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Article

Construction of fullerenes and Pogorelov polytopes with 5-, 6- and one 7-gonal face

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Abstract: A Pogorelov polytope is a combinatorial simple 3-polytope realizable in the Lobachevsky (hyperbolic) space as a bounded right-angled polytope. It has no 3- and 4-gons and may have any prescribed numbers of k-gons, $k \geq 7$. Any polytope with only 5-, 6- and at most one 7-gon is Pogorelov. For any other prescribed numbers of k-gons, $k \geq 7$, we give an explicit construction of a Pogorelov and a non-Pogorelov polytopes. Any Pogorelov polytope different from Löbel polytopes can be constructed from the 5- or the 6-barrel by cuttings off pairs of adjacent edges and connected sums with the 5-barrel along a 5-gon with the intermediate polytopes being Pogorelov. For fullerenes there is a stronger result. Any fullerene different from the 5-barrel and the $(5,0)$-nanotubes can be constructed by only cuttings off adjacent edges from the 6-barrel with all the intermediate polytopes having 5-, 6- and at most one additional 7-gon adjacent to a 5-gon. This result can not be literally extended to the latter class of polytopes. We prove that it becomes valid if we additionally allow connected sums with the 5-barrel and 3 new operations, which are compositions of cuttings off adjacent edges. We generalize this result to the case when the 7-gon may be isolated from 5-gons.

Keywords: Fullerenes; right-angled polytopes; truncation of edges; connected sum; k-belts; p-vector

MSC: 52B05, 52B10, 05C75, 05C76

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1. Introduction

By an *n-polytope* we mean a combinatorial convex n-dimensional polytope, that is a class of combinatorial equivalence of convex n-dimensional polytopes. For details on the theory of polytopes we recommend the books [1,2]. A 3-polytope P is called a *Pogorelov polytope* (see [3–5]), if it can be realized in Lobachevsky (hyperbolic) space $\mathbb{L}^3$ as a bounded polytope with right dihedral angles. An n-polytope is called simple if any its vertex is contained in exactly n facets. A *flag* polytope is a simple polytope such that any its set of pairwise intersecting facets has a non-empty intersection. A *k-belt* is a cyclic sequence of facets with empty common intersection such that two facets are adjacent if and only if they follow each other. It can be shown that a 3-polytope P is flag if and only if it is different from the simplex Δ^3 and has no 3-belts. Results by A.V. Pogorelov [6] and E.M. Andreev [7] imply that a 3-polytope P is a Pogorelov polytope if and only if it is flag and has no 4-belts. An example of Pogorelov polytopes is given by *fullerenes* – simple 3-polytopes with only 5- and 6-gonal faces. It follows from results by T. Doslic that fullerenes are flag [8] and have no 4-belts [9]. They are mathematical models for spherical-shaped carbon molecules discovered in 1985 by R.F. Curl [10], H.W. Kroto [11], and R.E. Smalley [12] (Nobel Prize 1996 in chemistry). Surveys on mathematical theory of fullerenes see in [13,14]. We also recommend a remarkable paper by W.P. Thurson [15], who gives a parametrization for the set of all fullerenes. Another example of Pogorelov polytopes is given by *k-barrels* (or Löbel polytopes (see [5,16,17]), see Fig. 1 for $k = 9$) – simple 3-polytopes with the boundary glued from two equal parts consisting of a k-gon surrounded by 5-gons.

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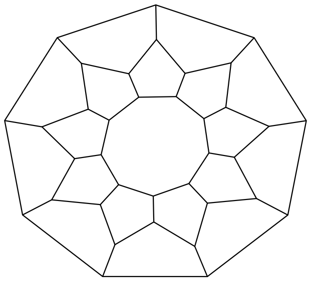

Figure 1. The 9-barrel.

A nice characterization of flag and Pogorelov polytopes is given by the following result.

Proposition 1 ([3,4]). *A simple* 3*-polytope is flag if and only if any its face is surrounded by a belt. A simple* 3*-polytope is a Pogorelov polytope if and only if any pair of its adjacent faces is surrounded by a belt.*

There are two operations transforming Pogorelov polytopes into Pororelov polytopes. First of them is a cutting off s subsequent edges of a k-gonal face, $2 \leq s \leq k-4$, of a simple 3-polytope by a single plane and is called an (s,k)*-truncation*, see Fig. 2(a). If the inverse operation is defined, we call it a *straightening along an edge*, see Fig. 2(b).

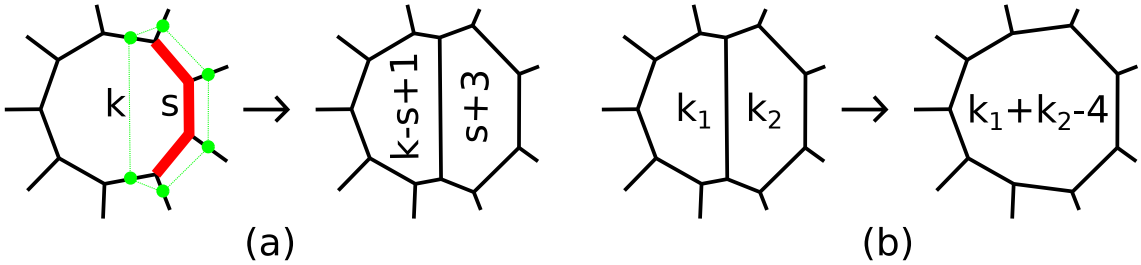

Figure 2. (**a**) An (s,k)-truncation. (**b**) A straightening along an edge.

If the k-gon in adjacent to an m_1- and an m_2-gon by edges next to cut edges, then we call the operation an $(s,k;m_1,m_2)$*-truncation* (see Fig. 3). We do not take into account an orientation of the surface of the polytope; hence we do not distinguish between $(s,k;m_1,m_2)$- and $(s,k;m_2,m_1)$-truncations.

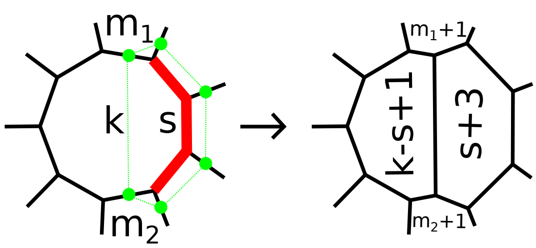

Figure 3. An $(s,k;m_1,m_2)$-truncation.

The second operation we need is a connected sum of 3-polytopes along k-gons surrounded by k-belts. It is the combinatorial analog of gluing of two polytopes along k-gonal faces orthogonal to adjacent faces.

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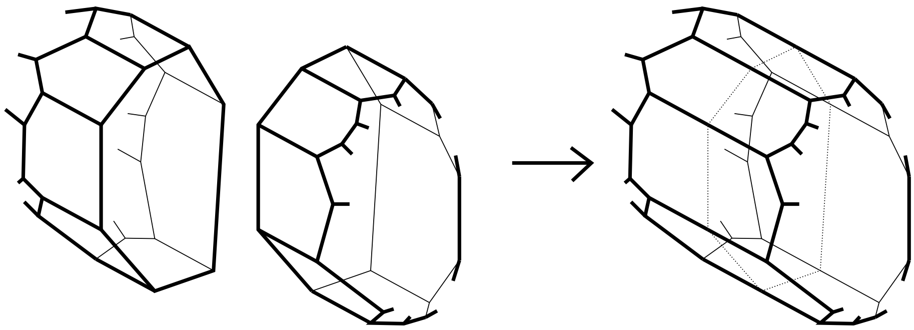

Figure 4. A connected sum of two polytopes along faces.

The existence of certain combinatorial types of 3-polytopes we usually verify using the Steinitz theorem (see [1,2]). We formulate it in the form (see, for example, [13,25]) convenient for our arguments.

Theorem 1 (Steinitz)**.** *A simple connected plane graph G is the graph of some convex* 3*-dimensional polytope if and only if any its face is bounded by a simple edge-cycle and boundary cycles of any two faces either do not intersect, or intersect by a vertex, or intersect by an edge.*

Moreover, there is a Whitney's theorem (see [1]), which states that a plane realization of the graph of a 3-polytope is combinatorially unique. Using the Steinitz theorem the following fact may be proved ([13], see also [4])

Theorem 2. *Let P be a connected* 3*-valent plane graph with each face bounded by a cycle with at least* 5 *and at most* 7 *edges, where the number of boundary cycles with* 7 *edges is at most one. Then this graph is a graph of a simple* 3*-polytope.*

In [13] the polytopes with 5-, 6- and one 7-gon are called 7*-disk-fullerenes*. Denote by $\mathcal{F}$ the family of fullerenes, by $\mathcal{P}_7$ the family of 7-disk-fullerenes, by $\mathcal{P}_{7,5}$ its subfamily consisting of polytopes with the 7-gon adjacent to a 5-gon, by $\mathcal{P}_{\leq 7,5}$ the family $\mathcal{F}\sqcup\mathcal{P}_{7,5}$, and by $\mathcal{P}_{\leq 7}$ the family $\mathcal{F}\sqcup\mathcal{P}_7$. In [4] the following generalization of Theorem 2 was proved.

Theorem 3. *Let $P\in\mathcal{P}_{\leq 7}$. Then P is a Pogorelov polytope.*

This result leads to a natural question. Let p_k be the number of k-gonal faces of a simple 3-polytope P. The collection $(p_k, k\geq 3)$ is called a *p-vector*. There Euler formula in the case of simple 3-polytopes implies the following formula (see [2]), which can be proved by a direct calculation:

$$3p_3+2p_4+p_5=12+\sum_{k\geq 7}(k-6)p_k. \tag{1}$$

V. Eberhard proved ([19], see also [2]) that for any finite collection of non-negative integers $(p_k, k\geq 3, k\neq 6)$ satisfying the equation (1) there exists a simple 3-polytope P with $p_k(P)=p_k$ for all $k\neq 6$. A flag polytope has no 3-gons. On the base of Eberhard's result it was proved in [18] that for any finite collection of non-negative integers $(p_k, k\geq 4, k\neq 6)$ satisfying the equation (1) there exists a flag polytope P with $p_k(P)=p_k$, $k\neq 3,6$. The proof used the construction of a simultaneous cutting off all the edges of a simple 3-polytope by different planes, see Fig. 5. This operation does not change the numbers p_k, $k\neq 6$, and increases the number p_6 by the number of edges.

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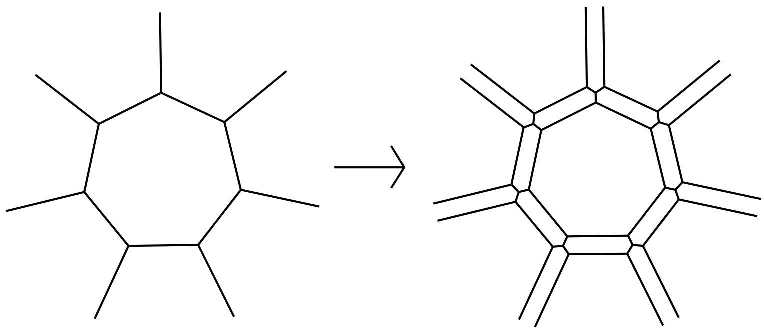

Figure 5. A cutting off all the edges of a polytope by different planes.

