b-t}{n}, t - \left(i + \frac{1}{2}\right) \frac{t-a}{n} \right] \tag{4.14}$$

then we have the estimate:

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - M_n(f;t) \right| \tag{4.15}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

$$\leq \begin{cases} \frac{1}{4\pi n}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{q}{2\pi n(q+1)^{\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2\pi n}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}; \end{cases}$$

$$\leq \begin{cases} \frac{1}{8\pi n}(b-a)^2\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{q}{2\pi n(q+1)^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2\pi n}(b-a)\|f''\|_{[a,b],1}; \end{cases}$$

for all $t\in(a,b)$, *where* $[f;c,d]$ *denotes the divided difference*

$$[f;c,d]:=\frac{f(c)-f(d)}{c-d}.$$

Proof. Applying Lemma 6 for the function f', we may write that

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{n}\sum_{i=0}^{n-1}f'\left(t+\left(i+\frac{1}{2}\right)\frac{(\tau-t)}{n}\right)\right|$$
$$\leq \begin{cases} \frac{|\tau-t|}{4n}\|f''\|_{[t,\tau],\infty} \\ \frac{|\tau-t|^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}n}\|f''\|_{[t,\tau],p} \\ \frac{1}{2n}\|f''\|_{[t,\tau],1}, \end{cases}$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

 doi:10.20944/preprints201803.0017.v1

Consequently, we have

$$\left|\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{n\pi}\sum_{i=0}^{n-1}PV\int_a^b f'\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)d\tau\right| \tag{4.16}$$

$$\leq\begin{cases}\dfrac{1}{4\pi n}PV\displaystyle\int_a^b|\tau-t|\,\|f''\|_{[t,\,\tau],\infty}\,d\tau & \text{if} \quad f''\in L_\infty[a,b]\,;\\[2ex] \dfrac{1}{2\pi(q+1)^{\frac{1}{q}}n}PV\displaystyle\int_a^b|\tau-t|^{\frac{1}{q}}\,\|f''\|_{[t,\,\tau],p}\,d\tau & \text{if} \quad f''\in L_p[a,b]\,,\\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{1}{2\pi n}PV\displaystyle\int_a^b\|f''\|_{[t,\,\tau],1}\,d\tau. \end{cases}$$

Since

$$\begin{aligned}
&PV\int_a^b f'\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)d\tau\\
&=\lim_{\varepsilon\to0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)\right)d\tau\\
&=\lim_{\varepsilon\to0+}\left[\frac{n}{i+\frac{1}{2}}f\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)\Bigg|_a^{t-\varepsilon}+\frac{n}{i+\frac{1}{2}}f\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)\Bigg|_{t+\varepsilon}^b\right]\\
&=\frac{n}{i+\frac{1}{2}}\left[f(t)-f\left(t+\left(i+\frac{1}{2}\right)\frac{a-t}{n}\right)+f\left(t+\left(i+\frac{1}{2}\right)\frac{b-t}{n}\right)-f(t)\right]\\
&=\frac{n}{i+\frac{1}{2}}\left[f\left(t+\left(i+\frac{1}{2}\right)\frac{b-t}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\frac{a-t}{n}\right)\right]\\
&=(b-a)\left[f;t+\left(i+\frac{1}{2}\right)\frac{b-t}{n},t-\left(i+\frac{1}{2}\right)\frac{t-a}{n}\right]
\end{aligned}$$

and

$$\begin{aligned}
PV\int_a^b|\tau-t|\,\|f''\|_{[t,\tau],\infty}\,d\tau&\leq\|f''\|_{[a,b],\infty}\,PV\int_a^b|\tau-t|\,d\tau\\
&=\|f''\|_{[a,b],\infty}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right],\\
PV\int_a^b|\tau-t|^{\frac{1}{q}}\,\|f''\|_{[t,\,\tau],p}\,d\tau&\leq\|f''\|_{[a,b],p}\,PV\int_a^b|\tau-t|^{\frac{1}{q}}\,d\tau\\
&=\frac{q\,\|f''\|_{[a,b],p}}{q+1}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]
\end{aligned}$$

 doi:10.20944/preprints201803.0017.v1

and

$$\begin{aligned} PV\int_a^b \|f''\|_{[t,\,\tau],1}\,d\tau &= PV\left[\int_a^t \|f''\|_{[\tau,t],1}\,d\tau + \int_t^b \|f''\|_{[t,\,\tau],1}\,d\tau\right] \\ &\le (t-a)\|f''\|_{[a,t],1} + (b-t)\|f''\|_{[t,\,b],1} \\ &\le \max(t-a,b-t)\left[\|f''\|_{[a,t],1} + \|f''\|_{[t,b],1}\right] \\ &= \left(\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right)\|f''\|_{[a,b],1} \end{aligned}$$

then by (4.16) we obtain:

$$\left|\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau - M_n(f;t)\right| \tag{4.17}$$

$$\le \begin{cases} \frac{1}{4\pi n}\left[\frac{1}{4}(b-a)^2 + \left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{q}{2\pi n(q+1)^{\frac{1}{q}}}\left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2\pi n}\left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}. \end{cases}$$

On the other hand, as for the function $f_0:(a,b)\to\mathbb{R}$, $f_0(t)=1$, we have

$$(Tf_0)(a,b;t) = \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\quad t\in(a,b),$$

then obviously

$$\begin{aligned} (Tf)(a,b;t) &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau \\ &= \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau + \frac{f(t)}{\pi}PV\int_a^b \frac{d\tau}{\tau-t} \end{aligned}$$

from where we get the equality:

$$(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau. \tag{4.18}$$

Finally, using (4.17) and (4.18), we deduce (4.15). □

Before we go further and point out another estimate of the remainder in approximating the Hilbert Transform for functions whose second derivatives are absolutely continuous, we need the following lemma.

Lemma 7. *Let $u : [a,b] \to \mathbb{R}$ be a function such that its derivative is absolutely continuous on $[a,b]$. Then one has the inequalities*

$$\left| \int_a^b u(s)\,ds - u\left(\frac{a+b}{2}\right)(b-a) \right| \tag{4.19}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24} \|u''\|_{[a,b],\infty} & \text{if } \; u'' \in L_\infty[a,b]; \\ \frac{|b-a|^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}} \|u''\|_{[a,b],p} & \text{if } \; u'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8} \|u''\|_{[a,b],1}. \end{cases}$$

A simple proof of this inequality may be done by using the identity:

$$\int_a^b u(s)\,ds - u\left(\frac{a+b}{2}\right)(b-a) \tag{4.20}$$
$$= \frac{1}{2}\int_a^{\frac{a+b}{2}} (s-a)^2 f''(s)\,ds + \frac{1}{2}\int_{\frac{a+b}{2}}^b (b-s)^2 f''(s)\,ds.$$

We omit the details.

The following lemma also holds.

Lemma 8. *Let $u : [a,b] \to \mathbb{R}$ be a differentiable function such that $u' : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. Then for any $t,\tau \in [a,b]$, $t \in \tau$ and $n \in \mathbb{N}$, $n \geq 1$, we have the inequality:*

$$\left| \frac{1}{\tau-t}\int_t^\tau u(s)\,ds - \frac{1}{n}\sum_{i=0}^{n-1} u\left(t + \left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right) \right| \tag{4.21}$$

$$\leq \begin{cases} \frac{(\tau-t)^2}{24n^2} \|u''\|_{[t,\tau],\infty} & \text{if } \; u'' \in L_\infty[a,b]; \\ \frac{|\tau-t|^{1+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}} n^2} \|u''\|_{[t,\tau],p} & \text{if } \; u'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{|\tau-t|}{8n^2} \|u''\|_{[t,\tau],1}. \end{cases}$$

Proof. Consider the equidistant division of $[t,\tau]$ (or $[\tau,t]$)

$$E_n : x_i = t + i\cdot\frac{\tau-t}{n}, \quad i = \overline{0,n}.$$

If we apply the inequality (4.19), we may state that

$$\left|\int_{x_i}^{x_{i+1}} u(s)\,ds - \frac{\tau-t}{n}\cdot u\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)\right|$$

$$\leq \begin{cases} \dfrac{(\tau-t)^3}{24n^3}\left\|u''\right\|_{[x_i,x_{i+1}],\infty} & \text{if} \quad u''\in L_\infty[a,b]; \\ \dfrac{|\tau-t|^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}n^{2+\frac{1}{q}}}\left\|u''\right\|_{[x_i,x_{i+1}],p} & \text{if} \quad u''\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{|\tau-t|^2}{8n^2}\left\|u''\right\|_{[x_i,x_{i+1}],1}. \end{cases}$$

Dividing by $|\tau-t|>0$ and using a similar argument to the one in Lemma 6, we conclude that the desired inequality (4.21) holds. □

The following theorem also holds.

Theorem 16 (Dragomir et al., 2004, [7])**.** *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function such that the second derivative f'' is absolutely continuous on $[a,b]$. Then*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-M_n(f;t)\right| \tag{4.22}$$

$$\leq \begin{cases} \dfrac{1}{24\pi n^2}\left[\dfrac{(b-a)^2}{12}+\left(t-\dfrac{a+b}{2}\right)^2\right](b-a)\left\|f'''\right\|_{[a,b],\infty} & \textit{if} \quad f'''\in L_\infty[a,b]; \\ \dfrac{1}{8(2q+1)^{\frac{1}{q}}\pi n^2}\left[(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}\right]\left\|f'''\right\|_{[a,b],p} & \textit{if} \quad f'''\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{8\pi n^2}\left[\dfrac{(b-a)^2}{4}+\left(t-\dfrac{a+b}{2}\right)^2\right]\left\|f'''\right\|_{[a,b],1}; \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{72\pi n^2}(b-a)^3\left\|f'''\right\|_{[a,b],\infty} & \textit{if} \quad f'''\in L_\infty[a,b]; \\ \dfrac{1}{8(2q+1)^{\frac{1}{q}}\pi n^2}(b-a)^{2+\frac{1}{q}}\left\|f'''\right\|_{[a,b],p} & \textit{if} \quad f'''\in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\|f'''\right\|_{[a,b],1}}{16\pi n^2}(b-a)^2. \end{cases}$$

 doi:10.20944/preprints201803.0017.v1

Proof. Applying Lemma 8 for the function f'', we may write that (see also Theorem 15)

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{n}\sum_{i=0}^{n-1}f'\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)\right| \tag{4.23}$$

$$\leq\begin{cases}\dfrac{(\tau-t)^2}{24n^2}\left\|f'''\right\|_{[t,\tau],\infty} & \text{if } f'''\in L_\infty[a,b];\\[2ex] \dfrac{|\tau-t|^{1+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}n^2}\left\|f'''\right\|_{[t,\tau],p} & \text{if } f'''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \dfrac{|\tau-t|}{8n^2}\left\|f'''\right\|_{[t,\tau],1},\end{cases}$$

for all $t,\tau\in[a,b]$, $t\neq\tau$.

Consequently, by (4.23), we may write that

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi n}\sum_{i=0}^{n-1}PV\int_a^b f'\left(t+\left(i+\frac{1}{2}\right)\frac{\tau-t}{n}\right)d\tau\right| \tag{4.24}$$

$$\leq\begin{cases}\dfrac{1}{24\pi n^2}PV\displaystyle\int_a^b|\tau-t|^2\left\|f'''\right\|_{[t,\tau],\infty}d\tau\\[2ex] \dfrac{1}{8\pi(2q+1)^{\frac{1}{q}}n^2}PV\displaystyle\int_a^b|\tau-t|^{1+\frac{1}{q}}\left\|f'''\right\|_{[t,\tau],p}d\tau\\[2ex] \dfrac{1}{8\pi n^2}PV\displaystyle\int_a^b|\tau-t|\left\|f'''\right\|_{[t,\tau],1}d\tau.\end{cases}$$

Since

$$\begin{aligned}PV\int_a^b|\tau-t|^2\left\|f'''\right\|_{[t,\tau],\infty}d\tau&\leq\left\|f'''\right\|_{[a,b],\infty}PV\int_a^b|\tau-t|^2d\tau\\&=\left\|f'''\right\|_{[a,b],\infty}\left[\frac{(t-a)^3+(b-t)^3}{3}\right]\\&=\left\|f'''\right\|_{[a,b],\infty}\left[\frac{(b-a)^2}{12}+\left(t-\frac{a+b}{2}\right)^2\right](b-a),\end{aligned}$$

$$\begin{aligned}PV\int_a^b|\tau-t|^{1+\frac{1}{q}}\left\|f'''\right\|_{[t,\tau],p}d\tau&\leq\left\|f'''\right\|_{[a,b],p}PV\int_a^b|\tau-t|^{1+\frac{1}{q}}d\tau\\&=\frac{q\left\|f'''\right\|_{[a,b],p}}{2q+1}\left[(t-a)^{2+\frac{1}{q}}+(b-t)^{2+\frac{1}{q}}\right]\end{aligned}$$

 doi:10.20944/preprints201803.0017.v1

and

$$\begin{aligned} PV\int_a^b |\tau - t| \|f'''\|_{[t,\tau],1} d\tau &\le \|f'''\|_{[a,b],1} PV\int_a^b |\tau - t| d\tau \\ &= \|f'''\|_{[a,b],1} \frac{(t-a)^2 + (b-t)^2}{2} \\ &= \|f'''\|_{[a,b],1} \left[\frac{(b-a)^2}{4} + \left(t - \frac{a+b}{2} \right)^2 \right], \end{aligned}$$

then, as in Theorem 15, by (4.24) we deduce the first part of (4.22). The second part is obvious. □

5. Estimates for Derivatives of Bounded Variation

5.1. **Some Integral Inequalities.** We start with the following lemma proved in [8] dealing with an Ostrowski type inequality for functions of bounded variation.

Lemma 9. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then, for all $x \in [a,b]$, we have the inequality:*

$$\left| u(x)(b-a) - \int_a^b u(t) dt \right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (u), \tag{5.1}$$

where $\bigvee_a^b (u)$ denotes the total variation of u on $[a,b]$.
The constant $\frac{1}{2}$ is the possible one.

Proof. For the sake of completeness and since this result will be essentially used in what follows, we give here a short proof.

Using the integration by parts formula for the Riemann-Stieltjes integral we have

$$\int_a^x (t-a) du(t) = u(x)(x-a) - \int_a^x u(t) dt$$

and

$$\int_x^b (t-b) du(t) = u(x)(b-x) - \int_x^b u(t) dt.$$

If we add the above two equalities, we get

$$u(x)(b-a) - \int_a^b u(t) dt = \int_a^x (t-a) du(t) + \int_x^b (t-b) du(t) \tag{5.2}$$

for any $x \in [a,b]$.

If $p : [c,d] \to \mathbb{R}$ is continuous on $[c,d]$ and $v : [c,d] \to \mathbb{R}$ is of bounded variation on $[c,d]$, then:

$$\left| \int_c^d p(x) dv(x) \right| \le \sup_{x \in [c,d]} |p(x)| \bigvee_c^d (u). \tag{5.3}$$

Using (5.2) and (5.3), we deduce

$$\begin{aligned}\left|u(x)(b-a)-\int_a^b u(t)\,dt\right| &\le \left|\int_a^x (t-a)\,du(t)\right|+\left|\int_x^b (t-b)\,du(t)\right| \\ &\le (x-a)\bigvee_a^x (u)+(b-x)\bigvee_x^b (u) \\ &\le \max\{x-a,b-x\}\left[\bigvee_a^x (u)+\bigvee_x^b (u)\right] \\ &= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b (u)\end{aligned}$$

and the inequality (5.1) is proved.

Now, assume that the inequality (5.2) holds with a constant $c>0$, i.e.,

$$\left|u(x)(b-a)-\int_a^b u(t)\,dt\right| \le \left[c(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b (u) \tag{5.4}$$

for all $x\in[a,b]$.

Consider the function $u_0:[a,b]\to\mathbb{R}$ given by

$$u_0(x)=\begin{cases}0 & \text{if} \quad x\in[a,b]\setminus\left\{\frac{a+b}{2}\right\} \\ \\ 1 & \text{if} \quad x=\frac{a+b}{2}.\end{cases}$$

Then u_0 is of bounded variation on $[a,b]$ and

$$\bigvee_a^b (u_0)=2,\quad \int_a^b u_0(t)\,dt=0.$$

If we apply (5.4) for u_0 and choose $x=\frac{a+b}{2}$, then we get $2c\ge 1$ which implies that $c\ge\frac{1}{2}$ showing that $\frac{1}{2}$ is the best possible constant in (5.1). □

The best inequality we can get from (5.1) is the following midpoint inequality.

Corollary 12. *With the assumptions in Lemma 9, we have*

$$\left|u\left(\frac{a+b}{2}\right)(b-a)-\int_a^b u(t)\,dt\right|\le\frac{1}{2}(b-a)\bigvee_a^b (u). \tag{5.5}$$

The constant $\frac{1}{2}$ is best possible.

Using the above Ostrowski type inequality we may point out the following result in estimating the finite Hilbert transform.

Theorem 17 (Dragomir, 2002, [9])**.** *Let $f:[a,b]\to\mathbb{R}$ be a function such that its derivative $f':[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}&\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{b-a}{\pi}[f;\lambda t+(1-\lambda)b,\lambda t+(1-\lambda)a]\right| \\ &\le\frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b (f'),\end{aligned} \tag{5.6}$$

for any $t \in (a,b)$ *and* $\lambda \in [0,1)$*, where* $[f;\alpha,\beta]$ *is the divided difference, i.e.,*

$$[f;\alpha,\beta] := \frac{f(\alpha)-f(\beta)}{\alpha-\beta}.$$

Proof. Since f' is bounded on $[a,b]$, it follows that f is Lipschitzian on $[a,b]$ and thus the finite Hilbert transform exists everywhere in (a,b).

As for the function $f_0 : (a,b) \to \mathbb{R}$, $f_0(t) = 1$, $t \in (a,b)$, we have

$$(Tf_0)(a,b;t) = \frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right), \quad t \in (a,b),$$

then obviously

$$(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau. \tag{5.7}$$

Now, if we choose in (5.1), $u = f'$, $x = \lambda c + (1-\lambda)d$, $\lambda \in [0,1]$, then we get

$$\begin{aligned}&\left|f(d)-f(c)-(d-c)f'(\lambda c+(1-\lambda)d)\right| \\ &\leq \left[\frac{1}{2}|d-c| + \left|\lambda c + (1-\lambda)d - \frac{c+d}{2}\right|\right]\left|\bigvee_c^d (f')\right|\end{aligned}$$

where $c,d \in (a,b)$, which is equivalent to

$$\left|\frac{f(d)-f(c)}{d-c} - f'(\lambda c+(1-\lambda)d)\right| \leq \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left|\bigvee_c^d (f')\right| \tag{5.8}$$

for any $c,d \in (a,b)$, $c \neq d$.

