Chaotic Itinerancy in an Associative Memory Model

S. Galatolo[*], L. Marangio[†], M. Monge[‡] and R. B. Liberalquino[§]

Abstract

We consider a random dynamical system arising in a model of associative memory. This system can be seen as a small (stochastic and deterministic) perturbation of a determinstic system having two weak attractors which are destroyed after the perturbation. We show, with a computer aided proof, that the system has a kind of chaotic itineracy. Typical orbits are globally chaotic, while they spend relatively long time visiting attractor's ruins.

Keywords: Chaotic itinerancy, computer aided proof, neural networks.

1 Introduction

Chaotic itinerancy is a concept used to refer to a dynamical behavior in which typical orbits visit a sequence of regions of the phase space called "quasi attractors" or "attractor ruins" in some irregular way. Informally speaking, during this itinerancy, the orbits visit a neighborhood of a quasi attractor (the attractor ruin) with a relatively regular and stable motion, for relatively long times and then the trajectory jumps to another quasi attractor of the system after a relatively small chaotic transient. This behavior was observed in several models and experiments related to the dynamics of neural networks and related to neurosciences (see [15]). In this itinerancy, the visit near some attractor was associated to the appearance of some macroscopic aspect of the system like the emergence of a perception or a memory, while the chaotic iterations from a quasi attractor to another are

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associated to an intelligent (history dependent with trial and errors) search for the next thought or perception (see [16]). This kind of phenomena was observed in models of the neural behavior which are fast-slow systems or random dynamical systems (sometimes modeling as a random system the fast-slow behavior).

As far as we know the concept had not a complete mathematical formalization though in [17] some mathematical scenario is presented, showing some situations in which these phenomena may appear. Often this phenomenon is associated to the presence of a some kind of "weak" attractor, as Milnor type attractors (see e.g. [17] or [14]) in the system and to small perturbations allowing typical trajectories to escape the attractor. Chaotic itinerancy was found in many applied contexts, a sistematic treatment of the literature is out of the scope of this paper, we invite the reader to consult [15] for a wider introduction to the subject and to its literature.

In this paper, our goal is to investigate and illustrate this concept from a mathematical point of view in some meaningful example. We consider a simple one-dimensional map derived by the literature on the subject (see Section 3). This map is a relatively simple discrete time random dynamical system on the interval, obtained from a neural network justified by the findings of [12] on modeling the neurocortex with a variant of Hopfield's asynchronous recurrent neural network presented in [9]. In Hopfield's network, memories are represented by stable attractors and an unlearning mechanism is suggested in [10] to account for unpinning of these states (see also, e.g., [11]). In the network presented in [12], however, these are replaced by Milnor attractors, which appear due to a combination of symmetrical and asymmetrical couplings and some resetting mechanism. A similar map is also obtained in [21], in the context of the BvP neuron driven by a sinusodial external stimulus. They belong to a family known as Arnold circle maps (named after [1]), which are useful in physiology (see [8, equation 3]).

The model we consider is made by a deterministic map T on the circle perturbed by a small additive noise. For a large enough noise, its associated random dynamical system exhibits an everywhere positive stationary density concentrated on a small region (see [22] for an analytical treatment), which can be attributed to the chaotic itinerancy of the neural network.

In the paper, with the help of a computer aided proof, we establish several results about the statistical and geometrical properties of the above system, with the goal to show that "the behavior of this system exhibit a kind of chaotic itineracy". We show that the system is (exponentially) mixing, hence globally chaotic. We also show a rigorous estimate of the density of probability (and then the frequency) of visits of typical trajectories near

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the attractors, showing that this is relatively high with respect to the density of probability of visits in other parts of the space. This is done by a computer aided rigorous estimate of the stationary probability density of the system. The computer aided proof is based on the approximation of the transfer operator of the real system by a finite rank operator which is rigorously computed and whose properties are estimated by the computer. The approximation error from the real system to the finite rank one is then managed using an appropriated functional analytic approach developped in [3] for random systems (see also [4], [5] and [6] for applications to deterministic dynamics or iterated functions systems).