It turns out that for a polytope with no 3-gons the cut polytope is flag. A Pogorelov polytope has no 3- and 4-gons, since any face of a flag polytope is surrounded by a belt. In [3,4] is was proved that for any finite collection of non-negative integers $(p_k, k \geq 7)$ there exists a Pogorelov polytope with $p_k(P) = p_k$, $k \geq 7$. Moreover, $p_5(P) = 12 + \sum_{k\geq 7}(k-6)p_k$. The proof is similar to the case of flag polytopes. Namely, for a polytope without 3- and 4-gons the cut polytope is a Pogorelov polytope.

Question. *Which restrictions on the numbers $(p_k, k \geq 7)$ imply that a polytope without 3- and 4-gons is a Pogorelov polytope?*

We have seen that the example is given by the restriction $p_7 \leq 1$, $p_k = 0$, $k \geq 8$.

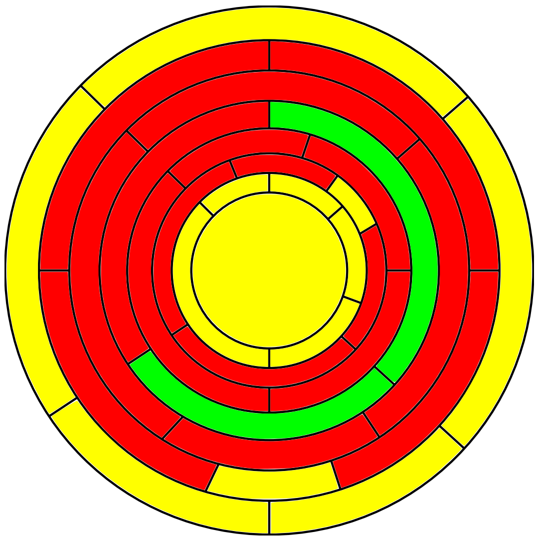

Figure 6. A graph of a polytope with 5-, 6- and two 7-gonal faces containing a 3-belt.

Example 1. *On Fig. 6 we present the graph of a simple 3-polytope (this can be easily checked using the Steinitz theorem) with 5-, 6- and two 7-gonal faces. This polytope has a 3-belt containing both 7-gons, hence it is not a Pogorelov polytope.*

The first main result of our paper is the answer to this question.

Theorem 4 (The first main result)**.** *For any finite collection of non-negative integers $(p_k, k \geq 7)$ with $\sum_{k\geq 7} p_k > 1$ or $p_7 = 0$ and $\sum_{k\geq 7} p_k = 1$ there exists a non-flag simple polytope P with $p_k(P) = p_k$, $k \geq 7$.*

Remark 1. *We will also give a slight modification of this construction producing a Pogorelov polytope with prescribed numbers p_k, $k \geq 7$, not using Ebrehard's result.*

Hence $\mathcal{P}_{\leq 7}$ is a natural subclass in the class of Pogorelov polytopes.

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It can be shown ([20], see also [4]) that an (s,k)-truncation sransforms a Pogorelov polytope into a Pogorelov polytope if and only if $2 \leq s \leq k-4$, and a connected sum of any two Pogorelov polytopes along faces is a Pogorelov polytope.

It is easy to see that k-barrels, $k \geq 5$, are irreducible polytopes with respect to operations of an (s,k)-truncation and a connected sum along faces in the class of Pogorelov polytopes. It follows from results in [20] that a simple 3-polytope P is a Pogorelov polytope if and only if either P is a k-barrel for some $k \geq 5$, or P can be obtained from q-barrels, $q \geq 5$, by a sequence of operations of an (s,k)-truncation, $2 \leq s \leq k-4$, and a connected sum along p-gons. In [4] the following stronger result was proved.

Theorem 5 ([4]). *A simple 3-polytope P is a Pogorelov polytope if and only if either P is a k-barrel, $k \geq 5$, or it can be obtained from the 5-, or the 6-barrel by a sequence of operations of a $(2,k)$-truncation, $k \geq 6$ (Fig. 7(a)), and operations of a connected sum with the 5-barrel along a 5-gon (Fig. 7(b)).*

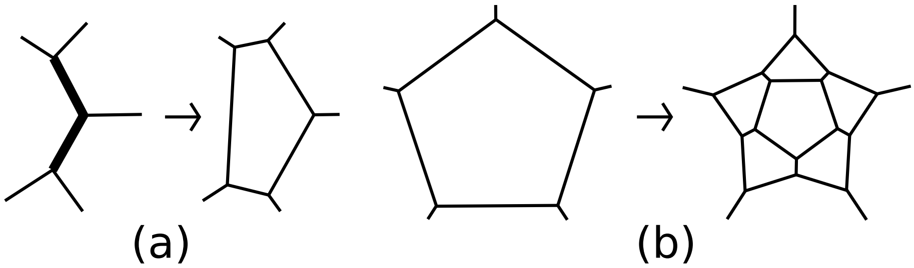

Figure 7. (**a**) A $(2,k)$-truncation. (**b**) A connected sum with the 5-barrel.

This result is related to classical result in the polytope theory. It was proved by V. Eberhard [19] and by M.Bruckner [21] (see also [2]), that a 3-polytope is simple if and only if it can be obtained from the 3-simplex by a sequence of operations of cutting off a vertex, an edge or a pair of two adjacent edges by a single plane. This result was used by a famous crystallographer E. S. Fedorov [22]. From a result by V.D. Volodin [23] it follows that a simple 3-polytope is flag if and only if it can be obtained from a 3-cube by a sequence of operations of an (s,k)-truncation, $1 \leq s \leq k-3$. In [18] this result was improved. Namely, a simple 3-polytope P is flag if and only if it can be obtained from the 3-cube by a sequence of $(2,k)$-truncations, $k \geq 6$. For fullerenes there are analogs of this result (see [4,24–27]). The starting point can be taken to be the 5- or the 6-barrel, but the difficulty is that the only (s,k)-truncation transforming fullerenes to fullerenes is a $(2,6;5,5)$-truncation, also called an *Endo-Kroto operation* [28]. This is a *growth operation*, that is it transforms a simple 3-polytope into a simple 3-polytope substituting a new *patch* (disk partitioned into polygons bounded by a simple edge-cycle on the surface of a simple polytope) with more faces and the same boundary for a patch of a polytope. It was proved in [29] that there is no finite sets of growth operations transforming fullerenes to fullerenes sufficient to construct any fullerene from a finite set of initial fullerenes (*seeds*). In [27] an infinite family of growth operations with this property was found. In [4,24–26] *finite sets* of growth operations sufficient to built any fullerene from a finite set of seeds was found *on account of allowing*, at intermediate steps, simple 3-polytopes with 5-, 6- and one 7-gon adjacent to some 5-gon. By Theorem 3 any such polytope is a Pogorelov polytope.

Let us formulate the most strong result in this direction improving Theorem 5 for a special class of polytopes. Let us introduce a special subfamily of fullerenes. The first polytope D_0 is the dodecahedron (the 5-barrel). D_5 is a connected sum of two copies of D_0. $D_{5(k+1)}$ is a connected sum of D_{5k} with D_0 along a 5-gon surrounded by 5-gons (see Fig. 8). The polytopes D_{5k}, $k > 0$, are called $(5,0)$-*nanotubes*. Denote the family of polytopes $\{D_{5k}, k \geq 0\}$ by $\mathcal{D}$.

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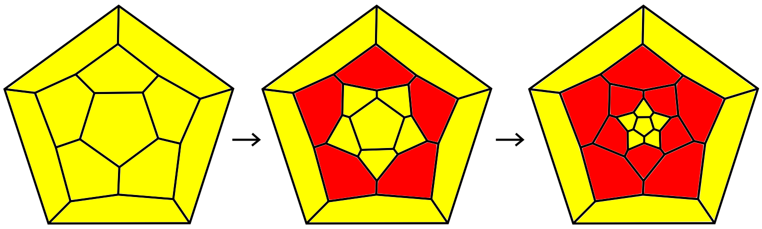

Figure 8. A construction of $(5,0)$-nanotubes.

Theorem 6 ([4]). *Any fullerene $P \in \mathcal{D}$ can not be obtained from a simple 3-polytope without 4-gons by a $(2,k)$-truncation, $k \geq 6$. Any fullerene $P \in \mathcal{F} \setminus \mathcal{D}$ can be obtained from the 6-barrel by a sequence of $(2,6;5,5)$-, $(2,6;5,6)$-, $(2,7;5,5)$-, and $(2,7;5,6)$-truncations in such a way that any intermediate polytope is either a fullerene or a polytope in $\mathcal{P}_{7,5}$.*

Nevertheless, not any polytope in $\mathcal{P}_{7,5}$ can be obtained by a connected sum with the 5-barrel or by a $(2,k)$-truncation from a polytope in $\mathcal{P}_{\leq 7,5}$. The example is given by the polytope with the graph drawn on Fig. 9. Indeed, a connected sum with the 5-barrel produces a 5-gon surrounded by 5-gons, and a $(2,k)$-truncation produces a 5-gon with one edge lying in an r-gon, $r = 5$ or 6, and intersecting by vertices a p- and a q- gon with $p, q \geq 6$. In the presented polytope P any such edge belongs to a 6-gon and intersects two 6-gons, which means that the polytope Q transforming to P contains two 7-gons.

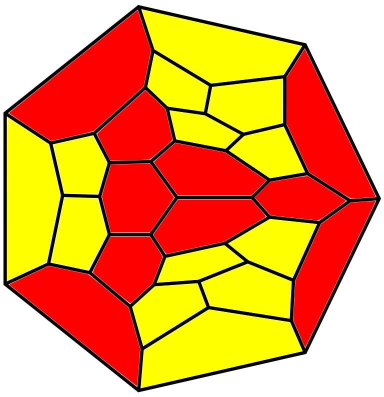

Figure 9. A polytope in $\mathcal{P}_{7,5}$, which can not be obtained from a polytope in $\mathcal{P}_{\leq 7,5}$ by a $(2,k)$-truncation or a connected sum with the 5-barrel.

Let us mention that a connected sum with the 5-barrel is evidently a growth operation. Also an $(s,k;m_1,m_2)$-truncation, $2 \leq s \leq k-4$ is a growth operation on the class of flag polytopes, since it substitutes the patch consisting of the new 5-gon, and the $(k-1)$-, (m_1+1)-, and (m_2+1)-gons for the patch consisting of the corresponding k-, m_1- and m_2-gons.

Our second main result gives the method to construct any polytope in $\mathcal{P}_{\leq 7,5} \setminus \mathcal{D}$ from the 6-barrel by a sequence of growth operations from the finite list in such a way that intermediate polytopes belong to the same family.