Using (5.8), we may write

$$\begin{aligned}&\left|\frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau - \frac{1}{\pi}PV\int_a^b f'(\lambda t+(1-\lambda)\tau)\,d\tau\right| \\ &\leq \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right] PV\int_a^b \left|\bigvee_\tau^t (f')\right| dt \\ &= \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\int_a^t \left(\bigvee_\tau^t (f')\right) dt + \int_t^b \left(\bigvee_t^\tau (f')\right) dt\right] \\ &\leq \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[(t-a)\bigvee_a^t (f') + (b-t)\bigvee_t^b (f')\right] \\ &\leq \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b (f').\end{aligned} \tag{5.9}$$

Since (for $\lambda \neq 1$)

$$\begin{aligned}
&\frac{1}{\pi} PV \int_a^b f'(\lambda t + (1-\lambda)\tau)\, d\tau \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] (f'(\lambda t + (1-\lambda)\tau)\, d\tau) \\
&= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_a^{t-\varepsilon} + \frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_{t+\varepsilon}^b \right] \\
&= \frac{1}{\pi} \cdot \frac{f(t) - f(\lambda t + (1-\lambda)a) + f(\lambda t + (1-\lambda)b) - f(t)}{1-\lambda} \\
&= \frac{b-a}{\pi} [f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a].
\end{aligned}$$

Using (5.9) and (5.7), we deduce the desired result (5.6). □

It is obvious that the best inequality we can get from (5.6) is the one for $\lambda = \frac{1}{2}$. Thus, we may state the following corollary.

Corollary 13. *With the assumptions of Theorem 17, we have*

$$\begin{aligned}
&\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - \frac{b-a}{\pi}\left[f; \frac{t+b}{2}, \frac{a+t}{2}\right] \right| \\
&\leq \frac{1}{2\pi}\left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] \bigvee_a^b (f').
\end{aligned} \tag{5.10}$$

The above Theorem 17 may be used to point out some interesting inequalities for the functions for which the finite Hilbert transforms $(Tf)(a,b;t)$ can be expressed in terms of special functions.

For instance, we have:

1) Assume that $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Then

$$(Tf)(a,b;t) = \frac{1}{\pi t} \ln\left[\frac{(b-t)a}{(t-a)b}\right], \quad t \in (a,b),$$

$$\begin{aligned}
&\frac{b-a}{\pi} \cdot [f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a] \\
&= -\frac{1}{\pi} \cdot \frac{b-a}{[\lambda t + (1-\lambda)b][\lambda t + (1-\lambda)a]},
\end{aligned}$$

$$\bigvee_a^b (f') = \int_a^b |f''(t)|\, dt = \frac{b^2-a^2}{a^2b^2}.$$

Using the inequality (5.6) we may write that

$$\begin{aligned}
&\left| \frac{1}{\pi t} \ln\left[\frac{(b-t)a}{(t-a)b}\right] - \frac{1}{\pi t} \ln\left(\frac{b-t}{t-a}\right) + \frac{b-a}{\pi[\lambda t + (1-\lambda)b][\lambda t + (1-\lambda)a]} \right| \\
&\leq \frac{1}{\pi}\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] \cdot \frac{b^2-a^2}{a^2b^2}
\end{aligned}$$

which is equivalent to:

$$\left|\frac{b-a}{\left[\lambda t+(1-\lambda) b\right]\left[\lambda t+(1-\lambda) a\right]}-\frac{1}{t} \ln \left(\frac{b}{a}\right)\right| \leq\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right] \cdot \frac{b^{2}-a^{2}}{a^{2} b^{2}} . \tag{5.11}$$

If we use the notations

$$\begin{aligned} L(a, b) &:=\frac{b-a}{\ln b-\ln a} && \text{(the logarithmic mean)} \\ A_{\lambda}(x, y) &:=\lambda x+(1-\lambda) y && \text{(the weighted arithmetic mean)} \\ G(a, b) &:=\sqrt{a b} && \text{(the geometric mean)} \\ A(a, b) &:=\frac{a+b}{2} && \text{(the arithmetic mean)} \end{aligned}$$

then by (5.11) we deduce

$$\left|\frac{1}{A_{\lambda}(t, b) A_{\lambda}(t, a)}-\frac{1}{t L(a, b)}\right| \leq\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+|t-A(a, b)|\right] \frac{2 A(a, b)}{G^{4}(a, b)},$$

giving the following proposition:

Proposition 1. *With the above assumption, we have*

$$\begin{aligned} &\left|t L(a, b)-A_{\lambda}(t, b) A_{\lambda}(t, a)\right| \\ &\leq \frac{2 A(a, b)}{G^{4}(a, b)}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right] t A_{\lambda}(t, b) A_{\lambda}(t, a) L(a, b) \end{aligned} \tag{5.12}$$

for any $t \in(a, b)$, $\lambda \in[0,1)$.

In particular, for $t=A(a, b)$ and $\lambda=\frac{1}{2}$, we get

$$\left|A(a, b) L(a, b)-\frac{(A(a, b)+a)(A(a, b)+b)}{4}\right| \leq \frac{1}{2} \cdot \frac{A^{2}(a, b)}{G^{4}(a, b)} \cdot \frac{(A(a, b)+a)(A(a, b)+b)}{4} L(a, b) . \tag{5.13}$$

2) Assume that $f:[a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, $f(x)=\exp (x)$. Then

$$(T f)(a, b ; t)=\frac{\exp (t)}{\pi}[E i(b-t)-E i(a-t)],$$

where

$$E i(z):=P V \int_{-\infty}^{z} \frac{\exp (t)}{t} d t, \quad z \in \mathbb{R}.$$

Also, we have:

$$\begin{aligned} &\frac{b-a}{\pi}[\exp ; \lambda t+(1-\lambda) b, \lambda t+(1-\lambda) a] \\ &=\frac{1}{\pi} \cdot \frac{\exp (\lambda t+(1-\lambda) b)-\exp (\lambda t+(1-\lambda) a)}{1-\lambda}, \end{aligned}$$

$$\bigvee_a^b (f') = \int_a^b |f''(t)|\, dt = \exp(b) - \exp(a).$$

Using the inequality (5.6) we may write:

$$\left| \exp(t) \left[Ei(b-t) - Ei(a-t) - \ln\left(\frac{b-t}{t-a}\right)\right] - \frac{\exp(\lambda t + (1-\lambda) b) - \exp(\lambda t + (1-\lambda) a)}{1-\lambda} \right| \leq \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right] \left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] [\exp(b) - \exp(a)] \tag{5.14}$$

for any $t \in (a,b)$.

If in (5.14) we make $\lambda = \frac{1}{2}$ and $t = \frac{a+b}{2}$, we get

$$\left| \exp\left(\frac{a+b}{2}\right) \mathrm{Ei}\left(\frac{b-a}{2}\right) - 2\left[\exp\left(\frac{a+3b}{4}\right) - \exp\left(\frac{3a+b}{4}\right)\right]\right| \leq \frac{1}{4}(b-a)[\exp(b) - \exp(a)],$$

which is equivalent to:

$$\left| \mathrm{Ei}\left(\frac{b-a}{2}\right) - 2\left[\exp\left(\frac{b-a}{4}\right) - \exp\left(-\frac{b-a}{4}\right)\right]\right| \leq \frac{1}{4}(b-a)\left[\exp\left(\frac{b-a}{2}\right) - \exp\left(-\frac{b-a}{2}\right)\right].$$

If in this inequality we make $\frac{b-a}{2} = z > 0$, then we get

$$\left| Ei(z) - 2\left[\exp\left(\frac{z}{2}\right) - \exp\left(-\frac{z}{2}\right)\right]\right| \leq \frac{1}{2} z [\exp(z) - \exp(-z)] \tag{5.15}$$

for any $z > 0$.

Consequently, we may state the following proposition.

Proposition 2. *With the above assumptions, we have*

$$\left| Ei(z) - 4\sinh\left(\frac{1}{2}z\right)\right| \leq z \sinh(z) \tag{5.16}$$

for any $z > 0$.

The reader may get other similar inequalities for special functions if appropriate examples of functions f are chosen.

5.2. **A Quadrature Formula for Equidistant Divisions.** The following lemma is of interest in itself.

Lemma 10. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. Then for all $n \geq 1$, $\lambda_i \in [0,1)$ $(i = 0, \dots, n-1)$ and $t, \tau \in [a,b]$ with $t \neq \tau$, we have the inequality:*

$$\left| \frac{1}{\tau - t} \int_t^\tau u(s)\, ds - \frac{1}{n} \sum_{i=0}^{n-1} u\left[t + (i + 1 - \lambda_i) \frac{\tau - t}{n}\right]\right| \leq \frac{1}{n}\left[\frac{1}{2} + \max_{i=\overline{0,n-1}} \left|\lambda_i - \frac{1}{2}\right|\right] \left|\bigvee_t^\tau (u)\right|. \tag{5.17}$$

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$E_n : x_i = t + i \cdot \frac{\tau - t}{n}, \quad i = \overline{0,n}. \tag{5.18}$$

Then the points $\xi_i = \lambda_i \left[t + i \cdot \frac{\tau-t}{n}\right] + (1-\lambda_i)\left[t + (i+1)\cdot\frac{\tau-t}{n}\right]$ $\left(\lambda_i \in [0,1],\ i = \overline{0,n-1}\right)$ are between x_i and x_{i+1}. We observe that we may write for simplicity $\xi_i = t + (i+1-\lambda_i)\frac{\tau-t}{n}$ $\left(i=\overline{0,n-1}\right)$. We also have

$$\xi_i - \frac{x_i + x_{i+1}}{2} = \frac{\tau - t}{2n}(1-2\lambda_i), \ \xi_i - x_i = (1-\lambda_i)\frac{\tau-t}{n}$$

and

$$x_{i+1} - \xi_i = \lambda_i \cdot \frac{\tau - t}{n}$$

for any $i = \overline{0,n-1}$.

If we apply the inequality (5.1) on the interval $[x_i, x_{i+1}]$ and the intermediate point ξ_i $\left(i=\overline{0,n-1}\right)$, then we may write that

$$\begin{aligned}
&\left|\frac{\tau-t}{n} u\left(t + (i+1-\lambda_i)\frac{\tau-t}{n}\right) - \int_{x_i}^{x_{i+1}} u(s)\,ds\right| \\
&\leq \left[\frac{1}{2}\cdot\frac{|\tau-t|}{n} + \left|\frac{\tau-t}{2n}(1-2\lambda_i)\right|\right]\left|\bigvee_{x_i}^{x_{i+1}}(u)\right|.
\end{aligned} \tag{5.19}$$

Summing, we get

$$\begin{aligned}
&\left|\int_t^\tau u(s)\,ds - \frac{\tau-t}{n}\sum_{i=0}^{n-1} u\left[t + (i+1-\lambda_i)\frac{\tau-t}{n}\right]\right| \\
&\leq \frac{|\tau-t|}{2n}\sum_{i=0}^{n-1}\left[1 + |1-2\lambda_i|\right]\left|\bigvee_{x_i}^{x_{i+1}}(u)\right| \\
&= \frac{|\tau-t|}{n}\left[\frac{1}{2} + \max_{i=\overline{0,n-1}}\left|\lambda_i - \frac{1}{2}\right|\right]\left|\bigvee_t^\tau(u)\right|,
\end{aligned}$$

which is equivalent to (5.17). □

We may now state the following theorem in approximating the finite Hilbert transform of a differentiable functions with the derivative of bounded variation on $[a,b]$.

Theorem 18 (Dragomir, 2002, [9]). *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is of bounded variation on $[a,b]$. If $\boldsymbol{\lambda} = (\lambda_i)_{i=\overline{0,n-1}}$, $\lambda_i \in [0,1)$ $\left(i=\overline{0,n-1}\right)$ and*

$$S_n(f;\boldsymbol{\lambda},t) := \frac{b-a}{\pi n}\sum_{i=0}^{n-1}\left[f;(i+1-\lambda_i)\frac{b-t}{n}+t,(i+1-\lambda_i)\frac{a-t}{n}+t\right], \tag{5.20}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

then we have the estimate:

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-S_n(f;\boldsymbol{\lambda},t)\right| \tag{5.21}$$
$$\leq\frac{b-a}{n\pi}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f')$$
$$\leq\frac{b-a}{n\pi}\bigvee_a^b(f').$$

Proof. Applying Lemma 10 for the function f', we may write that

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\frac{1}{n}\sum_{i=0}^{n-1}f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]\right| \tag{5.22}$$
$$\leq\frac{1}{n}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]\left|\bigvee_t^\tau(f')\right|$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

Consequently, we have

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi n}\sum_{i=0}^{n-1}PV\int_a^b f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau\right| \tag{5.23}$$
$$\leq\frac{1}{n\pi}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]PV\int_a^b\left|\bigvee_t^\tau(f')\right|d\tau$$
$$\leq\frac{1}{n\pi}\left[\frac{1}{2}+\max_{i=\overline{0,n-1}}\left|\lambda_i-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f').$$

On the other hand

$$PV\int_a^b f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau \tag{5.24}$$
$$=\lim_{\varepsilon\to0+}\left[\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right]\left(f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau\right)$$
$$=\lim_{\varepsilon\to0+}\left[\frac{n}{i+1-\lambda_i}f\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right)\Big|_a^{t-\varepsilon}\right.$$
$$\left.+\frac{n}{i+1-\lambda_i}f\left(t+(i+1-\lambda_i)\frac{\tau-t}{n}\right)\Big|_{t+\varepsilon}^b\right]$$
$$=\frac{n}{i+1-\lambda_i}\left[f\left(t+(i+1-\lambda_i)\frac{b-t}{n}\right)-f\left(t+(i+1-\lambda_i)\frac{a-t}{n}\right)\right]$$
$$=(b-a)\left[f;t+(i+1-\lambda_i)\frac{b-t}{n},(i+1-\lambda_i)\frac{a-t}{n}+t\right].$$

Since (see for example (5.7)),

$$(Tf)(a,b;t)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$

for $t \in (a,b)$, then by (5.23) and (5.24) we deduce the desired estimate (5.21). □

Remark 7. *For $n = 1$, we recapture the inequality (5.6).*

Corollary 14. *With the assumptions of Theorem 18, we have*

$$(Tf)(a,b;t) = \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) + \lim_{n\to\infty} S_n(f;\boldsymbol{\lambda},t) \tag{5.25}$$

uniformly by rapport of $t \in (a,b)$ and $\boldsymbol{\lambda}$ with $\lambda_i \in [0,1)$ $(i \in \mathbb{N})$.

Remark 8. *If one needs to approximate the finite Hilbert Transform $(Tf)(a,b;t)$ in terms of*

$$\frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) + S_n(f;\boldsymbol{\lambda},t)$$

with the accuracy $\varepsilon > 0$ (ε small), then the theoretical minimal number n_ε to be chosen is:

$$n_\varepsilon := \left[\frac{b-a}{\varepsilon\pi} \bigvee_a^b (f')\right] + 1 \tag{5.26}$$

where $[\alpha]$ is the integer part of α.

It is obvious that the best inequality we can get in (5.21) is for $\lambda_i = \frac{1}{2}$ $\left(i = \overline{0,n-1}\right)$ obtaining the following corollary.

Corollary 15. *Let f be as in Theorem 18. Define*

$$M_n(f;t) := \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; \left(i+\frac{1}{2}\right)\frac{b-t}{n} + t, \left(i+\frac{1}{2}\right)\frac{a-t}{n} + t\right]. \tag{5.27}$$

Then we have the estimate

$$\begin{aligned} &\left|(Tf)(a,b;t) - \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) - M_n(f;t)\right| \\ &\leq \frac{b-a}{2n\pi}\left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] \bigvee_a^b (f') \end{aligned} \tag{5.28}$$

for any $t \in (a,b)$.

5.3. **A More General Quadrature Formula.** We may state the following lemma.

Lemma 11. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$, $0 = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = 1$ and $\nu_i \in [\mu_i, \mu_{i+1}]$, $i = \overline{0,n-1}$. Then for any $t,\tau \in [a,b]$ with $t \neq \tau$, we have the inequality:*

$$\begin{aligned} &\left|\frac{1}{\tau - t}\int_t^\tau u(s)\,ds - \sum_{i=0}^{n-1} (\mu_{i+1} - \mu_i)\, u[(1-\nu_i)t + \nu_i\tau]\right| \\ &\leq \left[\frac{1}{2}\Delta_n(\boldsymbol{\mu}) + \max_{i=\overline{0,n-1}}\left|\nu_i - \frac{\mu_i + \mu_{i+1}}{2}\right|\right] \left|\bigvee_t^\tau (u)\right|, \end{aligned} \tag{5.29}$$

where $\Delta_n(\boldsymbol{\mu}) := \max_{i=\overline{0,n-1}} (\mu_{i+1} - \mu_i)$.

Proof. Consider the division of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$I_n : x_i := (1-\mu_i)\,t + \mu_i\tau \quad \left(i=\overline{0,n}\right). \tag{5.30}$$

Then the points $\xi_i := (1-\nu_i)\,t+\nu_i\tau \quad \left(i=\overline{0,n-1}\right)$ are between x_i and x_{i+1}. We have

$$x_{i+1}-x_i = \left(\mu_{i+1}-\mu_i\right)(\tau-t) \quad \left(i=\overline{0,n-1}\right)$$

and

$$\xi_i - \frac{x_i+x_{i+1}}{2} = \left(\nu_i - \frac{\mu_i+\mu_{i+1}}{2}\right)(\tau-t) \quad \left(i=\overline{0,n-1}\right).$$

Applying the inequality (5.1) on $[x_i,x_{i+1}]$ with the intermediate points ξ_i $\left(i=\overline{0,n-1}\right)$, we get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} u(s)\,ds - \left(\mu_{i+1}-\mu_i\right)(\tau-t)\,u\left[(1-\nu_i)\,t+\nu_i\tau\right]\right|\\ &\leq \left[\frac{1}{2}\left(\mu_{i+1}-\mu_i\right)|\tau-t| + |\tau-t|\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_{x_i}^{x_{i+1}}(u)\right|\end{aligned}$$

for any $i=\overline{0,n-1}$. Summing over i, using the generalized triangle inequality and dividing by $|t-\tau|>0$, we obtain

$$\begin{aligned}&\left|\frac{1}{\tau-t}\int_a^b u(s)\,ds - \sum_{i=0}^{n-1}\left(\mu_{i+1}-\mu_i\right)u\left[(1-\nu_i)\,t+\nu_i\tau\right]\right|\\ &\leq \sum_{i=0}^{n-1}\left[\frac{1}{2}\left(\mu_{i+1}-\mu_i\right)+\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_{x_i}^{x_{i+1}}(u)\right|\\ &\leq \left[\frac{1}{2}\Delta_n(\boldsymbol{\mu}) + \max_{i=\overline{0,n-1}}\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_t^{\tau}(u)\right|\end{aligned}$$

and the inequality (5.29) is proved. □

The following theorem holds.