The paper is structured as follows. In the first section, we review basic definitions related to random dynamical systems, in particular, the Perron-Frobenius operator, which will play a major role. In section 2, we present our example along with an explanation of the method used to study it. Numerical results are presented in section 3 and the mathematical model of the neural network is presented in the appendix.

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2 Random dynamical systems

This section follows [20] and [13]. We denote by $(X, \mathcal{X}, p)$ a probability space and by $(M, \mathcal{A}, \mu)$ the corresponding space of sequences (over $\mathbb{N}$ or $\mathbb{Z}$), with the product σ-algebra and probability measure. Also, we denote by f the shift map on M, $f(\{x_i\}_{i\in I}) = \{x_{i+1}\}_{i\in I}$, where $I = \mathbb{N}_0$ or $\mathbb{Z}$.

Definition 1. Let $(N, \mathcal{B})$ be a measurable space. Endow $M \times N$ with the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$. A *random transformation* over f is a measurable transformation of the form

$$F : M \times N \to M \times N, \quad F(x, y) = (f(x), F_x(y)),$$

where $x \mapsto F_x$ depends only on the zeroth coordinate of x.

Suppose we have a (bounded, measurable) function $\phi : N \to \mathbb{R}$. Given a random orbit $\{y_i\}_{i\in I}$, for which we know the value of $y_0 = v \in N$, we may ask what is the expected value for $\phi(y_1)$, that is, $\mathbb{E}(\phi(y_1)|y_0 = v)$. Since the iterate depends on an outcome $x \in M$, which is distributed according to μ, this can be calculated as

$$U\phi(v) = \int_M \phi(F_x(v))\, d\mu(x).$$

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The equation above defines the *transition operator* associated with the random transformation F as an operator in the space $L^\infty(m)$ of bounded measurable functions. Dually, we can consider its *adjoint transition operator*[1] that acts in the space of probability measures η on N, defined by

$$U^*\eta(B) = \int (F_x)_*\eta(B)\, d\mu(x) = \int \eta(F_x^{-1}(B))\, d\mu(x).$$

Of particular importance are the fixed points of the operator U^*.

Definition 2. A probability measure η for N is called *stationary* for the random transformation F if $U^*\eta = \eta$.

We recall the deterministic concept of invariant measures.

Definition 3. If $h : C \to C$ is a measurable mapping in the measurable space $(C, \mathcal{C})$, we say ν is an *invariant measure* for h if $\nu(h^{-1}(E)) = \nu(E)$ for any measurable $E \subset C$.

In the one-sided case ($M = X^{\mathbb{N}}$), invariant measures and stationary measures for F are related by the following proposition.

Proposition 1 ([20, proposition 5.4])**.** *A probability measure η on N is stationary for F if and only if $\mu \times \eta$ is invariant for F.*

Another concept from deterministic dynamical systems that can be naturally extended to random dynamical systems is that of ergodicity.

Definition 4. Suppose η is stationary for F. We say that η is *ergodic* for F if either

1. every (bounded measurable) function ϕ that is η-stationary, i.e. $U\phi = \phi$, is constant in some full η-measure set;

2. every set B that is η-stationary, i.e. whose characteristic function χ_B is η-stationary, has full or null η-measure.

In fact, both conditions are equivalent (see [20, proposition 5.10]). In the one-sided case, the following proposition relates ergodicity of a random dynamical system with the deterministic concept.

Proposition 2. *A stationary measure η is ergodic for F if and only if $\mu \times \eta$ is an ergodic F-invariant measure.*

[1] These operators are related by $\int \phi\, d(U^*\eta) = \int (U\phi)\, d\eta$, for every bounded measurable $\phi : N \to \mathbb{R}$ (see [20, lemma 5.3]).

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Suppose N admits a Lebesgue measure m. It's useful to study the measures which are absolutely continuous to it by considering the effect of U^* upon their densities. This is possible, for example, if $(F_x)_* m$ is absolutely continuous to m for every $x \in X$.