Theorem 7 (The second main result). *Any polytope in $\mathcal{P}_{\leq 7,5} \setminus \mathcal{D}$ can be obtained from the 6-barrel by a sequence of growth operations each being either a connected sum with the 5-barrel, a $(2,6;5,5)$-,$(2,6;5,6)$-, $(2,7;5,5)$-, $(2,7;5,6)$-truncation, or one of the operations O_1, O_2, O_3 drawn on Fig. 10 in such a way that intermediate polytopes also belong to $\mathcal{P}_{\leq 7,5} \setminus \mathcal{D}$. Any of the operations O_1, O_2, O_3 is a composition of $(2,6;5,6)$-, $(2,7;5,5)$-, $(2,7;5,6)$-truncations such that intermediate polytopes are Pogorelov polytopes with 5-, 6-, and at most two 7-gonal faces.*

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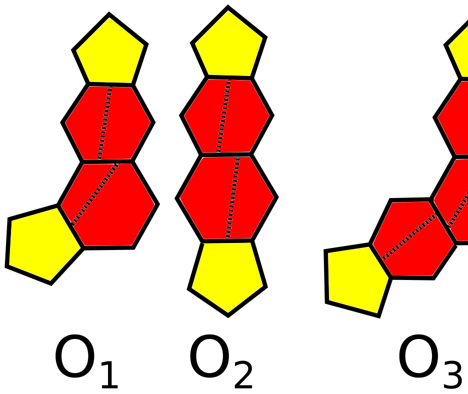

Figure 10. Three growth operations. Dotted lines denote edges arising during the operation.

The third main result concerns all the polytopes in $\mathcal{P}_7$. There are polytopes $P \in \mathcal{P}_7$, which can not be obtained by any of the operations used in Theorem 7 from any polytope $Q \in \mathcal{P}_{\leq 7}$. To obtain an example we can cut off all the edges of any polytope in $\mathcal{P}_7$ several times. The resulting polytope still belongs to $\mathcal{P}_7$, but it has the non-hexagonal faces far from each other. Then it can be obtained from some polytope $Q \in \mathcal{P}_{\leq 7}$ only by a $(2,7;5,5)$-truncation. But Q should have two 7-gons. A contradiction. To generalize Theorem 8 to the class $\mathcal{P}_{\leq 7}$ and a finite set of growth operations we add a $(2,7;6,6)$-truncation and allow intermediate polytopes to have two 7-gons.

Theorem 8 (The third main result)**.** *Any polytope in $\mathcal{P}_{\leq 7} \setminus \mathcal{D}$ can be obtained from the 6-barrel by a sequence of growth operations each being either a connected sum with the 5-barrel, a $(2,6;5,5)$-,$(2,6;5,6)$-, $(2,7;5,5)$-, $(2,7;5,6)$-, $(2,7;6,6)$-truncation, or one of the operations O_1, O_2, O_3 in such a way that intermediate polytopes are Pogorelov polytopes not in $\mathcal{D}$ with 5-, 6- and at most two 7-gonal faces.*

2. Proof of the main results

Proof of the first main result (Theorem 4). We will develop the idea of Example 1 corresponding to the case $p_7 = 2$, $p_k = 0$, $k \geq 8$. First let us take the disk drawn on Fig. 11(a). Let β be its boundary circle. If $p_7 = 0$, $p_8 = 1$, and $p_k = 0$, $k \geq 9$, then add to F_1 two 2-valent vertices on β to become a 8-gon, and to F_2 and F_3 one 2-valent vertex to become 6-gons. Then glue to the boundary of the disk a copy of the disk lying inside the 3-belt $\mathcal{B} = (F_1, F_2, F_3)$ to obtain a graph of a polytope due to the Steinitz theorem. This graph can be also obtained by adding to the figure the image of the graph inside the belt under the circle inversion interchanging the boundary circles of $\mathcal{B}$.

Now let either $\sum_{k\geq 9} p_k > 0$, or $\sum_{k\geq 9} p_k = 0$ and $(p_7, p_8) \notin \{(2,0),(0,1)\}$. For each $k \geq 7$ with $p_k \neq 0$ take p_k k-gons and arrange all the polygons in a descending order of numbers of edges. Add to F_1 vertices of valency 2 on β to become the first polygon. If $\sum_{k\geq 7} p_k \geq 3$, do the same for F_2, F_3 and the second, the third polygons. Else take 6-gons instead of lacking polygons. Let m_1, m_2, m_3 be the numbers of edges of F_1, F_2 and F_3. The number ν of 2-valent vertices on β is equal to $m_1 + m_2 + m_3 - 16$. Then $\nu \geq 5$, since either $m_1 \geq 9$, $m_2, m_3 \geq 6$, or $m_1 = 8$, $m_2 \geq 7$, $m_3 \geq 6$, or $m_1 = 7 = m_2 = m_3$. Also any face has at least one 2-valent vertex on β. If there are still polygons not in use, we form from them a ν-belt of faces around $\mathcal{B}$, taking 6-gons for lacking polygons intersecting 2 edges on the boundary of $\mathcal{B}$, and 5-gons for lacking polygons intersecting one edge, if necessary. Each face of the new belt $\mathcal{B}_1$ has at least one 2-valent vertex on the outer bundary circe β_1, hence the number ν_1 of 2-valent vertices on β_1 is not smaller than $\nu \geq 5$. Repeat this argument until all the polygons are in use. Now add one new belt consisting only of 5- and 6-gons, where each 5-gon intersects the boundary of the previous disk by one edge, and each 6-gon by two edges. We obtain a new disk with the boundary faces having 2 edges on the boundary circle, where the number b of boundary faces is it least 5 (see Fig. 11(b) for the case $(p_7, p_8, p_9) = (0,2,1)$, $p_k = 0$, $k \geq 10$.).

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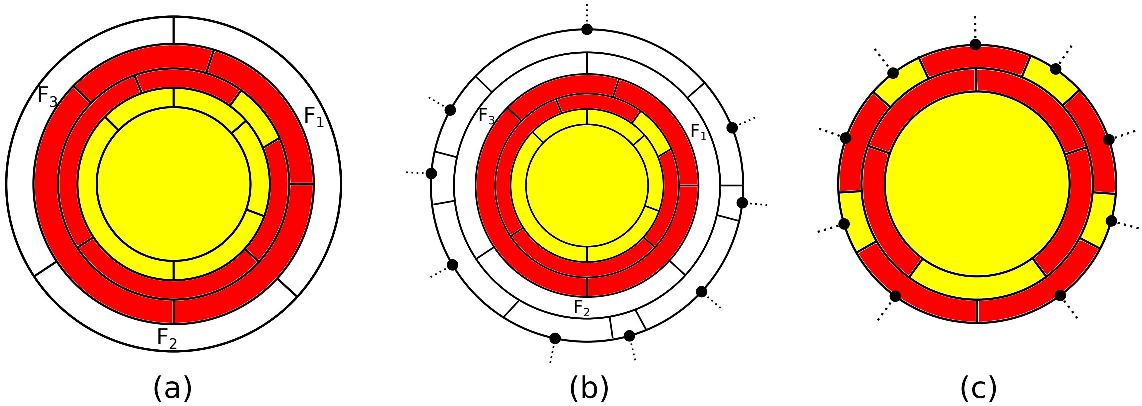

Figure 11. (**a**) An initial disk. (**b**) An addition of belts. (**c**) A construction of the complementary disk.

Let us build another disk with the identical boundary neighbourhood. First take a 5-gon. Add a 5-belt of faces around it consisting of c pentagons and d hexagons, $c + d = 5$. This belt has $\mu = c + 2d = 5 + d$ vertices of valency 2 on the outer boundary circle, and each face has at least one 2-valent vertex. If $b \leq 10$ then take $d = b - 5$, $c = 10 - b$. Else take $c = 0$, $d = 5$ and add a new belt of faces around the obtained disk, where 3-valent vertices on the boundary circle γ of the disk correspond to 6-gons of the belt (we say that they are of the first type), and edges of γ connecting 2-valent vertices correspond to 5-gons and 6-gons (of the second type). In the new belt any face has at least one 2-valent vertex on the outer boundary circle γ_1, and the total number μ_1 of the 2-valent vertices on γ_1 is equal to μ plus the number of 6-gons of the second type. If the value of μ_1 can not reach the number b by varying the number of 6-gons of the second type, then make this value maximal possible and add new belts in the same manner. In the end we add the last belt without 6-gons of the second type to obtain the desired disk.

Now glue both disks together to obtain a 2-sphere with a 3-valent graph on it. We claim that this graph is a graph of a simple 3-polytope. Indeed, any face by construction is a disk bounded by a simple edge-cycle. Two faces intersect if and only if either one of them is the centre of one of the disks and the other belongs to the belt surrounding it, or they are subsequent faces of the same belt, or they belong to subsequent belts. In the first two cases it is evident that the faces intersect by an edge. In the last case this is also true, since by construction any face of a new belt in each disk intersects any face of the previous belt either by the empty set, or by an edge, and the same is true for faces of the boundary belts of disks. This finishes the proof of the theorem. □

Corollary 1. *A slight modification of the proof of Theorem 4 gives a new explicit construction of a Pogorelov polytope with given numbers $(p_k, k \geq 7)$ different from constructions based on Eberhard's [19] and Grünbaum's [32] constructions of polytopes with given p-vectors and an operation of a cutting off all the edges.*

Construction 1. *For $\sum_{k\geq 7} p_k = 0$ take any fullerene. Let $\sum_{k\geq 7} p_k > 0$. For each $p_k \neq 0$, $k \geq 7$, take p_k k-gons and arrange all the polygons in a linear order. If there are more than one polygon, add around the first polygon a belt of polygons from the remaining list, taking 5-gons for missing faces, if necessary. If not all polygons are in use, add new belts by the same manner, taking 6-gons for lacking polygons intersecting 2 edges on the boundary of the previous belt, and 5-gons for lacking polygons intersecting one edge. In the end add around the disk the last belt of 5- and 6-gons with 3-valent vertices on the boundary of the disk corresponding to 6-gons, and the edges on the boundary of the disk connecting 2-valent vertices corresponding to 5-gons. We have the disk with $b \geq 7$ boundary faces each having 2 edges on the boundary circle. The number of faces in added belts does not decrease, in particular each belt has at least 7 faces. Take the second disk with the same boundary neighbourhood constructed above. In this disk the number of faces in added belts also does not decrease, in particular each belt has at least 5 faces. Glue the two disks along the boundaries to obtain a 2-sphere with a plane graph corresponding to a simple 3-polytope with prescribed numbers p_k, $k \geq 7$. We claim that this polytope is a Pogorelov polytope.*

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Proof. We will prove that P has no 3- and 4-belts. First observe that a 3- or a 4-belt can not contain the centre of one of the two disks in construction, since any two non-subsequent faces of the belt surrounding the centre are not adjacent in the polytope and do not intersect the same face outside this belt by construction. The polytope P outside the centres of the disks consists of the belts added in construction. Let us call them *levels*. In each disk arrange levels in the order they were added. Let us call the top level of a disk a *boundary level*.

