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Article

ON THE EXPANSION OF SPACE AND HOW TO MEASURE IT

Firmin J. Oliveira [1,†]

1 East Asian Observatory / James Clerk Maxwell Telescope; outreach@eaobservatory.org or firmjay@hotmail.com

† Current address: P. O. Box 10882, Hilo, Hawaii 96721

Abstract: We describe the effect of the expansion of space on the wavelength of the light beam in a Fabry-Pérot interferometer. For an instrument such as the Laser Interferometer Gravitational-Wave Observatory (LIGO), which has high sensitivity and a long period of light storage, the wavelength λ_L of laser photons are redshifted due to the expansion of space in each cavity by an amount $\delta\lambda$ given by $\delta\lambda/\lambda_L = H_0\tau_s \approx 8.8 \times 10^{-21}$, where $H_0 \approx 2.2 \times 10^{-18}\,\mathrm{s}^{-1}$ is the Hubble constant and $\tau_s \approx 4\,\mathrm{ms}$ is the light storage time for the cavity. Since τ_s is based on the cavity *finesse* $\mathcal{F}$ which depends on the laser beam full width at half maximum (FWHM) $\delta\omega$ of each cavity, we show that a difference in finesses between the LIGO arm cavities produces a signal $h_H(t)$ at the anti-symmetric output port given by $h_H(t) = 2a_1H_0\left(\frac{1}{\delta\omega_X(t)} - \frac{1}{\delta\omega_Y(t)}\right)$, where $\delta\omega_X(t)$ and $\delta\omega_Y(t)$ are the beam FWHM at time t, respectively, for the X and Y arm cavities and a_1 is a beam proportionality constant to be determined expermentally. Assuming $a_1 \approx 1$, then for cavity beams FWHM of $\delta\omega(t) \approx (523.2 \pm 31)\,\mathrm{rad.\,s}^{-1}$ the output signal has the range $|\,h_H(t)\,| \leq 1 \times 10^{-21}$, which is detectable by advanced LIGO.

Keywords: universe expansion; Hubble constant; cavity finesse; cosmological redshift; strain

1. Introduction

The Hubble expansion of the universe[1] should be observable in a local frame via the expansion of space. Although gravity in the Galaxy is strong enough that it keeps all massive objects bounded, and material objects are held together by gravitational and electromagnetic forces, photons have a vacuum speed $c \approx 3 \times 10^{10}\,\mathrm{cm\,s}^{-1}$ which is greater than the escape velocity for the Galaxy ($v_e = \sqrt{2GM/R} \approx \sqrt{2G \times 1.5 \times 10^{12}\,\mathrm{M}_\odot / 8\,\mathrm{kpc}} \approx 1.3 \times 10^8\,\mathrm{cm\,s}^{-1}$). Therefore it is reasonable to inquire whether the wavelength of photons moving freely in a vacuum anywhere in the Galaxy will be redshifted by the expansion of space. Because General Relativity (GR) is based on fields, all phenomena are determined by local effects alone. Thus the expansion of the universe, a global effect, is described by the local expansion of space at each point in the universe. In the Friedman-Lemaître-Robertson-Walker (FLRW)[2–5] cosmological model, the isotropic expansion of space is measured by the scale factor R which is a solution to the Friedman equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3}, \tag{1}$$

where $\dot{R} = dR/dt$ is the time derivative of R, G is Newton's gravitation constant, ρ is the local matter density, $k = \{-1, 0, 1\}$ is the curvature constant and Λ is the cosmological constant. In an interferometer cavity, such as in the Laser Interferometer Gravitational-Wave Observatory (LIGO)[6], there is a high vacuum, so the matter density composed of molecules and photons is assumed negligible, implying $\rho \approx 0$, and the curvature is assumed flat with $k = 0$. Then only the Λ term remains in (1) and its solution takes the form

$$\frac{R(t)}{R(t_e)} = e^{\sqrt{\Lambda/3}\,c(t-t_e)} = 1 + \sqrt{\Lambda/3}\,c\,(t-t_e) + (\Lambda/6)\,c^2\,(t-t_e)^2 + ..., \tag{2}$$

