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A (P, V)-EXTENSION OF HURWITZ-LERCH ZETA FUNCTION AND ITS PROPERTIES

GAUHAR RAHMAN[1], KOTTAKKARAN SOOPPY NISAR[2,⋆] AND SHAHID MUBEEN[3]

[1] G. Rahman: Department of Mathematics, International Islamic University, Islamabad, Pakistan, email: gauhar55uom@gmail.com

[2] Department of Mathematics, College of Arts & Science-Wadi Addawaser, Prince Sattam bin Abdulaziz University, Saudi Arabia, email: n.sooppy@psau.edu.sa

[3] Shahid Mubeen: Department of Mathematics, University of Sargodha, Sargodha, Pakistan, email: smjhanda@gmail.com

⋆ *Corresponding Author*.

Abstract. In this paper, we define a (p, v)-extension of Hurwitz-Lerch Zeta function by considering an extension of beta function defined by Parmar et al. [J. Classical Anal. 11 (2017) 81106]. We obtain its basic properties which include integral representations, Mellin transformation, derivative formulas and certain generating relations. Also, we establish the special cases of the main results.

keywords: Extended beta function, hypergeometric functions, Hurwitz-Lerch Zeta functions, Mellin transform, transformation formula and summation formula.

MSC[2010]: 11B68, 33C20, 33C45, 33C65, 33C05

1. Introduction

The Hurwitz-Lerch Zeta function and its properties are appear in various recent research papers $\Big($see e.g., [10] pp. 27, [24], pp. 121 and [25], pp. 194 $\Big)$. The definition and integral representation of Hurwitz-Lerch Zeta function respectively given by

$$\Phi(z,\sigma,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^{\sigma}}, \tag{1.1}$$

$$(a\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ s\in\mathbb{C}, \text{when } |z|<1;\Re(\sigma)>1 \text{ when } |z|=1).$$

$$\Phi(z,\sigma,a)=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}\frac{t^{\sigma-1}e^{-at}}{1-ze^{-t}}dt=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}\frac{t^{\sigma-1}e^{-(a-1)t}}{e^t-z}dt \tag{1.2}$$

$$(\Re(\sigma)>0, \Re(a)>0, \text{when } |z|\leq 1 (z\neq 1);\Re(\sigma)>1 \text{ when } z=1).$$

Many extension and generalizations of (1.1) found in the literature (see .g., [3, 7, 10, 11]). In [13], authors gave the following generalization and integral representation:

$$\Phi_{\zeta}^{*}(z,\sigma,a)=\sum_{n=0}^{\infty}\frac{(\zeta)_n}{n!}\frac{z^n}{(n+a)^{\sigma}}, \tag{1.3}$$

$$(\zeta\in\mathbb{C}, a\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ s\in\mathbb{C}, \text{when } |z|<1, \Re(\sigma-\zeta)>1 \text{ when } |z|=1).$$

$$\Phi_{\zeta}^{*}(z,\sigma,a)=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}\frac{t^{\sigma-1}e^{-at}}{(1-ze^{-t})^{\zeta}}dt=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}\frac{t^{\sigma-1}e^{-(a-\zeta)t}}{(e^t-z)^{\zeta}}dt \tag{1.4}$$

$$(\Re(\sigma)>0, \Re(a)>0, \text{when } |z|\leq 1\, (z\neq 1);\Re(\sigma)>1 \text{ when } z=1).$$

The extension of (1.3) and its integral representation are given in [12]:

$$\Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a)=\sum_{n=0}^{\infty}\frac{(\varrho)_n(\varsigma)_n}{(\zeta)_n n!}\frac{z^n}{(n+a)^{\sigma}}, \tag{1.5}$$

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$$(\varrho,\, \varsigma,\, \zeta \in \mathbb{C}\,, a \in \mathbb{C}\setminus\mathbb{Z}_0^-\,,\ s\in\mathbb{C}\,, \text{when}\,|z|<1\,, \Re(\sigma+\zeta-\varrho-\varsigma)>1\,\text{when}\,|z|=1).$$

$$\Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a)=\frac{1}{\Gamma(\sigma)}\int_0^\infty t^{\sigma-1}e^{-at}\,{}_2F_1\Big(\varrho,\varsigma;\zeta;ze^{-t}\Big)dt \tag{1.6}$$

$$(\Re(\sigma)>0\,, \Re(a)>0\,, \text{when}\,|z|\leq 1\,(z\neq 1)\,; \Re(\sigma)>1\,\text{when}\,z=1).$$