Theorem 19 (Dragomir, 2002, [9])**.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function such that its derivative f' is of bounded variation on $[a,b]$. If $0=\mu_0<\mu_1<\cdots<\mu_{n-1}<\mu_n=1$ and $\nu_i\in\left[\mu_i,\mu_{i+1}\right]$, $\left(i=\overline{0,n-1}\right)$, then*

$$(Tf)(a,b;t) = \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) + \frac{1}{\pi}Q_n(\boldsymbol{\mu},\boldsymbol{\nu},t) + W_n(\boldsymbol{\mu},\boldsymbol{\nu},t) \tag{5.31}$$

for any $t\in(a,b)$, where

$$\begin{aligned}&Q_n(\boldsymbol{\mu},\boldsymbol{\nu},t)\\ &:= \mu_1 f'(t)(b-a) + (b-a)\sum_{i=1}^{n-2}\Big\{\left(\mu_{i+1}-\mu_i\right)\\ &\times\left[f;(1-\nu_i)\,t+\nu_i b,(1-\nu_i)\,t+\nu_i a\right]\Big\} + \left(1-\mu_{n-1}\right)\left[f(b)-f(a)\right]\end{aligned} \tag{5.32}$$

if $\nu_0 = 0,\ \nu_{n-1} = 1,$

$$Q_n(\boldsymbol{\mu},\boldsymbol{\nu},t) := \mu_1 f'(t)(b-a) + (b-a)\sum_{i=1}^{n-1}(\mu_{i+1}-\mu_i) \times [f;(1-\nu_i)t+\nu_i b,(1-\nu_i)t+\nu_i a] \tag{5.33}$$

if $\nu_0 = 0,\ \nu_{n-1} < 1,$

$$Q_n(\boldsymbol{\mu},\boldsymbol{\nu},t) := (b-a)\sum_{i=1}^{n-2}(\mu_{i+1}-\mu_i) \times [f;(1-\nu_i)t+\nu_i b,(1-\nu_i)t+\nu_i a] + (1-\mu_{n-1})[f(b)-f(a)] \tag{5.34}$$

if $\nu_0 > 0,\ \nu_{n-1} = 1$ *and*

$$Q_n(\boldsymbol{\mu},\boldsymbol{\nu},t) := (b-a)\sum_{i=1}^{n-1}(\mu_{i+1}-\mu_i)[f;(1-\nu_i)t+\nu_i b,(1-\nu_i)t+\nu_i a] \tag{5.35}$$

if $\nu_0 > 0,\ \nu_{n-1} < 1.$
In all cases, the remainder satisfies the estimate:

$$\begin{aligned}|W_n(\boldsymbol{\mu},\boldsymbol{\nu},t)| &\le \frac{1}{\pi}\left[\frac{1}{2}\Delta_n(\boldsymbol{\mu}) + \max_{i=\overline{0,n-1}}\left|\nu_i - \frac{\mu_i+\mu_{i+1}}{2}\right|\right] \\ &\quad\times\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\bigvee_a^b(f') \\ &\le \frac{1}{\pi}\Delta_n(\boldsymbol{\mu})\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\left|\bigvee_a^b(f')\right| \\ &\le \frac{1}{\pi}\Delta_n(\boldsymbol{\mu})(b-a)\bigvee_a^b(f').\end{aligned} \tag{5.36}$$

Proof. If we apply Lemma 11 for the function f', we may write that

$$\begin{aligned}&\left|\frac{f(\tau)-f(t)}{\tau-t} - \sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i) f'[(1-\nu_i)t+\nu_i\tau]\right| \\ &\le \left[\frac{1}{2}\Delta_n(\boldsymbol{\mu}) + \max_{i=\overline{0,n-1}}\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]\left|\bigvee_t^\tau(f')\right|\end{aligned}$$

for any $t,\tau\in[a,b],\ t\neq\tau$.

Taking the PV in both sides, we may write that

$$\begin{aligned}&\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau \right.\\ &\left.-\frac{1}{\pi}PV\int_a^b\left(\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i) f'[(1-\nu_i)t+\nu_i\tau]\right)d\tau\right| \\ &\le \frac{1}{\pi}\left[\frac{1}{2}\Delta_n(\boldsymbol{\mu}) + \max_{i=\overline{0,n-1}}\left|\nu_i-\frac{\mu_i+\mu_{i+1}}{2}\right|\right]PV\int_a^b\left|\bigvee_t^\tau(f')\right|d\tau.\end{aligned} \tag{5.37}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

If $\nu_0 = 0$, $\nu_{n-1} = 1$, then

$$\begin{aligned}
&PV \int_a^b \left(\sum_{i=0}^{n-1} \left(\mu_{i+1} - \mu_i\right) f' \left[(1 - \nu_i) t + \nu_i \tau \right] \right) d\tau \\
&= PV \int_a^b \mu_1 f'(t)\, d\tau + \sum_{i=1}^{n-2} \left(\mu_{i+1} - \mu_i\right) PV \int_a^b f' \left[(1 - \nu_i) t + \nu_i \tau \right] d\tau \\
&+ \left(1 - \mu_{n-1}\right) PV \int_a^b f'(\tau)\, d\tau \\
&= \mu_1 f'(t)(b-a) + (b-a) \sum_{i=1}^{n-2} \left(\mu_{i+1} - \mu_i\right) \left[f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a \right] \\
&+ \left(1 - \mu_{n-1}\right) \left[f(b) - f(a) \right].
\end{aligned}$$

If $\nu_0 = 0$, $\nu_{n-1} < 1$, then

$$\begin{aligned}
&PV \int_a^b \left(\sum_{i=0}^{n-1} \left(\mu_{i+1} - \mu_i\right) f' \left[(1 - \nu_i) t + \nu_i \tau \right] \right) d\tau \\
&= \mu_1 f'(t)(b-a) + (b-a) \sum_{i=1}^{n-1} \left(\mu_{i+1} - \mu_i\right) \left[f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a \right].
\end{aligned}$$

If $\nu_0 > 0$, $\nu_{n-1} = 1$, then

$$\begin{aligned}
&PV \int_a^b \left(\sum_{i=0}^{n-1} \left(\mu_{i+1} - \mu_i\right) f' \left[(1 - \nu_i) t + \nu_i \tau \right] \right) d\tau \\
&= (b-a) \sum_{i=1}^{n-2} \left(\mu_{i+1} - \mu_i\right) \left[f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a \right] + \left(1 - \mu_{n-1}\right) \left[f(b) - f(a) \right].
\end{aligned}$$

and, finally, if $\nu_0 > 0$, $\nu_{n-1} < 1$, then

$$\begin{aligned}
&PV \int_a^b \left(\sum_{i=0}^{n-1} \left(\mu_{i+1} - \mu_i\right) f' \left[(1 - \nu_i) t + \nu_i \tau \right] \right) d\tau \\
&= (b-a) \sum_{i=1}^{n-1} \left(\mu_{i+1} - \mu_i\right) \left[f; (1-\nu_i) t + \nu_i b, (1-\nu_i) t + \nu_i a \right].
\end{aligned}$$

Since

$$PV \int_a^b \left| \bigvee_t^\tau (f') \right| d\tau \le \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \bigvee_a^b (f')$$

and

$$(Tf)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right),$$

then by (5.37) we deduce (5.31). □

6. Estimates for Absolutely Continuous Derivatives

6.1. **Ostrowski type inequalities.** For the sake of completeness, we state and prove the following lemma providing some Ostrowski type inequalities for absolutely continuous functions (see [15], [16] and [17]).

Lemma 12. *Let $u : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then we have:*

$$\left| u(x)(b-a) - \int_a^b u(t)\,dt \right| \tag{6.1}$$

$$\leq \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|u'\|_{[a,b],\infty} & \text{if } u' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[(x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}}\right] \|u'\|_{[a,b],p} & \text{if } u' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \|u'\|_{[a,b],1}, & \text{if } u' \in L[a,b] \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesgue norms, i.e., for $c < d$

$$\|h\|_{[c,d],\infty} := ess \sup_{t\in[c,d]} |h(t)|$$

and

$$\|h\|_{[c,d],r} := \left(\int_c^d |h(t)|^r\,dt\right)^{\frac{1}{r}}, \quad r \geq 1.$$

Proof. Using the integration by parts formula, we have

$$\int_a^x (t-a)\,u'(t)\,dt = u(x)(x-a) - \int_a^x u(t)\,dt$$

and

$$\int_x^b (t-b)\,u'(t)\,dt = u(x)(b-x) - \int_x^b u(t)\,dt.$$

If we add the above two equalities, we get

$$u(x)(b-a) - \int_a^b u(t)\,dt = \int_a^x (t-a)\,u'(t)\,dt + \int_x^b (t-b)\,u'(t)\,dt \tag{6.2}$$

for any $x \in [a,b]$.

Taking the modulus, we have

$$\left| u(x)(b-a) - \int_a^b u(t)\,dt \right| \leq \int_a^x (t-a)\,|u'(t)|\,dt + \int_x^b (t-b)\,|u'(t)|\,dt$$
$$=: M(x).$$

Now, it is obvious that

$$M(x) \leq \|u'\|_{[a,x],\infty} \int_a^x (t-a)\,dt + \|u'\|_{[x,b],\infty} \int_x^b (b-t)\,dt$$
$$= \|u'\|_{[a,x],\infty} \cdot \frac{(x-a)^2}{2} + \|u'\|_{[x,b],\infty} \cdot \frac{(b-x)^2}{2}$$

$$\le \|u'\|_{[a,b],\infty}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]$$
$$= \|u'\|_{[a,b],\infty}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right],$$

proving the first part of (6.1).

Using Hölder's integral inequality, we may write:

$$M(x) \le \|u'\|_{[a,x],p}\left(\int_a^x (t-a)^q\,dt\right)^{\frac{1}{q}} + \|u'\|_{[x,b],p}\left(\int_x^b (b-t)^q\,dt\right)^{\frac{1}{q}}$$
$$= \|u'\|_{[a,x],p}\cdot\left[\frac{(x-a)^{q+1}}{q+1}\right]^{\frac{1}{q}} + \|u'\|_{[x,b],p}\cdot\left[\frac{(b-x)^{q+1}}{q+1}\right]^{\frac{1}{q}}$$
$$\le \|u'\|_{[a,b],p}\frac{1}{(q+1)^{\frac{1}{q}}}\left[(x-a)^{1+\frac{1}{q}}+(b-x)^{1+\frac{1}{q}}\right],$$

proving the second part of (6.1).

Finally, we observe that

$$M(x) \le (x-a)\|u'\|_{[a,x],1} + (b-x)\|u'\|_{[x,b],1}$$
$$\le \max\{x-a,b-x\}\left[\|u'\|_{[a,x],1}+\|u'\|_{[x,b],1}\right]$$
$$= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\|u'\|_{[a,b],1}$$

and the lemma is proved. □

The best inequalities we can get from (6.1) are embodied in the following corollary.

Corollary 16. *With the assumptions of Lemma 12, we have*

$$\left|u\left(\frac{a+b}{2}\right)(b-a)-\int_a^b u(t)\,dt\right| \tag{6.3}$$
$$\le \begin{cases} \frac{1}{4}(b-a)^2\|u'\|_{[a,b],\infty} & \text{if } u'\in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|u'\|_{[a,b],p} & \text{if } u'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(b-a)\|u'\|_{[a,b],1}. & \text{if } u'\in L[a,b] \end{cases}$$

The following theorem providing an estimate for the finite Hilbert transform, holds.

Theorem 20 (Dragomir, 2002, [10])**.** *Let $f:[a,b]\to\mathbb{R}$ be a function so that its derivative $f':[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequalities*

$$\left|(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)-\frac{b-a}{\pi}[f;\lambda t+(1-\lambda)b,\lambda t+(1-\lambda)a]\right| \tag{6.4}$$

$$\leq \begin{cases} \frac{1}{\pi}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right] \\ \quad\times\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{1}{\pi}\frac{q}{(q+1)^{1+\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}}+(1-\lambda)^{1+\frac{1}{q}}\right] \\ \quad\times\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, & \text{if } f''\in L[a,b] \end{cases}$$

for any $t\in(a,b)$ and $\lambda\in[0,1)$, where $[f;\alpha,\beta]$ is the divided difference, i.e.,

$$[f;\alpha,\beta]=\frac{f(\alpha)-f(\beta)}{\alpha-\beta}.$$

Proof. Since f' is bounded on $[a,b]$, it follows that f is Lipschitzian on $[a,b]$ and thus the finite Hilbert transform exists everywhere in (a,b). As for the function $f_0:(a,b)\to\mathbb{R}$, $f_0(t)=1$, $t\in(a,b)$, we have

$$(Tf_0)(a,b;t)=\frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\ t\in(a,b),$$

then, obviously,

$$\begin{aligned}(Tf)(a,b;t)&=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)+f(t)}{\tau-t}d\tau\\&=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau+\frac{f(t)}{\pi}PV\int_a^b\frac{d\tau}{\tau-t},\end{aligned}$$

from where we get the identity

$$(Tf)(a,b;t)-\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau. \tag{6.5}$$

Now, if we choose in (6.1), $u=f'$, $x=\lambda c+(1-\lambda)d$, $\lambda\in[0,1]$, $c,d\in[a,b]$ then we get

$$|f(d)-f(c)-(d-c)f'(\lambda c+(1-\lambda)d)|$$

$$\leq \begin{cases} (d-c)^2\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f''\|_{[c,d],\infty} & \text{if } f''\in L_\infty[a,b]; \\ \frac{|d-c|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}}+(1-\lambda)^{1+\frac{1}{q}}\right]\left|\|f''\|_{[c,d],p}\right| & \text{if } f''\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ |d-c|\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left|\|f''\|_{[c,d],1}\right|, \end{cases}$$

which is equivalent to:

$$\left|\frac{f(d)-f(c)}{d-c}-f'(\lambda c+(1-\lambda)d)\right| \tag{6.6}$$
$$\leq\begin{cases}|d-c|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f''\|_{[c,d],\infty} & \text{if } f''\in L_\infty[a,b];\\ \frac{|d-c|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}}+(1-\lambda)^{1+\frac{1}{q}}\right]\left|\|f''\|_{[c,d],p}\right| & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left|\|f''\|_{[c,d],1}\right|.\end{cases}$$

Using (6.6), we may write

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\frac{1}{\pi}PV\int_a^b f'(\lambda t+(1-\lambda)\tau)\,d\tau\right| \tag{6.7}$$
$$\leq\begin{cases}\frac{1}{\pi}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]PV\int_a^b|t-\tau|\left|\|f''\|_{[t,\tau],\infty}\right|d\tau\\ \frac{1}{\pi}\cdot\frac{1}{(q+1)^{\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}}+(1-\lambda)^{1+\frac{1}{q}}\right]PV\int_a^b|t-\tau|^{\frac{1}{q}}\left|\|f''\|_{[t,\tau],p}\right|d\tau\\ \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]PV\int_a^b\left|\|f''\|_{[t,\tau],1}\right|d\tau\end{cases}$$

$$\leq\begin{cases}\frac{1}{\pi}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f''\|_{[a,b],\infty}PV\int_a^b|t-\tau|\,d\tau\\ \frac{1}{\pi}\cdot\frac{1}{(q+1)^{\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}}+(1-\lambda)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}PV\int_a^b|t-\tau|^{\frac{1}{q}}d\tau\\ \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]PV\left[\int_a^t\|f''\|_{[\tau,t],1}d\tau+\int_t^b\|f''\|_{[t,\tau],1}d\tau\right]\end{cases}$$

$$\leq\begin{cases}\frac{1}{\pi}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f''\|_{[a,b],\infty}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\\ \frac{1}{\pi}\cdot\frac{1}{(q+1)^{\frac{1}{q}}}\left[\lambda^{1+\frac{1}{q}}+(1-\lambda)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p}\\ \qquad\times\frac{q}{(q+1)}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\\ \frac{1}{\pi}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}.\end{cases}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Since (note that $\lambda \neq 1$)

$$\begin{aligned} &\frac{1}{\pi} PV \int_a^b f'(\lambda t + (1-\lambda)\tau)\, d\tau \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] (f'(\lambda t + (1-\lambda)\tau)\, d\tau) \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \left[\frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_a^{t-\varepsilon} + \frac{1}{1-\lambda} f(\lambda t + (1-\lambda)\tau) \Big|_{t+\varepsilon}^b \right] \\ &= \frac{1}{\pi} \frac{f(t) - f(\lambda t + (1-\lambda) a) + f(\lambda t + (1-\lambda) a) - f(t)}{1-\lambda} \\ &= \frac{b-a}{\pi} [f; \lambda t + (1-\lambda) b, \lambda t + (1-\lambda) a], \end{aligned}$$

then by (6.5) and (6.7) we deduce the desired inequality (6.4). □

The best inequality one may obtain from (6.4) is embodied in the following corollary.

Corollary 17. *With the assumptions of Theorem 20, one has the inequality*

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right) - \frac{b-a}{\pi}\left[f; \frac{t+b}{2}, \frac{a+t}{2}\right] \right| \tag{6.8}$$

$$\leq \begin{cases} \dfrac{1}{4\pi}\left[\dfrac{1}{4}(b-a)^2 + \left(t - \dfrac{a+b}{2}\right)^2\right] \|f''\|_{[a,b],\infty} & \text{if } f'' \in L_\infty[a,b]; \\ \dfrac{1}{\pi} \dfrac{q}{2^{\frac{p}{q}}(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}}\right] \|f''\|_{[a,b],p} & \text{if } f'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{2\pi}\left[\dfrac{1}{2}(b-a) + \left|t - \dfrac{a+b}{2}\right|\right] \|f''\|_{[a,b],1}, & \text{if } f'' \in L[a,b]; \end{cases}$$

for any $t \in (a,b)$.