Let (F_i, F_j, F_k) be a 3-belt. Since adjacent faces should belong to the same or adjacent levels, and a 3-belt can not belong to one level, two faces, say F_i and F_j, lie on one level L_1, and F_k on another level L_2. If L_2 is next to L_1 in one disk, or both levels are boundary, then F_k intersects at most two faces, which should intersect it by a common vertex. A contradiction. If L_1 is next to L_2, then F_i and F_j are subsequent faces of the level. By construction there are at least 5 faces on L_2, each having a 2-valent vertex on the circle between L_1 and L_2, whence the edge $F_i \cap F_j$ intersects F_k. A contradiction. Thus, P has no 3-belts.

Let (F_i, F_j, F_k, F_l) be a 4-belt. Since it can not belong to one level, assume that F_i and F_j lie on adjacent levels L_2 and L_1. Without loss of generality assume that either both levels are boundary, or L_2 is next to L_1 in one disk. Then F_i intersects at most two faces on L_1, which should intersect it by a common vertex. Since $F_i \cap F_l \neq \emptyset$, and $F_j \cap F_l = \emptyset$, F_l lies either on L_2, or on the third level L_3. In the first case F_l and F_i are subsequent in L_2 and F_j is one of the two faces intersecting F_i on L_1. The second face intersects F_l. The face F_k should intersect both F_j and F_l, hence it lies on L_1 or L_2. If it lies on L_2, it is a subsequent to F_l and can not intersect F_j. If it lies on L_1, it is one of the two faces intersecting F_l on L_1, and it does not intersect F_i. Then it does not intersect F_j. A contradiction. Now let F_l lie on L_3. Since F_k intersects both F_j and F_l, it lies on L_2. If L_1 and L_2 belong to the same disk, then L_3 is either next to L_2, or both L_2 and L_3 are boundary levels. Then F_i and F_k should be adjacent, since they both intersect F_l on L_2. A contradiction. If L_1 and L_2 are boundary levels, then F_i and F_k should be adjacent, since they both intersect F_j on L_2. A contradiction. Hence P has no 4-belts and it is a Pogorelov polytope. □

Example 2. *For the case $p_7 = 2$, $p_k = 0$, $k \geq 8$, the first disk is drawn on Fig. 12. The second disk is drawn on Fig. 11(c).*

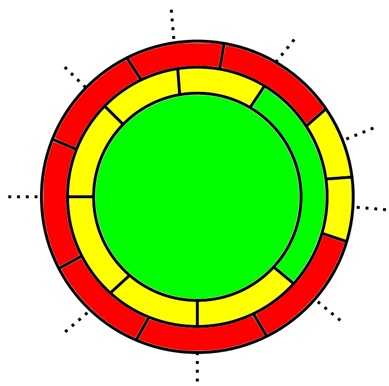

Figure 12. The first disk for the case $p_7 = 2$, $p_k = 0$, $k \geq 8$. The second disk is drawn on Fig. 11(c).

Remark 2. *Construction 1 of Pogorelov polytopes with given numbers $(p_k, k \geq 7)$ can be generalized by taking two disks of the first type and substituting several belts of 5- and 6-gons for the last belt of the disk with shorter boundary circle to make the lengths of the boundary circles equal. Then for the case $p_7 = 2$, $p_k = 0$, $k \geq 8$, the modified construction can produce the 7-barrel.*

Now we proceed to prove the second and the third main result. We call by a *k-loop* a cyclic sequence of faces with adjacent subsequent faces. Since any face of a flag 3-polytope is surrounded by a belt, if a Pogorelov polytope contains a 5-gon surrounded by 5-gons, these 6 faces together form a patch, which we denote C_1, see Fig. 13(a).

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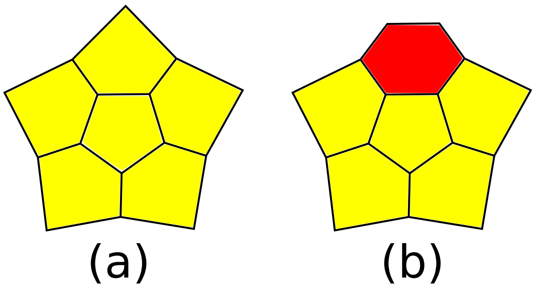

Figure 13. (**a**) A patch C_1. (**b**) Another patch.

For a k-belt $\mathcal{B} = (F_{i_1}, \dots, F_{i_k})$ the set $\bigcup_{j=1}^{k} F_{i_j}$ is homeomorphic to a cylinder. Each its boundary component has a boundary code $(\alpha_1, \dots, \alpha_k)$ corresponding to the number of edges of faces lying on this component. We will need the following result. For fullerenes it follows from results in [30,31] (see also [24,25], and [4, Theorem 2.12.1]). For polytopes in $\mathcal{P}_7$ it was proved in [4, Theorem 3.2.6].

Theorem 9. *Let $P \in \mathcal{P}_{\le 7}$. Then any 5-belt either surrounds a face and has on this side the boundary code $(1,1,1,1,1)$, or surrounds a patch obtained by addition of $r \ge 0$ 5-belts of 6-gons around the patch C_1 and has on this side the boundary code $(2,2,2,2,2)$.*

Proof of the second main result (Theorem 7). We start with the following

Lemma 1. *Let a polytope $P \in \mathcal{P}_{\le 7}$ contain a patch C_1. Then either P is the 5-barrel, or P is obtained from some polytope $Q \in \mathcal{P}_{\le 7}$ by a connected sum with the 5-barrel producing this patch. In particular, if $P \in \mathcal{F}$, then $P \in \mathcal{D}$, and if $P \in \mathcal{P}_7$, then P is obtained from a fullerene containing a patch drawn on Fig. 13(b) by a sequence of connected sums with the 5-barrel, where the first connected sum is along the central 5-gon of the patch, and all the other connected sums are along the central 5-gon of the arising patch C_1.*

Proof. First note that the patch C_1 is surrounded by a 5-belt on a Pogorelov polytope. Indeed, it is surrounded by a 5-loop. If two non-subsequent faces intersect, without loss of generality these are F_i and F_j drawn on Fig. 14(a). But they are non-subsequent faces of the 6-belt surrounding the adjacent 5-gons F_k and F_l. A contradiction. Thus, C_1 is surrounded by a 5-belt. If this belt contains no 5-gons, then we can apply an operation inverse to a connected sum with the 5-barrel, see Fig. 14(b). It is well defined by the Steinitz theorem and produces a polytope in $\mathcal{P}_{\le 7}$. Let one of the faces of the belt be a 5-gon, see Fig. 14(c). We claim that for $P \ne D_0$ the patch consisting of C_1 and an additional 5-gon is surrounded by a 5-belt $\mathcal{B} = (F_i, F_j, F_k, F_l, F_r)$. Indeed, faces (F_l, F_r, F_i, F_j) belong to the 5-belt surrounding C_1, whence they are distinct and $F_l \cap F_i = \varnothing = F_r \cap F_j$. Faces F_l and F_j are non-subsequent in the 6-belt surrounding two 5-gons, whence $F_l \cap F_j = \varnothing$. Faces F_i and F_k belong to the belt surrounding F_j. They are distinct, since F_j has at least 5 edges. They are adjacent if and only if F_j has exactly 5 edges. In this case the 4-loop (F_i, F_k, F_l, F_r) can not be a 4-belt, whence $F_k \cap F_r \ne \varnothing$, since $F_i \cap F_l = \varnothing$. Then $P = D_0$.

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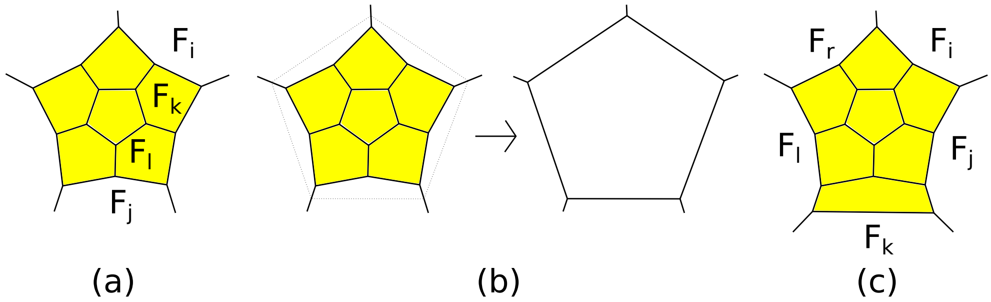

Figure 14. (**a**) A patch C_1. (**b**) An operation inverse to a connected sum. (**c**) A non-existing patch.

Thus, for $P \neq D_0$ we have $F_i \cap F_k = \varnothing$, and $F_k \cap F_r = \varnothing$ by a similar argument, and $\mathcal{B}$ is a 5-belt. By Theorem 9 either this belt surrounds a 5-gon, or each face of the belt has two edges on the outer part of the boundary ∂P of P. In the first case F_k is a 4-gon, and in the second case both F_j and F_l are 7-gons. A contradiction. The lemma is proved. $\square$

Denote the patches arising after operations of a $(2,6;5,5)$-, $(2,6;5,6)$-, $(2,7;5,5)$- or $(2,7;5,6)$-truncation, or operations O_1, O_2, or O_3, by $D_{2,6;5,5}$, $D_{2,6;5,6}$, $D_{2,7;5,5}$, $D_{2,7;5,6}$, D_1, D_2, D_3 respectively (see Fig. 15). We do not take into account the orientation. Therefore, we do not distinguish between a patch and its mirror image.

By Theorem 2 and Lemma 1 a polytope P in the class $\mathcal{A}$ can be obtained from a polytope Q in the class B by an operation of a connected sum with the 5-barrel, or of a $(2,6;5,5)$-, $(2,6;5,6)$-, $(2,7;5,5)$-, $(2,7;5,6)$- truncation, or O_1, O_2, O_3, if and only if P contains respectively a patch C_1, $D_{2,6;5,5}$, $D_{2,6;5,6}$, $D_{2,7;5,5}$, $D_{2,7;5,6}$, D_1, D_2, D_3, where $A, B = \mathcal{P}_{\leq 7}$ for a connected sum, a $(2,6;5,5)$-truncation, and operations O_1, O_2, O_3; $(A,B) = (\mathcal{P}_7, \mathcal{F})$ for a $(2,6;5,6)$-truncation; $(A,B) = (\mathcal{F}, \mathcal{P}_7)$ for a $(2,7;5,5)$-truncation; and $A, B = \mathcal{P}_7$ for a $(2,7;5,6)$-truncation. Let us call a polytope $P \in \mathcal{P}_{\leq 7}$ *irreducible*, if it can not be obtained from a polytope in $\mathcal{P}_{\leq 7}$ by these operations. Otherwise let us call P *reducible*. First we will prove that only the 5- and the 6-barrel are irreducible, and then we will explain how to avoid polytopes in $\mathcal{D}$.

It can be proved that a collection of faces of a polytope $P \in \mathcal{P}_{\leq 7}$ with the same combinatorics as in any of these patches indeed forms the corresponding patch. For the first 6 patches this follows from the fact that the collection of faces consists of two adjacent faces and some faces of the belt surrounding them. For the patch D_3 this argument works for the collection without the top face and the collection without the bottom face. These faces should be distinct, for otherwise a 4-belt arises, and they should be non-adjacent, for otherwise a 5-belt with both boundary codes different from $(1,1,1,1,1)$ and $(2,2,2,2,2)$ arises (see more details in [4, Lemma 4.0.1]).