In [21], more generalization and integral representation are given in the following form,

$$\Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a;p)=\sum_{n=0}^{\infty}\frac{(\varrho)_n B(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}, \tag{1.7}$$

where $(p\geq 0\,, \varrho,\, \varsigma,\, \zeta\in\mathbb{C}\,, a\in\mathbb{C}\setminus\mathbb{Z}_0^-\,,\ s\in\mathbb{C}$ when $|z|<1$, $\Re(\sigma+\zeta-\varrho-\varsigma)>1$, when $|z|=1)$ and

$$\Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a)=\frac{1}{\Gamma(\sigma)}\int_0^\infty t^{\sigma-1}e^{-at}F_p\Big(\varrho,\varsigma;\zeta;ze^{-t}\Big)dt \tag{1.8}$$

where $p\geq 0\,, \Re(\sigma)>0\,\Re(a)>0\,, \text{when}|z|\leq 1\,(z\neq 1)\,; \Re(\sigma)>1$ when $z=1$, and F_p is the extended hypergeometric function (see [5]).

Now, we recall the following:

The extended beta function $B(\delta_1,\delta_2;p)$ due to [4] is

$$B(\delta_1,\delta_2;p)=B_p(\delta_1,\delta_2)=\int_0^1 t^{\delta_1-1}(1-t)^{\delta_2-1}e^{-\frac{p}{t(1-t)}}\,dt, \tag{1.9}$$

where $\Re(p)>0, \Re(x)>0, \Re(y)>0$ respectively. When $p=0$, then $B(\delta_1,\delta_2;0)=B(\delta_1,\delta_2)$.

In [23] Parmar et al. defined extended beta function as:

$$B_v(\delta_1,\delta_2;p)=B(\delta_1,\delta_2;p,v)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{\delta_1-\frac{3}{2}}(1-t)^{\delta_2-\frac{3}{2}}K_{v+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big)dt, \tag{1.10}$$

where $\Re(x)>0$, $\Re(y)>0$ and $K_{v+\frac{1}{2}}\Big(.\Big)$ is modified Bessel function of order v. Obviously, when $v=0$ then $B_0(\delta_1,\delta_2;p)=B(\delta_1,\delta_2;p)$ is the extended beta function (see[4]). They also defined the extended hypergeometric function and its integral representation as

$$\begin{aligned}F_v(\sigma_1,\sigma_2;\sigma_3;z;p)={}_2F_1\Big(\sigma_1,\sigma_2;\sigma_3;z;p,v\Big)&=\sum_{n=0}^{\infty}(\sigma_1)_n\frac{B_v(\sigma_2+n,\sigma_3-\sigma_2;p)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}(\sigma_1)_n\frac{B(\sigma_2+n,\sigma_3-\sigma_2;p,v)}{B(\sigma_2,\sigma_3-\sigma_2)}\frac{z^n}{n!}\end{aligned} \tag{1.11}$$

where $p,v\geq 0$, $\sigma_1,\sigma_2,\sigma_3\in\mathbb{C}$ and $|z|<1$.

The use of (1.10) found in [8, 9, 16]. Very recently, Gauhar et al. [15], introduced a new extension of Hurwitz-Lerch Zeta function which is defined by

$$\begin{aligned}\Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a;p,\lambda)&=\Phi^{\lambda}_{\varrho,\varsigma;\zeta}(z,\sigma,a;p)\\&=\sum_{n=0}^{\infty}\frac{(\varrho)_n B(\varsigma+n,\zeta-\varsigma;p,\lambda)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\\&=\sum_{n=0}^{\infty}\frac{(\varrho)_n B^{\lambda}_p(\varsigma+n,\zeta-\varsigma)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\end{aligned} \tag{1.12}$$