6.2. **A Quadrature Formula.** The following lemma is of interest in itself.

Lemma 13. *Let* $u : [a,b] \to \mathbb{R}$ *be an absolutely continuous function on* $[a,b]$*. Then for all* $n \geq 1$, $\lambda_i \in [0,1)$ $(i = 0, \ldots, n-1)$ *and* $t, \tau \in [a,b]$ *with* $t \neq \tau$*, we have the*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

inequality

$$\left|\frac{1}{\tau-t}\int_t^\tau u(s)\,ds-\frac{1}{n}\sum_{i=0}^{n-1}u\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]\right| \tag{6.9}$$

$$\leq\begin{cases}\frac{|t-\tau|}{n}\left[\frac{1}{4}+\frac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\frac{1}{2}\right)^2\right]\|u'\|_{[t,\tau],\infty}; \\ \frac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i^{1+\frac{1}{q}}+(1-\lambda_i)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\|u'\|_{[t,\tau],p} \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n}\left[\frac{1}{2}+\max\left|\lambda_i-\frac{1}{2}\right|\right]\|u'\|_{[t,\tau],1};\end{cases}$$

$$\leq\begin{cases}\frac{|t-\tau|}{2n}\|u'\|_{[t,\tau],\infty}; \\ \frac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\|u'\|_{[t,\tau],p} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n}\|u'\|_{[t,\tau],1},\end{cases}$$

where

$$\|u'\|_{[t,\tau],p}:=\left|\int_t^\tau |u'(s)|^p\,ds\right|^{\frac{1}{p}},\quad p\geq 1$$

and

$$\|u'\|_{[t,\tau],\infty}:=ess\sup_{\substack{s\in[t,\tau]\\(s\in[\tau,t])}}|u'(s)|.$$

Proof. Consider the equidistant division of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$E_n: x_i=t+i\cdot\frac{\tau-t}{n},\quad i=\overline{0,n}. \tag{6.10}$$

Then the points $\xi_i:=\lambda_i\left[t+i\cdot\frac{\tau-t}{n}\right]+(1-\lambda_i)\left[t+(i+1)\cdot\frac{\tau-t}{n}\right]$ $\left(\lambda_i\in[0,1),\ i=\overline{0,n-1}\right)$ are between x_i and x_{i+1}. We observe that we may write for simplicity $\xi_i = t+(i+1-\lambda_i)\frac{\tau-t}{n}$ $\left(i=\overline{0,n-1}\right)$. We also have

$$\xi_i-\frac{x_i+x_{i+1}}{2}=\frac{\tau-t}{n}\left(\frac{1}{2}-\lambda_i\right);\ \xi_i-x_i=(1-\lambda_i)\frac{\tau-t}{n}$$

and

$$x_{i+1}-\xi_i=\lambda_i\cdot\frac{\tau-t}{n}$$

for any $i=\overline{0,n-1}$.

 doi:10.20944/preprints201803.0017.v1

If we apply the inequality (6.1) on the interval $[x_i, x_{i+1}]$ and the intermediate points ξ_i $\left(i = \overline{0, n-1}\right)$, then we may write that

$$\left|\frac{\tau - t}{n} u\left[t + (i + 1 - \lambda_i)\frac{\tau - t}{n}\right] - \int_{x_i}^{x_{i+1}} u(s)\,ds\right| \tag{6.11}$$

$$\leq \begin{cases} \left[\frac{1}{4}\frac{(t-\tau)^2}{n^2} + \frac{(t-\tau)^2}{4n^2}(1-2\lambda_i)^2\right]\|u'\|_{[x_i,x_{i+1}],\infty} & \text{if } u' \in L_\infty[a,b], \\ \frac{1}{(q+1)^{\frac{1}{q}}}\left[\frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}}\lambda_i^{1+\frac{1}{q}} + \frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}}(1-\lambda_i)^{1+\frac{1}{q}}\right]\|u'\|_{[x_i,x_{i+1}],p} & \text{if } u' \in L_p[a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}\frac{|\tau - t|}{n} + \frac{|\tau - t|}{n}\left|\frac{1}{2} - \lambda_i\right|\right]\|u'\|_{[t,\tau],1}; \end{cases}$$

Summing (6.3), we get:

$$\left|\int_t^\tau u(s)\,ds - \frac{\tau - t}{n}\sum_{i=0}^{n-1} u\left[t + (i + 1 - \lambda_i)\frac{\tau - t}{n}\right]\right| \tag{6.12}$$

$$\leq \begin{cases} \frac{(t-\tau)^2}{n^2}\sum_{i=0}^{n-1}\left[\frac{1}{4} + \left(\lambda_i - \frac{1}{2}\right)^2\right]\|u'\|_{[x_i,x_{i+1}],\infty}; \\ \frac{|t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}\, n^{1+\frac{1}{q}}}\sum_{i=0}^{n-1}\left[\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}}\right]\|u'\|_{[x_i,x_{i+1}],p} \\ \qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|t-\tau|}{n}\sum_{i=0}^{n-1}\left[\frac{1}{2} + \left|\lambda_i - \frac{1}{2}\right|\right]\|u'\|_{[x_i,x_{i+1}],1}. \end{cases}$$

However,

$$\sum_{i=0}^{n-1}\left[\frac{1}{4} + \left(\lambda_i - \frac{1}{2}\right)^2\right]\|u'\|_{[x_i,x_{i+1}],\infty} = \|u'\|_{[t,\tau],\infty}\left[\frac{1}{4}n + \sum_{i=0}^{n-1}\left(\lambda_i - \frac{1}{2}\right)^2\right], \tag{6.13}$$

$$\begin{aligned} &\sum_{i=0}^{n-1}\left[\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}}\right]\|u'\|_{[x_i,x_{i+1}],p} \\ &\leq \left(\sum_{i=0}^{n-1}\left[\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}}\right]^q\right)^{\frac{1}{q}}\left(\sum_{i=0}^{n-1}\|u'\|^p_{[x_i,x_{i+1}],p}\right)^{\frac{1}{p}} \\ &= \|u'\|_{[t,\tau],p}\left[\sum_{i=0}^{n-1}\left(\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \end{aligned} \tag{6.14}$$

and

$$\sum_{i=0}^{n-1}\left[\frac{1}{2} + \left|\lambda_i - \frac{1}{2}\right|\right]\|u'\|_{[x_i,x_{i+1}],1} \tag{6.15}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

$$\leq \left[\frac{1}{2}+\max\left|\lambda_i-\frac{1}{2}\right|\right]\sum_{i=0}^{n-1}\|u'\|_{[x_i,x_{i+1}],1}=\left[\frac{1}{2}+\max\left|\lambda_i-\frac{1}{2}\right|\right]\|u'\|_{[t,\tau],1}.$$

Now, using (6.12)-(6.15), we deduce the first part of (6.9).

The second part is obvious. □

We may now state the following theorem in approximating the finite Hilbert transform of a differentiable function whose derivative is absolutely continuous.

Theorem 21 (Dragomir, 2002, [10]). *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function so that its derivative f' is absolutely continuous on $[a,b]$. If $\lambda=(\lambda_i)_{i=\overline{0,n-1}}$, $\lambda_i\in[0,1)$ $\left(i=\overline{0,n-1}\right)$ and*

$$S_n(f;\lambda,t):=\frac{b-a}{n\pi}\sum_{i=0}^{n-1}\left[f;t+(i+1-\lambda_i)\frac{b-t}{n},t-(i+1-\lambda_i)\frac{t-a}{n}\right] \tag{6.16}$$

then we have

$$(Tf)(a,b;t)=\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+S_n(f;\lambda,t)+R_n(f;\lambda,t) \tag{6.17}$$

and the remainder $R_n(f;\lambda,t)$ satisfies the estimate:

$$|R_n(f;\lambda,t)| \tag{6.18}$$

$$\leq\frac{1}{\pi}\times\begin{cases}\frac{1}{n}\left[\frac{1}{4}+\frac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\frac{1}{2}\right)^2\right]\\ \quad\times\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b];\\ \\ \frac{1}{n}\cdot\frac{q}{(q+1)^{\frac{1}{q}+1}}\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i^{1+\frac{1}{q}}+(1-\lambda_i)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\\ \quad\times\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{n}\left[\frac{1}{2}+\max\left|\lambda_i-\frac{1}{2}\right|\right]\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1},\end{cases}$$

$$\leq\frac{1}{\pi}\times\begin{cases}\frac{1}{4n}\|f''\|_{[a,b],\infty}(b-a)^2 & \text{if } f''\in L_\infty[a,b];\\ \\ \frac{q}{(q+1)^{\frac{1}{q}+1}n}(b-a)^{1+\frac{1}{q}}\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \frac{1}{n}(b-a)\|f''\|_{[a,b],1},\end{cases}$$

Proof. Applying Lemma 13 for the function f', we may write that

$$\left|\frac{f(t)-f(\tau)}{\tau-t}-\frac{1}{n}\sum_{i=0}^{n-1}f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]\right| \tag{6.19}$$

$$\leq\begin{cases}\dfrac{|t-\tau|}{n}\left[\dfrac{1}{4}+\dfrac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\dfrac{1}{2}\right)^2\right]\|f''\|_{[t,\tau],\infty}\,;\\[2ex] \dfrac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}n}\left[\dfrac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i^{1+\frac{1}{q}}+(1-\lambda_i)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\|f''\|_{[t,\tau],p}\\[2ex] \dfrac{1}{n}\left[\dfrac{1}{2}+\max\left|\lambda_i-\dfrac{1}{2}\right|\right]\|f''\|_{[t,\tau],1}\end{cases}$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

Taking the PV, we may write

$$\left|\frac{1}{\pi}PV\int_a^b\frac{f(t)-f(\tau)}{\tau-t}d\tau-\frac{1}{n\pi}\sum_{i=0}^{n-1}PV\int_a^b f'\left[t+(i+1-\lambda_i)\frac{\tau-t}{n}\right]d\tau\right| \tag{6.20}$$

$$\leq\frac{1}{\pi}\times\begin{cases}\dfrac{1}{n}\left[\dfrac{1}{4}+\dfrac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i-\dfrac{1}{2}\right)^2\right]PV\int_a^b|t-\tau|\,\|f''\|_{[t,\tau],\infty}\,d\tau;\\[2ex] \dfrac{1}{n(q+1)^{\frac{1}{q}}}\left[\dfrac{1}{n}\sum_{i=0}^{n-1}\left(\lambda_i^{1+\frac{1}{q}}+(1-\lambda_i)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}PV\int_a^b|t-\tau|^{\frac{1}{q}}\,\|f''\|_{[t,\tau],p}\,d\tau\\[2ex] \dfrac{1}{n}\left[\dfrac{1}{2}+\max\left|\lambda_i-\dfrac{1}{2}\right|\right]PV\int_a^b\|f''\|_{[t,\tau],1}\,d\tau.\end{cases}$$

However,

$$\begin{aligned}PV\int_a^b|t-\tau|\,\|f''\|_{[t,\tau],\infty}\,d\tau&\leq\|f''\|_{[a,b],\infty}\,PV\int_a^b|t-\tau|\,d\tau\\&=\|f''\|_{[a,b],\infty}\left[\frac{(t-a)^2+(b-t)^2}{2}\right],\end{aligned}$$

$$\begin{aligned}PV\int_a^b|t-\tau|^{\frac{1}{q}}\,\|f''\|_{[t,\tau],p}\,d\tau&\leq\|f''\|_{[a,b],p}\,PV\int_a^b|t-\tau|^{\frac{1}{q}}\,d\tau\\&=\|f''\|_{[a,b],p}\left[\frac{(t-a)^{\frac{1}{q}+1}+(b-t)^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right]\\&=\frac{q}{(q+1)}\left[(t-a)^{\frac{1}{q}+1}+(b-t)^{\frac{1}{q}+1}\right]\|f''\|_{[a,b],p}\,,\end{aligned}$$

$$\begin{aligned}PV\int_a^b \|f''\|_{[t,\tau],1}\,d\tau &= PV\left[\int_a^t \|f''\|_{[\tau,t],1}\,d\tau + \int_t^b \|f''\|_{[t,\tau],1}\,d\tau\right]\\ &\le \max\{t-a,b-t\}\|f''\|_{[a,b],1}\\ &= \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}\end{aligned}$$

and using the inequality (6.20) we obtain the desired estimate (6.18). □

The following particular case which may be easily numerically implemented holds.

Corollary 18. *let f be as in Theorem 21. Define*

$$S_{M,n}(f;t) := \frac{b-a}{n\pi}\sum_{i=0}^{n-1}\left[f;t+\left(i+\frac{1}{2}\right)\frac{b-t}{n},t-\left(i+\frac{1}{2}\right)\frac{t-a}{n}\right].$$

Then we have the representation:

$$(Tf)(a,b;t) = \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) + S_{M,n}(f;t) + R_{M,n}(f;t)$$

and the remainder $R_{M,n}(f;t)$ satisfies the estimate

$$|R_{M,n}(f;t)| \tag{6.21}$$

$$\le \frac{1}{\pi n}\times\begin{cases}\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|f''\|_{[a,b],\infty} & \text{if } f''\in L_\infty[a,b];\\[2mm] \frac{1}{2^{\frac{1}{q}}(q+1)^{\frac{1}{q}+1}}\left[(t-a)^{1+\frac{1}{q}}+(b-t)^{1+\frac{1}{q}}\right]\|f''\|_{[a,b],p} & \text{if } f''\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{1}{2}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]\|f''\|_{[a,b],1}, & \end{cases}$$

for any $t\in(a,b)$.

7. Inequalities for Convex Derivatives

7.1. An Inequality on the Interval (a,b). The following result holds.

Theorem 22 (Dragomir, 2002, [11])**.** *Assume that the differentiable function $f:(a,b)\to\mathbb{R}$ is such that f' is convex on (a,b). Then the Hilbert transform $(Tf)(a,b;\cdot)$ exists in every point $t\in(a,b)$ and:*

$$\begin{aligned}&\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right)-f\left(\frac{t+a}{2}\right)\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\ &\le (Tf)(a,b;t)\\ &\le \frac{1}{2\pi}\left[f(b)-f(a)+(b-a)f'(t)\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned} \tag{7.1}$$

for any $t\in(a,b)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Proof. The existence of the Hilbert transform in each point $t \in (a,b)$ follows by the fact that f is locally Lipschitzian on (a,b).

Since f' is convex, we have, by the Hermite-Hadamard inequality, that

$$f'\left(\frac{t+\tau}{2}\right) \leq \frac{1}{\tau - t}\int_t^{\tau} f'(u)\,du \leq \frac{f'(t)+f'(\tau)}{2} \tag{7.2}$$

for all $t,\tau \in (a,b)$, $t \neq \tau$, giving

$$f'\left(\frac{t+\tau}{2}\right) \leq \frac{f(\tau)-f(t)}{\tau - t} \leq \frac{f'(t)+f'(\tau)}{2} \tag{7.3}$$

for all $t,\tau \in (a,b)$, $t \neq \tau$.

Applying the PV in t, i.e., $\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)(\cdot)$, we get

$$PV\int_a^b f'\left(\frac{t+\tau}{2}\right)d\tau \leq PV\int_a^b \frac{f(\tau)-f(t)}{\tau - t}d\tau \leq PV\int_a^b \frac{f'(t)+f'(\tau)}{2}d\tau. \tag{7.4}$$

Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left(\frac{t+\tau}{2}\right)d\tau\right)\\
&=\lim_{\varepsilon\to 0+}\left[\int_a^{t-\varepsilon} f'\left(\frac{t+\tau}{2}\right)d\tau+\int_{t+\varepsilon}^b f'\left(\frac{t+\tau}{2}\right)d\tau\right]\\
&=\lim_{\varepsilon\to 0+}2\left[\left(f\left(\frac{2t-\varepsilon}{2}\right)-f\left(\frac{t+a}{2}\right)\right)+\left(f\left(\frac{t+b}{2}\right)-f\left(\frac{2t+\varepsilon}{2}\right)\right)\right]\\
&=2\left[f\left(\frac{t+b}{2}\right)-f\left(\frac{t+a}{2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left[\frac{f'(t)+f'(\tau)}{2}d\tau\right]\\
&=\frac{1}{2}\lim_{\varepsilon\to 0+}\left[f'(t)(t-\varepsilon-a)+f'(t)(b-t-\varepsilon)+f(t-\varepsilon)-f(a)+f(b)-f(t+\varepsilon)\right]\\
&=\frac{1}{2}\left[f(b)-f(a)+(b-a)f'(t)\right],
\end{aligned}$$

then by (7.4), we may state that:

$$\begin{aligned}
\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right)-f\left(\frac{t+a}{2}\right)\right] &\leq \frac{1}{\pi}PV\int_a^b \frac{f(\tau)-f(t)}{\tau - t}d\tau\\
&\leq \frac{1}{2\pi}\left[f(b)-f(a)+(b-a)f'(t)\right]
\end{aligned} \tag{7.5}$$

for all $t \in (a,b)$.

As for the function $f_0(t)=1$, $t\in(a,b)$, we have

$$(Tf)(a,b;t)=\frac{1}{\pi}\ln\left(\frac{b-t}{t-a}\right),\quad t\in(a,b),$$

then obviously

$$(7.6)\qquad (Tf_0)(a,b;t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + \frac{f(t)}{\pi} \ln\left(\frac{b-t}{t-a}\right)$$

for any $t \in (a,b)$.

Finally, by (7.5) and (7.6), we may obtain (7.14). □

The inequality (7.1) in Theorem 22 may be used to obtain different analytic inequalities for functions $f : [a,b] \to \mathbb{R}$ whose derivatives are convex on (a,b) and the Hilbert Transform $(Tf)(a,b;\cdot)$ is known.

For example, the following proposition holds.

Proposition 3 (Dragomir, 2002, [11])**.** *For any $a,b \in \mathbb{R}$, $a < b$ and $t \in (a,b)$, we have the inequality:*

$$(7.7)\qquad \ln\left(\frac{b-t}{t-a}\right) + 2\left(e^{\frac{b-t}{2}} - e^{\frac{a-t}{2}}\right) \le E_i(b-t) - E_i(a-t) \le \ln\left(\frac{b-t}{t-a}\right) + \frac{1}{2}\left[e^{b-t} - e^{a-t} + (b-a)\right],$$

where E_i is defined in (7.8).