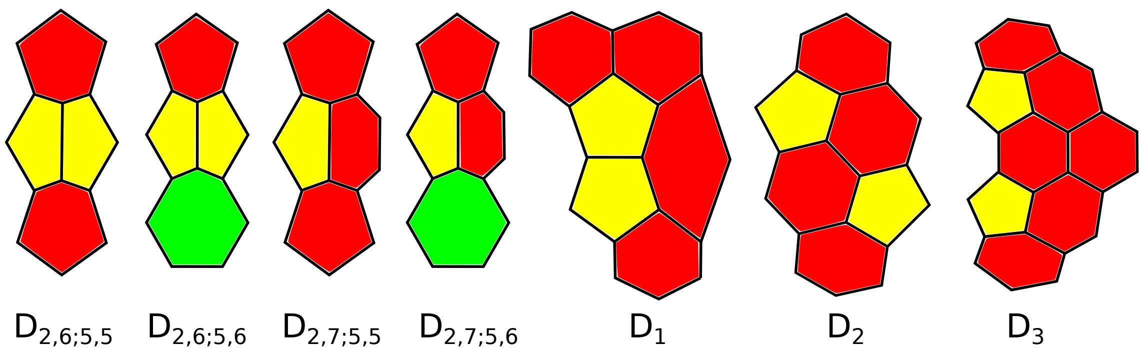

Figure 15. Patches arising after operations.

Lemma 2. *Let $P \in \mathcal{P}_{7,5}$ be irreducible. Then the 7-gon can not be adjacent to 5-gons by 3 subsequent edges.*

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Proof. The 7-gon is surrounded by a 7-belt. If 3 of its subsequent faces F_a, F_b, F_c are 5-gons then the 4 faces F_u, F_v, F_w, F_t adjacent to them and lying in the outer part of ∂P are 5-gons (see Fig. 16), for otherwise P contains one of the patches $D_{2,6;5,6}$ or $D_{2,7;5,6}$. All these seven 5-gons are distinct, since they belong to the patch formed by two adjacent 5-gons F_v and F_w and the 6-belt $\mathcal{B}$ surrounding them. Consider the 6-th face of $\mathcal{B}$. It is different from the 7-gon, since these two faces are non-subsequent in the 6-belt surrounding the 5-gons (F_c, F_w). It cannot be a 5-gon, for otherwise the patch C_1 appears. Therefore, it is a 6-gon. Consider the 5-loop $\mathcal{B}_1 = (F_i, F_j, F_k, F_l, F_r)$ arising on the boundary of $\mathcal{B}$, where F_i is the 7-gon. Any two non-subsequent faces of this loop do not intersect, since they are adjacent to the same face of this loop by non-adjacent edges. Then $\mathcal{B}_1$ is a 5-belt. Since on the side of the belt $\mathcal{B}$ it has the boundary code $(3, 2, 2, 2, 2)$, and F_i has on the other side 2 edges, by Theorem 9 the other boundary code is $(2, 2, 2, 2, 2)$, P contains the patch C_1 and is obtained by a connected sum with the 5-barrel by Lemma 1. The lemma is proved.

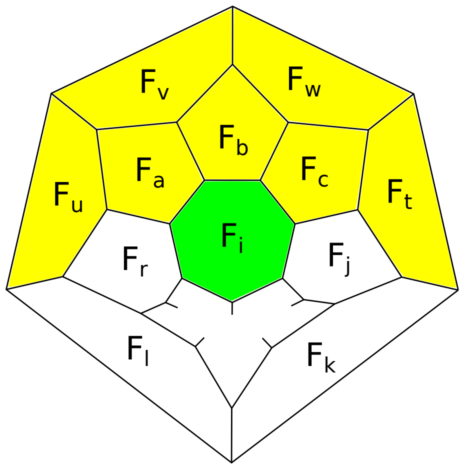

Figure 16. The 7-gon adjacent to 3 subsequent 5-gons.

□

Lemma 3. *Let $P \in \mathcal{P}_{7,5}$ be irreducible. Then the 7-gon can not be adjacent to 5-gons by 2 subsequent edges.*

Proof. The 7-gon is surrounded by a 7-belt. If 2 of its subsequent faces F_i and F_j are 5-gons then the 3 faces F_b, F_c, F_d adjacent to them and lying in the outer part of ∂P are 5-gons (see Fig. 17(a)), for otherwise P contains one of the patches $D_{2,6;5,6}$ or $D_{2,7;5,6}$. All these five 5-gons are distinct since belong to the patch formed by the 5-gon F_l and the 5-belt $\mathcal{B}$ surrounding it. Consider the 5-th face F_c of $\mathcal{B}$. It does not intersect the 7-gon, since these two faces are non-subsequent faces of the 6-belt surrounding the 5-gons (F_j, F_l). It cannot be a 5-gon, for otherwise the patch C_1 appears. Therefore, it is a 6-gon. The faces F_a and F_e are 6-gons by Lemma 2. Also F_b and F_d are 6-gons, for otherwise the patch $D_{2,6;5,5}$ appears. The face F_f is not the 7-gon, since the 7-gon and F_c are not adjacent. If F_f is a 5-gon, we obtain the patch D_2 (see Fig. 17(b)). If F_f is a 6-gon, we obtain the patch D_3 (see Fig. 17(c)). The lemma is proved.

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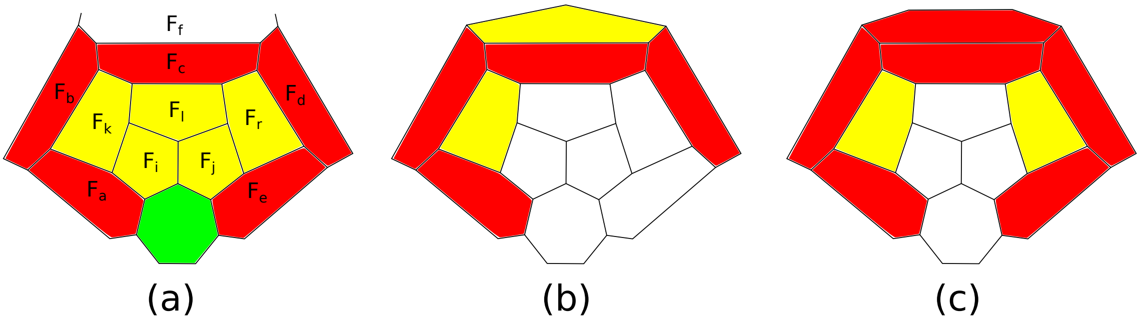

Figure 17. (**a**) The 7-gon adjacent to 2 subsequent 5-gons. (**b**) The patch D_2. (**c**) The patch D_3.

□

Lemma 4. *Any polytope $P \in \mathcal{P}_{7,5}$ is reducible.*

Proof. Let a polytope $P \in \mathcal{P}_{7,5}$ be irreducible. By definition the 7-gon F is adjacent to at least one 5-gon, say F_j. By Lemma 3 the faces F_i and F_k adjacent to F by the edges next to $F \cap F_j$ are 6-gons. The rest two faces adjacent to F_j are 5-gons, for otherwise the patch $D_{2,7;5,6}$ appears. We obtain the picture drawn on Fig. 18(a). The faces F_b and F do not intersect, since they are non-subsequent in the belt surrounding F_j and F_q. If F_b is a 6-gon, then F_a and F_c are also 6-gons, for otherwise the patch $D_{2,6;5,5}$ appears. Then P contains the patch D_1 (see Fig. 18(b)). Thus, F_b is a 5-gon (see Fig. 18(c)). The faces F_a, F_c, F_d are different from F, since $F_b \cap F = \varnothing$. If both F_a and F_c are 6-gons, then either F_d is a 5-gon and we obtain the patch $D_{2,6;5,5}$, or F_d is a 6-gon and we obtain the patch D_1. If both F_a and F_c are 5-gons, then F_d is a 6-gon, for otherwise we obtain the patch C_1. Also F_u and F_v are 6-gons, for otherwise the patch $D_{2,6;5,5}$ appears. Thus we obtain the scheme drawn on Fig. 18(d). The face F_w is different from F, for otherwise $(F_j, F_q, F_b, F_d, F_w)$ is a 5-belt, since any two non-subsequent faces of this 5-loop are adjacent to some face of this loop by non-subsequent edges. But this belt has both boundary codes different from $(1,1,1,1,1)$ and $(2,2,2,2,2)$, which contradicts Theorem 9. Like in the proof of Lemma 3 we see that either F_w is a 5-gon, and we obtain the patch D_2, or F_w is a 6-gon and we obtain the patch D_3.

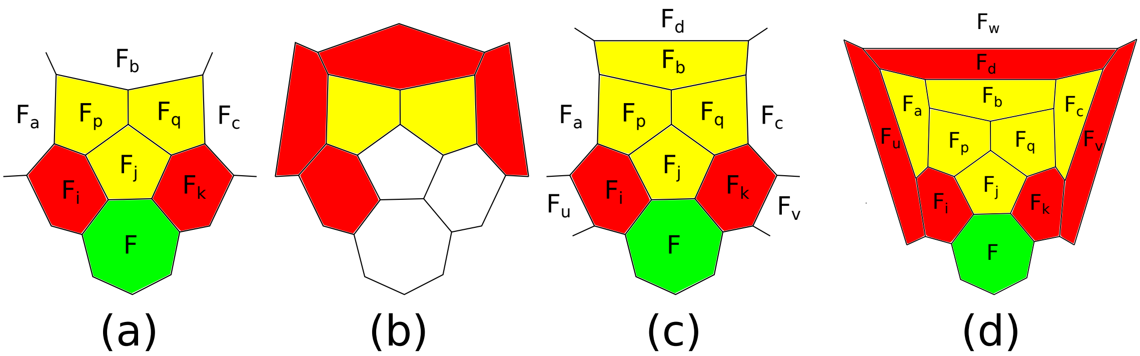

Figure 18. (**a**) The 7-gon adjacent to a 5-gon. (**b**) The patch D_1. (**c**) The case when F_b is a 5-gon. (**d**) The case when F_a and F_c are 5-gons.

Now we can assume that one of the faces F_a and F_c is a 5-gon and the other is a 6-gon. Since we do not take into account the orientation, without loss of generality assume that F_a is a 5-gon and F_c is a 6-gon (Fig. 19(a)). If F_d is a 6-gon, then F_u is also a 6-gon, for otherwise we obtain the patch $D_{2,6;5,5}$. Then we have the patch D_1 (Fig. 19(b)). Thus, F_d is a 5-gon and we obtain Fig. 19(c). The face F_t is different from F, for otherwise $(F_j, F_q, F_b, F_d, F_t)$ is a 5-belt, since any two non-subsequent faces of this 5-loop are adjacent to some face of this loop by non-subsequent edges. But this belt has both boundary codes different from $(1,1,1,1,1)$ and $(2,2,2,2,2)$.

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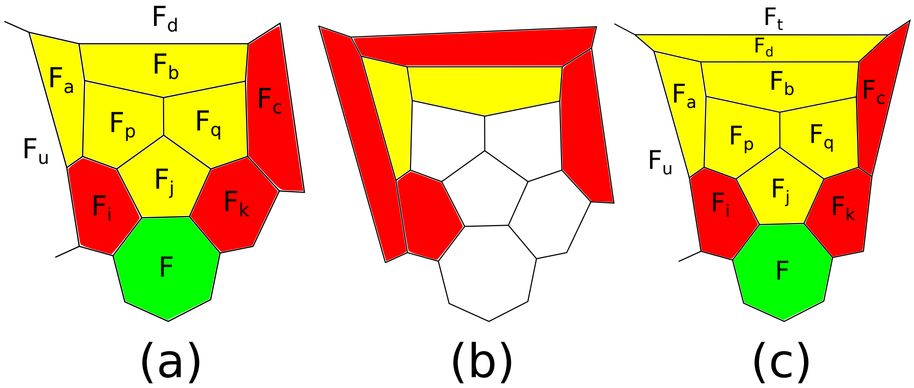

Figure 19. (**a**) The case when F_a is a 5-gon and F_c is a 6-gon. (**b**) The patch D_1. (**c**) The case when F_d is a 5-gon.

If F_u is a 5-gon, we obtain Fig. 20(a). All the 5-gons are distinct, since they consist of adjacent faces F_a, F_p and some faces of the 6-belt surrounding them. We have a 5-loop (F_s, F, F_k, F_c, F_t), which is a 5-belt, since any two non-subsequent faces of this 5-loop are adjacent to some face of this loop by non-subsequent edges. But this belt has both boundary codes different from $(1,1,1,1,1)$ and $(2,2,2,2,2)$, which contradicts Theorem 9. Hence F_u is a 6-gon and we obtain Fig. 20(b). Then if F_t is a 5-gon, we obtain the patch $D_{2,6;5,5}$, and if F_t is a 6-gon, we obtain the patch D_1 (or, more precisely, its mirror image, which we do not distinguish from it), see Fig. 20(c).

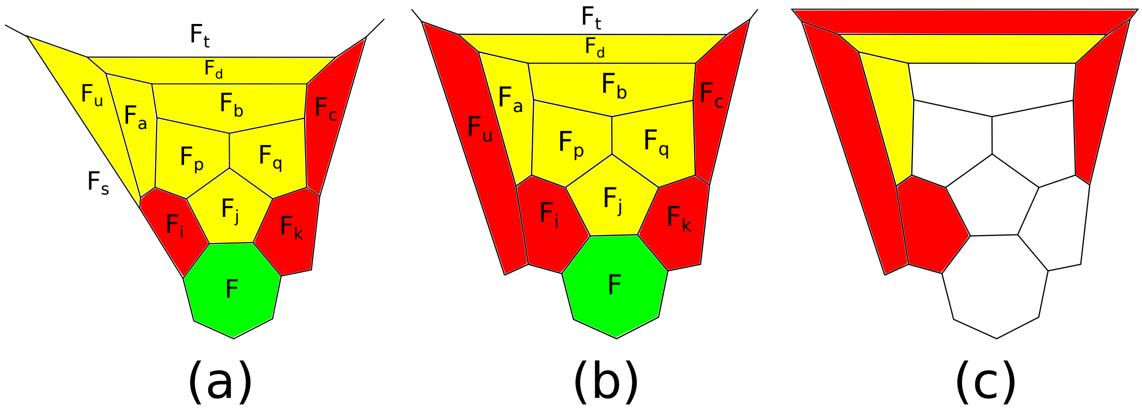

Figure 20. (**a**) The case when F_u is a 5-gon. (**b**) The case when F_u is a 6-gon. (**c**) The patch D_1.

□

Thus, any irreducible polytope in $\mathcal{P}_{\leq 7,5}$ is a fullerene. Now we will prove the result, which will be useful also for $\mathcal{P}_{\leq 7}$. For fullerenes it was proved in [4, Thorem 4.0.2 1].

Lemma 5. *Let P be a fullerene or a polytope in $\mathcal{P}_7$ with the 7-gon surrounded by 6-gons. If P has two adjacent 5-gons, then either P is the 5- or the 6-barrel, or it can be obtained from a fullerene or a polytope in $\mathcal{P}_7$ respectively by one of the operations: a connected sum with the 5-barrel, a $(2,6;5,5)$-truncation, O_1, O_2, O_3.*

Proof. We need to prove that P contains one of the corresponding patches. Assume that it is not true. Consider two adjacent 5-gons F_i and F_j. Then the edge $F_i \cap F_j$ intersects by one of its edges some 5-gon F_k, for otherwise the patch $D_{2,6;5,5}$ appears. If this patch consisting of three 5-gons with a common vertex, is surrounded by 6-gons, then P contains the patch D_1. Hence one of the faces around the patch is a 5-gon. If it intersects only one of the three 5-gons, then the edge of intersection should intersect by a vertex a new 5-gon adjacent to two 5-gons of the patch, for otherwise the patch $D_{2,6;5,5}$ appears. Therefore without loss of generality assume that the edge $F_i \cap F_j$ intersects two 5-gons F_k

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and F_l by vertices (see Fig. 21(a)). Then each pair of faces (F_p, F_q) and (F_u, F_v) contains at least one 6-gon, for otherwise the patch C_1 appears. Up to a mirror symmetry corresponding to the change of an orientation of the polytope, we have two possibilities: F_p, F_v are 5-gons (Fig. 21(b)), or F_p, F_u are 5-gons (Fig. 21(c)).

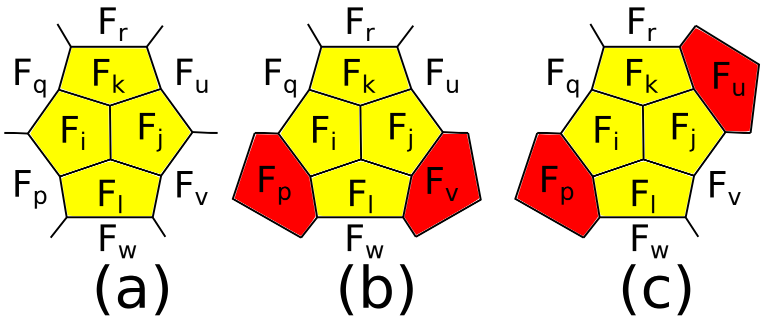

Figure 21. (**a**) Four 5-gons. (**b**) F_p and F_v are 5-gons. (**c**) F_p and F_u are 5-gons.

In the first case F_w is a 6-gon, for otherwise the patch $D_{2,6;5,5}$ appears. Then F_u and F_q are 5-gons, for otherwise the patch D_1 appears. Then F_q and F_u are 5-gons, for otherwise the patch D_1 appears. F_r is a 6-gon, for otherwise the patch C_1 appears (see Fig. 22(a)). Also faces F_s and F_t are 6-gons, for otherwise the patch $D_{2,6;5,5}$ appears. Faces F_a and F_b are distinct, since they are adjacent to F_s by distinct edges. Then one of them is not a 7-gon. If it is a 5-gon, we obtain the patch D_2 (Fig. 22(b)). If it is a 6-gon we obtain the patch D_3 (Fig. 22(c)).

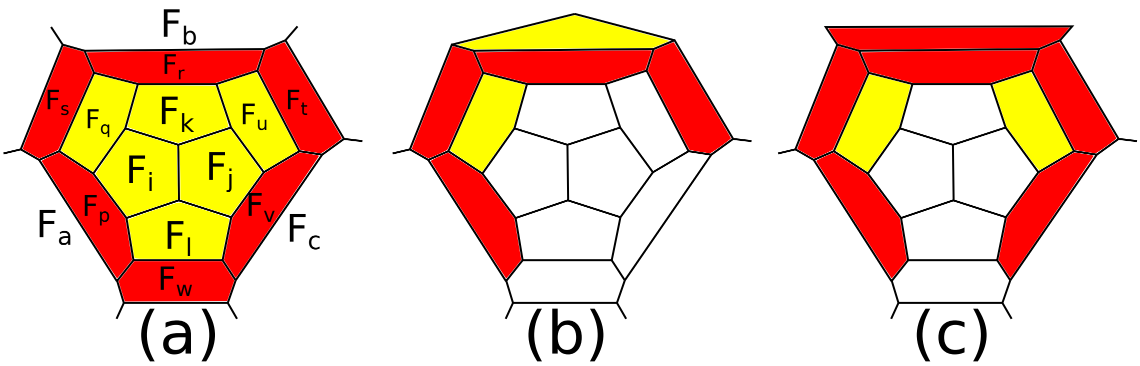

Figure 22. (**a**) F_p and F_v are 5-gons. (**b**) The patch D_2. (**c**) The patch D_3.

In the second case each pair of faces (F_q, F_r) and (F_v, F_w) contains at least one 5-gon, for otherwise the patch D_1 appears. If F_w is a 5-gon, then F_v is also a 5-gon, for otherwise the patch $D_{2,6;5,5}$ appears. Therefore we can assume that F_v is a 5-gon, and similarly F_q is a 5-gon, see Fig. 23(a). The 6-loop $(F_p, F_q, F_k, F_u, F_v, F_w)$ is a 6-belt, since any two non-subsequent faces of this loop are non-subsequent faces of the 6-belt surrounding one of the 3 pairs of adjacent 5-gons F_i, F_j, F_l. If F_w is a 5-gon, then we obtain a patch D drawn on Fig. 23(b). If both faces F_s and F_t are 6-gons, we obtain the patch D_1. If F_s is a 5-gon, then F_t is a 5-gon, for otherwise we obtain the patch $D_{2,6;5,5}$. Thus, we can assume that F_t is a 5-gon, see Fig. 23(c). The faces $(F_a, F_r, F_b, F_t, F_s)$ form a 5-loop in the complement of the patch D in the boundary of P. They are pairwise distinct, since any two non-subsequent faces of this loop are adjacent to some its face by distinct edges. Now we have the 4-loop (F_r, F_b, F_s, F_a). $F_r \cap F_s = \emptyset$, since these two faces are non-subsequent in the belt surrounding (F_p, F_q). Since P has no 4-belts, $F_a \cap F_b \neq \emptyset$. Since P has no 3-belts, $F_a \cap F_b \cap F_s$ and $F_a \cap F_b \cap F_r$ are vertices, and all the faces in the 4-loop are 5-gons. Then P is the 6-barrel. If F_w is a 6-gon, then F_t is also a 6-gon (see Fig. 23(d)), for otherwise the patch $D_{2,6;5,5}$ appears. Then we obtain the patch D_1. The lemma is proved.