$p\geq 0\,, \lambda>0\,, \varrho,\, \varsigma,\, \zeta\in\mathbb{C}\,, a\in\mathbb{C}\setminus\mathbb{Z}_0^-\,, \sigma\in\mathbb{C}$ when $|z|<1$, $\Re(\sigma+\zeta-\varrho-\varsigma)>1$ when $|z|=1$ where $B^{\lambda}_p(\delta_1,\delta_2)$ is an extended beta function defined in [26] by

$$B^{\lambda}_p(\delta_1,\delta_2)=B(\delta_1,\delta_2;p,\lambda)=\int_0^1 t^{\delta_1-1}(1-t)^{\delta_2-1}E_\lambda\Big(-\frac{p}{t(1-t)}\Big)dt, \tag{1.13}$$

where $\Re(x) > 0$, $\Re(y) > 0$ and $E_\lambda\Big(.\Big)$ is Mittag-Leffler function defined by

$$E_\lambda\Big(z\Big) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)}. \tag{1.14}$$

For various extensions and generalizations of special functions the readers are refer to the recent work (see e.g., [1, 2, 6, 14, 17, 19, 20, 22]).

2. (p, v)-Extension of Hurwitz-Lerch Zeta function and its properties

This section introduce a new extension in terms of extended beta function (1.10) called the (p, v)-Extension of Hurwitz-Lerch Zeta function and is defined by

$$\begin{aligned}\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p) &= \Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a;p,v)\\ &= \sum_{n=0}^{\infty} \frac{(\varrho)_n B(\varsigma+n,\zeta-\varsigma;p,v)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\\ &= \sum_{n=0}^{\infty} \frac{(\varrho)_n B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\end{aligned} \tag{2.1}$$

$p, v \geq 0$, $\varrho, \varsigma, \zeta \in \mathbb{C}, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, $\Re(\sigma+\zeta-\varrho-\varsigma) > 1$ when $|z| = 1$.

Remark 2.1. *The following special cases can be derived from* (2.1)*.*

Case 1. If we consider $\varrho = 1$, then (2.1) will lead to another extension of generalized Hurwitz-Lerch Zeta function $\Phi^{1,1}_{v,\varsigma;\zeta}(z,\sigma,a;p)$ introduced earlier in [18] with $\zeta = \sigma = 1$.

$$\begin{aligned}\Phi^{1,1}_{v,\varsigma;\zeta}(z,\sigma,a;p) &= \Phi_{v,1,\varsigma;\zeta}(z,\sigma,a;p)\\ &= \sum_{n=0}^{\infty} \frac{B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\end{aligned} \tag{2.2}$$

$p, v \geq 0$, $\varsigma, \zeta \in \mathbb{C}$, $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, $\Re(\sigma+\zeta-\varrho-\varsigma) > 1$ when $|z| = 1$.

Case 2. If we set $\varrho = \zeta = 1$ in (2.1), then we get a new extension of Hurwitz-Lerch Zeta function $\Phi^*_{v,\varsigma}(z,\sigma,a;p)$ which is the extension defined in [13] as:

$$\begin{aligned}\Phi^*_{v,\varsigma}(z,\sigma,a;p) &= \Phi_{v,1,\varsigma;1}(z,\sigma,a;p)\\ &= \sum_{n=0}^{\infty} \frac{B_v(\varsigma+n,1-\varsigma;p)}{B(\varsigma,1-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\end{aligned} \tag{2.3}$$

$p, v \geq 0$, $\varsigma \in \mathbb{C}, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, $\Re(\sigma+\zeta-\varrho-\varsigma) > 1$ when $|z| = 1$.

Case 3. A limit case of extended Hurwitz-Lerch Zeta function $\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)$ defined by (2.1) is given by

$$\begin{aligned}\Phi^*_{v,\varsigma;\zeta}(z,\sigma,a;p) &= \lim_{|\varrho|\to\infty}\Big\{\Phi_{v,\varrho,\varsigma;\zeta}\Big(\frac{z}{\varrho},\sigma,a;p\Big)\Big\}\\ &= \sum_{n=0}^{\infty} \frac{B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\end{aligned} \tag{2.4}$$

$p, v \geq 0$, $\varsigma \in \mathbb{C}, a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, $\Re(\sigma+\zeta-\varrho-\varsigma) > 1$ when $|z| = 1$.