Proof. If we consider the function $f(t) = e^t$, $t \in (a,b)$, then f' is convex on (a,b),

$$(Tf)(a,b;t) = \frac{e^t}{\pi}\left[E_i(b-t) - E_i(a-t)\right],$$

where E_i is defined by

$$(7.8)\qquad E_i(z) := PV \int_{-\infty}^{z} \frac{e^t}{t} dt, \quad z \in \mathbb{R},$$

$$\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right) - f\left(\frac{t+a}{2}\right)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{2}{\pi}\left[e^{\frac{t+b}{2}} - e^{\frac{t+a}{2}}\right] + \frac{e^t}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$

and

$$\frac{1}{2\pi}\left[f(b) - f(a) + (b-a) f'(t)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right) = \frac{1}{2\pi}\left[e^b - e^a + (b-a)e^t\right] + \frac{e^t}{\pi}\ln\left(\frac{b-t}{t-a}\right).$$

Using (7.1) and dividing by e^t, we deduce (7.7). □

The following inequality also holds.

Proposition 4 (Dragomir, 2002, [11])**.** *For any $x > 0$, we have the inequality*

$$(7.9)\qquad 2\sinh\left(\frac{1}{2}x\right) \le E_i(x) \le \frac{1}{2}\sinh(x) + \frac{1}{2}x.$$

 doi:10.20944/preprints201803.0017.v1

Proof. If in (7.7) we put $t = \frac{a+b}{2}$, then we deduce:

$$2\left(e^{\frac{b-a}{4}} - e^{-\frac{b-a}{4}}\right) \leq E_i\left(\frac{b-a}{2}\right) - E_i\left(-\frac{b-a}{2}\right)$$
$$\leq \frac{1}{2}\left[e^{\frac{b-a}{2}} - e^{-\frac{b-a}{2}} + b - a\right].$$

If we denote $x := \frac{b-a}{2}$, then we get

$$2\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right) \leq E_i(x) - E_i(-x) \leq \frac{1}{2}\left[e^{x} - e^{-x} + 2x\right]. \tag{7.10}$$

However,

$$-E_i(-x) = E_i(x),$$
$$2\left(e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}\right) = 4\sinh\left(\frac{1}{2}x\right)$$

and

$$\frac{1}{2}\left(e^{x} - e^{-x} + 2x\right) = \sinh(x) + x$$

and then, by (7.10), we deduce (7.9). □

If we choose another function, for instance, $f : (a,b) \subset (0,\infty) \to \mathbb{R}$, $f(t) = -\frac{1}{t}$, then obviously f' is convex on (a,b), and we may state the following result as well.

Proposition 5 (Dragomir, 2002, [11])**.** *For any $0 < a < b$ and $t \in (a,b)$, we have the inequality:*

$$\frac{2tG^2}{G^2 + t^2} \leq L \leq \frac{t^2 + 2At + G^2}{4t}, \quad \textit{for any} \quad t \in (a,b), \tag{7.11}$$

where $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and $L = \frac{b-a}{\ln b - \ln a}$ (the logarithmic mean).

Proof. For the function $f : (a,b) \to \mathbb{R}$, $f(t) = -\frac{1}{t}$, we have

$$(Tf)(a,b;t) = \frac{1}{\pi t}\left[\ln\left(\frac{b}{a}\right) - \ln\left(\frac{b-t}{t-a}\right)\right],$$

$$\frac{2}{\pi}\left[f\left(\frac{t+b}{2}\right) - f\left(\frac{t+a}{2}\right)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$
$$= \frac{4}{\pi} \cdot \frac{b-a}{(t+a)(t+b)} - \frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right),$$

and

$$\frac{1}{2\pi}\left[f(b) - f(a) + (b-a) f'(t)\right] + \frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)$$
$$= \frac{b-a}{2\pi}\left[\frac{1}{ab} + \frac{1}{t^2}\right] - \frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right).$$

Now, if we use (7.1), we may write:

$$\frac{4}{\pi} \cdot \frac{b-a}{(t+a)(t+b)} - \frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right) \leq \frac{1}{\pi t}\ln\left(\frac{b}{a}\right) - \frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right)$$
$$\leq \frac{b-a}{2\pi}\left(\frac{t^2 + ab}{abt^2}\right) - \frac{1}{\pi t}\ln\left(\frac{b-t}{t-a}\right),$$

which is equivalent to:

$$\frac{4t}{(t+a)(t+b)} \leq \frac{\ln b - \ln a}{b-a} \leq \frac{t^2+ab}{2tab}.$$

Using the fact that $L := \frac{b-a}{\ln b - \ln a}$, we deduce (7.11). □

Corollary 19. *We have the inequality*

$$G \leq L \leq \frac{G+A}{2}. \tag{7.12}$$

Remark 9. *The first inequality is a well known result as the following sequence of inequalities hold*

$$G \leq L \leq I \leq A.$$

The second inequality is equivalent with:

$$L(a,b) \leq \left[A\left(\sqrt{a},\sqrt{b}\right)\right]^2, \tag{7.13}$$

which is interesting in itself.

7.2. **An Inequality on an Equidistant Division of** (a,b)**.** The following lemma is interesting in itself.

Lemma 14. *Let* $g:[a,b]\to\mathbb{R}$ *be a convex function. Then for* $n\geq 1$ *and* $t,\tau\in[a,b]$, $t\neq\tau$, *we have the inequality:*

$$\begin{aligned} &\frac{1}{n}\sum_{i=0}^{n-1} g\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{t-\tau}{n}\right] \\ &\leq \frac{1}{\tau-t}\int_t^\tau g(u)\,du \\ &\leq \frac{1}{2n}\sum_{i=0}^{n-1}\left[g\left(t+i\cdot\frac{\tau-t}{n}\right)+g\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]. \end{aligned} \tag{7.14}$$

Proof. Consider the equidistant partitioning of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$E_n : x_i = t + i\cdot\frac{\tau-t}{n},\quad i=\overline{0,n}. \tag{7.15}$$

Then, applying the Hermite-Hadamard inequality, we may write that:

$$g\left(\frac{x_i+x_{i+1}}{2}\right) \leq \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} g(u)\,du \leq \frac{g(x_i)+g(x_{i+1})}{2}$$

i.e.,

$$\begin{aligned} g\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{t-\tau}{n}\right) &\leq \frac{n}{\tau-t}\int_{x_i}^{x_{i+1}} g(u)\,du \\ &\leq \frac{1}{2}\left[g\left(t+i\cdot\frac{\tau-t}{n}\right)+g\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]. \end{aligned}$$

Dividing by n and summing over i from 0 to $n-1$, we deduce the desired inequality (7.14). □

The following generalization of Theorem 22 holds.

Theorem 23 (Dragomir, 2002, [11]). *Assume that $f:(a,b)\to\mathbb{R}$ fulfills the hypothesis of Theorem 22. Then for all $n\geq 1$, we have the double inequality:*

$$\begin{aligned}(7.16)\quad &\frac{b-a}{n\pi}\sum_{i=0}^{n-1}\left[f;t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\ &\leq (Tf)(a,b;t)\\ &\leq \frac{f(b)-f(a)+f'(t)(b-a)}{2n\pi}+\frac{b-a}{n\pi}\sum_{i=1}^{n-1}\left[f;t-i\cdot\frac{t-a}{n},t+i\cdot\frac{b-t}{n}\right]\\ &+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

for any $t\in(a,b)$, where $[f;c,d]$ denotes the divided difference $\frac{f(c)-f(d)}{c-d}$.

Proof. If we write the inequality (7.14) for f', then we have

$$\begin{aligned}(7.17)\quad &\frac{1}{n}\sum_{i=0}^{n-1}f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]\\ &\leq\frac{f(\tau)-f(t)}{\tau-t}\\ &\leq\frac{1}{2n}\sum_{i=0}^{n-1}\left[f'\left(t+i\cdot\frac{\tau-t}{n}\right)+f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)\right]\\ &=\frac{1}{2n}\left[f'(t)+\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)+\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right)+f'(\tau)\right]\\ &=\frac{1}{2n}\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right],\end{aligned}$$

since it is obvious that

$$\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)=\sum_{i=0}^{n-2}f'\left(t+(i+1)\cdot\frac{\tau-t}{n}\right).$$

Applying the PV over t, i.e., $\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)$ to the inequality (7.17), we deduce

$$\begin{aligned}(7.18)\quad &\frac{1}{n}\sum_{i=1}^{n-1}PV\int_a^b f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]d\tau\\ &\leq PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\\ &\leq\frac{1}{2n}PV\int_a^b\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau.\end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Now, as

$$\begin{aligned}
&PV\int_a^b f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]d\tau\\
&=\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left(f'\left[t+\left(i+\frac{1}{2}\right)\cdot\frac{\tau-t}{n}\right]d\tau\right)\\
&=\lim_{\varepsilon\to 0+}\frac{n}{i+\frac{1}{2}}\left[f\left(t-\left(i+\frac{1}{2}\right)\cdot\frac{\varepsilon}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{a-t}{n}\right)\right.\\
&\left.+f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{\varepsilon}{n}\right)\right]\\
&=\frac{n}{i+\frac{1}{2}}\left[f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right)-f\left(t+\left(i+\frac{1}{2}\right)\cdot\frac{a-t}{n}\right)\right]\\
&=(b-a)\left[f;t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right],
\end{aligned}$$

and

$$\begin{aligned}
&PV\int_a^b\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau\\
&=\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)\left[f'(t)+f'(\tau)+2\sum_{i=1}^{n-1}f'\left(t+i\cdot\frac{\tau-t}{n}\right)\right]d\tau\\
&=\lim_{\varepsilon\to 0+}\left[f'(t)(t-\varepsilon-a)+f'(t)(b-t-\varepsilon)+f(t-\varepsilon)-f(a)+f(b)-f(t+\varepsilon)\right.\\
&\left.+2\sum_{i=1}^{n-1}\frac{n}{i}\left[f\left(t-\frac{i\varepsilon}{n}\right)-f\left(t+i\cdot\frac{a-t}{n}\right)+f\left(t+i\cdot\frac{b-t}{n}\right)-f\left(t+\frac{i\varepsilon}{n}\right)\right]\right]\\
&=f(b)-f(a)+f'(t)(b-a)+2(b-a)\sum_{i=1}^{n-1}\left[f;t-i\cdot\frac{t-a}{n},t+i\cdot\frac{b-t}{n}\right],
\end{aligned}$$

then by (7.18) we deduce

$$\begin{aligned}
&\frac{b-a}{n}\sum_{i=0}^{n-1}\left[f;t-\left(i+\frac{1}{2}\right)\cdot\frac{t-a}{n},t+\left(i+\frac{1}{2}\right)\cdot\frac{b-t}{n}\right]\\
&\leq PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau\\
&\leq\frac{f(b)-f(a)+f'(t)(b-a)}{2n}+\frac{b-a}{n}\sum_{i=1}^{n-1}\left[f;t-i\cdot\frac{t-a}{n},t+i\cdot\frac{b-t}{n}\right].
\end{aligned}\tag{7.19}$$

Using the identity (7.6) and the inequality (7.19), we obtain the desired result (7.16). □

7.3. **The Case of Non-equidistant Partitioning.** The following lemma holds

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Lemma 15. *Let $g:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $t,\tau\in[a,b]$ with $t\neq\tau$. If $0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=1$, then we have the inequality:*

$$\begin{aligned}(7.20)\quad &\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\,g\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right]\\ &\le\frac{1}{\tau-t}\int_t^\tau g(u)\,du\\ &\le\frac{1}{2}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\left\{g\left[(1-\lambda_i)t+\lambda_i\tau\right]+g\left[(1-\lambda_{i+1})t+\lambda_{i+1}\tau\right]\right\}.\end{aligned}$$

Proof. Consider the partitioning of $[t,\tau]$ (if $t<\tau$) or $[\tau,t]$ (if $\tau<t$) given by

$$I_n: x_i=(1-\lambda_i)t+\lambda_i\tau,\quad\left(i=\overline{0,n}\right).$$

Then, obviously,

$$\frac{x_i+x_{i+1}}{2}=\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau,\quad\left(i=\overline{0,n-1}\right)$$

and

$$x_{i+1}-x_i=(\tau-t)(\lambda_{i+1}-\lambda_i),\quad\left(i=\overline{0,n-1}\right).$$

Applying the Hermite-Hadamard inequality on $[x_i,x_{i+1}]$ $\left(i=\overline{0,n-1}\right)$, we may write that

$$\begin{aligned}&g\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right]\\ &\le\frac{1}{(\tau-t)(\lambda_{i+1}-\lambda_i)}\int_{x_i}^{x_{i+1}}g(u)\,du\\ &\le\frac{1}{2}\left\{g\left[(1-\lambda_i)t+\lambda_i\tau\right]+g\left[(1-\lambda_{i+1})t+\lambda_{i+1}\tau\right]\right\}\end{aligned}$$

for any $i=\overline{0,n-1}$.

If we multiply with $\lambda_{i+1}-\lambda_i>0$ and sum over i from 0 to $n-1$, we deduce the desired inequality (7.20). □

The following theorem holds.

Theorem 24 (Dragomir, 2002, [11])**.** *Assume that $f:(a,b)\to\mathbb{R}$ fulfills the hypothesis of Theorem 22. Then for all $n\ge1$, and $0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n=1$, we have the inequality*

$$\begin{aligned}(7.21)\quad &\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\frac{b-a}{\pi}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\\ &\times\left[f;\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot b,\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot a\right]\\ &\le(Tf)(a,b;t)\\ &\le\frac{1}{2\pi}\left\{\lambda_1(b-a)f'(t)+(1-\lambda_{n-1})\left[f(b)-f(a)\right]\right\}\\ &+\frac{b-a}{2\pi}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\left[f;(1-\lambda_i)t+\lambda_ib,(1-\lambda_i)t+\lambda_ia\right]+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\end{aligned}$$

for any $t\in(a,b)$.

 doi:10.20944/preprints201803.0017.v1

Proof. If we write the inequality (7.20) for f', then we have

$$\begin{aligned}
(7.22)\quad & \sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right] \\
& \le \frac{f(\tau)-f(t)}{\tau-t} \\
& \le \frac{1}{2}\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\left\{f'\left[(1-\lambda_i)t+\lambda_i\tau\right]+f'\left[(1-\lambda_{i+1})t+\lambda_{i+1}\tau\right]\right\} \\
& = \frac{1}{2}\left[\lambda_1 f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'\left[(1-\lambda_i)t+\lambda_i\tau\right]\right. \\
& \left.+\sum_{i=0}^{n-2}(\lambda_{i+1}-\lambda_i)\, f'\left[(1-\lambda_{i+1})t+\lambda_{i+1}\tau\right]+(1-\lambda_{n-1})\, f'(\tau)\right] \\
& = \frac{1}{2}\left[\lambda_1 f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'\left[(1-\lambda_i)t+\lambda_i\tau\right]\right. \\
& \left.+\sum_{i=1}^{n-1}(\lambda_i-\lambda_{i-1})\, f'\left[(1-\lambda_i)t+\lambda_i\tau\right]+(1-\lambda_{n-1})\, f'(\tau)\right] \\
& = \frac{1}{2}\left[\lambda_1 f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, f'\left[(1-\lambda_i)t+\lambda_i\tau\right]+(1-\lambda_{n-1})\, f'(\tau)\right].
\end{aligned}$$

Applying the PV over t, i.e., $\lim_{\varepsilon\to 0+}\left(\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^b\right)$ to the inequality (7.22), we deduce

$$\begin{aligned}
(7.23)\quad & \sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\, PV\int_a^b f'\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right]d\tau \\
& \le PV\int_a^b \frac{f(\tau)-f(t)}{\tau-t}d\tau \\
& \le \frac{1}{2}\left[\lambda_1(b-a)\, f'(t)+\sum_{i=1}^{n-1}(\lambda_{i+1}-\lambda_i)\, PV\int_a^b f'\left[(1-\lambda_i)t+\lambda_i\tau\right]d\tau\right. \\
& \left.+(1-\lambda_{n-1})(f(b)-f(a))\right].
\end{aligned}$$

Since

$$\begin{aligned}
& PV\int_a^b f'\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot\tau\right]d\tau \\
& = \frac{2}{\lambda_i+\lambda_{i+1}}\left(f\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot b\right]\right. \\
& \left.-f\left[\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot a\right]\right) \\
& = (b-a)\left[f;\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot b,\left(1-\frac{\lambda_i+\lambda_{i+1}}{2}\right)t+\frac{\lambda_i+\lambda_{i+1}}{2}\cdot a\right]
\end{aligned}$$

and

$$PV\int_a^b f'\left[\left(1-\lambda_i\right)t+\lambda_i\tau\right]d\tau=\left(b-a\right)\left[f;\left(1-\lambda_i\right)t+\lambda_i b,\left(1-\lambda_i\right)t+\lambda_i a\right],$$

then by (7.23) we deduce the desired inequality (7.21). □

Remark 10. *It is obvious that for* $\lambda_i=\frac{i}{n}\ \left(i=\overline{0,n}\right)$, *we recapture the inequality (7.16).*

The following corollary also holds.

Corollary 20. *Assume that* $f:(a,b)\to\mathbb{R}$ *fulfills the hypothesis of Theorem 22. Then for* $n\geq 1$ *we have:*

$$\begin{aligned}
(7.24)\quad &\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)\\
&+\frac{b-a}{\pi}\left[\frac{1}{2^{n-1}}\left[f;\left(1-\frac{1}{2^n}\right)t+\frac{1}{2^n}b,\left(1-\frac{1}{2^n}\right)t+\frac{1}{2^n}a\right]\right]\\
&+\frac{b-a}{\pi}\sum_{i=1}^{n-1}\frac{1}{2^{n-1}}\left[f;\left(1-\frac{3}{2^{n-i}}\right)t+\frac{3}{2^{n-i}}b,\left(1-\frac{3}{2^{n-i}}\right)t+\frac{3}{2^{n-i}}a\right]\\
&\leq\left(Tf\right)(a,b;t)\\
&\leq\frac{1}{2\pi}\left\{\frac{(b-a)f'(t)}{2^{n-1}}+\frac{1}{2}\left[f(b)-f(a)\right]\right\}\\
&+\frac{b-a}{2\pi}\left(\frac{1}{2^{n-2}}-1\right)\left[f;\left(1-\frac{1}{2^{n-1}}\right)t+\frac{1}{2^{n-1}}b,\left(1-\frac{1}{2^{n-1}}\right)t+\frac{1}{2^{n-1}}a\right]\\
&+3\cdot\frac{b-a}{2\pi}\sum_{i=2}^{n-1}\frac{1}{2^{n-i+1}}\left[f;\left(1-\frac{1}{2^{n-i}}\right)t+\frac{1}{2^{n-i}}b,\left(1-\frac{1}{2^{n-i}}\right)t+\frac{1}{2^{n-i}}a\right]\\
&+\frac{f(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right),
\end{aligned}$$

for any $t\in(a,b)$.