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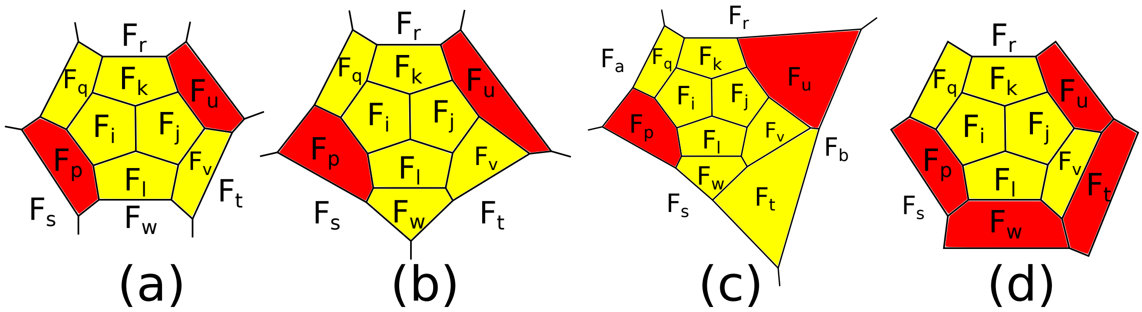

Figure 23. (**a**) F_p and F_u are 5-gons. (**b**) F_w is a 5-gon. (**c**) F_t is a 5-gon. (**d**) F_w is a 6-gon.

□

We are ready to prove the following result.

Lemma 6. *Only the* 5*- and the* 6*-barrel are irreducible polytopes in* $\mathcal{P}_{\leq 7,5}$.

Proof. The 5- and the 6-barrel are evidently irreducible. Any polytope in $\mathcal{P}_{7,5}$ is reducible by Lemma 4. If P is a fullerene different from the 5- and the 6-barrel and has adjacent 5-gons, then it is reducible by Lemma 5. If a fullerene has no adjacent 5-gons, then any its 5-gon belongs to a patch $D_{2,7;5,5}$. Hence P is reducible. □

Now we will show how to avoid polytopes in $\mathcal{D}$.

Lemma 7. *Let P be a polytope in $\mathcal{P}_{\leq 7} \setminus \mathcal{D}$. If it can be reduced to a polytope in $\mathcal{D}$, then it can also be reduced to a polytope $Q \in \mathcal{P}_{\leq 7} \setminus \mathcal{D}$.*

Proof. For a polytope D_{5k}, $k \geq 0$, the operation of a connected sum with the 5-barrel can be applied only along the central 5-gon of a patch C_1, for otherwise two 7-gons appear. This operation transforms D_{5k} into $D_{5(k+1)}$. The only other operations that can be applied to the polytope D_{5k} are a $(2,6;5,5)$-truncation, if $k = 1$, O_1 or O_2, if $k = 2$, O_3, if $k = 3$, and a $(2;6;5,6)$-truncation, if $k \geq 2$. In all the cases any of the operations makes the transformation of the patches drawn on Fig. 24 (a). Then the polytope P also contains the patch D_1 and can be reduced to a polytope $Q \in \mathcal{P}_{\leq 7}$ containing the patch $D_{2,6;5,5}$ (see Fig. 24(b)). We have $Q \notin \mathcal{D}$ and the lemma is proved.

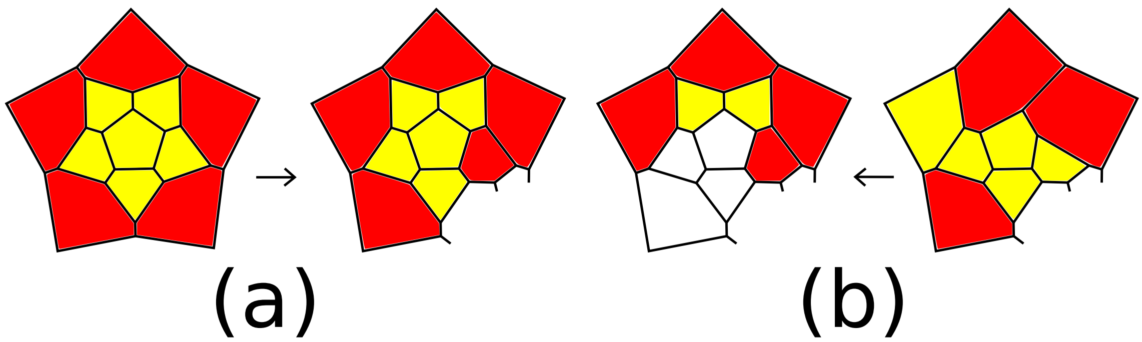

Figure 24. (**a**) A transformation of a patch. (**b**) A reduction.

□

Lemma 7 implies that any polytope in $\mathcal{P}_{\leq 7,5} \setminus \mathcal{D}$ can be reduced to the 6-barrel by a sequence of operations in such a way that intermediate polytopes also belong to $\mathcal{P}_{\leq 7,5} \setminus \mathcal{D}$. This finishes the proof of Theorem 7. □

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Proof of the third main result (Theorem 8). Consider a polytope $P \in \mathcal{P}_{\leq 7}$. If $P \in \mathcal{P}_{\leq 7,5}$, then the theorem follows from Theorem 7. If $P \in \mathcal{P}_7 \setminus \mathcal{P}_{\leq 7,5}$, and P has two adjacent 5-gons, then the theorem follows from Lemma 5 and Lemma 7. Thus it remains to consider the case of polytopes with the 7-gon and all the 5-gons isolated. By a *thick path* we call a sequence of faces $(F_{i_1}, \dots, F_{i_k})$ such that any two subsequent faces are adjacent. It is easy to see that any two faces of a simple 3-polytope can be connected by a thick path. Let us call a *length of the thick path* consisting of k faces the number $k-1$. We will use the idea presented in [27] and [24] for fullerenes. Consider the 7-gon and the shortest thick path among all thick paths connecting it to 5-gons. Then all the faces except for the first and the last are 6-gons. Since the path is the shortest, each 6-gon can not intersect the next and the previous faces by adjacent edges. We say that the path goes "forward" in the 6-gon, if these edges of intersection are opposite. If they are not opposite and not adjacent, then the path "turns left" or "turns right", depending on the orientation of the boundary of the polytope. In the shortest path all the 6-gons are distinct and non-subsequent faces are not adjacent. Moreover, there can not be two subsequent turns to the same side, and it is possible to modify the shortest path to have no more than one turn (see details in [27] and [24]).

Lemma 8. *Let Γ be the shortest path among all thick paths connecting the 7-gon with 5-gons in a polytope $P \in \mathcal{P}_7$ with the 7-gon and all the 5-gons isolated. If Γ hat no turns, then it is contained in the patch drawn on Fig. 25(a). If it has one turn, then it is contained in the patch drawn on Fig. 26(a).*

Proof. The path Γ itself forms a patch on the polytope P. To prove that Γ is contained in the desired patch it is sufficient to show that all the faces on each figure are distinct on the polytope and the faces are adjacent on the polytope if and only if they are adjacent on the figure. Let Γ have length k. Let us call a *distance between faces* of a disk on a figure the length of the shortest thick path connecting them on the figure. If two faces are distinct or non-adjacent on the figure and the distance between them is at most 3, then they are respectively distinct or non-adjacent on the polytope, since either they are adjacent, if the distance is 1, or are non-subsequent faces of the belt surrounding a face or a pair of adjacent faces, if the distance is 2 or 3. Thus, if two faces on the figure are distinct or non-adjacent, but the corresponding condition is not valid on the polytope, then the distance between them is at least 4. We claim that for any two faces on each figure there is a thick path Γ_1 of length at most $k+2$ with the same ends as Γ containing both faces. Indeed, each figure consists of faces lying in the union of the face $F_{j_{k+1}}$ and two thick paths of lengths k and $k+1$: Γ and $(F_{i_0}, F_{j_1}, \dots, F_{j_k}, F_{i_k})$ for the first figure, and $(F_{i_0}, F_{j_1}, \dots, F_{j_s}, F_{i_{s+1}}, \dots, F_{i_k})$ and $(F_{i_0}, F_{i_1}, \dots, F_{i_s}, F_{j_{s+1}}, \dots, F_{j_k}, F_{i_k})$ for the second. If both faces lie on the same path, we can take this path. If they lie on different paths, then take the path of length $k+1$. Then the face C lying on the other path is adjacent to two subsequent faces (A, B) of the first path. Substitute the segment (A, C, B) for (A, B) to obtain the new path of length $k+2$. If one of the faces is $F_{j_{k+1}}$, then take the path containing the second face. If it has length k, then simply add the segment $(F_{i_k}, F_{j_{k+1}}, F_{i_k})$. If it has length $k+1$, then substitute $(F_{j_k}, F_{j_{k+1}}, F_{i_k})$ for (F_{j_k}, F_{i_k}) to obtain the desired path.

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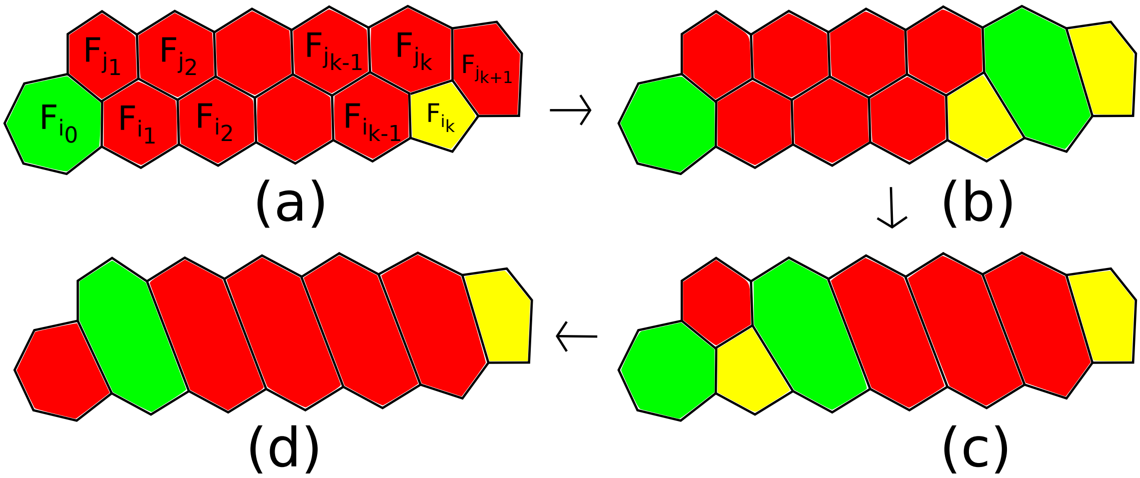

Figure 25. (**a**) The initial patch. (**b**), (**c**) Transformations of the patch. (**d**) The resulting patch.

Let two distinct or non-adjacent faces of one of the figures respectively coincide or be adjacent on the polytope. Take a thick path Γ_1 of length at most $k+2$ containing them. Since the faces coincide or are adjacent on the polytope, we can shorten the path deleting the segment between these faces. This segment consists of at least 3 intermediate faces, whence the new path has length at most $k-1$ and is shorter than Γ. A contradiction. Thus, the lemma is proved. □

Now reduce the obtained patch to the corresponding patch drawn on Fig. 25(d) or Fig. 26(e) by straightenings along edges inverse to $(2,7;5,5)$-, $(2,7;5,6)$-, and $(2,7;6,6)$-truncations (see Fig. 25(b),(c) or Fig. 26(b)-(d) respectively). Then P is obtained from the polytope Q with the last patch substituted for the first patch in P by the corresponding truncations. Also Q or any intermediate polytope contains a 7-gon, hence it does not belong to $\mathcal{D}$.