Remark 2.2. *It is clear that, if $v = 0$ then* (2.1) *will reduces* (1.7)

$$\begin{aligned}\Phi_{0,\varrho,\varsigma;\zeta}(z,\sigma,a;p) &= \Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a;p)\\ &= \sum_{n=0}^{\infty} \frac{(\varrho)_n B(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)n!}\frac{z^n}{(n+a)^\sigma}\end{aligned} \tag{2.5}$$

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$p, v \geq 0,\ \varrho, \varsigma, \zeta \in \mathbb{C}, a \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C}$ *when* $|z| < 1$, $\Re(\sigma + \zeta - \varrho - \varsigma) > 1$ *when* $|z| = 1$. *Similarly for* $p = 0$, (2.5) *converts to* (1.5) *and for* $\varrho = \zeta = 1$ *and* $p = 0$, (2.5) *turns to* (1.3). *Taking* $\varrho = \varsigma = \zeta = 1$ *and* $p = 0$ *in* (2.5) *results* (1.1).

3. Integral representations and derivative formulas

In this section, we define various integral representations and a derivative formula for (2.1).

Theorem 3.1. *The following integral representation holds true:*

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-at} F_v\Big(\varrho, \varsigma; \zeta; ze^{-t}; p\Big) dt \tag{3.1}$$

$$(p \geq 0, \lambda > 0, \varsigma \in \mathbb{C}, a \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C} \text{ when } |z| < 1,\ \Re(\sigma + \zeta - \varrho - \varsigma) > 1 \text{ when } |z| = 1).$$

Proof. Consider the Eulerian integral

$$\frac{1}{(n+a)^\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-(n+a)t} dt \tag{3.2}$$

where $\min\{\Re(\sigma), \Re(a)\} > 0; n \in \mathbb{N}_0$ in (2.1), we have

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p) = \sum_{n=0}^{\infty} \frac{(\varrho)_n B_v(\varsigma + n, \zeta - \varsigma; p)}{B(\varsigma, \zeta - \varsigma) n!} \Big(\frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-(n+a)t} dt\Big). \tag{3.3}$$

Interchanging the order of summation and integration under the suitable conditions given in 3.1, we have

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p, \lambda) = \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1} e^{-(n+a)t} \Big(\sum_{n=0}^{\infty} \frac{(\varrho)_n B_v(\varsigma + n, \zeta - \varsigma; p)}{B(\varsigma, \zeta - \varsigma) n!}\Big) dt,$$

by using (1.11) we get the desired result. □

Theorem 3.2. *The following formula holds true:*

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p) = \frac{1}{B(\varsigma, \zeta - \varsigma)} \int_0^\infty \frac{x^{\varsigma-1}}{(1+x)^\varsigma} K_{v+\frac{1}{2}}\Big(p(2 + x + \frac{1}{x})\Big) \Phi_\varrho^*\Big(\frac{zx}{1+x}, \sigma, a\Big) dx \tag{3.4}$$

$$(p \geq 0,\ \lambda > 0, \Re(\zeta) > \Re(\varsigma) > 0)$$

and

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p) = \frac{1}{B(\varsigma, \zeta - \varsigma)} \int_0^\infty \frac{t^{\sigma-1} e^{-at} x^{\varsigma-1}}{(1+x)^\varsigma} K_{v+\frac{1}{2}}\Big(p(2 + x + \frac{1}{x})\Big) \Big(1 - \frac{zxe^{-t}}{1+x}\Big)^{-\varrho} dx \tag{3.5}$$

$$(p \geq 0,\ \lambda > 0, \Re(\zeta) > \Re(\varsigma) > 0, \min\{\Re(\sigma), \Re(a)\} > 0),$$

provided that both the integrals converges.

Proof. Consider the following (see [21])

$$B_v(\sigma_1, \sigma_2; p) = \int_0^\infty \frac{x^{\sigma_1-1}}{(1+x)^{\sigma_1+\sigma_2}} K_{v+\frac{1}{2}}\Big(p(2 + x + \frac{1}{x})\Big) dx,\ \Re(p) > 0. \tag{3.6}$$

Substituting $\sigma_1 = \varsigma + n$ and $\sigma_2 = \zeta - \varsigma$ in (3.6), we have

$$B_v(\varsigma + n, \zeta - \varsigma; p) = \int_0^\infty \frac{x^{\varsigma+n-1}}{(1+x)^{\zeta+n}} K_{v+\frac{1}{2}}\Big(p(2 + x + \frac{1}{x})\Big) dx,\ \Re(p) > 0. \tag{3.7}$$