Definition 5. The *Perron-Frobenius operator* with respect to the random transformation F is the operator[2]

$$L : L^1(m) \to L^1(m), \quad Lf = \frac{dU^*(fm)}{dm}.$$

We can use the Perron-Frobenius operator to define a mixing property.

Definition 6. We say that a stationary measure $\eta = h\,dm$ is *mixing* for F if the Perron-Frobenius operator L satisfies

$$\lim_{n\to\infty} \int L^n(f) g\,dm = \int f\,dm \int gh\,dm$$

for every $f \in L^1(m)$ and $g \in L^\infty(m)$.[3]

Since L and U are dual, the mixing condition can be restated as

$$\lim_{n\to\infty} \int U^n(g) f\,d\eta = \int g\,d\eta \int f\,d\eta$$

for every $f \in L^1(\eta)$ and $g \in L^\infty(\eta)$, which is similar to the usual definition of decay of correlations.

If F is mixing, then F is ergodic for η (see [13, theorem 4.4.1]). In our example, we shall verify the following stronger condition.

$$\|L^n|_V\|_{L^1(m)\to L^1(m)} \to 0, \qquad V = \left\{ f \in L^1(m) : \int f\,dm = 0 \right\}. \tag{1}$$

Remark 1. *It suffices to verify that* $\|L^n|_V\|_{L^1(m)\to L^1(m)} < 1$ *for some* $n \in \mathbb{N}$, *because* $\|L_\xi\|_{L^1} = 1$.

Proposition 3. *If the Perron-Frobenius operator L of a random dynamical system F satisfies (1), then L admits a unique stationary density h and F is mixing.*

[2] This is an extension to the deterministic case, in which the Koopman operator $Uf = f \circ F$ is used instead of the transition operator.

[3] This definition is adapted from [2, equation (1.5)].

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Proof. $\|L\|_{L^1(m)} = 1$ because L is a Markov operator [13, remark 3.2.2]. Take a density $f \in L^1(m)$. We claim that $L^n(f) \to h$ for some density h. On the contrary, there would be $\epsilon > 0$ and a subsequence $\{L^{n_k}(f)\}_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N} : \|L^{n_{k+1}}(f) - L^{n_k}(f)\|_{L^1(m)} \geq \epsilon.$$

$L^{n_{k+1}-n_k}(f) - f \in V$ because $L^{n_{k+1}-n_k}(f)$ is a density, and $2\|L|_V\|^{n_k}_{L^1(m)} \geq \epsilon$ for every $k \in \mathbb{N}$, a contradiction.
h is stationary because L is bounded, and unique because $g - h \in V$ implies $\|L^n(g) - h\|_{L^1(m)} \to 0$ for any density g. Similarly, the mixing property follows from $f - (\int f\, dm)h \in V$. $\square$

Remark 2. *Any density f must converge exponentially fast to the stationary density h. Precisely, given N and $\alpha < 1$ such that $\|L^N\|_{L^1(m)} \leq \alpha$, the fact that $f - h \in V$ implies that $\|L^n(f) - h\|_{L^1(m)} \leq C\lambda^n$ for some $C > 0$ and $\lambda = \alpha^{1/N} < 1$.*

In the following, we will use (1) as the definition of mixing property.

3 A neuroscientifically motivated example

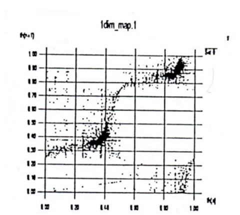

Figure 1: Data plot suggesting the model we studied ([19]).

In [12], a neural network showing successive memory recall without instruction, inspired in Hopfield's asynchronous neural network and the structure of the mammalian neocortex, is presented.

A macrovariable, related to the "activity" of the network (see figure 1, similar plots appeared also in [12] and [18]) was observed to evolve as a noisy one dimensional map in the case that the network receives no external stimulus.[4] This was regarded in [18] as a rule of successive association of memory, exhibiting chaotic dynamics.