The proof follows by Theorem 24 applied for $\lambda_0=0$, $\lambda_i=\frac{2^i}{2^n}$, $i=\overline{1,n}$. We omit the details.

8. Inequalities for Products

8.1. **Some Basic Inequalities.** The following lemma holds.

Lemma 16 (Dragomir, 2003, [12])**.** *If* f *and* g *are locally Hölder continuous on* $[a,b]$, *then* fg *is also locally Hölder continuous on* $[a,b]$ *and:*

$$\begin{aligned}
(8.1)\quad &T(fg)(a,b;t)\\
&=f(t)T(g)(a,b;t)+g(t)T(f)(a,b;t)\\
&-\frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau
\end{aligned}$$

for any $t\in(a,b)$.

Proof. Assume that for a subinterval $[c,d]\subseteq[a,b]$, we have

$$|f(s)-f(u)|\leq L_1|s-u|^{r_1}\quad\text{for any}\quad s,u\in[c,d]; \tag{8.2}$$

$$|g(s)-g(u)|\leq L_2|s-u|^{r_2}\quad\text{for any}\quad s,u\in[c,d]. \tag{8.3}$$

Then

$$\begin{aligned}|f(s)g(s)-f(u)g(u)| &= |f(s)g(s)-f(s)g(u)+f(s)g(u)-f(u)g(u)|\\ &\leq |f(s)||g(s)-g(u)|+|g(u)||f(s)-f(u)|\\ &\leq M_1L_1|s-u|^{r_1}+M_2L_2|s-u|^{r_2}\\ &\leq |s-u|^r\left[M_1L_1|s-u|^{r_1-r}+M_2L_2|s-u|^{r_2-r}\right]\\ &\leq |s-u|^r\left[M_1L_1|d-c|^{r_1-r}+M_2L_2|d-c|^{r_2-r}\right]\\ &= M|s-u|^r\end{aligned}$$

where

$$M_1:=\sup_{s\in[c,d]}|f(s)|,\quad M_2:=\sup_{u\in[c,d]}|g(u)|,\quad r=\min(r_1,r_2),$$

and

$$M=M_1L_1|d-c|^{r_1-r}+M_2L_2|d-c|^{r_2-r},$$

proving that fg is locally Hölder continuous on $[a,b]$.

Now, for any $t,\tau\in[a,b]$, we may write that

$$(f(\tau)-f(t))(g(\tau)-g(t))=f(\tau)g(\tau)+f(t)g(t)-f(t)g(\tau)-f(\tau)g(t)$$

giving

$$\begin{aligned}\frac{f(\tau)g(\tau)}{\tau-t} &= f(t)\cdot\frac{g(\tau)}{\tau-t}+g(t)\cdot\frac{f(\tau)}{\tau-t}-\frac{f(t)g(t)}{\tau-t}\\ &\quad+\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}\end{aligned}$$

for any $t,\tau\in[a,b]$, $t\neq\tau$.

Consequently,

$$\begin{aligned}&T(fg)(a,b;t)\\ &=\frac{1}{\pi}PV\int_a^b\frac{f(\tau)g(\tau)}{\tau-t}d\tau\\ &=\frac{1}{\pi}f(t)PV\int_a^b\frac{g(\tau)}{\tau-t}d\tau+g(t)\frac{1}{\pi}PV\int_a^b\cdot\frac{f(\tau)}{\tau-t}d\tau\\ &-\frac{1}{\pi}f(t)g(t)PV\int_a^b\frac{d\tau}{\tau-t}+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau\\ &=f(t)T(g)(a,b;t)+g(t)T(f)(a,b;t)\\ &-\frac{f(t)g(t)}{\pi}\ln\left(\frac{b-t}{t-a}\right)+\frac{1}{\pi}PV\int_a^b\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}d\tau\end{aligned}$$

for any $t\in(a,b)$, and the identity (8.1) is proved. □

Theorem 25 (Dragomir, 2003, [12]). *Assume that f is of $L_1 - r_1-$Hölder type and g is of $L_2 - r_2-$Hölder type on $[a,b]$, where $L_1, L_2 > 0$, $r_1, r_2 \in (0,1]$. Then we have the inequality:*

$$\begin{aligned}(8.4)\quad & \Big| T(fg)(a,b;t) - f(t)T(g)(a,b;t) \\ & \qquad - g(t)T(f)(a,b;t) + \frac{1}{\pi} f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Big| \\ & \le \frac{L_1 L_2}{\pi(r_1+r_2)}\left[(b-t)^{r_1+r_2} + (t-a)^{r_1+r_2}\right] \le \frac{2L_1L_2(b-a)^{r_1+r_2}}{\pi(r_1+r_2)}\end{aligned}$$

for any $t \in (a,b)$.

Proof. Taking the modulus in (8.1), we may write

$$\begin{aligned}& \Big| T(fg)(a,b;t) - f(t)T(g)(a,b;t) \\ & \qquad - g(t)T(f)(a,b;t) + \frac{1}{\pi} f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Big| \\ & \le \frac{1}{\pi} PV \int_a^b \left|\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau - t}\right| d\tau \le \frac{1}{\pi} PV \int_a^b L_1L_2|\tau - t|^{r_1+r_2-1} d\tau \\ & = \frac{L_1L_2}{\pi}\left[\frac{(b-t)^{r_1+r_2} + (t-a)^{r_1+r_2}}{r_1+r_2}\right]\end{aligned}$$

and the first part of inequality (8.4) is proved. The second part is obvious. □

The best inequality we can get from (8.4) is embodied in the following corollary.

Corollary 21. *With the assumptions in Theorem 25, we have*

$$\begin{aligned}(8.5)\quad & \Big| T(fg)\left(a,b;\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right) \\ & \qquad - g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\Big| \\ & \le \frac{L_1L_2(b-a)^{r_1+r_2}}{\pi(r_1+r_2)2^{r_1+r_2-1}}.\end{aligned}$$

The following corollary also holds.

Corollary 22. *If f and g are Lipschitzian with the constants K_1 and K_2, then we have the inequality*

$$\begin{aligned}& \left| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) + \frac{1}{\pi} f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\right| \\ (8.6)\quad & \le \frac{K_1K_2}{\pi}\left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] \le \frac{K_1K_2}{2\pi}(b-a)^2\end{aligned}$$

for any $t \in (a,b)$. In particular, for $t = \frac{a+b}{2}$, we have

$$\left| T(fg)\left(a,b;\frac{a+b}{2}\right) - f\left(\frac{a+b}{2}\right)T(g)\left(a,b;\frac{a+b}{2}\right) - g\left(\frac{a+b}{2}\right)T(f)\left(a,b;\frac{a+b}{2}\right)\right|$$

 doi:10.20944/preprints201803.0017.v1

$$\leq \frac{K_1 K_2}{4\pi} (b-a)^2 . \tag{8.7}$$

8.2. **Further Estimates.** The following theorem also holds.

Theorem 26 (Dragomir, 2003, [12])**.** *Assume that f and g are absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\Big| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) + \frac{1}{\pi} f(t) g(t) \ln\left(\frac{b-t}{t-a}\right) \Big| \tag{8.8}$$

$$\leq \frac{1}{\pi} \times \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right] \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \\ \qquad \text{if } f' \in L_\infty[a,b],\ g' \in L_\infty[a,b]; \\ \frac{\delta}{\delta+1}\left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}\right] \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\gamma} \\ \qquad \text{if } f' \in L_\infty[a,b],\ g' \in L_\gamma[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ (b-a)\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1} \qquad \text{if } f' \in L_\infty[a,b],\ g' \in L_1[a,b]; \\ \frac{\beta}{\beta+1}\left[(b-t)^{1+\frac{1}{\beta}} + (t-a)^{1+\frac{1}{\beta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],\infty} \\ \qquad \text{if } f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \text{ and } g' \in L_\infty[a,b]; \\ \frac{\beta\delta}{\beta+\delta}\left[(b-t)^{\frac{\delta+\beta}{\beta+\delta}} + (t-a)^{\frac{\delta+\beta}{\beta+\delta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],\gamma} \\ \text{if } f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \text{ and } g' \in L_\gamma[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \beta\left[(b-t)^{\frac{1}{\beta}} + (t-a)^{\frac{1}{\beta}}\right] \|f'\|_{[a,b],\alpha} \|g'\|_{[a,b],1} \\ \qquad \text{if } f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \text{ and } g' \in L_1[a,b]; \\ (b-a)\|f'\|_{[a,b],1}\|g'\|_{[a,b],\infty} \qquad \text{if } f' \in L_1[a,b],\ g' \in L_\infty[a,b]; \\ \delta\left[(b-t)^{1+\frac{1}{\delta}} + (t-a)^{1+\frac{1}{\delta}}\right] \|f'\|_{[a,b],1} \|g'\|_{[a,b],\gamma} \\ \qquad \text{if } f' \in L_1[a,b],\ g' \in L_\gamma[a,b],\ \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \end{cases}$$

Proof. Since f and g are absolutely continuous on $[a,b]$, we may write that

$$f(\tau) - f(t) = \int_t^\tau f'(u)\,du \quad \text{and} \quad g(\tau) - g(t) = \int_t^\tau g'(u)\,du$$

which implies:

$$|f(\tau) - f(t)| \leq \begin{cases} \|f'\|_{[\tau,t],\infty} |\tau - t| & \text{if } f' \in L_\infty[a,b]; \\ \|f'\|_{[\tau,t],\alpha} |\tau - t|^{\frac{1}{\beta}} & \text{if } f' \in L_\alpha[a,b], \\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \|f'\|_{[\tau,t],1} \end{cases} \tag{8.9}$$

and

$$|g(\tau)-g(t)| \leq \begin{cases} \|g'\|_{[\tau,t],\infty}|\tau-t| & \text{if } g' \in L_\infty[a,b]; \\ \|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\delta}} & \text{if } g' \in L_\gamma[a,b], \\ & \gamma>1,\ \frac{1}{\gamma}+\frac{1}{\delta}=1; \\ \|g'\|_{[\tau,t],1}. \end{cases} \tag{8.10}$$

Using the identity (8.2), we get

$$\begin{aligned} &\Big| T(fg)(a,b;t) - f(t)T(g)(a,b;t) \\ &\qquad - g(t)T(f)(a,b;t) + \frac{1}{\pi}f(t)g(t)\ln\left(\frac{b-t}{t-a}\right)\Big| \\ &\leq \frac{1}{\pi}PV\int_a^b \left|\frac{(f(\tau)-f(t))(g(\tau)-g(t))}{\tau-t}\right| d\tau =: I. \end{aligned} \tag{8.11}$$

Then we have, by using (8.9) or (8.10), that

$$I \leq \frac{1}{\pi} \times \begin{cases} PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],\infty}|\tau-t|\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],\alpha}|\tau-t|^{\frac{1}{\delta}}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],1}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],\infty}|\tau-t|^{\frac{1}{\beta}}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\beta}+\frac{1}{\delta}-1}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],1}|\tau-t|^{\frac{1}{\beta}-1}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],\infty}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\delta}-1}\,d\tau \\ PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],1}|\tau-t|^{-1}\,d\tau. \end{cases} \tag{8.12}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

However,

$$\begin{aligned} &PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],\infty}|\tau-t|\,d\tau \\ &\leq \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}\left[\frac{(b-t)^2+(t-a)^2}{2}\right] \\ &= \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right], \end{aligned}$$

$$\begin{aligned} &PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],\alpha}|\tau-t|^{\frac{1}{\delta}}\,d\tau \\ &\leq \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\gamma}\left[\frac{(b-t)^{1+\frac{1}{\delta}}+(t-a)^{1+\frac{1}{\delta}}}{\frac{1}{\delta}+1}\right] \\ &= \frac{\delta}{\delta+1}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\gamma}\left[(b-t)^{1+\frac{1}{\delta}}+(t-a)^{1+\frac{1}{\delta}}\right], \end{aligned}$$

$$PV\int_a^b \|f'\|_{[\tau,t],\infty}\|g'\|_{[\tau,t],1}\,d\tau \leq \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1}(b-a),$$

$$\begin{aligned} &PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],\infty}|\tau-t|^{\frac{1}{\beta}}\,d\tau \\ &\leq \|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],\infty}\cdot\frac{\beta}{\beta+1}\left[(b-t)^{\frac{1}{\beta}+1}+(t-a)^{\frac{1}{\beta}+1}\right], \end{aligned}$$

$$\begin{aligned} &PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\beta}+\frac{1}{\delta}-1}\,d\tau \\ &\leq \|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],\gamma}\frac{1}{\frac{1}{\beta}+\frac{1}{\delta}}\left[(b-t)^{\frac{1}{\beta}+\frac{1}{\delta}}+(t-a)^{\frac{1}{\beta}+\frac{1}{\delta}}\right] \\ &= \frac{\beta\delta}{\beta+\delta}\|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],\gamma}\left[(b-t)^{\frac{\delta+\beta}{\beta+\delta}}+(t-a)^{\frac{\delta+\beta}{\beta+\delta}}\right], \end{aligned}$$

$$\begin{aligned} &PV\int_a^b \|f'\|_{[\tau,t],\alpha}\|g'\|_{[\tau,t],1}|\tau-t|^{\frac{1}{\beta}-1}\,d\tau \\ &\leq \|f'\|_{[a,b],\alpha}\|g'\|_{[a,b],1}\,\beta\left[(b-t)^{\frac{1}{\beta}}+(t-a)^{\frac{1}{\beta}}\right], \end{aligned}$$

$$PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],\infty}\,d\tau \leq (b-a)\|f'\|_{[a,b],1}\|g'\|_{[a,b],\infty}$$

and

$$\begin{aligned} &PV\int_a^b \|f'\|_{[\tau,t],1}\|g'\|_{[\tau,t],\gamma}|\tau-t|^{\frac{1}{\delta}-1}\,d\tau \\ &\leq \|f'\|_{[a,b],1}\|g'\|_{[a,b],\gamma}\,\delta\left[(b-t)^{1+\frac{1}{\delta}}+(t-a)^{1+\frac{1}{\delta}}\right]. \end{aligned}$$

For the last inequality we cannot point out a bound as above.

Using (8.11) and (8.12), we deduce the desired inequality (8.8). □

The following lemma also holds.

Lemma 17 (Dragomir, 2003, [12])**.** *Let $f:[a,b]\to\mathbb{R}$ be locally Hölder continuous on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ so that g' is absolutely continuous on $[a,b]$. Then we have the identity:*

$$\begin{aligned} (8.13)\qquad & T(fg)(a,b;t) \\ &= f(t)\,T(g)(a,b;t)+g(t)\,T(f)(a,b;t)-\frac{1}{\pi}f(t)\,g(t)\ln\left(\frac{b-t}{t-a}\right) \\ &+\frac{1}{\pi}\left[\int_a^b f(\tau)\,d\tau-(b-a)\,f(t)\right]g'(t) \\ &-\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau(u-\tau)\,g''(u)\,du\right)d\tau \end{aligned}$$

for any $t\in(a,b)$.

Proof. We use the following identity:

$$\int_\alpha^\beta\varphi(u)\,du=\varphi(\alpha)(\beta-\alpha)-\int_\alpha^\beta(u-\beta)\,\varphi'(u)\,du$$

which holds for any absolutely continuous function $\varphi:[\alpha,\beta]\to\mathbb{R}$.

Then we have

$$\begin{aligned} &\frac{1}{\pi}PV\int_a^b[f(\tau)-f(t)]\left[\frac{1}{\tau-t}(g(\tau)-g(t))\right]d\tau \\ &=\frac{1}{\pi}PV\int_a^b[f(\tau)-f(t)]\left[\frac{1}{\tau-t}\int_t^\tau g'(u)\,du\right]d\tau \\ &=\frac{1}{\pi}PV\int_a^b[f(\tau)-f(t)]\left[g'(t)-\frac{1}{\tau-t}\int_t^\tau(u-\tau)\,g''(u)\,du\right]d\tau \\ &=\frac{1}{\pi}\left[g'(t)\int_a^b f(\tau)\,d\tau-(b-a)\,f(t)\,g'(t)\right. \\ &\left.-PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau(u-\tau)\,g''(u)\,du\right)d\tau\right] \end{aligned}$$

$$\begin{aligned} &=\frac{1}{\pi}g'(t)\int_a^b f(\tau)\,d\tau-\frac{1}{\pi}(b-a)\,f(t)\,g'(t) \\ &-\frac{1}{\pi}PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}\left(\int_t^\tau(u-\tau)\,g''(u)\,du\right)d\tau. \end{aligned}$$

Using (8.1), we deduce (8.13). □

The following theorem holds.