Using (3.7) in (2.1), we have

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p) = \frac{1}{B(\varsigma, \zeta - \varsigma)} \int_0^\infty \frac{x^{\varsigma-1}}{(1+x)^\varsigma} K_{v+\frac{1}{2}}\Big(p(2 + x + \frac{1}{x})\Big) \Big(\sum_{n=0}^{\infty} \frac{(\varrho)_n}{n!} \frac{(xz)^n}{(1+x)^n (n+a)^\sigma}\Big) dx. \tag{3.8}$$

By using (1.3), we get the required result (3.4).
Now from (3.1) and (3.8), we have

$$\Phi_{v,\varrho,\varsigma;\zeta}(z, \sigma, a; p)$$

$$=\frac{1}{\Gamma(\sigma)}\int_0^\infty t^{\sigma-1}e^{-(n+a)t}\Big(\sum_{n=0}^\infty \frac{(\varrho)_n B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)n!}\Big)dt \tag{3.9}$$

$$=\frac{1}{\Gamma(\sigma)B(\varsigma,\zeta-\varsigma)}\int_0^\infty\int_0^\infty \frac{x^{\varsigma-1}t^{\sigma-1}e^{-at}}{(1+x)^\zeta}K_{v+\frac{1}{2}}\Big(p(2+x+\frac{1}{x})\Big)\Big(\sum_{n=0}^\infty \frac{(\varrho)_n}{n!}\frac{(xze^{-t})^n}{(1+x)^n(n+a)^\sigma}\Big)dtdx. \tag{3.10}$$

By using the binomial expansion

$$(1-zt)^{-\alpha}=\sum_{n=0}^\infty \frac{(\alpha)_n(zt)^n}{n!},$$

we get the desired result. □

Theorem 3.3. *The following formula holds true:*

$$\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)=\frac{1}{\Gamma(\varrho)}\int_0^\infty t^{\varrho-1}e^{-t}\Phi^*_{\varsigma;\zeta}(zt,\sigma,a;p)dt \tag{3.11}$$

($p\geq 0$, $\lambda>0$, $\Re(\varrho)>0$, $\Re(a)>0$, when $|z|\leq 1 (z\neq 1)$; $\Re(\sigma)>1$, when $z=1$), where $\Phi^*_{v,\varsigma;\zeta}(zt,\sigma,a;p)$ is the limiting case defined by (2.4).

Proof. Using the integral representation of Pochhammer symbol $(\varrho)_n$ given in [25]:

$$(\varrho)_n=\frac{1}{\Gamma(\varrho)}\int_0^\infty t^{\varrho+n-1}e^{-t}dt \tag{3.12}$$

in (2.1) and interchanging the order of summation and integration under the suitable convergence conditions, we get

$$\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p,\lambda)=\frac{1}{\Gamma(\varrho)}\int_0^\infty t^{\varrho-1}e^{-t}\sum_{n=0}^\infty \frac{B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{(zt)^n}{n!(n+a)^\sigma}dt. \tag{3.13}$$

Now, by using (2.4), we get the desired proof. □

Next, we derive the derivative formula of (2.1).

Theorem 3.4. *The following formula holds true*

$$\frac{d^n}{dz^n}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)\}=\frac{(\varrho)_n(\varsigma)_n}{(\zeta)_n}\Phi_{v,\varrho+n,\varsigma+n;\zeta+n}(z,\sigma,a+n;p),\ (n\in\mathbb{N}_0). \tag{3.14}$$

Proof. Differentiation of (2.1) with respect to z yields

$$\frac{d}{dz}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)\}=\sum_{n=1}^\infty \frac{(\varrho)_n}{(n-1)!}\frac{B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^{n-1}}{(n+a)^\sigma}. \tag{3.15}$$

Replacing n by $n+1$ and applying the identities

$$B(b,c-b)=\frac{c}{b}B(b+1,c-b)\ \textit{and}\ (a)_{n+1}=a(a+1)_n$$

in the r.h.s of (3.15), we get

$$\frac{d}{dz}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)\}=\frac{\varrho\varsigma}{\zeta}\Phi_{v,\varrho+1,\varsigma+1;\zeta+1}(z,\sigma,a+1;p). \tag{3.16}$$

Again, differentiating (3.16) with respect to z, we obtain

$$\frac{d^2}{dz^2}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)\}=\frac{\varrho(\varrho+1)\varsigma(\varsigma+1)}{\zeta(\zeta+1)}\Phi_{v,\varrho+2,\varsigma+2;\zeta+2}(z,\sigma,a+2;p). \tag{3.17}$$