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where $t_e \leq t$, t_e is the time of emission of the light and t is the time of observation of the light. It can be shown[3] that the cosmological redshift z of the wavelength of light is given by (2), $1 + z = R(t)/R(t_e)$. For $t - t_e = \Delta t$ a small time interval, we drop all second order and higher terms from (2), and the expression for the cosmological redshift takes the form

$$z = \frac{R(t)}{R(t_e)} - 1 \approx \sqrt{\Lambda/3}\, c\, \Delta t. \tag{3}$$

It can be shown[7,8] that the Hubble constant H_0 is related to Λ by

$$H_0 \approx \sqrt{\Lambda/3}\, c. \tag{4}$$

From (3) and (4), we can expect a change in light wavelength $\delta\lambda$ given by

$$\frac{\delta\lambda}{\lambda_L} = z \approx \sqrt{\Lambda/3}\, c\, \Delta t \approx H_0 \Delta t, \tag{5}$$

where λ_L is the initial wavelength (laser nominal wavelength). From cosmological observations[9] we use a value of $H_0 \approx 67.8\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

2. How the expansion of space affects the light in an interferometer

The two perpendicular arms L_x and L_y of a LIGO facility[6] are each of length $L = 4\,\mathrm{km}$. At the ends of each arm are suspended test mirrors which are highly reflective, allowing photons from the input laser to be stored coherently in the cavity between the mirrors. The system has a *finesse*[10] of $\mathcal{F} \approx 450$, implying an amplification of $A \approx 300$, equivalent to increasing the effective length of each arm to

$$L_e = A\,L \approx 1200\,\mathrm{km}. \tag{6}$$

The effective travel time τ_s of a photon traveling at vacuum speed c within an arm cavity, undergoing multiple reflections, ignoring latencies at the reflection surface of each test mirror, is given by

$$\tau_s = \frac{L_e}{c} \approx 4\,\mathrm{ms}. \tag{7}$$

If the cavity field is not replenished, τ_s is the time for the field intensity to be attenuated by $1/e$. Substituting $\Delta t = \tau_s$ from (7) into (5) we obtain the maximum wavelength change $\delta\lambda_{max}$ given by

$$\frac{\delta\lambda_{max}}{\lambda_L} = H_0\,\tau_s \approx \left(2.2 \times 10^{-18}\,\mathrm{s^{-1}}\right)(0.004\,\mathrm{s}) \approx 8.8 \times 10^{-21}. \tag{8}$$

When a gravitational wave interacts with the LIGO instrument the strain h of the wave is related to the change in LIGO arm lengths L_x and L_y given by[6]

$$\delta L\,(t) = \delta L_x\,(t) - \delta L_y\,(t) = h\,(t)\;L. \tag{9}$$

From the measured strain of the GW150914 event, (9) implies that

$$\mid \frac{\delta L\,(t)}{L} = h\,(t) \mid \leq 1.0 \times 10^{-21}. \tag{10}$$

However, from (8), in any 4 ms time interval the cavity photons will have a range of drift given by $0 \leq \delta\lambda/\lambda_L \leq 8.8 \times 10^{-21}$ with an average drift of $\approx 4.4 \times 10^{-21}$. The average drift in phase of cavity photons is more than 4 times larger than the reported strain associated with the GW150914 event. Just prior to the gravitational wave interaction with the LIGO instrument, the photons stored in the

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arm cavities will be out of phase with the detector reference frequency $\omega_L = 2\pi c/\lambda_L$, where λ_L is the nominal laser wavelength. From (5) we get, for $0 \leq t \leq \tau_s$,

$$\omega\left(t\right) = \omega_L\left(1 - H_0\, t\right), \tag{11}$$

which is a linear sweep in the cavity frequency each time period τ_s. When a gravitational wave modulates the instrument, the measurement of the disturbance at the output detector will have both the cavity and wave signals intermixed.

3. Condition for resonance in a Fabry-Pérot cavity

From (Ref. [11], Eq. 9 and Eq. 10), the equation for the dynamics in a Fabry-Pérot cavity is given by

$$E\left(t\right) = t_a E_{in}\left(t\right) + r_a r_b\, e^{-2ik\delta L(t)} E\left(t - 2T\right), \tag{12}$$

where t is the time, t_a is the transmissivity of the input test mirror **a**, r_a is the reflectivity of input test mirror **a**, r_b is the reflectivity of end test mirror **b**, $E_{in}(t)$ is the input field amplitude, $E(t)$ is the cavity field amplitude, $i = \sqrt{-1}$, $k = \omega/c$ is the wave number where ω is the nominal laser frequency, $\delta L(t)$ is the variation in the cavity length and