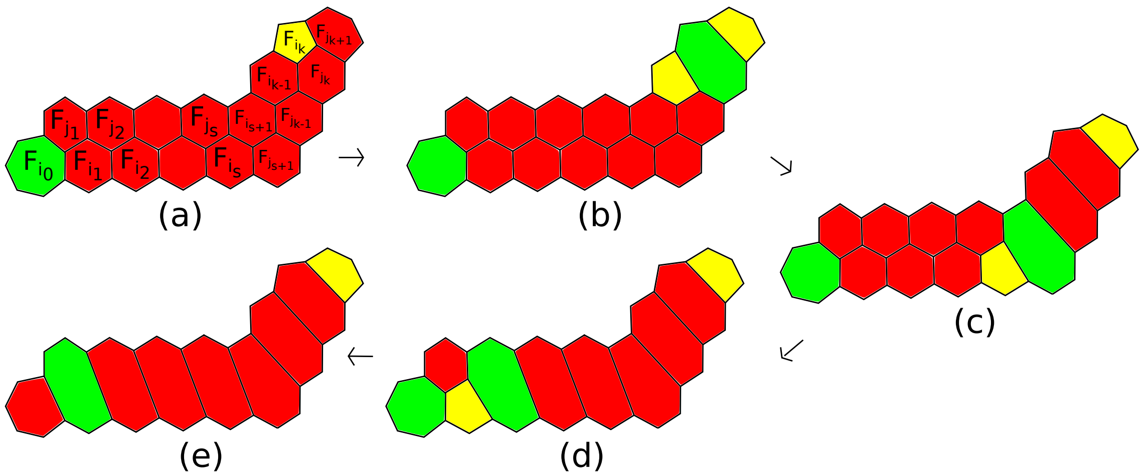

Figure 26. (**a**) The initial patch. (**b**), (**c**), (**d**) Transformations of the patch. (**e**) The resulting patch.

This finishes the proof of the theorem. □

3. Prospects

In Introduction we have enough discussed the place of our results in the context of studies in this direction. Let us mention the arising prospects.

1. The result of Theorem 8 may be strenghtened. It seems that the operation of a $(2,7;6,6)$-truncation can be excluded. Also, it seems to be an opened question, whether there is a finite set of growth

operations transforming the family $\mathcal{P}_{\leq 7}$ to itself sufficient to reduce any polytope in $\mathcal{P}_7$ with all the non-hexagons isolated to some polytope in $\mathcal{P}_{\leq 7}$. Let us remind that due to results in [29] there are no finite sets of growth operations transforming fullerenes to fullerenes sufficient to reduce any fullerene with all 5-gons isolated to some fullerene.

2. There arise further questions about p-vectors of Pogorelov polytopes. For example, for given numbers $(p_k, k \geq 7)$ for which values of p_6 a Pogorelov polytope realizing this p-vector exists?
3. To apply the construction of fullerenes and Pogorelov polytopes by operations presented in this article to problems on combinatorics of polytopes, toric topology (see [33]), and hyperbolic geometry. For example, to give a new prove of the 4-color theorem for special classes of Pogorelov polytopes. Or for a given Pogorelov polytope to enumerate all *characteristic mappings* sending the faces to vectors in $\mathbb{Z}^3$ (or $\mathbb{Z}_2^3$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$) such that for any triple of faces intresecting in a vertex their images form a basis in $\mathbb{Z}^3$ (respectively in $\mathbb{Z}_2^3$). Such functions correspond to 6-dimensional manifolds with an action of the compact torus T^3 and 3-dimensional hyperbolic manifolds (see [3,5]). In [3] it was proved that these manifolds are uniquely determined by their cohomology and respectively $\mathbb{Z}_2$-cohomology rings. There is a question to describe transformation of differential-geometric and algebraic-topological properties of the manifolds under transformation of polytopes.

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Abbreviations

The following abbreviations are used in this manuscript:

$\mathcal{F}$	the family of fullerenes
$\mathcal{P}_7$	the family of simple 3-polytopes with 5-, 6- and one 7-gonal face
$\mathcal{P}_{7,5}$	the subfamily in $\mathcal{P}_7$ consisting of polytopes with the 7-gon adjacent to a 5-gon
$\mathcal{P}_{\leq 7,5}$	$\mathcal{F} \sqcup \mathcal{P}_{7,5}$
$\mathcal{P}_{\leq 7}$	$\mathcal{F} \sqcup \mathcal{P}_7$
$\mathcal{D}$	the family of polytopes consisting of the dodecahedron and the $(5,0)$-nanotubes

The author is a Young Russian Mathematics award winner.

References

1. Ziegler, G. M. Lectures on polytopes. 7th ed. *Grad. Texts in Math.*, vol. *152*; Springer-Verlag, New York, USA, 2007.
2. Grünbaum, B. Convex polytopes. 2nd ed. *Grad. Texts in Math.*, vol. *221*; Springer-Verlag, New York, USA, 2003.
3. Buchstaber, V.M.; Erokhovets, N.Yu.; Masuda, M.; Panov, T. E.; Park, S. Cohomological rigidity of manifolds defined by 3-dimensional polytopes. *Russian Math. Surveys* **2017**, *72:2*, 199-256.
4. Buchstaber, V.M.; Erokhovets, N.Yu. Constructions of families of three-dimensional polytopes, characteristic patches of fullerenes, and Pogorelov polytopes. *Izv. Math.* **2017**, *81:5*, 901-972, DOI 10.1070/IM8665.
5. Vesnin, A.Yu. Right-angled polyhedra and hyperbolic 3-manifolds. *Russian Math. Surveys* **2017**, *72:2*, 335-374.
6. Pogorelov, A.V. A regular partition of Lobachevskian space. *Math. Notes* **1967**, *1:1*, 3-5, DOI: 10.1007/BF01221716.
7. Andreev, E. M. On convex polyhedra in Lobachevskii spaces. *Math. USSR-Sb.* **1970**, *10:3*, 413-440, DOI:10.1070/SM1970v010n03ABEH001677.
8. Dŏslić, T. On lower bounds of number of perfect matchings in fullerene graphs. *J. Math. Chem.* **1998**, *24:4*, 359-364.
9. Dŏslić, T. Cyclical edge-connectivity of fullerene graphs and $(k, 6)$-cages. *J. Math. Chem.* **2003**, *33:2*, 103-112.

10. Curl, R.F. Dawn of the fullerenes: experiment and conjecture. Nobel lecture, 1996. In *Nobel lectures: chemistry*; Grethe, I.; Hackensack, NJ, 2003; pp. 11-32, ISBN 981-02-4958-6; https://www.nobelprize.org/nobel prizes/chemistry/laureates/1996/curl-lecture.html.
11. Kroto, H.W. Symmetry, space, stars and C60. Nobel lecture, 1996. In *Nobel lectures: chemistry*; Grethe, I.; Hackensack, NJ, 2003; pp. 44-79, ISBN 981-02-4958-6; https://www.nobelprize.org/nobel prizes/chemistry/laureates/1996/kroto-lecture.html.
12. Smalley, R.E. Discovering the fullerenes. Nobel lecture, 1996. In *Nobel lectures: chemistry*; Grethe, I.; Hackensack, NJ, 2003; pp. 89-102, ISBN 981-02-4958-6; https://www.nobelprize.org/nobel prizes/chemistry/laureates/1996/smalley-lecture.html.
13. Deza M.; Dutour Sikirić, M.; Shtogrin, M. I. Fullerenes and disc-fullerenes. *Russian Math. Surveys* **2013**, *68:4*, 665-720.
14. Andova, V; Kardoš, F.; Škrekovski, R. Mathematical aspects of fullerenes. *Ars Math. Contemp.* **2016**, *11:2*, 353-379.
15. Thurston, W. P. Shapes of polyhedra and triangulations of the sphere. *Geom. Topol. Monogr.* **1998**, *1*, 511-549, DOI: 10.2140/gtm.1998.1.511.
16. Vesnin, A.Yu. Three-dimensional hyperbolic manifolds of Löbell type. *Siberian Math. J.* **1987**, *28:5*, 731-734.
17. Vesnin, A.Yu. Volumes of hyperbolic Löbell 3-manifolds. *Math. Notes* **1998**, *64:1*, 15-19.
18. Buchstaber, V. M.; Erokhovets, N. Yu. Truncations of simple polytopes and applications. *Proc. Steklov Inst. Math.* **2015**, *289*, 104-133.
19. Eberhard, V. Zur Morphologie der Polyeder. Teubner: Leipzig, 1891.
20. Inoue, T. Organizing volumes of right-angled hyperbolic polyhedra. *Algebr. Geom. Topol.* **2008**, *8:3*, 1523-1565.
21. Brückner, M. Vielecke und Vielflache. Theorie und Geschichte. Teubner: Leipzig, 1900.
22. Fedorov, E. S. Foundations of morphology and systematics of polyhedra, *Zap. Imperat. S.-Petersb. Mineralog. Obshch.* **1983**, *Ser. 2, 30*, 241-341. (Russian)
23. Volodin, V. D. Combinatorics of flag simplicial 3-polytopes. *Russian Math. Surveys* **2015**, *70:1*, 168-170.
24. Buchstaber, V. M.; Erokhovets, N. Construction of fullerenes. *arXiv* **2015**, arXiv: 1510.02948.
25. Buchstaber, V.M.; Erokhovets, N.Yu. Fullerenes, Polytopes and Toric Topology. In Volume 35 *Combinatorial and Toric Homotopy: Introductory Lectures* of Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.; World Sci. Publ.: River Edge, NJ, USA, 2017; pp. 67–178, ISBN 978-981-3226-58-6.
26. Buchstaber, V. M.; Erokhovets, N. Yu. Finite sets of operations sufficient to construct any fullerene from C_{20}. *Struct. Chem.* **2017**, *28:1*, 225-234.
27. Hasheminezhad, M.; Fleischner, H.; McKay, B. D. A universal set of growth operations for fullerenes. *Chem. Phys. Lett.* **2008**, *464:1-3*, 118-121.
28. Endo, M.; Kroto, H. W. Formation of carbon nanofibers. *J. Phys. Chem.* **1992**, *96:17*, 6941-6944.
29. Brinkmann,G.; Graver, J. E.; Justus, C. Numbers of faces in disordered patches. *J. Math. Chem.* **2009**, *45:2*, 263-278.
30. Kardoš F.; Škrekovski R. Cyclic edge-cuts in fullerene graphs. *J. Math. Chem.* **2008**, *44:1*, 121-132.
31. Kutnar K.; Marušič, D. On cyclic edge-connectivity of fullerenes. *Discrete Appl. Math.* **2008**, *156:10*, 1661-1669.
32. Grünbaum, B. Some analogues of Eberhard's theorem on convex polytopes. *Israel J. Math.* **1968**, *6:4*, 398-411.
33. Buchstaber, V. M.; Panov, T. E. Toric topology. *Math. Surveys Monogr.*, vol. 204, Amer. Math. Soc.: Providence, RI, USA, 2015; ISBN 978-1-4704-2214-1.