Continuing up to n-times gives the required proof. □

4. Mellin Transformation and generating relations

In this section, we define Mellin transformation and some generating relations for (2.1). Here, we recall the following:

$$\mathfrak{M}\{f(t);t\to r\}=\int_0^\infty t^{r-1}f(t)dt. \tag{4.1}$$

Theorem 4.1. *The Mellin transform of* (2.1) *is given by*

$$\mathfrak{M}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p);p\to r\}=\frac{2^{r-1}}{\sqrt{\pi}}\Gamma(\frac{r-v}{2})\Gamma(\frac{r+v+1}{2})\frac{B(\varsigma+r,\zeta-\varsigma+r)}{B(\varsigma,\zeta-\varsigma)}\Phi_{\varrho,\varsigma+r;\zeta+2r}(z,\sigma,a) \tag{4.2}$$

$$(\Re(r)>0 \textit{ and } \Re(\varsigma+r)>0).$$

Proof. The Mellin Transformation of (2.1) is given by:

$$\begin{aligned}\mathfrak{M}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p);p\to r\} &= \int_0^\infty p^{r-1}\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p)dp\\ &= \int_0^\infty p^{r-1}\Big(\sum_{n=0}^\infty\frac{(\varrho)_n}{n!}\frac{B_v(\varsigma+n,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^n}{(n+a)^\sigma}\Big)dp\\ &= \frac{1}{B(\varsigma,\zeta-\varsigma)}\sum_{n=0}^\infty\frac{(\varrho)_n}{n!}\frac{z^n}{(n+a)^\sigma}\\ &\times \int_0^\infty p^{r-1}B_v(\varsigma+n,\zeta-\varsigma;p)dp \end{aligned} \tag{4.3}$$

Applying the result (see [23]),

$$\int_0^\infty p^{r-1}B_v(\delta_1,\delta_2;p)dp=\frac{2^{r-1}}{\sqrt{\pi}}\Gamma(\frac{r-v}{2})\Gamma(\frac{r+v+1}{2})B(\delta_1+r,\delta_2+r), \tag{4.4}$$

we get

$$\begin{aligned}&\mathfrak{M}\{\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a;p);p\to r\}\\ &=\frac{2^{r-1}}{\sqrt{\pi}}\Gamma(\frac{r-v}{2})\Gamma(\frac{r+v+1}{2})\sum_{n=0}^\infty\frac{(\varrho)_n}{n!}\frac{z^n}{(n+a)^\sigma}\frac{B(\varsigma+n+r,\zeta+r-\varsigma)}{B(\varsigma,\zeta-\varsigma)}\\ &=\frac{2^{r-1}}{\sqrt{\pi}}\Gamma(\frac{r-v}{2})\Gamma(\frac{r+v+1}{2})\frac{B(\varsigma+r,\zeta-\varsigma+r)}{B(\varsigma,\zeta-\varsigma)}\sum_{n=0}^\infty\frac{(\varrho)_n(\varsigma+r)_n}{(\zeta+2r)_n}\frac{z^n}{n!(n+a)^\sigma}\end{aligned} \tag{4.5}$$

which by using (1.5) yields the required proof. □

Remark 4.1. *The special case of* (2.1) *for* $r=1$ *gives the following integral representation*

$$\int_0^\infty \Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a;p,\lambda)dp=\frac{1}{\sqrt{\pi}}\Gamma(\frac{1-v}{2})\Gamma(\frac{v+2}{2})\frac{B(\varsigma+1,\zeta-\varsigma+1)}{B(\varsigma,\zeta-\varsigma)}\Phi_{\varrho,\varsigma+1;\zeta+2}(z,\sigma,a) \tag{4.6}$$