This behavior can be modeled as a random dynamical system T_ξ, with a deterministic component given by an Arnold circle map (see figure 2) and a stochastic part given by a random additive noise. The system can be hence defined as

$$x_{n+1} = T(x_n) + \xi_n \pmod 1, \text{ where } T(v) = v + A\sin(4\pi v) + C, \quad (2)$$

[4]Its definition of can be found in [12, p. 6].

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for $A = 0.08$, $C = 0.1$ and ξ_n an i.i.d. sequence of random variables with a distribution assumed uniform over $[-\xi, \xi]$.

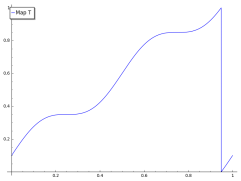

Figure 2: Deterministic component of (2).

The Perron-Frobenius operator (definition 5) associated to this system is given by $L_\xi = N_\xi L$ (a proof can be found in [13, p. 327]), where N_ξ is a convolution operator (equation 3) and L is the Perron-Frobenius operator of T.

$$N_\xi f(t) = \xi^{-1} \int_{-\xi/2}^{\xi/2} f(t - \tau). \tag{3}$$

It is well known that such an operator has an invariant probability density in L^1. In the following, we show that L_ξ can be rigorously approximated and a mixing property can be proved using the numerical estimates.

Also, a rigorous approximation of the stationary density of L_ξ is obtained, from which we can conclude the existence of a kind of chaotic itinerancy.

3.1 Rigorous approximation

Here we show how we can study the behavior of the Perron-Frobenius operator associated to (2) by approximating it by a finite rank operator. The finite rank approximation we use is known in literature as Ulam's method. For more details, see [3].

Suppose we're given a partition of $\mathbb{S}^1$ into intervals $\mathcal{I}_\delta = \{I_i\}_{i=1}^n$ and denote the characteristic function of I_i by χ_i. An operator $L : L^1(\mathbb{S}^1) \to L^1(\mathbb{S}^1)$ can be discretized by

$$L_\delta : L^1(\mathbb{S}^1) \to L^1(\mathbb{S}^1), \quad L_\delta = \pi_\delta L \pi_\delta,$$

where $\pi_\delta : L^1(\mathbb{S}^1) \to L^1(\mathbb{S}^1)$ is the projection

$$\pi_\delta h(x) = \sum_{i=1}^{n} \mathbb{E}(h|I_i)\chi_i.$$

This operator is completely determined by its restriction to the subspace generated by $\{\chi_1, \ldots, \chi_n\}$, and thus may be represented by a matrix in this base, which we call the Ulam matrix. In the following, we assume that $\delta > 0$ and $\mathcal{I}_\delta$ is a partition of $[0, 1] \pmod 1 \cong \mathbb{S}^1$ with diameter $< \delta$.

For computational purposes, the operator L_ξ is discretized as

$$L_{\delta,\xi} = \pi_\delta N_\xi \pi_\delta L \pi_\delta.$$

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This is simple to work with because it is the product of the discretized operators $\pi_\delta N_\xi \pi_\delta$ and $\pi_\delta L \pi_\delta$.

Let $V = \{f \in L^1(m) : \int f\, dm = 0\}$ and denote by f_ξ and $f_{\xi,\delta}$ the stationary probability densities for L_ξ and $L_{\delta,\xi}$, respectively. Suppose $\|L^n_{\delta,\xi}\| \leq \alpha < 1$ for some $n \in \mathbb{N}$. Since

$$\|L^i_\xi|_V\|_{L^1} \leq \|L^i_{\delta,\xi}|_V\|_{L^1} + \|L^i_{\delta,\xi}|_V - L^i_\xi|_V\|_{L^1} \tag{4}$$

and ([3, equation 2])

$$\|f_\xi - f_{\xi,\delta}\|_{L^1} \leq \frac{1}{1-\alpha}\|(L^n_{\delta,\xi} - L^n_\xi)f_\xi\|_{L^1}, \tag{5}$$

we search a good estimate of $\|(L^n_{\delta,\xi} - L^n_\xi)f_\xi\|_{L^1}$ to prove both mixing of T_ξ and give a rigorous estimate of $\|f_\xi - f_{\xi,\delta}\|_{L^1}$.