$$T = L/c \tag{13}$$

is the time for light to travel once along the nominal distance L between the cavity test mirrors **a** and **b**. Assuming $\delta L(t) = 0$ and Laplace transforming (12) yields the basic cavity response function

$$H(s) = \frac{\bar{E}\left(s\right)}{\bar{E}_{in}\left(s\right)} = \frac{t_a}{1 - r_a r_b\, e^{-2sT}}, \tag{14}$$

where over-bars denote Laplace transforms. If we consider the effect of the universe expansion upon the frequency of the laser light in the cavity, for one round trip time $2T$, the shift is toward lower frequency given by

$$\delta\omega_H = -2H_0 T\omega. \tag{15}$$

Including this shift effect into the cavity frequency control system, which corrects the laser frequency to compensate for changes in the cavity such as movements of the mirrors , from (Ref. [11], Eq. 20), for one round trip of the light in the cavity,

$$\delta\omega\left(t\right) - \delta\omega\left(t - 2T\right) = -2\left(\frac{\omega}{c}\right)\frac{d}{dt}\delta L\left(t\right) - 2H_0 T\omega, \tag{16}$$

where $v(t) = d\,\delta L(t)/dt$ is the relative velocity of the mirrors. Use (13) to substitute for $c = L/T$ in (16) and transform it to the Laplace domain, yielding

$$C(s)\frac{\delta\bar{\omega}\left(s\right)}{\omega} = -\frac{\delta\bar{L}\left(s\right)}{L} - \frac{H_0}{s}, \tag{17}$$

where

$$C(s) = \frac{1 - e^{-2sT}}{2sT} \tag{18}$$

is the normalized frequency-to-length transfer function. The cavity *free spectral range*

$$\delta\omega_{fsr} = \frac{\pi c}{L} \tag{19}$$

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determines the spacing between zeros of $C(s)$. A control system maintains resonance in the cavity by adjusting the input frequency ω by $\delta\omega(t)$. If these frequency changes are much less than the cavity free spectral range then $C(s) \approx 1$ and (17) reduces to

$$\frac{\delta\bar{\omega}\,(s)}{\omega} \approx -\frac{\delta\bar{L}\,(s)}{L} - \frac{H_0}{s}. \tag{20}$$

4. A toy model for an output signal from LIGO due to the expansion of space

We introduce a toy model of the laser light in the LIGO cavities which can produce a signal at the anti-symmetric port due to phase differences between the LIGO arms caused by the control system as it compensates for changes in the cavity laser frequency due to mirror movements. The control system maintains cavity resonance by shifting the cavity frequency by $\delta\omega_{adj}(t)$ according to (20), neglecting the expansion term, at regular time intervals based on the cavity round trip time $2T = 2L/c$ and the storage time τ_s. By (11), the maximum frequency shift during the storage time due to the universe expansion is

$$\delta\omega_{Hmax} = \omega_L\,(1 - H_0\tau_s) - \omega_L = -\omega_L H_0 \tau_s \tag{21}$$

where

$$\tau_s = \frac{2T\mathcal{F}}{\pi}, \tag{22}$$

where the finesse is given by[12]

$$\mathcal{F} = \frac{\delta\omega_{\text{fsr}}}{\delta\omega} \approx \frac{\pi\,(r_a r_b)^{1/4}}{1 - \sqrt{r_a r_b}}. \tag{23}$$

Here the cavity *free spectral range* is given by (19) and $\delta\omega$ is the laser beam full width at half maximum (FWHM). Assume that the frequency variation $\delta\omega_{Hadj}$ in a cavity given by (21) creates a coherent sub-beam which is imbedded in the main beam, having a subbeam central frequency

$$\omega_H = \omega_L - \frac{|\ \delta\omega_{Hmax}\ |}{2} \tag{24}$$

and a FWHM beamwidth of

$$\delta\omega_H \approx a_1\ |\ \delta\omega_{Hmax}\ |, \tag{25}$$

where a_1 is a beam proportionality constant to be determined experimentally. Since the cavity laser power is constantly being maintained for resonance by adjustments $\delta\omega_{adj}(t)$ to the cavity input beam, the subbeam centered on ω_H is also indirectly being maintained by effective adjustments $\delta\omega_{Hadj}(t)$ as older photons exiting the mirrors are replenished by newer photons from the main beam.