Corollary 4.1. *The following Mellin transform of the extended Hurwitz-Lerch Zeta function is given in (see e.g., [21]) by*

$$\mathfrak{M}\{\Phi_{\varrho,\varsigma;\zeta}(z,\sigma,a;p);p\to r\}=\frac{\Gamma(r)B(\varsigma+r,\zeta-\varsigma+r)}{B(\varsigma,\zeta-\varsigma)}\Phi_{\varrho,\varsigma+r;\zeta+2r}(z,\sigma,a) \tag{4.7}$$

$$(\Re(r)>0 \textit{ and } \Re(\varsigma+r)>0).$$

Theorem 4.2. *The following generating function for* (2.1) *holds true:*

$$\sum_{n=0}^\infty(\varrho)_n\Phi_{v,\varrho+n,\varsigma;\zeta}(z,\sigma,a;p)\frac{t^n}{n!}=(1-t)^{-\varrho}\Phi_{v,\varrho+n,\varsigma;\zeta}(\frac{z}{1-t},\sigma,a;p) \tag{4.8}$$

$$(p,v\geq 0\ \varrho,\varsigma,\zeta\in\mathbb{C}\ and\ |t|<1).$$

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Proof. Let the left hand side of (4.8) be denoted by S_1, the from (2.1), we have

$$S_1=\sum_{n=0}^{\infty}(\varrho)_n\Big\{\sum_{k=0}^{\infty}(\varrho+n)_k\frac{B_v(\varsigma+k,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^k}{k!(k+a)^\sigma}\Big\}\frac{t^n}{n!} \tag{4.9}$$

Interchanging the order of summations and using the identity $(\varrho)_n(\varrho+n)_k=(\varrho)_k(\varrho+k)_n$ in (4.9), we have

$$S_1=\sum_{k=0}^{\infty}(\varrho)_k\frac{B_v(\varsigma+k,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\Big\{\sum_{n=0}^{\infty}(\varrho+k)_n\frac{t^n}{n!}\Big\}\frac{z^k}{k!(k+a)^\sigma} \tag{4.10}$$

Now, using the following binomial expansion

$$(1-t)^{-\varrho-k}=\sum_{n=0}^{\infty}(\varrho+k)_n\frac{t^n}{n!}\ (|t|<1),$$

and interpreting in term of (2.1) as a function of the form $\Phi_{v,\varrho+n,\varsigma;\zeta}(\frac{z}{1-t},\sigma,a;p)$, which completes the proof of Theorem (4.2). □

Theorem 4.3. *The following generating function for* (2.1) *holds true:*

$$\sum_{n=0}^{\infty}\frac{(\sigma)_n}{n!}\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma+n,a;p)t^n=\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a-t;p) \tag{4.11}$$

$$(p,v\geq 0,\ \ \varrho,\varsigma,\zeta\in\mathbb{C}\ \textit{and}\ |t|<a;\ \ \sigma\neq 1).$$

Proof. Applying (2.1) to the r.h.s of (4.11), we have

$$\begin{aligned}\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a-t;p)&=\sum_{k=0}^{\infty}(\varrho)_k\frac{B_v(\varsigma+k,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^k}{k!(k+a-t)^\sigma}\\&=\sum_{k=0}^{\infty}(\varrho)_k\frac{B_v(\varsigma+k,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^k}{k!(k+a)^\sigma}\Big(1-\frac{t}{k+a}\Big)^{-\sigma}.\end{aligned} \tag{4.12}$$

Using the following binomial expansion

$$(1-t)^{-\varrho-k}=\sum_{n=0}^{\infty}(\varrho+k)_n\frac{t^n}{n!}\ (|t|<1),$$

in (4.12), we have

$$\begin{aligned}\Phi_{v,\varrho,\varsigma;\zeta}(z,\sigma,a-t;p)&=\sum_{k=0}^{\infty}(\varrho)_k\frac{B_v(\varsigma+k,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^k}{k!(k+a)^\sigma}\Big\{\sum_{n=0}^{\infty}\frac{(\sigma)_n}{n!}\frac{t^n}{(k+\sigma)^n}\\&=\sum_{n=0}^{\infty}\frac{(\sigma)_n}{n!}\Big\{\sum_{k=0}^{\infty}(\varrho)_k\frac{B_v(\varsigma+k,\zeta-\varsigma;p)}{B(\varsigma,\zeta-\varsigma)}\frac{z^k}{k!(k+a)^{\sigma+n}}\Big\}t^n\end{aligned} \tag{4.13}$$

By making the use of (2.1), we get the required result. □

5. Concluding Remark

In this paper, we established an extension of extended Hurwitz-Lerch Zeta function and its various properties. If we consider $v=0$, then the results established here will reduce to the results studied by Parmar et al. [21]. Silmilarly, if we consider $v=0$ and then set $p=0$, we then obtained the results of generalized Hurwitz-Lerch Zeta function defined by Garg et al. [12].