Since the calculus of $\|L^i_{\delta,\xi}|_V\|$ is computationally complex, an alternative approach is used in [3] (see [7] for a previous application of a similar idea to deterministic dynamics). First, we use a coarser version of the operator, $L_{\delta_{\text{contr}},\xi}$, where δ_{contr} is a multiple of δ. Then, we determine $n_{\text{contr}} \in \mathbb{N}$ and constants $C_{i,\text{contr}}$, for $i < n_{\text{contr}}$, and $\alpha_{\text{contr}} < 1$ in order that

$$\|L^i_{\delta_{\text{contr}},\xi}\| \leq C_{i,\text{contr}}, \quad \|L^{n_{\text{contr}}}_{\delta_{\text{contr}},\xi}|_V\| \leq \alpha_{\text{contr}}. \tag{6}$$

Finally, the following lemma from [3] is used.

Lemma 1. *Let $\|L^i_{\gamma,\xi}|_V\|_{L^1} \leq C_i(\gamma)$; let σ be a linear operator such that $\sigma^2 = \sigma$, $\|\sigma\|_{L^1} \leq 1$, and $\sigma\pi_\gamma = \pi_\gamma\sigma = \pi_\gamma$; let $\Lambda = \sigma N_\xi \sigma L$. Then we have*

$$\|(L^n_{\gamma,\xi} - \Lambda^n)N_\xi\|_{L^1} \leq \frac{\gamma}{\xi}\cdot\left(2\sum_{i=0}^{n-1} C_i(\gamma) + 1\right) \tag{7}$$

It is applied to two cases.

1. $\gamma = \delta_{\text{contr}}$, $\sigma = \pi_\delta$ and $\Lambda = L_{\delta,\xi}$ implies

$$\|(L^n_{\delta_{\text{contr}},\xi} - L^n_{\delta,\xi})N_\xi\|_{L^1} \leq \frac{\delta}{\xi}\cdot\left(2\sum_{i=0}^{n-1} C_{i,\text{contr}} + 1\right).$$

This is used to obtain $n \in \mathbb{N}$, $\alpha < 1$ and $C_i, i < n$, such that

$$\|L^i_{\delta,\xi}|_V\| \leq C_i, \|L^n_{\delta,\xi}|_V\| \leq \alpha. \tag{8}$$

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2. $\gamma = \delta$, $\sigma = \mathrm{Id}$ and $\Lambda = L_\xi$ implies

$$\begin{aligned}\|L_\xi^{n+1}|_V\|_{L^1} &\leq \|L_{\delta,\xi}^n L_\xi|_V\|_{L^1} + \|(L_\xi^n - L_{\delta,\xi}^n)L_\xi\|_{L^1} \\ &\leq \alpha + \frac{\delta}{\xi}\Big(2\sum_{i=0}^{n-1} C_i(\delta) + 1\Big).\end{aligned}$$

By remark 1, we conclude that the mixing condition is satisfied whenever

$$\lambda = \alpha + \frac{\delta}{\xi}\Big(2\sum_{i=0}^{n-1} C_i(\delta) + 1\Big) < 1.$$

We remark that a simple estimate to (5) is given by ([3, equation 4])

$$\|f_\xi - f_{\xi,\delta}\|_{L^1} \leq \frac{1 + 2\sum_{i=0}^{n-1} C_i}{2(1-\alpha)}\,\delta\xi^{-1}\ \mathrm{Var}(\rho). \tag{9}$$

The analysis of data obtained from the numerical approximation $\tilde{f}$ of $f_{\delta,\xi}$, in particular its variance, allows the algorithm in [3] to improve greatly this bound, using interval arithmetic. In table 1, the value `l1apriori` obtained from the first estimate (9) and the best estimate `l1err` are compared.

4 Results

We verified mixing and calculated the stationary density for the one dimensional system (2) using the numerical tools from the `compinv-meas` project (see [3]), which implements the ideas presented in subsection 3.1. The data obtained is summarized in table 1.

For an explanation of the values calculated, refer to subsection 3.1. In the column `l1apriori`, we have the estimate (9) for the approximation error of the stationary density in L^1 and in `l1err`, the improved estimate as in [3, section 3.2.5].

In every case, we used following sizes of the partition.

δ	$=$	2^{-19}	Used to calculate the invariant density.
δ_{contr}	$=$	2^{-14}	Used to find the estimates in equation 6.
δ_{est}	$=$	2^{-12}	Used to estimate the L^1 error of the invariant density.

In figure 3, stationary densities obtained with this method are shown.

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ξ	n_{contr}	α_{contr}	α	$\sum C_i$	`l1apriori`	`l1err`
0.732×10^{-1}	126	0.027	0.05	56.64	0.313×10^{-2}	0.715×10^{-4}
0.610×10^{-1}	167	0.034	0.067	78.66	0.530×10^{-2}	0.105×10^{-3}
0.488×10^{-1}	231	0.051	0.1	120.56	0.106×10^{-1}	0.184×10^{-3}
0.427×10^{-1}	278	0.068	0.14	156.45	0.163×10^{-1}	0.268×10^{-3}
0.366×10^{-1}	350	0.087	0.19	213.17	0.273×10^{-1}	0.432×10^{-3}
0.305×10^{-1}	453	0.12	0.26	307.03	0.523×10^{-1}	0.813×10^{-3}
0.275×10^{-1}	532	0.14	0.32	380.64	0.776×10^{-1}	0.122×10^{-2}
0.244×10^{-1}	596	0.19	0.41	467.70	0.124	0.202×10^{-2}

Table 1: Summary of the L^1 bounds on the approximation error obtained for the range of noises ξ, where n_{contr} and α_{contr} are chosen in order to satisfy equation 6 so that the values α and $\sum C_i$ obtained through lemma 1 attempt to minimize the error `l1err` obtained through the algorithm in [3].

We also studied the system (2) in the case that $A = 0.07$ for the same range of noises (table 2). In figure 4, stationary densities obtained in this case are shown. We note that the same kind of "chaotic itinerancy" obtained in the main case is observed.

ξ	n_{contr}	α_{contr}	α	$\sum C_i$	`l1apriori`	`l1err`
0.732×10^{-1}	183	0.03	0.059	83.57	0.466×10^{-2}	0.255×10^{-4}
0.610×10^{-1}	237	0.046	0.089	119.31	0.822×10^{-2}	0.282×10^{-4}
0.488×10^{-1}	332	0.069	0.14	186.80	0.170×10^{-1}	0.323×10^{-4}
0.427×10^{-1}	406	0.087	0.18	244.95	0.267×10^{-1}	0.358×10^{-4}
0.366×10^{-1}	494	0.12	0.25	330.89	0.459×10^{-1}	0.419×10^{-4}
0.305×10^{-1}	500	0.3	0.46	419.92	0.974×10^{-1}	0.646×10^{-4}
0.275×10^{-1}	596	0.32	0.52	517.97	0.151	0.807×10^{-4}
0.244×10^{-1}	600	0.49	0.73	573.04	0.326	0.189×10^{-3}

Table 2: Summary of the L^1 bounds on the approximation error obtained for the range of noises ξ, for the system (2) with the alternative value $A = 0.07$.

5 Conclusion

We've shown how the numerical approach developed in [3] can be used to study dynamical properties for a one dimensional random dynamical system of interest in the areas of physiology and neural networks.