The phase change associated with the cavity subbeam beamwidth $\delta\omega_H$, from (21), (22) and (25), for a period of one round trip time $2T$ in the cavity is expressed by

$$\Delta\phi_H = \delta\omega_H\,(2T) = a_1\,(\omega_L H_0 \tau_s)\left(\frac{2L}{c}\right) = a_1\omega_L H_0 \left(\frac{2L\mathcal{F}}{\pi c}\right)\left(\frac{2L}{c}\right). \tag{26}$$

Although both LIGO arms are locked simultaneously[13], it is reasonable to assume that each cavity is tweaked with a different frequency change $\delta\omega(t)$ to stabilize the cavity resonance. This changes the cavity beamwidth FWHM, which affects the cavity finesse (23), resulting in frequency shifts in the subbeam central frequency ω_H and the subbeam beamwidth $\delta\omega_H$ due to the expansion of space, of $\delta\omega_{HX}$ for arm X and $\delta\omega_{HY}$ for arm Y. For arm X, by (26), the phase change for the subbeam beamwidth $\delta\omega_{HX}(t)$ per round trip cycle time $2T$ expressed as a function of time t is given by

$$\Delta\phi_{HX}\,(t) = \delta\omega_{HX}\,(t)\,(2T) = a_1\omega_L H_0 \left(\frac{2L\mathcal{F}_X\,(t)}{\pi c}\right)\left(\frac{2L}{c}\right), \tag{27}$$

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where $\mathcal{F}_X(t)$ is the finesse of the arm X cavity at time t. Similarly, for arm Y, the phase change for the subbeam beamwidth $\delta\omega_{HY}(t)$ per round trip cycle is given by

$$\Delta\phi_{HY}(t) = \delta\omega_{HY}(t)(2T) = a_1\omega_L H_0 \left(\frac{2L\mathcal{F}_Y(t)}{\pi c}\right)\left(\frac{2L}{c}\right), \tag{28}$$

where $\mathcal{F}_Y(t)$ is the finesse of the arm Y cavity at time t. Taking the differential of the phase differences between arms X and Y by subtracting (28) from (27) we obtain

$$\Delta\phi_{HXY}(t) = \Delta\phi_{HX}(t) - \Delta\phi_{HY}(t) = a_1\omega_L H_0 \left(\frac{4L^2}{\pi c^2}\right)(\mathcal{F}_X(t) - \mathcal{F}_Y(t)). \tag{29}$$

From (19) and (23) we get the difference between the cavity finesses

$$\mathcal{F}_X(t) - \mathcal{F}_Y(t) = \frac{\pi c}{L}\left(\frac{1}{\delta\omega_X(t)} - \frac{1}{\delta\omega_Y(t)}\right). \tag{30}$$

Substituting (30) into (29) we obtain for the subbeam phase differential between the cavities

$$\Delta\phi_{HXY}(t) = \left(\frac{4a_1 L\omega_L H_0}{c}\right)\left(\frac{1}{\delta\omega_X(t)} - \frac{1}{\delta\omega_Y(t)}\right). \tag{31}$$

Dividing both sides of (31) by $(2L\omega_L/c)$ yields the output signal due to the expansion of space

$$h_H(t) = \frac{\Delta\phi_{HXY}(t)}{2L\omega_L/c} = (2a_1 H_0)\left(\frac{1}{\delta\omega_X(t)} - \frac{1}{\delta\omega_Y(t)}\right). \tag{32}$$

Since the LIGO anti-symmetric port is zero when the interferometer is in its resonant "locked" state, if the control system is not compensating for the frequency sweep due to the expansion of space then the subbeam induced signal (32) would be detectable at the anti-symmetric port.

Assume $a_1 \approx 1$. With $2H_0 \approx 4.40 \times 10^{-18}\text{s}^{-1}$ and for a signal range between $\pm 1 \times 10^{-21}$, this implies from (32) that

$$\left| \frac{1}{\delta\omega_X(t)} - \frac{1}{\delta\omega_Y(t)} \right| \leq 2.273 \times 10^{-4}\,\text{s rad}^{-1}. \tag{33}$$

LIGO's FWHM beamwidth can be estimated from (19), $\delta\omega = \delta\omega_{\text{fsr}}/\mathcal{F} \approx 2.355 \times 10^5/450 \approx 523.2\,\text{rad. s}^{-1}$. For example, with $\delta\omega_X(t1) \approx 554.2\,\text{rad. s}^{-1}$ and $\delta\omega_Y(t1) \approx 492.2\,\text{rad. s}^{-1}$, for some time $t1$, (33) is satisfied. In general, if the LIGO beamwidth FWHM varies as $\delta\omega \approx (523.2 \pm 31)\,\text{rad. s}^{-1}$ which is equivalent to $\delta\nu \approx (83.3 \pm 4.9)\,\text{Hz}$, a variation of about $\pm 6\,\%$, then the output signal $h_H(t)$ would vary within $\pm 1 \times 10^{-21}$. This range of variance for $\delta\omega$ is comparable to the design frequency noise limit of $\pm 5\,\%$ ($10\,\%$ maximum) for advanced LIGO[14].

5. Discussion

Since the control system maintains the cavity resonance by correcting the input laser frequency, the signal output given by (32) would be proportional to the control system *servoing* of the cavity frequency in each LIGO arm. This suggests that by servoing the laser beam frequency for each cavity in a known manner by $\delta\omega_{Xadj}(t)$ for the X cavity and $\delta\omega_{Yadj}(t)$ for the Y cavity, generating laser beamwidths $\delta\omega_X(t)$ and $\delta\omega_Y(t)$, respectively, a predictable signal $h_H(t)$ due to the expansion of space would be generated at the output port. In Figure 1 is shown the predicted output $h_H(t)$ when the beamwidths in the X and Y cavities are modulated at a rate of $f = 1/2\tau_s \approx 130.8\,\text{Hz}$ by the $\sin(x)$ function

$$\delta\omega(t) = \Delta\omega + 31\sin(\pi t/\tau_s + \phi_0), \tag{34}$$

where $\Delta\omega = 523.2\,\text{rad. s}^{-1}$, $\phi_0 = 0$ for the X arm beam and $\phi_0 = \pi$ for the Y arm beam.

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In Figure 2 is shown the predicted output $h_H(t)$ when the beamwidths in the X and Y cavities are modulated by the $\mathrm{sinc}(x)$ function at a rate of $f = 1/\tau_s \approx 261.6\,\mathrm{Hz}$ centered on time $t_p = 0.05\,\mathrm{s}$ given by

$$\delta\omega\,(t) = \Delta\omega + 60\,\mathrm{sinc}(2\pi\,(t - t_p)/\tau_s + \phi_0), \tag{35}$$

where $\phi_0 = 0$ for the X arm subbeam and $\phi_0 = \pi$ for the Y arm subbeam.

Whether or not the local expansion of space is real is uncertain. But an experimental facility such as LIGO can provide answer.

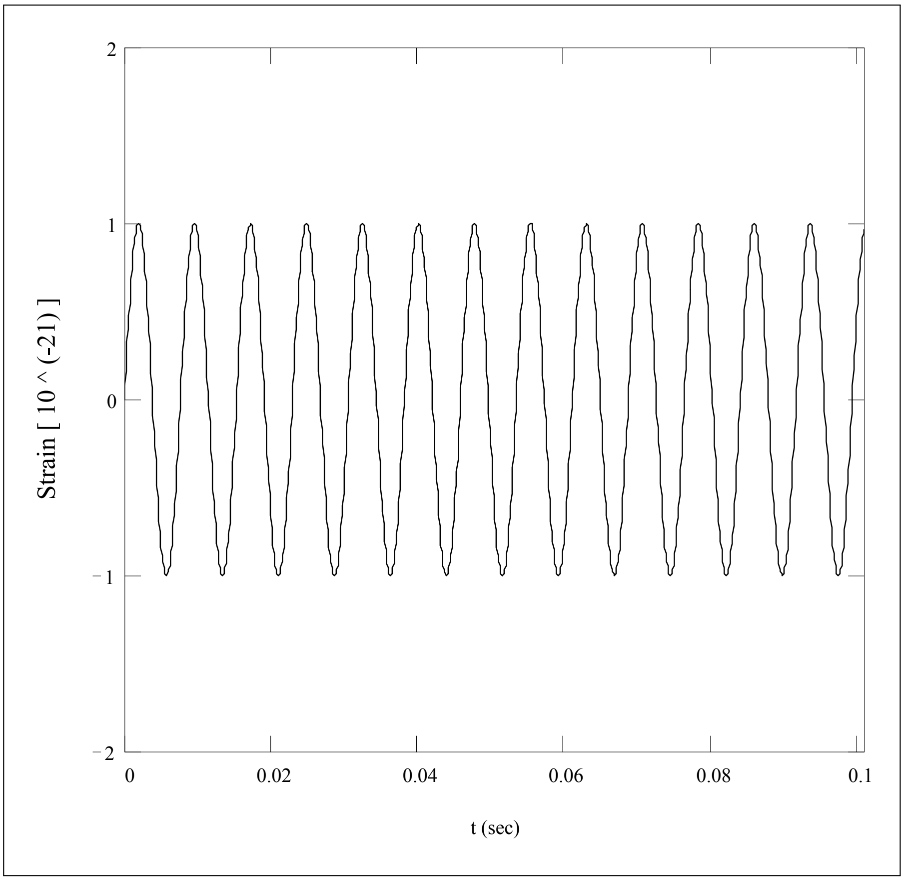

Figure 1. Predicted expansion of space signal output due to modulation of X and Y arms cavity beamwidths by $\delta\omega(t) = 523.2 + 31\sin(821.9\,t + \phi_0)$, where $\phi_0 = 3.142$ for the X beam and 0 for the Y beam.

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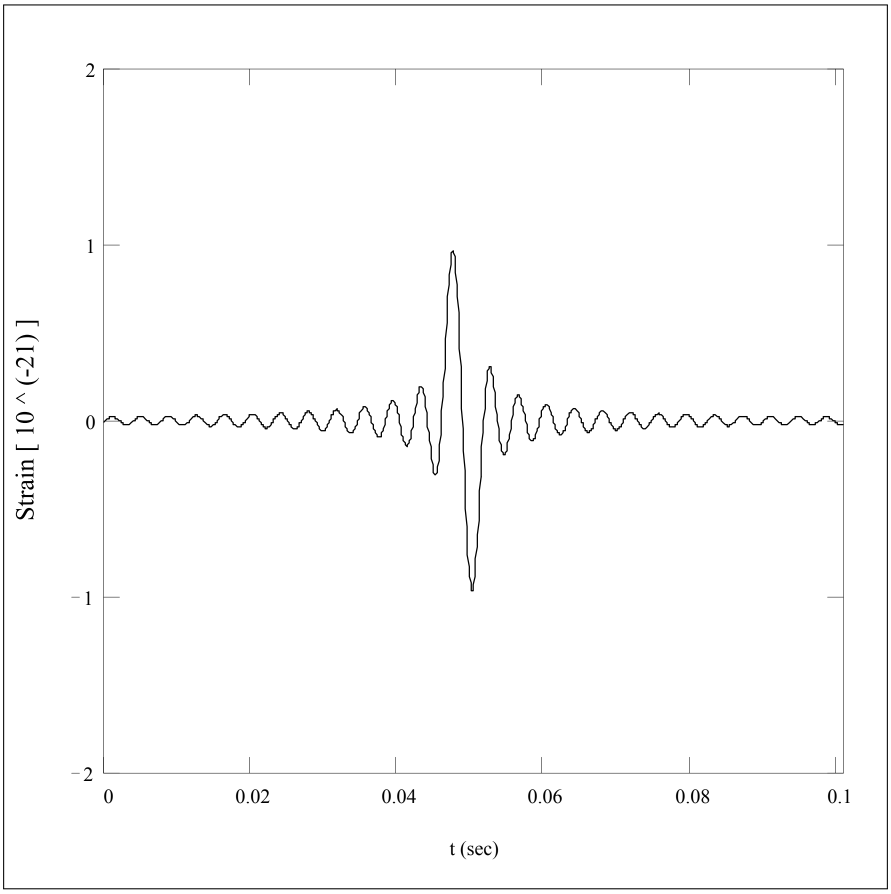

Figure 2. Predicted expansion of space signal output due to modulation of X and Y arms cavity beamwidths by $\delta\omega(t) = 523.2 + 60\,\mathrm{sinc}(1643.8\,(t - 0.05) + \phi_0)$, where $\phi_0 = 3.142$ for the X subbeam and 0 for the Y subbeam.

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