References

[1] M. Arshad, S. Mubeen, K. S. Nisar, G. Rahman, *Extended Wright-Bessel function and its properties*, Commun. Korean Math. Soc. 33 (2018), No. 1, 143-155.

[2] M. Arshad, J. Choi S. Mubeen, K. S. Nisar, G. Rahman, *A new extension of the Mittag-Leffler function*, Commun. Korean Math. Soc. 2018, (Accepted).

[3] M. A. Chaudhry, S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall, (CRC Press Company), Boca Raton, London, New York and Washington, D. C., 2001.

[4] M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair,*Extension of Eulers beta function*, J. Comput. Appl. Math. 78 (1997) 1932.

[5] M. A. Chaudhry, A. Qadir, H. M. Srivastava and R. B. Paris, *Extended Hypergeometric and Confluent Hypergeometric functions*, Appl. Math. Comput., 159 (2004) 589-602

[6] J. Choi, A. K. Rathie, R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Mathematical J. 36 (2014), No. 2, pp. 357-385.

[7] J. Choi, D. S. Jang, and H. M. Srivastava, *A generalization of the Hurwitz-Lerch Zeta function*, Integral Transforms Spec. Funct., 19, (2008), 65-79.

[8] S.A. Dar and R.B. Paris, *A (p,ν)-extension of the Appell function $F_1()$ and its properties*, arXiv:1711.07780v1 [math.CA].

[9] S.A. Dar and R.B. Paris, *A (p,ν)-extension of Srivastavas triple hypergeometric function H_B and its properties*, arXiv:1711.07809v1 [math.CA].

[10] A. Erdélyi,W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol.1, McGraw-Hill, NewYork, Toronto, London, 1953.

[11] A. Erdélyi,W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol.1, McGraw-Hill, NewYork, Toronto, London, 1955.

[12] M. Garg, K. Jain, and S. L. Kalla, *A further study of general Hurwitz-Lerch zeta function*, Algebras Groups Geom., 25, (2008), 311-319.

[13] S. P. Goyal and R. K. Laddha, *On the generalized Zeta function and the gener- alized Lambert function*, Ganita Sandesh, 11, (1997), 99-108.

[14] G. Rahman, K. S. Nisar, S. Mubeen, *New generalization of extended beta and hypergeometric functions*, Preprints 2018, 2018020036 (doi: 10.20944/preprints201802.0036.v1).

[15] G. Rahman, K. S. Nisar, J. Choi, *An extension of the Whittaker function*, arXiv:1801.08032 [math.CA].

[16] G. Rahman, K. S. Nisar, M. Arshad, *A new extension of Hurwitz-Lerch Zeta function*, arXiv:1802.07823 [math.CA].

[17] S. Mubeen, G. Rahman, K. S. Nisar, J. Choi. M. Arshad, *An extended beta function and its properties*, Far East Journal of Mathematical Sciences, 102(2017), 1545-1557.

[18] S. D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, Applied Mathematics and Computation, 154, (2004), 725-733

[19] E. Özergin, M. A. Özarslan, Abdullah Altn, *Extension of gamma, beta and hypergeometric functions*, J. Comput. Appl. Math., 235 (2011), 4601-4610.

[20] M. A. Özarslan, B. Ylmaz, *The extended Mittag-Leffler function and its properties*, Journal of Inequalities and Applications, 2014:85, (2014).

[21] R. K. Parmar, *On a certain extension of Hurwitz-Lerch Zeta function*, Analele Universitäii de Vest, Timioara. Seria matematica-informatică, LII, 2(2014), 157-170.

[22] R. K. Parmar, *A new generalization of gamma, beta, hypergeometric and confluent hypergeometric functions*, LE MATEMATICHE Vol. LXVIII (2013) Fasc. II, pp. 3352

[23] R. K. Parmar, P. Chopra, R. B. Paris, *On an Extension of Extended Beta and Hypergeometric Functions*, J. Classical Anal. 11 (2017) 91106. [arXiv:1502.06200

[24] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer, Acedemic Publishers, Dordrecht, Boston and London, 2001.

[25] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science, Publishers, Amsterdam, London and New York, 2012.

[26] M. Shadab, S. Jabee and J. Choi, *An extension of beta function and its application*, Far East Journal of Mathematical Sciences, 103(2018), No. 1, pp 235-251.