Theorem 27 (Dragomir, 2003, [12])**.** *Assume that $f:[a,b]\to\mathbb{R}$ is of $H-r-$Hölder type and $g:[a,b]\to\mathbb{R}$ is such that g' is absolutely continuous on $[a,b]$. Then we* *have the inequality:*

$$\left| T(fg)(a,b;t) - f(t)T(g)(a,b;t) - g(t)T(f)(a,b;t) \right.$$
$$\left. + \frac{1}{\pi} f(t) g(t) \ln\left(\frac{b-t}{t-a}\right) - \frac{1}{\pi}\left[\int_a^b f(\tau)\,d\tau - (b-a) f(t)\right] g'(t) \right| \tag{8.14}$$

$$\leq \frac{H}{\pi} \begin{cases} \frac{1}{2(r+2)}\left[(b-t)^{r+2} + (t-a)^{r+2}\right] \|g''\|_{[a,b],\infty} & \text{if } g'' \in L_\infty[a,b]; \\ \frac{q}{(rq+q+1)(q+1)^{\frac{1}{q}}}\left[(b-t)^{r+\frac{1}{q}+1} + (t-a)^{r+\frac{1}{q}+1}\right] \|g''\|_{[a,b],p} & \text{if } g'' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(r+1)}\left[(b-t)^{r+1} + (t-a)^{r+1}\right] \|g''\|_{[a,b],1}. \end{cases}$$

Proof. Using the identity (8.13), we deduce that the left side in (8.14) is upper bounded by

$$\begin{aligned} I &:= \frac{1}{\pi} PV \int_a^b |f(\tau) - f(t)| \frac{1}{|\tau - t|} \left| \int_t^\tau (u-\tau) g''(u)\,du \right| d\tau \\ &\leq \frac{H}{\pi} PV \int_a^b |\tau - t|^{r-1} \left| \int_t^\tau (u-\tau) g''(u)\,du \right| d\tau =: J. \end{aligned}$$

We observe that

$$\left| \int_t^\tau (u-\tau) g''(u)\,du \right| \leq \|g''\|_{[t,\tau],\infty} \frac{(\tau - t)^2}{2}$$

if $g'' \in L_\infty[a,b]$,

$$\left| \int_t^\tau (u-\tau) g''(u)\,du \right| \leq \|g''\|_{[t,\tau],p} \left| \int_t^\tau |t-\tau|^q\,d\tau \right|^{\frac{1}{q}} = \|g''\|_{[t,\tau],p} \frac{|t-\tau|^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}$$

if $g'' \in L_p[a,b]$ and, finally,

$$\left| \int_t^\tau (u-\tau) g''(u)\,du \right| \leq |t-\tau| \, \|g''\|_{[t,\tau],1}.$$

Consequently, we have

$$J \leq \frac{H}{\pi} \times \begin{cases} PV \int_a^b |\tau - t|^{r-1} \cdot \frac{(\tau-t)^2}{2} \cdot \|g''\|_{[t,\tau],\infty} \, d\tau \\ \\ PV \int_a^b \frac{|\tau - t|^{r-1} \cdot |t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|g''\|_{[t,\tau],p} \, d\tau \\ \\ PV \int_a^b |\tau - t|^{r-1} \cdot |t - \tau| \, \|g''\|_{[t,\tau],1} \, d\tau \end{cases}$$

$$\leq \frac{H}{\pi} \times \begin{cases} \frac{1}{2} \|g''\|_{[a,b],\infty} \left[\frac{(b-t)^{r+2} + (t-a)^{r+2}}{r+2} \right] \\ \\ \frac{1}{(q+1)^{\frac{1}{q}}} \|g''\|_{[a,b],p} \left[\frac{(b-t)^{r+\frac{1}{q}+1} + (t-a)^{r+\frac{1}{q}+1}}{r+\frac{1}{q}+1} \right] \\ \\ \|g''\|_{[a,b],1} \cdot \left[\frac{(b-t)^{r+1} + (t-a)^{r+1}}{r+1} \right], \end{cases}$$

which proves the inequality (8.14). □

The following lemma also holds.

Lemma 18. *Assume that f and g are as in Lemma 17. Then we have the identity:*

$$\begin{aligned} T(fg)(a,b;t) &\\ = f(t)\,T(g)(a,b;t) + g(t)\,T(f)(a,b;t) - \frac{1}{\pi} f(t)\,g(t) \ln\left(\frac{b-t}{t-a}\right) &\\ + \frac{1}{\pi}\left[\int_a^b f(\tau)\,g'(\tau)\,d\tau - [g(b) - g(a)]\,f(t)\right] &\\ - \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} \left(\int_t^\tau (u-t)\,g''(u)\,du\right) d\tau & \end{aligned} \tag{8.15}$$

for any $t \in (a,b)$.

Proof. In this case, we use the following identity:

$$\int_\alpha^\beta \varphi(u)\,du = \varphi(\beta)(\beta - \alpha) - \int_\alpha^\beta (u-\alpha)\,\varphi'(u)\,du$$

which holds for any absolutely continuous function $\varphi : [\alpha, \beta] \to \mathbb{R}$.

Then, as above, we have

$$\begin{aligned} &\frac{1}{\pi} PV \int_a^b [f(\tau) - f(t)] \left[\frac{1}{\tau - t}(g(\tau) - g(t))\right] d\tau \\ &= \frac{1}{\pi} PV \int_a^b [f(\tau) - f(t)] \left[\frac{1}{\tau - t} \int_t^\tau g'(u)\,du\right] d\tau \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi} PV \int_a^b \left[f(\tau) - f(t) \right] \left[g'(t) - \frac{1}{\tau - t} \int_t^\tau (u-\tau) g''(u)\, du \right] d\tau \\
&= \frac{1}{\pi} \left[\int_a^b f(\tau) g'(\tau)\, d\tau - f(t)\, PV \int_a^b g'(\tau)\, d\tau \right. \\
&\left. - PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} \left(\int_t^\tau (u-t) g''(u)\, du \right) d\tau \right] \\
&= \frac{1}{\pi} \left[\int_a^b f(\tau) g'(\tau)\, d\tau - [g(b) - g(a)] f(t) \right. \\
&\left. - PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} \left(\int_t^\tau (u-t) g''(u)\, du \right) d\tau \right],
\end{aligned}
$$

proving the identity (8.15). □

The following result also holds.

Theorem 28 (Dragomir, 2003, [12]). *With the assumptions in Theorem 27, we have:*

$$
\begin{aligned}
(8.16) \quad & \Bigg| T(fg)(a,b;t) - f(t)\, T(g)(a,b;t) - g(t)\, T(f)(a,b;t) \\
& + \frac{1}{\pi} f(t) g(t) \ln \left(\frac{b-t}{t-a} \right) - \frac{1}{\pi} \left[\int_a^b f(\tau) g'(\tau)\, d\tau - [g(b) - g(a)] f(t) \right] \Bigg|
\end{aligned}
$$

$$
\leq \frac{H}{\pi} \begin{cases}
\frac{1}{2(r+2)} \left[(b-t)^{r+2} + (t-a)^{r+2} \right] \|g''\|_{[a,b],\infty} & \text{if } g'' \in L_\infty [a,b]; \\
\frac{q}{(rq+q+1)(q+1)^{\frac{1}{q}}} \left[(b-t)^{r+\frac{1}{q}+1} + (t-a)^{r+\frac{1}{q}+1} \right] \|g''\|_{[a,b],p} & \text{if } g'' \in L_p [a,b], \\
 & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\
\frac{1}{(r+1)} \left[(b-t)^{r+1} + (t-a)^{r+1} \right] \|g''\|_{[a,b],1}. &
\end{cases}
$$

Proof. The proof follows in a similar manner to the one in Theorem 27 by the use of Lemma 18. We omit the details. □

9. Estimates Via Taylor's Expansion

9.1. **Inequalities on the Whole Interval** $[a,b]$**.** The following result holds.

Theorem 29 (Dragomir, 2005, [13]). *Let* $f : [a,b] \to \mathbb{R}$ *be such that* $f^{(n-1)}$ $(n \geq 1)$ *is absolutely continuous on* $[a,b]$*. Then we have the bounds:*

(9.1)

$$
\left| (Tf)(a,b;t) - f(t) \ln \left(\frac{b-t}{t-a} \right) - \sum_{k=1}^{n-1} \frac{f^{(k)}(t)}{k!} \cdot \left[\frac{(b-t)^k + (-1)^{k+1} (t-a)^k}{k} \right] \right|
$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

$$\leq \begin{cases} \dfrac{\left\|f^{(n)}\right\|_{[a,b],\infty}}{n \cdot n!}\left[(b-t)^n+(t-a)^n\right], & \text{if } f^{(n)} \in L_\infty[a,b]; \\ \dfrac{q\left\|f^{(n)}\right\|_{[a,b],p}\left[(b-t)^{n-1+\frac{1}{q}}+(t-a)^{n-1+\frac{1}{q}}\right]}{(n-1)!\left[(n-1)q+1\right]^{1+\frac{1}{q}}}, & \text{if } f^{(n)} \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\|f^{(n)}\right\|_{[a,b],1}}{(n-1)\cdot(n-1)!}\left[(b-t)^{n-1}+(t-a)^{n-1}\right], & \end{cases}$$

$$\leq \begin{cases} \dfrac{(b-a)^n}{n \cdot n!}\left\|f^{(n)}\right\|_{[a,b],\infty}, & \text{if } f^{(n)} \in L_\infty[a,b]; \\ \dfrac{q(b-a)^{n-1+\frac{1}{q}}}{(n-1)!\left[(n-1)q+1\right]^{1+\frac{1}{q}}}\left\|f^{(n)}\right\|_{[a,b],p}, & \text{if } f^{(n)} \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{1}{(n-1)\cdot(n-1)!}(b-a)^{n-1}\left\|f^{(n)}\right\|_{[a,b],1}, & \end{cases}$$

for any $t \in (a,b)$.

Proof. Start with Taylor's formula for a function $g: I \to \mathbb{R}$ (I is a compact interval) with the property that $g^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on I, then we have

$$g(x)=\sum_{k=0}^{n-1}\frac{(x-a)^k}{k!}g^{(k)}(a)+\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}g^{(n)}(t)\,dt,$$

where $a,x \in \mathring{I}$ ($\mathring{I}$is the interior of I). This implies that

$$\left|g(x)-\sum_{k=0}^{n-1}\frac{(x-a)^k}{k!}g^{(k)}(a)\right| \leq \frac{1}{(n-1)!}\left|\int_a^x |x-t|^{n-1}\left|g^{(n)}(t)\right|dt\right| =: \frac{1}{(n-1)!}\cdot M(x)$$

for any $a,x \in \mathring{I}$.

Before we estimate $M(x)$, let us introduce the following notations

$$\|h\|_{[a,x],p} := \left|\int_a^x |h(t)|^p\,dt\right|^{\frac{1}{p}} \quad \text{if } p \geq 1$$

and

$$\|h\|_{[a,x],\infty} := ess \sup_{\substack{t\in[a,x] \\ t\in([x,a])}} |h(t)|,$$

where $a,x \in \mathring{I}$.

It is obvious now that

$$M(x) \leq \sup_{\substack{t\in[a,x] \\ (t\in[x,a])}}\left|g^{(n)}(t)\right|\left|\int_a^x |x-t|^{n-1}\,dt\right| = \left\|g^{(n)}\right\|_{[a,x],\infty}\frac{|x-a|^n}{n!}$$

for any $a,x \in \mathring{I}$.

 doi:10.20944/preprints201803.0017.v1

Using Hölder's integral inequality, we may state that

$$M(x) \leq \left|\int_a^x \left|g^{(n)}(t)\right|^p dt\right|^{\frac{1}{p}} \left|\int_a^x |x-t|^{(n-1)q} dt\right|^{\frac{1}{q}} = \left\|g^{(n)}\right\|_{[a,x],p} \frac{|x-a|^{n-1+\frac{1}{q}}}{[(n-1)q+1]^{\frac{1}{q}}}$$

for any $a, x \in \mathring{I}$.

Also, we observe that

$$M(x) \leq |x-a|^{n-1} \left|\int_a^x \left|g^{(n)}(t)\right| dt\right| = \left\|g^{(n)}\right\|_{[a,x],1} |x-a|^{n-1}$$

for all $a, x \in \mathring{I}$.

In conclusion, we may state the following inequality which will be used in the sequel

$$\left|g(x) - \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} g^{(k)}(a)\right| \tag{9.2}$$

$$\leq \begin{cases} \frac{|x-a|^n}{n!} \left\|g^{(n)}\right\|_{[a,x],\infty} & \text{if} \quad g^{(n)} \in L_\infty(\mathring{I}); \\ \frac{|x-a|^{n-1+\frac{1}{q}}}{(n-1)!\,[(n-1)q+1]^{\frac{1}{q}}} \left\|g^{(n)}\right\|_{[a,x],p} & \text{if} \quad g^{(n)} \in L_p(\mathring{I}), \\ & \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{|x-a|^{n-1}}{(n-1)!} \left\|g^{(n)}\right\|_{[a,x],1} \end{cases}$$

for any $a, x \in \mathring{I}$.

Now, let us note for the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = 1$, we have that

$$(Tf_0)(a,b;t) = \ln\left(\frac{b-t}{t-a}\right), \ t \in (a,b),$$

and then

$$\begin{aligned} (Tf)(a,b;t) &= PV \int_a^b \frac{f(\tau) - f(t) + f(t)}{\tau - t} d\tau \\ &= PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau + f(t) \ln\left(\frac{b-t}{t-a}\right), \end{aligned}$$

giving the equality

$$(Tf)(a,b;t) - f(t)\ln\left(\frac{b-t}{t-a}\right) = PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau. \tag{9.3}$$

Writing (9.2) for $g=f$, $x=\tau$, $a=t$, we get

$$\left|\frac{f(\tau)-f(t)}{\tau-t}-\sum_{k=1}^{n-1}\frac{(\tau-t)^{k-1}}{k!}f^{(k)}(t)\right| \tag{9.4}$$

$$\leq\begin{cases}\dfrac{|\tau-t|^{n-1}}{n!}\left\|f^{(n)}\right\|_{[t,\tau],\infty} & \text{if } f^{(n)}\in L_\infty[a,b];\\ \dfrac{|\tau-t|^{n-2+\frac{1}{q}}}{(n-1)![(n-1)q+1]^{\frac{1}{q}}}\left\|f^{(n)}\right\|_{[t,\tau],p} & \text{if } f^{(n)}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \dfrac{|\tau-t|^{n-2}}{(n-1)!}\left\|f^{(n)}\right\|_{[t,\tau],1}\end{cases}$$

for any $t,\tau\in(a,b)$, $t\neq\tau$.

If we take the PV in (9.4), then we may write

$$\left|PV\int_a^b\frac{f(\tau)-f(t)}{\tau-t}d\tau-\sum_{k=1}^{n-1}\frac{f^{(k)}(t)}{k!}PV\int_a^b(\tau-t)^{k-1}d\tau\right| \tag{9.5}$$

$$\leq PV\int_a^b\left|\frac{f(\tau)-f(t)}{\tau-t}-\sum_{k=1}^{n-1}\frac{(\tau-t)^{k-1}}{k!}f^{(k)}(t)\right|d\tau$$

$$\leq\begin{cases}\dfrac{1}{n!}PV\int_a^b|\tau-t|^{n-1}\left\|f^{(n)}\right\|_{[t,\tau],\infty}d\tau\\ \dfrac{1}{(n-1)![(n-1)q+1]^{\frac{1}{q}}}PV\int_a^b|\tau-t|^{n-2+\frac{1}{q}}\left\|f^{(n)}\right\|_{[t,\tau],p}d\tau\\ \dfrac{1}{(n-1)!}PV\int_a^b|\tau-t|^{n-2}\left\|f^{(n)}\right\|_{[t,\tau],1}d\tau\end{cases}$$

$$\leq\begin{cases}\dfrac{1}{n!}\left\|f^{(n)}\right\|_{[a,b],\infty}PV\int_a^b|\tau-t|^{n-1}d\tau\\ \dfrac{1}{(n-1)![(n-1)q+1]^{\frac{1}{q}}}\left\|f^{(n)}\right\|_{[a,b],p}PV\int_a^b|\tau-t|^{n-2+\frac{1}{q}}d\tau\\ \dfrac{1}{(n-1)!}\left\|f^{(n)}\right\|_{[a,b],1}PV\int_a^b|\tau-t|^{n-2}d\tau.\end{cases}$$

However,

$$PV\int_a^b|\tau-t|^{n-1}d\tau=\frac{1}{n}\left[(b-t)^n+(t-a)^n\right],$$

$$PV\int_a^b|\tau-t|^{n-2+\frac{1}{q}}d\tau=\frac{q}{[(n-1)q+1]}\left[(b-t)^{n-1+\frac{1}{q}}+(t-a)^{n-1+\frac{1}{q}}\right],$$

$$PV\int_a^b |\tau - t|^{n-2}\, d\tau = \frac{1}{n-1}\left[(b-t)^{n-1} + (t-a)^{n-1}\right]$$

and

$$PV\int_a^b (\tau - t)^{k-1}\, d\tau = \frac{1}{k}\left[(b-t)^{k} + (-1)^{k+1}(t-a)^{k}\right]$$

and then by (9.5) we deduce the desired result (9.1). □

It is obvious that the best inequality one would deduce from (9.1) is the one for $t = \frac{a+b}{2}$, getting the following corollary.

Corollary 23. *With the assumptions of Theorem 29, we have*

$$\left|(Tf)\left(a,b;\frac{a+b}{2}\right) - \sum_{k=1}^{n-1}\frac{(b-a)^k}{2^k\cdot k\cdot k!}\left[1+(-1)^{k+1}\right]f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{9.6}$$

$$\leq \begin{cases} \dfrac{(b-a)^n}{2^{n-1}\cdot n\cdot n!}\left\|f^{(n)}\right\|_{[a,b],\infty}, & \text{if } f^{(n)}\in L_\infty[a,b]; \\[2ex] \dfrac{q(b-a)^{n-1+\frac{1}{q}}}{2^{n-2+\frac{1}{q}}(n-1)!\left[(n-1)q+1\right]^{1+\frac{1}{q}}}\left\|f^{(n)}\right\|_{[a,b],p}, & \text{if } f^{(n)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{(b-a)^{n-1}}{2^{n-2}\cdot(n-1)\cdot(n-1)!}\left\|f^{(n)}\right\|_{[a,b],1}. \end{cases}$$

It is important to note that for small intervals, we basically have the following representation:

Corollary 24. *Assume that* $f\in C^\infty[a,b]$ *and* $0<b-a\leq 1$. *Then*

$$(Tf)(a,b;t) = f(t)\ln\left(\frac{b-t}{t-a}\right) + \sum_{k=1}^{\infty}\frac{f^{(k)}(t)}{k!}\cdot\left[\frac{(b-t)^k+(-1)^{k+1}(t-a)^k}{k}\right]$$

and the convergence is uniform on $[a,b]$.

9.2. **The Composite Case.** The following lemma holds.