In particular, we established mixing of the system and a rigorous estimate of its stationary density, which allowed us to observe that the trajectories concentrate in certain "weakly attracting" and "low chaotic" regions of the

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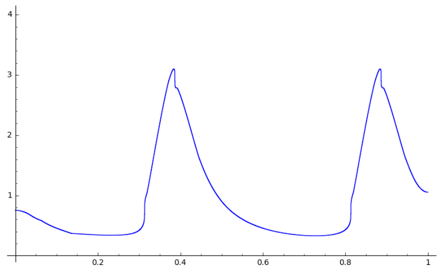
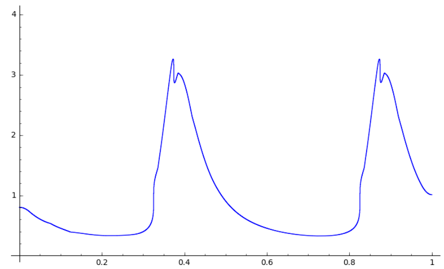
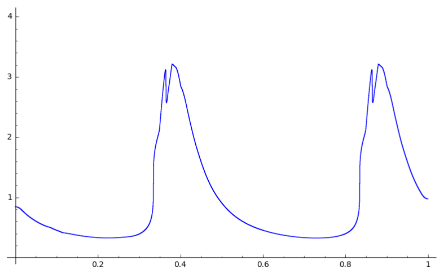


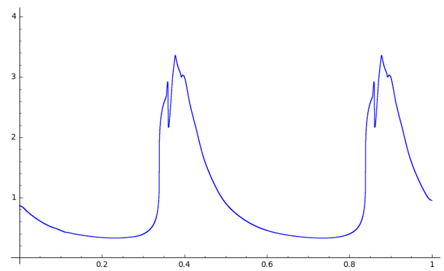
(a) $\xi = 0.732 \times 10^{-1}$



(b) $\xi = 0.488 \times 10^{-1}$

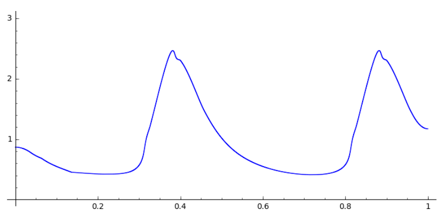


(c) $\xi = 0.305 \times 10^{-1}$

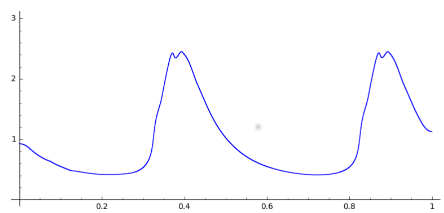


(d) $\xi = 0.214 \times 10^{-1}$

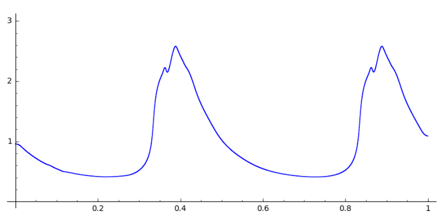
Figure 3: Approximated stationary densities $f_{\xi,\delta}$ for T_ξ, with $\delta = 2^{-19}$ and $A = 0.08$.



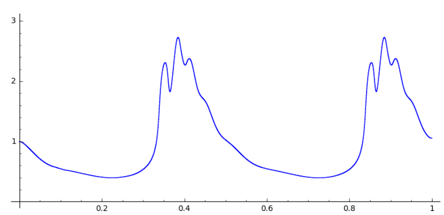
(a) $\xi = 0.732 \times 10^{-1}$



(b) $\xi = 0.488 \times 10^{-1}$



(c) $\xi = 0.305 \times 10^{-1}$



(d) $\xi = 0.214 \times 10^{-1}$

Figure 4: Approximated stationary densities $f_{\xi,\delta}$ for T_ξ, with $\delta = 2^{-19}$ and $A = 0.07$

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space, in concordance with the concept of chaotic itinerancy. The concept itself still had not a complete mathematical formalization, and deeper understanding of the systems where it was found is important to extract ist characterizing mathematical aspects.

The work we have done is only preliminary, to get some first rigorous evidence of the chaotic itineracy in the system. Further investigations are important to understand the phenomenon more deeply. In first place it would be important to understand more precisely the nature of the phenomenon: rigorously computing Lyapunov exponents and other chaos indicators. It would be also important to investigate the robustness of the behavior of the system under various kinds of perturbations, including the zero noise limit. Another important direction is to refine the model to adapt it better to the experimental data shown in Figure 1, with a noise intensity which depend on the point.

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