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Article

On the analysis of mixed-index time fractional differential equation systems

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Abstract: In this paper we study the class of mixed-index time fractional differential equations in which different components of the problem have different time fractional derivatives on the left hand side. We prove a theorem on the solution of the linear system of equations, which collapses to the well-known Mittag-Leffler solution in the case the indices are the same, and also generalises the solution of the so-called linear sequential class of time fractional problems. We also investigate the asymptotic stability properties of this class of problems using Laplace transforms and show how Laplace transforms can be used to write solutions as linear combinations of generalised Mittag-Leffler functions in some cases. Finally we illustrate our results with some numerical simulations.

Keywords: time fractional differential equations; mixed-index problems; analytical solution; asymptotic stability

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1. Introduction

Time fractional and space fractional differential equations are increasingly used as a powerful modelling tool for understanding the role of heterogeneity in modulating function in such diverse areas as cardiac electrophysiology [1–3], brain dynamics [4], medicine [5], biology [6], [7], porous media [8], [9] and physics [10]. Time fractional models are typically used to model subdiffusive processes (anomalous diffusion [11], [12]), while space fractional models are often associated with modelling processes occurring in complex spatially heterogeneous domains [1].

Time fractional models typically have solutions with heavy tails as described by the Mittag-Leffler matrix function [13] that naturally occurs when solving time fractional linear systems. However such models are usually only described by a single fractional exponent, α, associated with the fractional derivative. The fractional exponent can allow the coupling of different processes that may be occurring in different spatial domains by using different fractional exponents for the different regimes. One natural application here would be the coupling of models describing anomalous diffusion of proteins on the plasma membrane of the cell with the behaviour of other proteins in the cytosol of the cell. Tian et al [14] addressed this problem by coupling a stochastic model (based on the Stochastic Simulation Algorithm [15]) for the plasma membrane with systems of ordinary differential equations describing reaction cascades within the cell. It may also be necessary to couple more than two models and so in this paper we introduce a formulation that focuses on coupling an arbitrary number of domains in which dynamical processes are occurring described by different anomalous diffusive processes. This leads us to consider the r index time fractional differential equation problem in Caputo form

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$$D_t^{\alpha_i} y_i = \sum_{j=1}^{r} A_{ij} y_j + F_i(y), \quad y_i(0) = z_i, \quad y_i \in \mathbb{R}^{m_i}, \, i = 1, \cdots, r, \tag{1}$$

or in vector form

$$D_t^{\alpha} y = A\, y + F(y).$$

Here the A_{ij} are $m_i \times m_j$ matrices, while A is the associated block matrix of dimension $\sum_{j=1}^{r} m_j$ and $\alpha = (\alpha_1, \cdots, \alpha_r)^\top$ has all components $\alpha_i \in (0, 1]$.

We believe that a modelling approach based on this formulation has not been fully developed before. We note that scalar linear sequential fractional problems have been considered whose solution can be described by multi-indexed Mittag-Leffler functions [16], and there are a number of articles on the numerical solution of multi-term fractional differential equations [17–19], and while mixed index problems can, in some cases, be written in the form of linear sequential problems, namely $\sum_{i=1}^{R} D_t^{\beta_i} y = f(y)$, we claim that it is inappropriate to do so in many cases.

Therefore in this paper we develop a new theorem that gives the analytical solution of equations such as (1) that reduces to the Mittag-Leffler expansion in the case that all the indices are the same (section 3) and generalises the class of linear sequential problems (section 3.1). We then analyse the asymptotic stability properties of these mixed index problems using Laplace transform techniques (section 3.2), relating our results with known results that have been developed in control theory. In section 3.2 we also show that, in the case that the α_i are all rational, the solutions to the linear problem can be written as a linear combination of generalised Mittag-Leffler functions, again using ideas from control theory and transfer functions. In section 3.3 we present some numerical simulations illustrating the results in this paper and give some discussion (in section 4) on how these ideas can be used to solve semi-linear problems either by extending the methodology of exponential integrators to Mittag-Leffler functions, or by writing the solution as sums of certain Mittag-Leffler expansions.

2. Materials and Methods

2.1. Analytical Solutions

We consider the linear system given in (1) with $r = 2$. It will be convenient to let

$$A = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}, \quad y^\top = (y_1^\top, y_2^\top), \quad z^\top = (z_1^\top, z_2^\top) \tag{2}$$

where A is $m \times m$, $m = m_1 + m_2$. We will call such a system a time fractional index-2 system. Here the Caputo time fractional derivative with starting point at $t = 0$ is defined (see Podlubny [20], for example), as

$$D_t^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^{\alpha}} ds, \quad 0 < \alpha < 1.$$

Furthermore, given a fixed mesh of size h then a first order approximation of the Caputo derivative [21] is given by

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$$D_t^\alpha y_n = \frac{1}{\Gamma(2-\alpha)h^\alpha} \sum_{j=1}^{n} (j^{1-\alpha} - (j-1)^{1-\alpha})(y_{n-j-1} - y_{n-j}).$$

If $\beta = \alpha$ then the solution to (1) is given by the Mittag-Leffler expansion

$$y(t) = E_\alpha(t^\alpha A)\, y(0), \quad E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+j\alpha)} \tag{3}$$

where $\Gamma(x)$ is the Gamma function.

If the problem is completely decoupled, say $A_2 = 0$, then from (3) the solution to (1) and (2) satisfies

$$\begin{aligned} y_1(t) &= E_\alpha(t^\alpha A_1)\, z_1 \\ D_t^\beta y_2 &= B_2 y_2 + B_1 E_\alpha(t^\alpha A_1)\, z_1. \end{aligned} \tag{4}$$

In order to solve (4), this requires us to solve problems of the form

$$D_t^\beta y_2 = B_2 y_2 + f(t). \tag{5}$$

Before making further headway, we need some additional background material.

Definition 1. *Generalisations of the Mittag-Leffler functions are given by*

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \mathcal{Re}(\alpha) > 0 \\ E_{\alpha,\beta}^{\gamma}(z) &= \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \gamma \in \mathbb{N}_0, \end{aligned}$$

where $(\gamma)_k$ is the Pochhammer symbol

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1).$$

Remark 1. $E_{\alpha,1}(z) = E_\alpha(z), \quad E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z), \quad E_1(z) = e^z.$

Lemma 1.
$$\left(\frac{d}{dz}\right)^n E_{\alpha,\beta}(z) = n! E_{\alpha,\beta+\alpha n}^{n+1}(z), \; n \in \mathbb{N}.$$

Lemma 2. *The Laplace transform of $E_{\alpha,\beta}(\lambda t^\alpha)$ satisfies*

$$X(s) = \frac{s^\alpha}{s^\beta(s^\alpha - \lambda)}. \tag{6}$$

Lemma 3. *The Caputo derivatives satisfy the following relationships.*

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(i) $D_t^\alpha I^\alpha y(t) = y(t)$

(ii) $I^\alpha D_t^\alpha y(t) = y(t) - y(0)$

(iii) $D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds = I^{1-\alpha} D_t y(t)$.

Lemma 4. *(See [20], for example.) The solution of the scalar, linear, non-homogeneous problem*

$$D_t^\alpha y(t) = \lambda y(t) + f(t), \quad y(0) = y_0 \tag{7}$$

is

$$y(t) = E_\alpha(\lambda t^\alpha) y_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha\alpha}(\lambda(t-s)^\alpha) f(s) ds. \tag{8}$$

Proof. Using the integral form from Lemma 3, (7) can be rewritten as

$$y(t) = y_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{y(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds.$$

We now apply a Picard-style iteration of the form

$$y_k(t) = y_0(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{y_{k-1}(s)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad k = 1, 2, \cdots$$

where $y_0(t) = y_0$, $\forall t$.

It can be shown that this iteration will converge to (8) - see [16]. □

Lemma 5.

$$1 + \int_0^t \lambda s^{\alpha-1} E_{\alpha\alpha}(\lambda s^\alpha) ds = E_\alpha(\lambda s^\alpha).$$

Proof. Use Definition 1 and integrate the left hand side term by term. □

Remark 2. *The function multiplying $f(s)$ in the integrand of (8), namely*

$$G_\alpha(t-s) = (t-s)^{\alpha-1} E_{\alpha\alpha}(\lambda(t-s)^\alpha),$$

can be viewed as a Green function. For example, when $\alpha = 1$, $G_1(t-s) = e^{\lambda(t-s)}$.

The generalisation of the class of problems given by (7) to the systems case takes the form

$$D_t^\alpha y(t) = Ay(t) + F(t), \quad y(0) = y_0, \quad y \in \mathbb{R}^m. \tag{9}$$

In the case that $F(t) = 0$ the solution of the linear homogeneous system is

$$y(t) = E_\alpha(t^\alpha A) y_0. \tag{10}$$

From this we can prove

Theorem 1. *(See [20], for example.) The solution of (9) is given by*

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$$y(t) = E_\alpha(t^\alpha A)y_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha\alpha}((t-s)^\alpha A)F(s)ds. \tag{11}$$

Proof. We can use the idea of a Green function. But first of all it is trivial to see from Definition 1 that

$$\frac{d}{dz}(E_\alpha(z^\alpha A)) = Az^{\alpha-1}E_{\alpha\alpha}(z^\alpha A). \tag{12}$$

Now the solution of (9) can be written as

$$y(t) = y_0 + \int_0^t G_\alpha(t-s)(Ay_0 + F(s))ds,$$

where $G_\alpha(t-s)$ is a matrix Green function, or alternatively

$$y(t) = \left(I + \int_0^t G_\alpha(t-s)A\,ds\right) y_0 + \int_0^t G_\alpha(t-s)F(s)ds.$$

Finally it is clear from (12) that

$$G_\alpha(t-s) = (t-s)^{\alpha-1}E_{\alpha\alpha}((t-s)^\alpha A)$$

is the Green function and the result is proved by using Lemma 5. □

2.2. Asymptotic Stability of Multi-index Systems

The first contribution to the asymptotic stability analysis of time fractional linear systems was by Matignon [24]. Given the linear system $D_t^\alpha y(t) = Ay(t)$ in Caputo form, then taking the Laplace transform and using the definition of the Caputo derivative gives

$$s^\alpha X(s) - s^{\alpha-1}X(0) = AX(s)$$

or

$$X(s) = \frac{1}{s}(I - s^{-\alpha}A)^{-1}X(0). \tag{13}$$

Here $X(s)$ is the Laplace transform of $y(t)$. If we write $w = s^\alpha$, then the matrix $s^\alpha I - A$ will be nonsingular if w is not an eigenvalue of A. In the w-domain this will happen if $\mathrm{Re}(\sigma(A)) \leq 0$, where $\sigma(A)$ denotes the spectrum of A. In the s-domain this will happen if $|\mathrm{Re}(\sigma(A))| \geq \frac{\alpha\pi}{2}$. That is, the eigenvalues of A lie in the complex plane minus the sector subtended by angle $\alpha\pi$ symmetric about the positive real axis - see Figure 1.

In fact Laplace transforms are a very powerful technique for studying the asymptotic stability of mixed index fractional systems. Deng et al. [25] studied the stability of linear time fractional systems with delays using Laplace transforms. Given the delay system

$$\frac{d^{\alpha_i} y_i}{dt^{\alpha_i}} = \sum_{j=1}^m a_{ij} y_j(t - \tau_{ij}), \quad i = 1, \cdots, m \tag{14}$$

then the Laplace transforms results in

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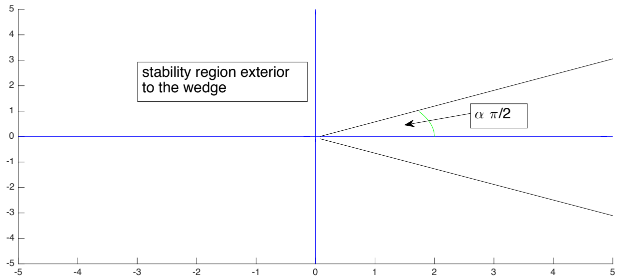

Figure 1. Stability region for single index problem

$$\begin{aligned} \Delta(s)\,X &= b \\ \Delta(s) &= Diag(s^{\alpha_1},\cdots,s^{\alpha_m}) - L \\ L_{ij} &= a_{ij}e^{-s\tau_{ij}}, \quad i,j = 1,\cdots,m. \end{aligned} \tag{15}$$

Hence, Deng et al. proved:

Theorem 2. *If all the zeros of the characteristic polynomial of $\Delta(s)$ have negative real part then the zero solution of (14) is asymptotically stable.*

Deng et al. also proved a very nice result in the case that all the indices $\alpha_1,\cdots,\alpha_m$ are rational.

Theorem 3. *Consider (14) with no delays and all the $\alpha_i \in (0,1)$ and are rational. In particular let*

$$\alpha_i = \frac{u_i}{v_i}, \quad gcd(u_i, v_i) = 1$$

and let M be the lowest common multiple of all the denominators and set $\gamma = \frac{1}{M}$. Then the problem will be asymptotically stable if all the roots, λ, of

$$p(\lambda) = Det(D - A) = 0, \quad D = diag(\lambda^{Ma_1},\cdots,\lambda^{Ma_m})$$

satisfy $|\arg(\lambda)| > \gamma\frac{\pi}{2}$.

Remark 3. *If $\alpha_i = \alpha$, $i = 1,\cdots,m$ then Theorem 3 reduces to the result of Matignon. The proof of Theorem 3 comes immediately from (15) where $p(\lambda)$ is the characteristic polynomial of $\Delta(s)$.*

Remark 4. *A nice survey on the stability (both linear and nonlinear) of fractional differential equations is given in Li and Zhang [26], while Saberi Najafi et al. [27] has extended some of these stability results to distributed order fractional differential equations with respect to an order density function. Zhang et al [28] consider the stability of nonlinear fractional differential equations.*

Remark 5. *Radwan et al. [29] note that the stability analysis of mixed index problems reduces to the study of the roots of the characteristic equation*

$$\sum_{i=1}^{m} \theta_i\, s^{\alpha_i} = 0, \quad 0 < \alpha_i \leq 1. \tag{16}$$

In the case that the α_i are arbitrary real numbers, the study of the roots of (16) is difficult. By letting $s = e^z$, we can cast this in the framework of quasi (or exponential) polynomials (Rivero et al. [30]). The zeros of exponential polynomials have been studied by Ritt [31].

The general form of an exponential polynomial with constant coefficients is

$$f(z) = \sum_{j=0}^{k} a_j e^{\alpha_j z}.$$

An analogue of the fact that a polynomial of degree k can have up to k roots is expressed by a Theorem due to Tamarkin, Pólya and Schwengler (see [31]).

Theorem 4. *Let P be the smallest convex polygon containing the values $\alpha_1, \cdots, \alpha_k$ and let the sides of P be $s_1, \cdots, s_k$. Then there exist k half strips with half rays parallel to the outer normal to b_i that contain all the zeros of f. If $|b_i|$ is the length of b_i, then the number of zeros in the i^{th} half strip with modulus less than or equal to r is asymptotically $\frac{r|b_i|}{2\pi}$.*

3. Results

3.1. The Solution of Mixed Index Linear Systems

The main focus of this paper is to consider generalisations of (9) where there are differing values of α on the left hand side. In its general form, we will let $y^\top = (y_1^\top, \cdots, y_r^\top) \in \mathbb{R}^m$ where $y_i \in \mathbb{R}^{m_i}$ and $m = \sum_{i=1}^{r} m_i$. We will also assume $F(t)^\top = (F_1(t)^\top, \cdots, F_r(t)^\top)$ and that A can be written in block form $A = (A_{ij})_{i,j=1}^{r}$, $A_{ij} \in \mathbb{R}^{m_i \times m_j}$. We will also let $\alpha = (\alpha_1, \cdots, \alpha_r)$ and consider a class of linear, non-homogeneous multi-indexed systems of FDEs of the form

$$D_t^\alpha y(t) = Ay(t) + F(t) \tag{17}$$

that we interpret as the system

$$D_t^{\alpha_i} y_i(t) = \sum_{j=1}^{r} A_{ij} y_j(t) + F_i(t), \quad i = 1, \cdots, r. \tag{18}$$

The index of the system is said to be r.

In the case that $F = 0$, then by letting

$$E_i = D^{\alpha_i} - A_{1i}$$

we can rewrite (17) as

$$M\, y = 0, \tag{19}$$

where M is the block matrix, whose determinant must be zero, with

$$\begin{aligned} M_{ii} &= E_i, \quad i = 1, \cdots, r \\ M_{ij} &= -A_{ij}, \quad i \neq j. \end{aligned}$$

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Thus, in the case all $m_i = 1$, so that the individual components are scalar and so $m = r$, (19) implies $\mathrm{Det}(M)\, y_r = 0$.

For example, when $r = 2$ this becomes

$$(E_1 E_2 - A_{21} A_{12})\, y_2 = 0$$

or

$$(D^{\alpha_1+\alpha_2} - A_{22} D^{\alpha_1} - A_{11} D^{\alpha_2} + \mathrm{Det}(A))\, y_2 = 0;$$

while for $r = 3$ this gives, after some simplification,

$$\begin{aligned} D^{\alpha_1+\alpha_2+\alpha_3}\, y_3 \;&-\; A_{11} D^{\alpha_2+\alpha_3}\, y_3 - A_{22} D^{\alpha_1+\alpha_3}\, y_3 - A_{33} D^{\alpha_1+\alpha_2}\, y_3 \\ &+\; (A_{22}A_{33} - A_{23}A_{32}) D^{\alpha_1}\, y_3 + (A_{11}A_{33} - A_{13}A_{31}) D^{\alpha_2}\, y_3 \\ &+\; (A_{11}A_{22} - A_{12}A_{21}) D^{\alpha_3}\, y_3 - \mathrm{Det}(A) = 0. \end{aligned}$$

Clearly there is a general formula for arbitrary r in terms of the cofactors of A. In particular, it can be fitted into the framework of linear sequential FDEs [16,20–23]. These take the form

$$D_t^{\beta_0} y_1(t) + \sum_{j=1}^{p} a_j D_t^{\beta_j} y_1(t) = d y_1(t) + f(t), \quad \beta_0 > \beta_1 > \cdots \beta_p. \tag{20}$$

However, this characterisation is not particularly simple, useful, or computationally expedient. Furthermore when the m_i are not 1, so that the individual components are not scalar, then there is no simple representation such as (20) and new approaches are needed. Before we consider this new approach we note the converse, namely that (20) can always be written in the form of (17) for a suitable matrix A with a special structure. In particular we can write (20) in the form of (17) with $p = r - 1$ as an r dimensional, r index problem with $\alpha = (\beta_0, \beta_1, \cdots, \beta_p)$, and

$$A = \begin{pmatrix} d & -a_1 & -a_2 & \cdots & -a_p \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & & \cdots & & 1 \end{pmatrix}, \quad F(t) = (f(t), 0, \cdots, 0)^\top.$$

For completeness we note in the case that $d = 0$ and $f(t) = 0$, an explicit solution to this problem was given in Podlubny [20]. This can be found by considering the transfer function (see section 3.2) given by

$$H(s) = \frac{1}{s^{\beta_0} + a_1 s^{\beta_1} + \cdots + a_p s^{\beta_p}}.$$

By finding the poles of this function and converting back to the untransformed domain, Podlubny gives the solution as

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$$
\begin{aligned}
y_1(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\cdots+k_{p-2}=m \\ k_i \geq 0}} \binom{m}{k_0 \cdots k_{p-2}} \prod_{i=0}^{p-2} (a_{p-i})^{k_i} \times \\
&\epsilon_m(t, -a_1; \beta_0-\beta_1, \beta_0 + \sum_{j=0}^{p-2} (\beta_1-\beta_{p-j})k_j + 1)
\end{aligned}
$$

160 where

$$
\begin{aligned}
\epsilon_k(t, y; \alpha, \beta) &= t^{k\alpha+\beta-1} E^k_{\alpha,\beta}(yt^\alpha) \\
E^k_{\alpha,\beta}(z) &= \sum_{i=0}^{\infty} \frac{(i+k)!\, z^i}{i!\,\Gamma(\alpha(i+k)+\beta)}.
\end{aligned}
$$

161 We now return to the index-2 problem (1) and (2). We first claim that the solution takes the matrix
162 form

$$
\begin{aligned}
y_1 &= \alpha_{00} + \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \alpha_{n,j+1} \frac{t^{n\alpha+j(\beta-\alpha)}}{\Gamma(1+n\alpha+j(\beta-\alpha))} z \\
y_2 &= \beta_{00} + \sum_{n=1}^{\infty} \sum_{j=1}^{n} \beta_{n,j} \frac{t^{n\alpha+j(\beta-\alpha)}}{\Gamma(1+n\alpha+j(\beta-\alpha))} z,
\end{aligned} \tag{21}
$$

163 where the $\alpha_{n,j}$, $\beta_{n,j}$ are appropriate matrices, of size $m_1 \times m$ and $m_2 \times m$, respectively, that are to be
164 determined.

165 We now use the fact that

$$
\begin{aligned}
D^\alpha_t \frac{t^{n\alpha+j(\beta-\alpha)}}{\Gamma(1+n\alpha+j(\beta-\alpha))} &= \frac{1}{\Gamma(1+(n-1)\alpha+j(\beta-\alpha))} t^{(n-1)\alpha+j(\beta-\alpha)} \\
D^\beta_t \frac{t^{n\alpha+j(\beta-\alpha)}}{\Gamma(1+n\alpha+j(\beta-\alpha))} &= \frac{1}{\Gamma(1+(n-1)\alpha+(j-1)(\beta-\alpha))} t^{(n-1)\alpha+(j-1)(\beta-\alpha)}.
\end{aligned} \tag{22}
$$

166 Using (21) and (22) the left hand side of (1) is

$$
\begin{aligned}
D^\alpha_t y_1 &= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \alpha_{n,j+1} \frac{t^{(n-1)\alpha+j(\beta-\alpha)}}{\Gamma(1+(n-1)\alpha+j(\beta-\alpha))} z \\
D^\beta_t y_2 &= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \beta_{n,j+1} \frac{t^{(n-1)\alpha+j(\beta-\alpha)}}{\Gamma(1+(n-1)\alpha+j(\beta-\alpha))} z
\end{aligned}
$$

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that can be written in matrix form as

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{pmatrix}\alpha_{n+1,j+1}\\ \beta_{n+1,j+1}\end{pmatrix}\frac{t^{n\alpha+j(\beta-\alpha)}}{\Gamma(1+n\alpha+j(\beta-\alpha))}z. \tag{23}$$

If we define

$$\alpha_{n,n+1}=0,\quad \beta_{n0}=0,\quad n=1,2,\cdots \tag{24}$$

then the right hand side of (1) is

$$A\left(\begin{pmatrix}\alpha_{00}\\ \beta_{00}\end{pmatrix}+\sum_{n=1}^{\infty}\sum_{j=0}^{n}\begin{pmatrix}\alpha_{n,j+1}\\ \beta_{nj}\end{pmatrix}\frac{t^{n\alpha+j(\beta-\alpha)}}{\Gamma(1+n\alpha+j(\beta-\alpha))}\right)z. \tag{25}$$

Equating (23) and (25) we find along with (24) that for $n=0,1,2,\cdots$

$$\begin{pmatrix}\alpha_{00}\\ \beta_{00}\end{pmatrix}=I_m,\ \begin{pmatrix}\alpha_{n+1,j+1}\\ \beta_{n+1,j+1}\end{pmatrix}=A\begin{pmatrix}\alpha_{n,j+1}\\ \beta_{nj}\end{pmatrix},\quad j=0,1,\cdots,n. \tag{26}$$

In order to get a succinct representation of the solution based on (21) and (26), it will be convenient to write

$$p_n(t)=\left(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)},\frac{t^{(n-1)\alpha+\beta}}{\Gamma(1+(n-1)\alpha+\beta)},\cdots,\frac{t^{n\beta}}{\Gamma(1+n\beta)}\right)^{\top}\otimes I_m,\quad n=1,2,\cdots$$

so $p_n(t)\in\mathbb{R}^{m(n+1)\times m}$, and let $p_0(t)=I_m$.

We will also define the matrices

$$\begin{aligned}L_n &= \begin{pmatrix}\alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} & 0\\ 0 & \beta_{n1} & \cdots & \beta_{n\,n-1} & \beta_{nn}\end{pmatrix}\in\mathbb{R}^{m\times m(n+1)},\ n=1,2,\cdots\\ L_0 &= I_m\end{aligned}$$

where 0 represents appropriately-sized zero matrices. Now we note that the recursive relation (26) is equivalent to

$$\begin{pmatrix}\alpha_{n1} & \cdots & \alpha_{nn}\\ \beta_{n1} & \cdots & \beta_{nn}\end{pmatrix}=A\,L_{n-1},\quad n=1,2,\cdots. \tag{27}$$

Thus we can state the following theorem.

Theorem 5. *The solution of the fractional index-2 system*

$$D_t^{\alpha,\beta}\,y(t)=A\,y(t),\ y(0)=z$$

is given by

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$$y(t) = \sum_{n=0}^{\infty} L_n\, p_n(t)\, z, \tag{28}$$

179 *where for $n = 1, 2, \cdots$*

$$\begin{aligned} L_n &= \begin{pmatrix} \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} & 0 \\ 0 & \beta_{n1} & \cdots & \beta_{n\,n-1} & \beta_{nn} \end{pmatrix}, \quad \begin{pmatrix} \alpha_{n1} & \cdots & \alpha_{nn} \\ \beta_{n1} & \cdots & \beta_{nn} \end{pmatrix} = A\, L_{n-1}, \\ L_0 &= I_m \\ p_n(t) &= \left(\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \frac{t^{(n-1)\alpha+\beta}}{\Gamma(1+(n-1)\alpha+\beta)}, \cdots, \frac{t^{n\beta}}{\Gamma(1+n\beta)} \right)^{\top} \otimes I_m. \end{aligned} \tag{29}$$

180 **Remark 6.** *In the case $\alpha = \beta$,*

$$p_n(t) = \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} (1, \cdots, 1)^{\top} \otimes I_m,$$

$$L_n\, p_n(t) = \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \sum_{j=1}^{n} \begin{pmatrix} \alpha_{nj} \\ \beta_{nj} \end{pmatrix}$$

181 *and with*

$$\sum_{j=1}^{n} \begin{pmatrix} \alpha_{nj} \\ \beta_{nj} \end{pmatrix} = A \sum_{j=1}^{n-1} \begin{pmatrix} \alpha_{n-1,j} \\ \beta_{n-1,j} \end{pmatrix}$$

182 *then (28) reduces, as expected, to*

$$y(t) = E_{\alpha}(t^{\alpha} A) z.$$

183 **Remark 7.** *It will be convenient to define the matrix*

$$P_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} L_n p_n(t)$$

184 *so that the solution (28) can be expressed as*

$$y(t) = P_{\alpha,\beta}(t) y_0. \tag{30}$$

185 **Remark 8.** *If the fractional index-2 system has initial condition $y(t_0) = z$ then the solution is*

$$y(t) = P_{\alpha,\beta}(t - t_0) z. \tag{31}$$

186 We note that in solving (9) an equivalent solution to (11) is

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$$\begin{aligned} y(t) &= E_\alpha(t^\alpha A)y_0 + I_\alpha(G_\alpha(t-s)F(s))ds, \\ G_\alpha(t-s) &= E_\alpha((t-s)^\alpha A), \end{aligned}$$

where G_α is the Green function satisfying

$$D_t^\alpha G_\alpha(t-s) = AG_\alpha(t-s). \tag{32}$$

This leads us to give a general result on the solution of the mixed index problem with a time-dependent forcing function

$$D_t^{\alpha,\beta} y(t) = Ay(t) + F(t),$$

but first we need the following definition.

Definition 2. *Let $y(t) = (y_1^\top(t), y_2^\top(t))^\top$, then define*

$$I_t^{\alpha,\beta} y(s)ds = \left(I_t^\alpha y_1^\top(s)ds, I_t^\beta y_2^\top(s)ds \right)^\top.$$

Theorem 6. *The solution to the fractional index-2 problem*

$$D_t^{\alpha,\beta} y(t) = Ay(t) + F(t), \quad y(0) = y_0 \tag{33}$$

is given by

$$y(t) = P_{\alpha,\beta}(t)y_0 + I_t^{\alpha,\beta}\left(P_{\alpha,\beta}(t-s)F(s)ds\right). \tag{34}$$

Proof. The result follows from $D_t^{\alpha,\beta} P_{\alpha,\beta}(t) = AP_{\alpha,\beta}(t)$, together with the above discussion. □

We now turn to analysing the asymptotic stability of linear fractional index-2 systems.

3.2. Study of Asymptotic Stability

Recalling Theorem 4, we note that if the α_i are rational and with M the lowest common multiple of the denominators, this reduces to the polynomial

$$\sum_{i=1}^{M} \theta_i W^i = 0, \quad W = s^{\frac{1}{M}}.$$

This leads us to think about stability from a control theory point of view. Thus given the system

$$\sum_{j=0}^{n} a_j D^{\alpha_j} y = \sum_{j=0}^{M} b_j D^{\beta_j} y \tag{35}$$

where

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$$\alpha_n > \cdots > \alpha_0, \quad \beta_M > \cdots \beta_0$$

then the solution of (35) can be written in terms of the transfer function

$$G(s) = \frac{\sum_{j=0}^{m} b_j s^{\beta_j}}{\sum_{j=0}^{n} a_j s^{\alpha_j}} := \frac{Q(s)}{P(s)}, \tag{36}$$

where s is the Laplace variable (see Rivero et al. [30], Petras [32]).

In the case of the so-called commensurate form in which

$$\alpha_k = k\alpha, \quad \beta_k = k\beta,$$

then

$$G(s) = \frac{\sum_{k=0}^{m} b_k (s^{\beta})^k}{\sum_{k=0}^{n} a_k (s^{\alpha})^k} := \frac{Q(s^{\beta})}{P(s^{\alpha})}. \tag{37}$$

Clearly, if $\frac{\beta}{\alpha}$ is rational with $\alpha \geq \beta$ and

$$\beta = \frac{q}{p}\alpha, \quad q, p \in \mathbb{Z}^+, \quad w = s^{\frac{\alpha}{p}}$$

then (37) can be written as

$$G(w) := \frac{Q(w^q)}{P(w^p)}, \quad p, q \in \mathbb{Z}^+, \quad q \leq p.$$

Cěrmák and Kisela [33] considered the specific problem

$$D^{\alpha} y + a D^{\beta} y + b y = 0, \quad y \in \mathbb{R}, \tag{38}$$

where $\alpha = pK$, $\beta = qK, K$ real $\in (0,1), p, q \in \mathbb{Z}^+, p \geq q$. In this case the appropriate stability polynomial is $P(\lambda) := \lambda^p + a\lambda^q + b$, where $\lambda = s^K$. Based on Theorem 3, (38) is asymptotically stable if all the roots of $P(\lambda)$ satisfy $|\arg(\lambda)| > K\frac{\pi}{2}$.

By setting $\lambda = re^{iK\frac{\pi}{2}}$ and substituting into $P(\lambda) = 0$ and equating real and imaginary parts, it is easily seen that

$$\begin{aligned} r^p \cos\frac{pK\pi}{2} + a\, r^q \cos\frac{qK\pi}{2} + b &= 0 \\ r^p \sin\frac{pK\pi}{2} + a\, r^q \sin\frac{qK\pi}{2} &= 0. \end{aligned}$$

This leads to the following result, given in Cěrmák and Kisela [33].

Theorem 7. *Equation (38) is asymptotically stable with $\alpha > \beta > 0$ real and $\frac{\alpha}{\beta}$ rational if*

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$$\begin{aligned} \beta &< 2, \quad \alpha-\beta<2 \\ b &> 0, \quad a > \frac{-\sin\frac{\alpha\pi}{2}}{(\sin\frac{\beta\pi}{2})^{\frac{\beta}{\alpha}}\,(\sin\frac{(\alpha-\beta)\pi}{2})^{\frac{\alpha-\beta}{\alpha}}}\, b^{\frac{\alpha-\beta}{\alpha}}. \end{aligned}$$

We now follow this idea but for arbitrarily sized systems in our mixed index format, and this leads to slight modifications to (38). We first make a slight simplification and take $m_1 = m_2$ and we also assume that A_2 is nonsingular, then problem (1) leads to

$$y_2 = A_2^{-1}\,(D^\alpha\, I - A_1)\, y_1$$

and substituting into the equation for y_1 gives

$$\begin{gathered} (D^{\alpha+\beta}\, I - B_2 D^\alpha\, I - \bar{A}_1 D^\beta\, I + B_2\bar{A}_1 - B_1 A_2)\, A_2^{-1}\, y_1 = 0 \\ \bar{A}_1 = A_2^{-1} A_1 A_2. \end{gathered}$$

This leads us to consider the roots of the characteristic function

$$P(\lambda) := \mathrm{Det}(D^{\alpha+\beta} I - B_2 D^\alpha I - \bar{A}_1 D^\beta I + B_2\bar{A}_1 - B_1 A_2) = 0. \tag{39}$$

In the scalar case this gives an extension to (38) where the characteristic equation is

$$P(\lambda) = \lambda^{\alpha+\beta} - B_2\lambda^\alpha - A_1\lambda^\beta + \mathrm{Det}(A). \tag{40}$$

Now reverting to Laplace transforms of (1) and (2) then

$$\begin{aligned} s^\alpha X_1(s) - s^{\alpha-1} X_1(0) &= A_1 X_1(s) + A_2 X_2(s) \\ s^\beta X_2(s) - s^{\beta-1} X_2(0) &= B_1 X_1(s) + B_2 X_2(s). \end{aligned}$$

This can be written in systems form as

$$(D_1 - A)X(s) = D_2 X(0), \tag{41}$$

where

$$D_1 = \begin{pmatrix} s^\alpha I & 0 \\ 0 & s^\beta I \end{pmatrix}, \quad D_2 = \begin{pmatrix} s^{\alpha-1} I & 0 \\ 0 & s^{\beta-1} I \end{pmatrix}$$

or alternatively as

$$X(s) = \frac{1}{s}(I - D_1^{-1}A)^{-1} X(0). \tag{42}$$

This can now be considered as a generalised eigenvalue problem. From (41) we require $D_1 - A$ to be nonsingular. That is

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$$\begin{pmatrix} s^\alpha I - A_1 & -A_2 \\ -B_1 & s^\beta I - B_2 \end{pmatrix} v = 0 \implies v = 0.$$

Let us write $v = (v_1^\top, v_2^\top)^\top$ and assume $\alpha \geq \beta$ and that $s^\beta I - B_2$ is nonsingular, so that from the previous analysis this means

$$|\mathrm{Re}(\sigma(B_2))| \geq \frac{\beta\pi}{2}. \quad (43)$$

Hence

$$\begin{aligned} v_2 &= (s^\beta I - B_2)^{-1} B_1 v_1 \\ ((s^\alpha I - A_1) - A_2(s^\beta I - B_2)^{-1} B_1) v_1 &= 0. \end{aligned}$$

Thus (43) and

$$\mathrm{Det}((s^\alpha I - A_1) - A_2(s^\beta I - B_2)^{-1} B_1) = 0 \quad (44)$$

define the asymptotic stability boundary - see also (39).

In order to make this more specific, let $m_1 = m_2 = 1$ and

$$A = \begin{bmatrix} d & b \\ a & d \end{bmatrix}, \quad d < 0. \quad (45)$$

Note that $\sigma(A) = \{d \pm \sqrt{ab}\}$. Then (44) becomes

$$(s^\alpha - d)(s^\beta - d) - ab = 0. \quad (46)$$

Furthermore, let $b = -a = \theta$, so that the eigenvalues of A are $d \pm i\theta$ and (46) becomes

$$(s^\alpha - d)(s^\beta - d) + \theta^2 = 0. \quad (47)$$

If we now assume that

$$s = re^{i\frac{\pi}{2}},$$

which defines the asymptotic stability boundary (the imaginary axis) when $\alpha = \beta = 1$, then (47) becomes

$$\theta^2 = -(r^\alpha e^{i\frac{\pi\alpha}{2}} - d)\,(r^\beta e^{i\frac{\pi\beta}{2}} - d). \quad (48)$$

Now since θ and d are real, the imaginary part of the right hand side of (48) must be zero, so that

$$r^{\alpha+\beta} \sin\frac{\alpha+\beta}{2}\pi = d(r^\alpha \sin\frac{\alpha\pi}{2} + r^\beta \sin\frac{\beta\pi}{2}). \quad (49)$$

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Hence

$$-\theta^2 = r^{\alpha+\beta}\cos\frac{\alpha+\beta}{2}\pi - d(r^{\alpha}\cos\frac{\alpha\pi}{2} + r^{\beta}\cos\frac{\beta\pi}{2}) + d^2. \tag{50}$$

Equations (49) and (50) will define the asymptotic stability boundary with θ as a function of d. Rewriting (49) as

$$d = \frac{r^{\alpha+\beta}\sin\frac{\alpha+\beta}{2}\pi}{r^{\alpha}\sin\frac{\alpha\pi}{2} + r^{\beta}\sin\frac{\beta\pi}{2}}. \tag{51}$$

and substituting (50) leads after simplification to

$$\frac{\theta^2}{d^2} = \frac{1}{r^{\alpha+\beta}(\sin\frac{\alpha+\beta}{2}\pi)^2}\left[\sin\frac{\alpha+\beta}{2}\pi(\frac{r^{2\alpha}}{2}\sin\alpha\pi + \frac{r^{2\beta}}{2}\sin\beta\pi)\right.$$
$$\left. -\cos\frac{\alpha+\beta}{2}\pi(r^{2\alpha}\sin^2\frac{\alpha\pi}{2} + r^{2\beta}\sin^2\frac{\beta\pi}{2} + 2r^{\alpha+\beta}\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2})\right].$$

Using the relationships

$$\begin{aligned}\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)\\ \sin A\sin B + \cos A\cos B &= \cos(A-B)\end{aligned}$$

gives

$$\begin{aligned}\frac{\theta^2}{d^2} &= \frac{1}{2r^{\alpha+\beta}\sin^2\frac{\alpha+\beta}{2}\pi}\left((r^{2\alpha}+r^{2\beta})\left(\cos\frac{\alpha-\beta}{2}\pi - \cos\frac{\alpha+\beta}{2}\pi\right)\right.\\ &\qquad \left. -4r^{\alpha+\beta}\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}\cos\frac{\alpha+\beta}{2}\pi\right).\end{aligned} \tag{52}$$

Since

$$\cos\frac{\alpha-\beta}{2}\pi - \cos\frac{\alpha+\beta}{2}\pi = 2\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}$$

and letting $x = r^{\alpha-\beta}$, then we can write (52) as

$$\left(\frac{\theta}{d}\right)^2 = \frac{\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}{\sin^2\frac{\alpha+\beta}{2}\pi}\left(\frac{x^2+1}{x} - 2\cos\frac{\alpha+\beta}{2}\pi\right). \tag{53}$$

Furthermore, we can write (51) as

$$d = \frac{x^{\frac{\alpha}{\alpha-\beta}}\sin\frac{\alpha+\beta}{2}\pi}{x\sin\frac{\alpha\pi}{2} + \sin\frac{\beta\pi}{2}}. \tag{54}$$

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It is easily seen that as a function of x the minimum of (53) is when $x = 1$. Thus

$$\begin{aligned} \frac{\theta}{d} &\geq \frac{\sqrt{2\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}}{\sin\frac{\alpha+\beta}{2}\pi}\sqrt{1-\cos\frac{\alpha+\beta}{2}\pi} \\ &= \frac{2\sqrt{\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}\,\sin\frac{\alpha+\beta}{4}\pi}{2\sin\frac{\alpha+\beta}{4}\pi\cos\frac{\alpha+\beta}{4}\pi} \\ &= \frac{\sqrt{\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}}{\cos\frac{\alpha+\beta}{4}\pi}. \end{aligned}$$

Thus we have proved the following result.

Theorem 8. *Given the mixed index problem with A as in (45), the angle for asymptotic stability $\hat{\theta} = \arctan(\frac{\theta}{d})$ satisfies*

$$\tan\hat{\theta} \in \left[\frac{\sqrt{\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}}{\cos\frac{\alpha+\beta}{4}\pi}, \infty\right), \tag{55}$$

or in radians with $\tilde{\theta} = \frac{1}{\pi}\arctan(\frac{\theta}{d})$

$$\tilde{\theta} \in \frac{1}{\pi}\left[\arctan\frac{\sqrt{\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}}{\cos\frac{\alpha+\beta}{4}\pi}, \arctan\frac{\pi}{2}\right]$$

with the minimum occuring with

$$d = \frac{\sin\frac{\alpha+\beta}{2}\pi}{\sin\frac{\alpha\pi}{2}+\sin\frac{\beta\pi}{2}}. \tag{56}$$

Remark 9. *We have the following results for $\hat{\theta}$ in three particular cases:*

(i) $\alpha = \beta:\quad \hat{\theta} = \alpha\frac{\pi}{2}$, *since in this case* $(\frac{\theta}{d})^2 = \tan^2\frac{\alpha\pi}{2}$.

(ii) $\alpha+\beta = 1:\quad \hat{\theta} \in (\sqrt{\sin\alpha\pi}, \frac{\pi}{2})$, $\alpha \in [\frac{1}{2}, 1]$. *In the case* $\alpha+\beta = 1$ *we see from (53) that*

$$\left(\frac{\theta}{d}\right)^2 = \sin\alpha\pi\left(\frac{x^2+1}{2x}\right).$$

Letting $\alpha = \frac{1}{2}+\epsilon$ with $\epsilon > 0$ small, then $x = r^{2\epsilon}$. This means that $\frac{x^2+1}{2x}$, as a function of r, is very shallow apart from when r is near the origin or very large. Hence the asymptotic stability boundary will be almost constant over long periods of d when α and β are close together.

(iii) $\alpha = 2\beta:\quad \hat{\theta} \in [\frac{\sin\frac{\beta\pi}{2}\sqrt{2\cos\frac{\beta\pi}{2}}}{\cos\frac{3\beta\pi}{4}}, \frac{\pi}{2})$, $\beta \in (0, \frac{1}{2}]$.

Letting

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$$K = \frac{\sin\frac{\alpha\pi}{2}\sin\frac{\beta\pi}{2}}{\sin^2\frac{\alpha+\beta}{2}\pi}, \quad L = 2\cos\frac{\alpha+\beta}{2}\pi,\ \phi = \frac{\theta}{d},$$

we can write (53) and (54) as

$$x^2 - x(L + \frac{\phi^2}{K})x + 1 = 0 \tag{57}$$

$$x^{\frac{\alpha}{\alpha-\beta}} - x\,d_\alpha - d_\beta = 0, \tag{58}$$

where

$$d_\alpha = d\frac{\sin\frac{\alpha\pi}{2}}{\sin\frac{\alpha+\beta}{2}\pi}, \quad d_\beta = d\frac{\sin\frac{\beta\pi}{2}}{\sin\frac{\alpha+\beta}{2}\pi}.$$

Due to the nonlinearities in (58) it is hard to determine an explicit simple relation between ϕ and d except if $\alpha = 2\beta$. In this case we make use of the following Lemma.

Lemma 6. *If $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ then there is a solution*

$$\begin{array}{ll} x = 0, & b = d \\ x^2 - ax + b = 0, & a = c, b = d \\ x = \frac{d-b}{c-a}, & c \neq a \text{ and } (d-b)^2 = (c-a)(ad-bc). \end{array} \tag{59}$$

Proof. By subtraction of the two equations and substitution. □

In the case of (57) and (58) then (59) becomes

$$(1 + d_\beta)^2 = (P - d_\alpha)(Pd_\beta + d_\alpha), \quad P = L + \frac{\phi^2}{K},$$

that is

$$P^2 d_\beta - P d_\alpha(d_\beta - 1) - (d_\alpha^2 + (1 + d_\beta)^2) = 0.$$

Hence

$$2d_\beta P = d_\alpha(d_\beta - 1) \pm (1 + d_\beta)\sqrt{d_\alpha^2 + 4d_\beta}. \tag{60}$$

Note that

$$\phi^2 = K\,P - K\,L$$

and

$$d_\alpha d_\beta = d^2\,K.$$

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Some manipulation from (60) leads to

$$\phi^2 = \frac{1}{2}\left(\frac{d_\alpha}{d}\right)^2 \left(d_\beta - 1 \pm (1+d_\beta)\sqrt{1 + 4\frac{d_\beta}{d_\alpha^2}} - 2L\frac{d_\beta}{d_\alpha} \right).$$

Now since $\alpha = 2\beta$, this reduces to

$$\begin{aligned} \phi^2 &= \frac{1}{2}\left(\frac{\sin\beta\pi}{\sin\frac{3\beta}{2}\pi}\right)^2 \left(d_\beta - 1 \pm (1+d_\beta)\sqrt{1 + \frac{4}{d}\frac{\sin\frac{\beta}{2}\pi \sin\frac{3\beta}{2}\pi}{(\sin\beta\pi)^2}} - 2\frac{\cos\frac{3\beta}{2}\pi}{\cos\frac{\beta}{2}\pi} \right) \\ d_\beta &= d\frac{\sin\frac{\beta}{2}\pi}{\sin\frac{3\beta}{2}\pi}. \end{aligned} \quad (61)$$

By taking $\tilde{\theta} = \arctan(\phi)$ this gives an explicit relationship between $\tilde{\theta}$ and d for the case $\alpha = 2\beta$.

Remark 10. *Particular solutions are*

(i) $\beta = \frac{1}{2}$, $\alpha = 1$, $\tan\tilde{\theta} = \sqrt{(1+d)(1+\sqrt{1+\frac{2}{d}})}$

(ii) $\beta = \frac{1}{3}$, $\alpha = \frac{2}{3}$, $\tan\tilde{\theta} = \sqrt{\frac{3}{8}}\sqrt{(1+\frac{d}{2})\sqrt{1+\frac{8}{3d}} + \frac{d}{2} - 1}$.

It is clear from (61) that when $d = 0$ and $d = \infty$, then $\theta = \frac{\pi}{2}$ and then the angle will make an excursion from $\frac{\pi}{2}$ down to a minimum value and back to $\frac{\pi}{2}$ as d increases. For example, in the case of $\beta = \frac{1}{2}$, $\alpha = 1$ we can see from Remark 10(i) that the minimum value of the angle is when

$$d = \sqrt{2} - 1, \quad \tan\tilde{\theta} = \sqrt{\sqrt{2} + \sqrt{4 + 3\sqrt{2}}}.$$

Returning to (41) and taking $m_1 = m_2 = 1$ and

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

then the Laplace transform in (42) is

$$X(s) = \frac{1}{\mathrm{Det}(s)}\left(s^{\alpha+\beta-1}X(0) + \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} s^{\beta-1}X_2(0) + \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} s^{\alpha-1}X_1(0) \right) \quad (62)$$

where

$$\begin{aligned} \mathrm{Det}(s) &= s^{\alpha+\beta} - a_1 s^\beta - b_2 s^\alpha + D_A, \\ D_A &= a_1 b_2 - a_2 b_1 = \mathrm{Det}(A). \end{aligned}$$

Now if α and β are rational ($\alpha \leq \beta$)

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$$\alpha = \frac{m}{n}, \quad \beta = \frac{p}{q}, \quad m \le n,\ p \le q, \quad \text{positive integers}$$

and with $z = s^{\frac{1}{nq}}$, then

$$\text{Det}(z) = z^{mq+np} - a_1 z^{np} - b_2 z^{mq} + D_A. \tag{63}$$

Hence (62) gives

$$X_1(z) = \frac{1}{z^{(n-m)q}\text{Det}(z)} \left((z^{np} - b_2) X_1(0) + a_2 z^{np-mq} X_2(0) \right) \tag{64}$$

$$X_2(z) = \frac{1}{z^{(n-m)q}\text{Det}(z)} \left(b_1 X_1(0) + (z^{np} - a_1 z^{np-mq}) X_2(0) \right). \tag{65}$$

From Descartes rule of sign, then (63) will have at most 4 real zeros if $mq + np$ is even, and at most 5 real zeros if $mq + np$ is odd.

Now factorise

$$\text{Det}(z) = \Pi_{j=1}^{N}(z - \lambda_j), \quad N = mq + np,$$

where there are at most 4 real zeros if N is even and at most 5 real zeros if N is odd. Then using (64) and (65) we can write

$$X_i(s) = \frac{s^{\frac{1}{nq}}}{s^{1-\alpha+\frac{1}{nq}}} \sum_{j=1}^{N} \frac{A_j^{(i)}}{s^{\frac{1}{nq}} - \lambda_j}, \quad i = 1, 2$$

where the $A_j^{(i)}$ can be found by writing

$$\frac{p_i(z)}{\text{Det}(z)} = \sum_{j=1}^{N} \frac{A_j^{(i)}}{z - \lambda_j}, \quad i = 1,2$$

where

$$\begin{aligned} p_1(z) &= X_1(0) z^{np} + X_2(0) a_2 z^{np-mq} - b_2 X_1(0) \\ p_2(z) &= X_2(0) z^{np} - X_2(0) a_1 z^{np-mq} + b_1 X_1(0). \end{aligned}$$

Using Lemma 2 with

$$\tilde{\alpha} = \frac{1}{nq}, \quad \tilde{\beta} = 1 - \alpha + \tilde{\alpha}$$

leads to the following result.

296 **Theorem 9.** *The solution of the mixed index 2 problem with $\alpha = \frac{m}{n}$, $\beta = \frac{p}{q}$, $m \leq n$, $p \leq q$ all positive integers*
297 *is, with $N = mq + np$, given by*

$$\begin{aligned} y(t) &= \sum_{j=1}^{N} A_j E_{\frac{1}{nq},1-\alpha+\frac{1}{nq}}(\lambda_j t^{\frac{1}{nq}}) \\ A_j &= (A_j^{(1)}, A_j^{(2)})^\top, \end{aligned} \tag{66}$$

298 *where the λ_j are the zeros of (63) and the A_j are the coefficients in the partial fraction expansion.*

299 **Remark 11.** *In the case that $\alpha = \beta$ then (66) should collapse to the solution*

$$y(t) = E_\alpha(t^\alpha A)y(0), \tag{67}$$

300 *and this is not immediately clear. However, in this case, $mq = np$ and so*

$$D(z) = z^{2np} - (a_1 + b_2)z^{np} + D(A)$$

301 *which is a quadratic function in z^{np} while the equivalent p_1 and p_2 numerator functions are linear in z^{np}. Thus*
302 *in (66) N is replaced by 2, $\frac{1}{nq}$ is replaced by α, and $1 - \alpha + \frac{1}{nq}$ becomes 1. Thus (66) reduces to*

$$y(t) = \sum_{j=1}^{2} A_j E_\alpha(\lambda_j t^\alpha)$$

303 *that then becomes (67). On the other hand if α is rational and $\beta = K\alpha$, K a positive integer, then*

$$Det(s) = (s^\alpha)^{K+1} - a_1(s^\alpha)^K - b_2 s^\alpha + D_A. \tag{68}$$

If we factorise

$$Det(s) = \Pi_{j=1}^{K+1}(s^\alpha - \lambda_j)$$

304 *and find $A_j^{(1)}$, $A_j^{(2)}$, $j = 1, \cdots, K+1$ by*

$$\sum_{j=1}^{K+1} A_j \frac{1}{s^\alpha - \lambda_j} = \frac{1}{Det(s)} \left((s^\alpha)^K X(0) + \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} (s^\alpha)^{K-1} X_2(0) + \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} X_1(0) \right) \tag{69}$$

305 *then we have the following Corollary.*

306 **Corollary 1.** *The solution of the mixed index 2 problem with α rational and $\beta = K\alpha$, K a positive integer, is*
307 *given by*

$$y(t) = \sum_{j=1}^{K+1} A_j E_\alpha(\lambda_j t^\alpha),$$

308 *where the vectors A_j and "eigenvalues" λ_j satisfy (69).*

309 As a particular example, take $K = 2$, $\alpha = \frac{p}{q}$, then the λ_j and A_j in Corollary 1 satisfy

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$$D(z) := \Pi_{j=1}^{3}(z-\lambda_j) := z^3 - a_1 z^2 - b_2 z + D_A = 0$$

and

$$\sum_{j=1}^{3} A_j \frac{1}{z-\lambda_j} = \frac{1}{D(z)} \left(X_0 z^2 + \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} X_2(0) z + \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} X_1(0) \right).$$

In other words

$$\begin{aligned} A_1(z-\lambda_2)(z-\lambda_3) + A_2(z-\lambda_1)(z-\lambda_3) + A_3(z-\lambda_1)(z-\lambda_2) \\ = X_0 z^2 + \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} X_2(0) z + \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} X_1(0) \end{aligned}$$

or

$$[A_1 \;\; A_2 \;\; A_3] = \left[X_0, \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix} X_2(0), \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} X_1(0) \right] S^{-1}$$

with

$$S = \begin{bmatrix} 1 & -(\lambda_2+\lambda_3) & \lambda_2\lambda_3 \\ 1 & -(\lambda_1+\lambda_3) & \lambda_1\lambda_3 \\ 1 & -(\lambda_1+\lambda_2) & \lambda_1\lambda_2 \end{bmatrix}.$$

Clearly in the case described by Corollary 1, writing the solution as a linear combination of generalised Mittag-Leffler functions makes the evaluation of the solution much more computationally efficient.

3.3. Simulations

In this section we give a variety of asymptotic stability and dynamics results for different parameter values of the linear mixed index models.

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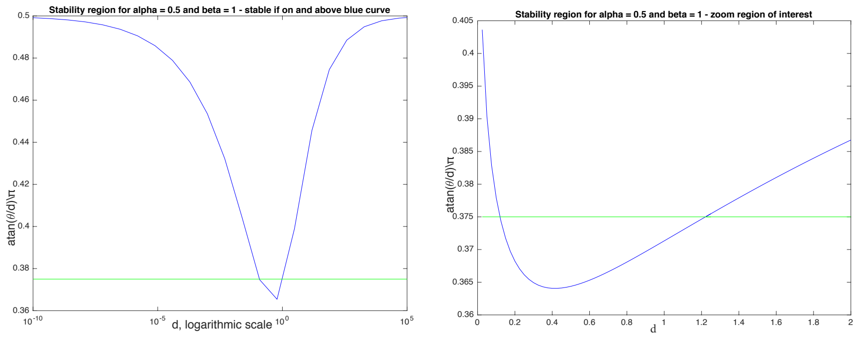

Figure 2. Stability region, above the blue line, for choosing d and θ, when the eigenvalues of A are $d \pm i\theta$, $\alpha = \frac{1}{2}, \beta = 1$. The logarithmic scale is explored in the right hand figure where the stability boundary dips below the angle $\frac{3\pi}{8}$.

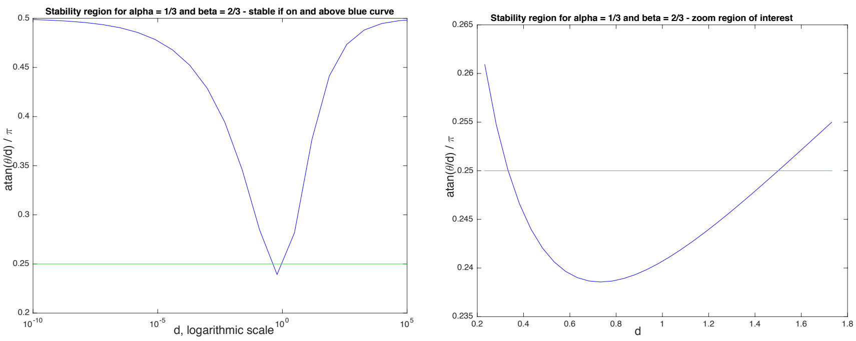

Figure 3. Stability region, above the blue line, for choosing d and θ, when the eigenvalues of A are $d \pm i\theta$, $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$. The logarithmic scale is explored in the right hand figure where the stability boundary dips below the angle $\frac{\pi}{4}$.

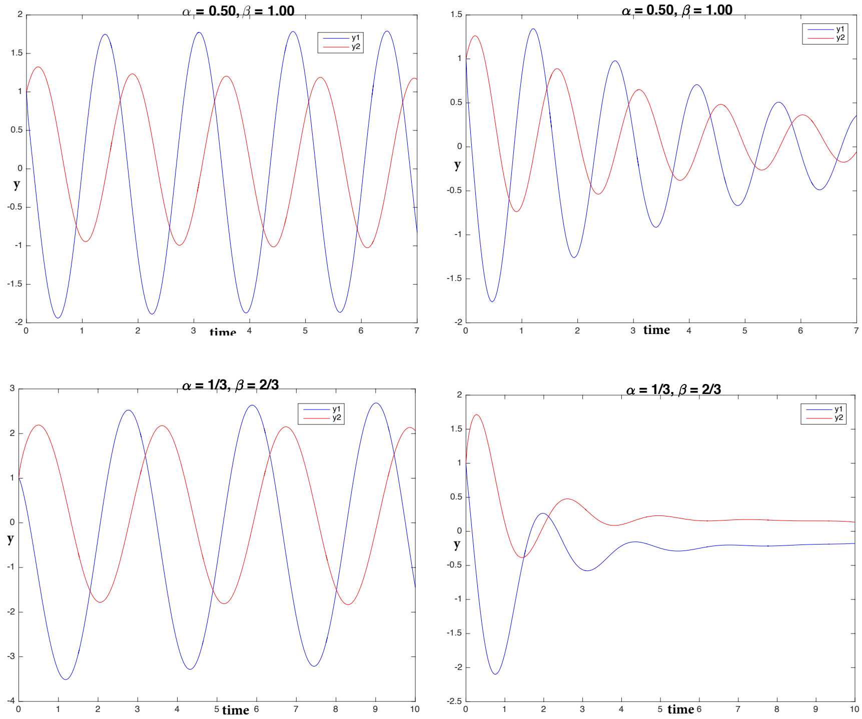

Figure 4. System Dynamics with $(\alpha, \beta) = (\frac{1}{2}, 1)$, top, and $(\alpha, \beta) = (\frac{1}{3}, \frac{2}{3})$, bottom. The left hand column shows sustained dynamics with $d = 1$ and θ chosen so that (d, θ) lies on the stability boundary. The right hand column corresponds to the same d but 0.3 has been added to the θ value.

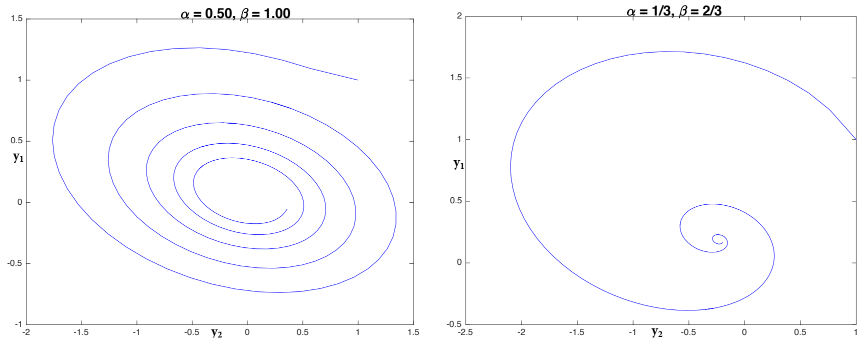

Figure 5. Phase Plots of y_1 versus y_2 for the decaying solutions in the right hand column of Figure 4.

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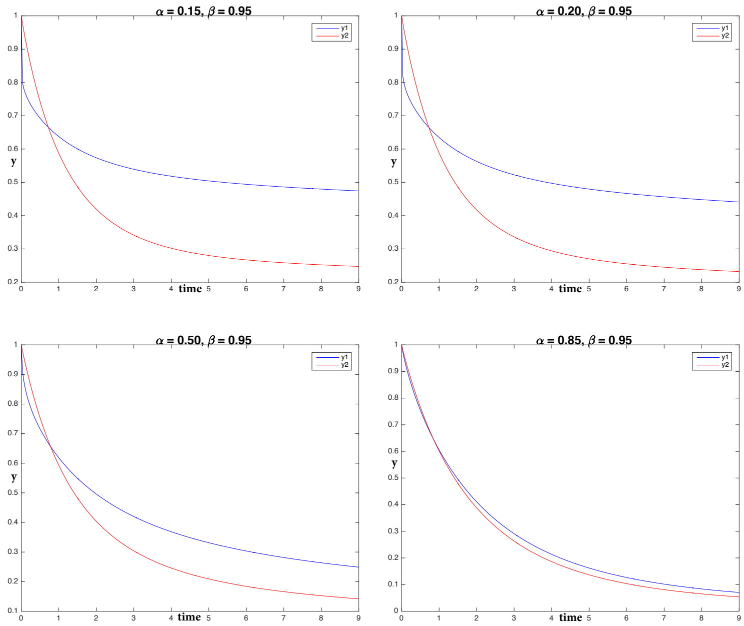

Figure 6. For A given by (71) with $d = -1$, $\theta = \frac{1}{2}$ so that the eigenvalues are $-\frac{3}{2}$, $-\frac{1}{2}$, showing the effect of variation of α with fixed β on the system dynamics.

320 ## 4. Discussion

321 In Figures 2 and 3 we plot the asymptotic stability boundary of the two dimensional, index-two
322 problem given by (1) where

$$A = \begin{pmatrix} d & -\theta \\ \theta & d \end{pmatrix}, \quad d > 0 \tag{70}$$

323 for the two cases considered in section 3.2, namely $\beta = 1$, $\alpha = \frac{1}{2}$ (Figure 2) and $\beta = \frac{2}{3}$, $\alpha = \frac{1}{3}$ (Figure
324 3). Since the eigenvalues of A are $d \pm i\theta$, we plot on the vertical axis the angle $\hat{\theta}$ in radians, where
325 $\hat{\theta} = \frac{1}{\pi}\arctan(\frac{\theta}{\lambda})$, as a function of d. In Figure 2 we see that $\hat{\theta} \in (\frac{1}{4}, \frac{1}{2})$ corresponding to an angle lying
326 between 45° and 90°, as expected from the theory. We also plot the angle, in green, corresponding to
327 the midpoint between these two extremes, i.e. $\frac{3}{8}\pi$. We see that for the most part the asymptotic stability
328 angle lies above this midpoint except for the values of d, as shown in the right hand figure.

329 In the case of Figure 3, we give a similar plot as Figure 2. We also plot in green the midpoint between
330 the two lines subtended by angles $\frac{1}{3}\pi$ and $\frac{1}{6}\pi$, namely $\frac{1}{4}\pi$. As with Figure 2 there is a small range of d
331 for which the asymptotic stability angle drops beneath $\frac{1}{4}\pi$. Furthermore, it is clear from Remark 9(ii)
332 that as α and β approach one another, the asymptotic stability boundary will be almost constant over
333 increasingly longer periods of d and will only asymptotically approach the angle $\frac{\pi}{2}$ for very small and
334 very large values of d.

335 In Figure 4 we confirm the asymptotic stability analysis showing sustained and decaying oscillations
336 with $\alpha = \frac{1}{2}, \beta = 1$ (top panel) and $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$ (bottom panel). In all four cases, $d = 1$ while for
337 the top panel we take $\theta = \sqrt{2(1+\sqrt{3})}$, $\theta = \sqrt{2(1+\sqrt{3})} + 0.3$, while for the bottom panel we take
338 $\theta = \frac{\sqrt{3}}{4}\sqrt{\sqrt{33}-1}$, $\theta = \frac{\sqrt{3}}{4}\sqrt{\sqrt{33}-1} + 0.3$.

In Figure 5 we present phase plots of y_1 versus y_2 for the two decaying oscillations cases. The figures confirm our theoretical results on the asymptotic stabiity boundary and also show the effects that the fractional indices have on the period of the solutions. As α approaches β we expect the oscillatory behaviour to disappear.

Finally, in Figure 6 we consider the problem

$$A = \begin{pmatrix} d & \theta \\ \theta & d \end{pmatrix}, \quad d < 0 \tag{71}$$

in which case the eigenvalues of A are $d \pm \theta$. We take $d = -1$, $\theta = \frac{1}{2}$ and present the solutions for four pairs of indices, namely $(\alpha, \beta) = (0.85, 0.95)$, $(0.5, 0.95)$, $(0.2, 0.05)$, $(0.15, 0.95)$. The simulations show that the components of the solution y_1 and y_2 seem to pick up "energy" from one another due to the coupling and that as the distance between α and β grows there is a greater separation between the two components. Finally, as α gets smaller, the solutions appear to "flat-line" more quickly.

5. Conclusions

In this paper we have studied mixed index fractional differential equations with coupling between the different components. We find an analytical expression for the solution of the linear system that generalises the Mittag-Leffler expansion of a matrix and the solution of linear sequential fractional differential equations. We can use this result to derive new numerical methods that generalise the concept of exponential methods used in the approximation of the Mittag-Leffler matrix function, see [34–36], for example, and exponential integrators [37], [38]. The second element would deal with developing numerical techniques for the integration component that incorporates the integral of a function times a Green function. We also use Laplace transform techniques to find the asymptotic stability domain in terms of the eigenvalues of the defining linear system. Finally we have also used Laplace transforms to get analytical expansions of the mixed index problem in terms of a sum of Mittag-Leffler or generalised Mittag-Leffler functions, in the case that the fractional indices are rational.

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References

1. A. Bueno-Orovio, D. Kay, V. Grau, B. Rodriguez, K. Burrage, Fractional diffusion models of cardiac electrical propagation: Role of structural heterogeneity in dispersion of repolarization, J. R. Soc. Interface 11 (97) Aug 6 (2014) 20140352.
2. N. Cusimano, A. Bueno-Orovio, I. Turner, and K. Burrage, On the order of the fractional Laplacian in determining the spatio-temporal evolution of a space-fractional model of cardiac electrophysiology, PLoS ONE Vol. 10 Dec 2 (2015) e0143938.
3. A. Bueno-Orovio, I. Teh, J. E. Schneider, K. Burrage, V. Grau, Anomalous diffusion in cardiac tissue as an index of myocardial microstructure, IEEE Trans. Med. Imaging, 35 (9), Sept. (2016) 2200–2207.
4. B. Henry, T. Langlands, Fractional cable models for spiny neuronal dendrites, Phys. Rev. Lett. 100 (12) Mar 28 (2008) 128103.

5. R. Magin, X. Feng, D. Baleanu, Solving the fractional order Bloch equation, Concepts in Magnetic Resonance 470 Part A 34A (2009) 16–23.
6. J. Klafter, B. White, M. Levandowsky, Microzooplankton feeding behavior 465 and the Lévy walk, Lecture Notes in Biomath. 89 (1990) 281–296.
7. N. Cusimano, K. Burrage, P. Burrage, Fractional models for the migration of biological cells in complex spatial domains, ANZIAM J. Electron. Suppl. 54 (2013) C250–C270.
8. S. Shen, F. Liu, Q. Liu, V. Anh, Numerical simulation of anomalous infiltration in porous media, Numerical Algorithms 68 (2015) 443–454.
9. J. Carcione, F. Sanchez-Sesma, F. Luzon, J. Perez Gavilan, Theory and simulation of time-fractional fluid diffusion in porous media, J. Phys. A 46 (2013) 345501.
10. R. Metzler, J. Klafter, I. M. Sokolov, Anomalous transport in external fields: Continuous time random walks and fractional diffusion equations extended, Phys. Rev. E 58(2) (1998) 1621–1633.
11. R. Klages, G. Radons, I. Sokolov, Anomalous transport, Wiley-VCH Verlag GmbH & Co., 2008.
12. R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (1) (2000) 1–77.
13. G. M. Mittag-Leffler, Sur la nouvelle function Ea, C.R. Acad. Sci. Paris 137 (1903) 554–558.
14. T. Tian, A. Harding, K. Inder, R. G. Parton, J. F. Hancock, Plasma membrane nanoswitches generate high-fidelity Ras signal transduction, Nat Cell Biol. 9(8) (2007) 905–914.
15. D. T. Gillespie, Exact stochastic simulation of coupled chemical reactions, J. Phys. Chem. 81(25) (1977) 2340–2361.
16. A. Kilbas, H. M Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies 204, Elsevier, Amsterdam 2006.
17. Q. Yu, F. Liu, I. Turner and K. Burrage, Numerical simulation of the fractional Bloch equations, J. Comp. Appl. Math. 255 (2014) 635–651.
18. F. Liu, M. M. Meerschaert, R. McGough, P. Zhuang and Q. Liu, Numerical methods for solving the multi-term time fractional wave equations, Fractional Calculus & Applied Analysis, 16(1) (2013) 9–25.
19. S. Qin, F. Liu, I. Turner, Q. Yu, Q. Yang and V. Vegh, Characterization of anomalous relaxation using the time-fractional Bloch equation and multiple echo T2*-weighted magnetic resonance imaging at 7T, Magnetic Resonance in Medicine, in press (accepted 29 February, 2016).
20. I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
21. A. Sunarto, J. Sulaiman, A. Saudi, Implicit finite difference solution for time-fractional diffusion equations using AOR method, Journal of Physics: Conference Series 495 (2014) 012032.
22. K. S. Miller, B. Ross, Fractional Green's functions, Indian J. Pure Appl. Math. 22(9) (1991) 763–767.
23. L. Vazquez, Fractional diffusion equations with internal degrees of freedom, J. Comp. Math. 21(4) (2003) 491–494.
24. D. Matignon, Stability result on fractional differential equations with applications to control processing. In Proceedings of IMACS-SMC, 963–968, Lille, France, 1998.
25. W. Deng, C. Li, J. Lu, Stability analysis of linear fractional differential system with multiple time delays, Nonnlinear Dyn., 48 (2007) 409–416.
26. C.P. Li, F.R. Zhang, A survey on the stability of fractional differential equations, Eur. Phys. J. Special Topics, 193, 27–47, 2011.
27. H. Saberi Najafi, A. Refaki Sheikhani, A. Ansari, Stability analysis of distributed order Fractional Differential Equations, Abstract and Applied Analysis, 1017 5323, 2011.
28. F. Zhang, C. Li, Y. Chen, Asymptotical stability of nonlinear fractional differential system with Caputo Derivative, Int. J. of Diff. Eqns, 2011.
29. A.G. Radwan, A.M. Soliman, A.S. Elwakil et al., Chaos, Sol. Frac. 40, 2317, 2009.
30. M. Rivero, S.V. Rogosin, J.A. Tenreiro Machado, J.J. Trujillo, Mathematical Problems in Engineering, 356215, 2013.
31. J. F. Ritt, On the zeros of exponential polynomials, Trans. Amer. Math. Soc. 31 (1929) 680–686.
32. I. Petras, Stability of fractional order systems with rational orders: a survey, Fractional Calculus and Applied Analysis, Vol. 12, No. 3, 2009.
33. J. Cěrmák, T. Kisela, Stability properties of two term fractional differential equations, Nonlinear Dyn., 80, 1673–1684, 2015.

34. R. Garrappa, M. Popolizio, Evaluation of generalized Mittag-Leffler functions on the real line, Advances in Computational Mathematics Vol 39 (1) July (2013) 205–225.
35. C. Zeng, Y. Q. Chen, Global Padé Approximations of the Generalized Mittag-Leffler Function and its Inverse, Fractional Calculus and Applied Analysis Vol 18(6) December (2015) 149–156.
36. R. Garrappa, Numerical evaluation of two and three parameters Mittag-Leffler functions, SIAM J Numer Anal. 53 (2015) 1350–1369.
37. R. B. Sidje. Expokit: a software package for computing matrix exponentials, ACM Trans. Math. Softw. 24 (1998) 130–156.
38. M. Hochbruck, A. Ostermann, Exponential integrators, Acta Numerica Vol 19 May (2010) 209–286.