Lemma 19. *Let* $g:[a,b]\to\mathbb{R}$ *be such that* $g^{(n-1)}$ $(n\geq 1)$ *is absolutely continuous on* $[a,b]$. *Then for any* $m\in\mathbb{N}$, $m\geq 1$, *we have the inequality:*

$$\left|\frac{1}{b-a}\int_a^b g(u)\,du - \sum_{i=0}^{m-1}\sum_{k=1}^{n}\frac{(b-a)^{k-1}}{m^k k!}\cdot g^{(k-1)}\left(a+i\cdot\frac{b-a}{m}\right)\right| \tag{9.7}$$

$$\leq \begin{cases} \dfrac{(b-a)^n}{m^n(n+1)!}\left\|g^{(n)}\right\|_{[a,b],\infty}, & \text{if } g^{(n)}\in L_\infty[a,b]; \\[2ex] \dfrac{(b-a)^{n-1+\frac{1}{q}}}{m^n n!(nq+1)^{\frac{1}{q}}}\left\|g^{(n)}\right\|_{[a,b],p}, & \text{if } g^{(n)}\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \dfrac{(b-a)^{n-1}}{m^n n!}\left\|g^{(n)}\right\|_{[a,b],1}. \end{cases}$$

Proof. Write Taylor's formula with the integral remainder for $\varphi(x)=\int_\alpha^x g(u)\,du$ and then choose $x=\beta$, to get:

$$\left|\int_\alpha^\beta g(u)\,du-\sum_{k=1}^n\frac{(\beta-\alpha)^k}{k!}g^{(k-1)}(\alpha)\right| \tag{9.8}$$

$$\leq\begin{cases}\dfrac{|\beta-\alpha|^{n+1}}{(n+1)!}\left\|g^{(n)}\right\|_{[\alpha,\beta],\infty}, & \text{if } g^{(n)}\in L_\infty[a,b];\\[2ex] \dfrac{|\beta-\alpha|^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}}\left\|g^{(n)}\right\|_{[\alpha,\beta],p}, & \text{if } g^{(n)}\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \dfrac{|\beta-\alpha|^n}{n!}\left\|g^{(n)}\right\|_{[\alpha,\beta],1}\end{cases}$$

for any $\alpha,\beta\in[a,b]$.

Now, if we consider the division

$$I_n: x_i=a+i\cdot\frac{b-a}{m},\quad i=\overline{0,m},$$

and apply (9.8) on the intervals $[x_i,x_{i+1}]$ $\left(i=\overline{0,m-1}\right)$, we can write:

$$\left|\int_{x_i,}^{x_{i+1}} g(u)\,du-\sum_{k=1}^n\frac{(b-a)^k}{m^k k!}\cdot g^{(k-1)}\left(a+i\cdot\frac{b-a}{m}\right)\right|$$

$$\leq\begin{cases}\dfrac{(b-a)^{n+1}}{m^{n+1}(n+1)!}\left\|g^{(n)}\right\|_{[x_i,x_{i+1}],\infty},\\[2ex] \dfrac{(b-a)^{n+\frac{1}{q}}}{m^{n+\frac{1}{q}}n!(nq+1)^{\frac{1}{q}}}\left\|g^{(n)}\right\|_{[x_i,x_{i+1}],p},\\[2ex] \dfrac{(b-a)^n}{m^n n!}\left\|g^{(n)}\right\|_{[x_i,x_{i+1}],1}.\end{cases}$$

Summing over i from 0 to $m-1$ and using the generalized triangle inequality, we deduce (9.8). □

The following main result holds.

Theorem 30 (Dragomir, 2005, [13]). *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ $(n\geq 0)$ is absolutely continuous on $[a,b]$. Then for any $m\in\mathbb{N}$, $m\geq 1$, we have:*

$$(Tf)(a,b;t)=f(t)\ln\left(\frac{b-t}{t-a}\right)+A_{n,m}(f,t)+R_{n,m}(f,t),$$

where

$$A_{n,m}(f,t) = \sum_{k=1}^{n} \frac{f^{(k)}(t)}{m^k k!} \cdot \left[\frac{(b-t)^k + (-1)^{k+1}(t-a)^k}{k} \right] + (b-a) \sum_{i=1}^{m-1} \sum_{k=1}^{n} \frac{1}{m^k k!} \times \left\{ \sum_{\nu=1}^{k-1} (-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1} \times \left[f^{(k-\nu)}; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right] + (-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)! \left[f; t + \frac{i}{m}(b-t), t - \frac{i}{m}(t-a) \right] \right\} \tag{9.9}$$

and the remainder $R_{n,m}(f,t)$ *satisfies the estimate*

$$|R_{n,m}(f,t)| \tag{9.10}$$

$$\leq \begin{cases} \dfrac{\left\| f^{(n+1)} \right\|_{[a,b],\infty}}{m^n (n+1)! \cdot (n+1)} \left[(b-t)^{n+1} + (t-a)^{n+1} \right], & \text{if } f^{(n+1)} \in L_\infty [a,b]; \\ \dfrac{q\left[(b-t)^{n+\frac{1}{q}} + (t-a)^{n+\frac{1}{q}} \right]}{m^n n! (nq+1)^{1+\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[a,b],p}, & \text{if } f^{(n+1)} \in L_p [a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{\left\| f^{(n+1)} \right\|_{[a,b],1}}{m^n n! \cdot n} \left[(b-t)^n + (t-a)^n \right]; \end{cases}$$

$$\leq \begin{cases} \dfrac{(b-a)^{n+1}}{m^n (n+1) \cdot (n+1)!}, \left\| f^{(n+1)} \right\|_{[a,b],\infty} & \text{if } f^{(n+1)} \in L_\infty [a,b]; \\ \dfrac{q(b-a)^{n+\frac{1}{q}}}{m^n n! (nq+1)^{1+\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[a,b],p}, & \text{if } f^{(n+1)} \in L_p [a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \dfrac{(b-a)^n}{m^n n! \cdot n} \left\| f^{(n+1)} \right\|_{[a,b],1}; \end{cases}$$

Proof. We have (see (9.3)) that:

$$(Tf)(a,b;t) - f(t)\ln\left(\frac{b-t}{t-a}\right) = PV \int_a^b \frac{f(\tau)-f(t)}{\tau - t} d\tau.$$

If we write the inequality (9.7) for $g = f'$, we get

$$\left| \frac{f(t)-f(\tau)}{\tau - t} - \sum_{i=0}^{m-1} \sum_{k=1}^{n} \frac{(\tau-t)^{k-1}}{m^k k!} \cdot f^{(k)}\left(t + i \cdot \frac{\tau - t}{m} \right) \right| \tag{9.11}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

$$\leq \begin{cases} \dfrac{|\tau - t|^n}{m^n (n+1)!} \left\| f^{(n+1)} \right\|_{[t,\tau],\infty}, & \text{if } f^{(n+1)} \in L_\infty [a,b]; \\ \dfrac{|\tau - t|^{n-1+\frac{1}{q}}}{m^n n! (nq+1)^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[t,\tau],p}, & \text{if } f^{(n+1)} \in L_p [a,b], \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \dfrac{|\tau - t|^{n-1}}{m^n n!} \left\| f^{(n+1)} \right\|_{[t,\tau],1}. \end{cases}$$

If we apply PV to (9.11), we may write that:

$$\text{(9.12)} \quad \left| PV \int_a^b \frac{f(t) - f(\tau)}{\tau - t} d\tau - \sum_{i=0}^{m-1} \sum_{k=1}^{n} PV \int_a^b \frac{(\tau - t)^{k-1}}{m^k k!} \cdot f^{(k)} \left(t + i \cdot \frac{\tau - t}{m} \right) d\tau \right|$$

$$\leq \begin{cases} \dfrac{1}{m^n (n+1)!} PV \int_a^b |\tau - t|^n \left\| f^{(n+1)} \right\|_{[t,\tau],\infty} d\tau, \\ \dfrac{1}{m^n n! (nq+1)^{\frac{1}{q}}} PV \int_a^b |\tau - t|^{n-1+\frac{1}{q}} \left\| f^{(n+1)} \right\|_{[t,\tau],p} d\tau, \\ \dfrac{1}{m^n n!} PV \int_a^b |\tau - t|^{n-1} \left\| f^{(n+1)} \right\|_{[t,\tau],1} d\tau, \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{m^n (n+1)!} \left\| f^{(n+1)} \right\|_{[a,b],\infty} PV \int_a^b |\tau - t|^n d\tau, \\ \dfrac{1}{m^n n! (nq+1)^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[a,b],p} PV \int_a^b |\tau - t|^{n-1+\frac{1}{q}} d\tau, \\ \dfrac{1}{m^n n!} \left\| f^{(n+1)} \right\|_{[a,b],1} PV \int_a^b |\tau - t|^{n-1} d\tau, \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{m^n (n+1)!} \left\| f^{(n+1)} \right\|_{[a,b],\infty} \left[\dfrac{(b-t)^{n+1} + (t-a)^{n+1}}{n+1} \right], \\ \dfrac{1}{m^n n! (nq+1)^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[a,b],p} \left[\dfrac{(b-t)^{n+\frac{1}{q}} + (t-a)^{n+\frac{1}{q}}}{n+\frac{1}{q}} \right], \\ \dfrac{1}{m^n n!} \left\| f^{(n+1)} \right\|_{[a,b],1} \left[\dfrac{(b-t)^n + (t-a)^n}{n} \right]. \end{cases}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 March 2018 doi:10.20944/preprints201803.0017.v1

Now, let us denote

$$I_{i,k} := PV \int_a^b (\tau - t)^{k-1} f^{(k)} \left(t + \frac{i}{m} (\tau - t) \right) d\tau,$$

where $i = 0, \ldots, m-1$, $k = 1, \ldots, n$.

For $i = 0$, we have

$$I_{0,k} := PV \int_a^b (\tau - t)^{k-1} f^{(k)} (t) \, d\tau = f^{(k)} (t) \cdot \frac{(b-t)^k + (-1)^{k+1} (t-a)^k}{k}$$

for any $k = 1, \ldots, n$.

For $k = 1, \ldots, n$ and $i = 1, \ldots, m-1$, we have

$$\begin{aligned} I_{i,k} = \lim_{\varepsilon \to 0+} \Bigg[\int_a^{t-\varepsilon} (\tau - t)^{k-1} f^{(k)} \left[t + \frac{i}{m} (\tau - t) \right] d\tau \\ + \int_{t+\varepsilon}^b (\tau - t)^{k-1} f^{(k)} \left[t + \frac{i}{m} (\tau - t) \right] d\tau \Bigg] \end{aligned} \tag{9.13}$$

$$\begin{aligned} &= \lim_{\varepsilon \to 0+} \Bigg[\frac{m}{i} f^{(k-1)} \left[t + \frac{i}{m} (\tau - t) \right] (\tau - t)^{k-1} \Big|_a^{t-\varepsilon} \\ &- \frac{m}{i} \int_a^{t-\varepsilon} (k-1) (\tau - t)^{k-2} f^{(k-1)} \left[t + \frac{i}{m} (\tau - t) \right] d\tau \\ &+ \frac{m}{i} f^{(k-1)} \left[t + \frac{i}{m} (\tau - t) \right] (\tau - t)^{k-1} \Big|_{t+\varepsilon}^b \\ &- \frac{m}{i} \int_{t+\varepsilon}^b (k-1) (\tau - t)^{k-2} f^{(k-1)} \left[t + \frac{i}{m} (\tau - t) \right] d\tau \Bigg] \\ &= \frac{m}{i} \left[f^{(k-1)} \left[t + \frac{i}{m} (b - t) \right] - f^{(k-1)} \left[t - \frac{i}{m} (t - a) \right] \right] \\ &- \frac{m}{i} (k-1) \, PV \int_a^b (\tau - t)^{k-2} f^{(k-1)} \left(t + \frac{i}{m} (\tau - t) \right) d\tau \\ &= (b-a) \left[f^{(k-1)}; t + \frac{i}{m} (b-t), t - \frac{i}{m} (t-a) \right] - \frac{m}{i} (k-1) \, I_{i,k-1} \end{aligned}$$

for any $k = 2, \ldots, n$.

For $k = 1$, we have

$$\begin{aligned} I_{i,1} &= PV \int_a^b f^{(1)} \left(t + \frac{i}{m} (\tau - t) \right) d\tau \\ &= \frac{m}{i} \left[f \left[t + \frac{i}{m} (b-t) \right] - f \left[t - \frac{i}{m} (t-a) \right] \right] \\ &= (b-a) \left[f; t + \frac{i}{m} (b-t), t - \frac{i}{m} (t-a) \right]. \end{aligned}$$

Using the recursive relation (9.13), we may write

$$
\begin{aligned}
(9.14)\quad & I_{i,k} \\
&= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\left(\frac{m}{i}\right)(k-1)\,I_{i,k-1} \\
&= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\left(\frac{m}{i}\right)(k-1) \\
&\times\left[(b-a)\left[f^{(k-2)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]-\left(\frac{m}{i}\right)(k-2)\,I_{i,k-2}\right] \\
&\quad= (b-a)\left[f^{(k-1)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\
&\quad-(b-a)\left(\frac{m}{i}\right)(k-1)\left[f^{(k-2)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\
&\quad+\left(\frac{m}{i}\right)^2(k-1)(k-2)\,I_{i,k-2} \\
&= \cdots\cdots\cdots\cdots = \\
&= (b-a)\sum_{\nu=1}^{k-1}(-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1} \\
&\times\left[f^{(k-\nu)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]+(-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!I_{i,1} \\
&= (b-a)\left[\sum_{\nu=1}^{k-1}(-1)^{\nu-1}(k-1)\cdots(k-\nu)\left(\frac{m}{i}\right)^{\nu-1}\right. \\
&\times\left[f^{(k-\nu)};t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right] \\
&\left.+(-1)^{k-1}\left(\frac{m}{i}\right)^{k-1}(k-1)!\left[f;t+\frac{i}{m}(b-t),t-\frac{i}{m}(t-a)\right]\right].
\end{aligned}
$$

Replacing $I_{i,k}$ in (9.12), we deduce the estimate (9.10) with $A_{m,n}$ as defined by (9.9). The theorem is thus proved. □

References

[1] R. P. Agarwal and S. S. Dragomir, An application of Hayashi's inequality for differentiable functions, *Computers. Math. Applic.,* **32** (6) (1996), 95-99.

[2] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation Theory*, Vol. 1, Birkhaüser Verlag, Basel, 1977.

[3] N. M. Dragomir, S. S. Dragomir and P. M. Farrell, Some inequalities for the finite Hilbert transform. *Inequality Theory and Applications.* Vol. I, 113–122, Nova Sci. Publ., Huntington, NY, 2001.

[4] N. M. Dragomir, S. S. Dragomir and P. M. Farrell, Approximating the finite Hilbert transform via trapezoid type inequalities. *Comput. Math. Appl.* **43** (2002), no. 10-11, 1359–1369.

[5] N. M. Dragomir, S. S. Dragomir, P. M Farrell and G. W. Baxter, On some new estimates of the finite Hilbert transform. *Libertas Math.* **22** (2002), 65–75.

[6] N. M. Dragomir, S. S. Dragomir, P. M Farrell and G. W. Baxter, A quadrature rule for the finite Hilbert transform via trapezoid type inequalities. *J. Appl. Math. Comput.* **13** (2003),no. 1-2, 67–84.

[7] N. M. Dragomir, S. S. Dragomir, P. M Farrell and G. W. Baxter, A quadrature rule for the finite Hilbert transform via midpoint type inequalities. *Fixed Point Theory and Applications.* Vol. 5, 11–22, Nova Sci. Publ., Hauppauge, NY, 2004.

[8] S. S. Dragomir, On the Ostrowski integral inequality for mappings of bounded variation and applications, *Math. Ineq.& Appl.*, **3**(1) (2001), 59-66.

[9] S. S. Dragomir, Approximating the finite Hilbert transform via an Ostrowski type inequality for functions of bounded variation. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 4, Article 51, 19 pp.

[10] S. S. Dragomir, Approximating the finite Hilbert transform via Ostrowski type inequalities for absolutely continuous functions. *Bull. Korean Math. Soc.* **39** (2002), no. 4, 543–559.

[11] S. S. Dragomir, Inequalities for the Hilbert transform of functions whose derivatives are convex. *J. Korean Math. Soc.* **39** (2002), no. 5, 709–729.

[12] S. S. Dragomir, Some inequalities for the finite Hilbert transform of a product. *Commun. Korean Math. Soc.* **18** (2003), no. 1, 39–57.

[13] S. S. Dragomir, Sharp error bounds of a quadrature rule with one multiple node for the finite Hilbert transform in some classes of continuous differentiable functions. *Taiwanese J. Math.* **9** (2005), no. 1, 95–109.

[14] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000 (ONLINE: httP://rgmia.vu.edu.au/monographs).

[15] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and to some numerical quadrature rules, *Comp. Math. Appl.*, **33** (1997), 15-20.

[16] S. S. Dragomir and S. Wang, A new inequality of Ostrowski type in L_1 norm and applications to some special means and some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[17] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p norm and applications to some numerical quadrature rules, *Indian J. of Math.,* **40** (3) (1998), 245-304.

[18] F. D. Gakhov, *Boundary Value Problems* (English translation), Pergamon Press, Oxford, 1966.

[19] H. Kober, A note on Hilbert's operator, *Bull. Amer. Math. Soc.,* **48** (1942), 421-427.

[20] W. Liu and X. Gao, Approximating the finite Hilbert transform via a companion of Ostrowski's inequality for function of bounded variation and applications. Appl. Math. Comput. 247 (2014), 373–385.

[21] W. Liu, X. Gao and Y. Wen, Approximating the finite Hilbert transform via some companions of Ostrowski's inequalities. *Bull. Malays. Math. Sci. Soc.* 39 (2016), no. 4, 1499–1513.

[22] W. Liu and N. Lu, Approximating the finite Hilbert transform via Simpson type inequalities and applications. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **77** (2015), no. 3, 107–122.

[23] S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators* (English translation), Springer Verlag, Berlin, 1986.

[24] S. Okada and D. Elliot, Hölder continuous functions and the finite Hilbert transform, *Mathematische Nach.,* **163** (1994), 250-272.

[25] S. Wang, X. Gao and N. Lu, A quadrature formula in approximating the finite Hilbert transform via perturbed trapezoid type inequalities. *J. Comput. Anal. Appl.* **22** (2017), no. 2, 239–246.

[26] S. Wang, N. Lu and X. Gao, A quadrature rule for the finite Hilbert transform via Simpson type inequalities and applications. *J. Comput. Anal. Appl.* **22** (2017), no. 2, 229–238

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa