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The Role of Inflation-indexed Bond in Optimal Management of Defined Contribution Pension Plan During the Decumulation Phrase

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Abstract: In this paper we investigate the optimal investment strategy for a defined contribution (DC) pension plan during the decumulation phrase which is risk-averse and pays close attention to inflation risk. The plan aims to maximize the expected constant relative risk aversion (CRRA) utility from the terminal wealth by investing the wealth in a financial market consisting of an inflation-indexed bond, an ordinary zero coupon bond and a risk-free asset. We derive the optimal investment strategy in closed-form using the dynamic programming approach by solving the corresponding Hamilton-Jacobi-Bellman (HJB) equation. Our theoretical and numerical results reveal that under some rational assumptions, an inflation-indexed bond do has significant advantage to hedge inflation risk.

Keywords: inflation-indexed bond; DC pension plan; stochastic optimal control; dynamic programming approach; HJB equation.

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1. Introduction and motivation

Asset allocation problem incorporating inflation risk for individual investors has been studied by many researchers. A closed form optimal investment strategy is given by Brennan and Xia[1], then Munk et al.[2] obtain the optimal strategy in a model with inflation uncertainty by dynamic programming method. Since differences in spending patterns and in price increases lead to unequal inflation experiences, Li et al.[3] consider an optimal investment and consumption problem of households under inflation inequality.

Inflation-indexed bond is defined as an financial instrument that delivers a defined payoff indexed by inflation at maturity time, which can be utilized to hedge against inflation risk. For an investment company, Nkeki and Nwozo[4] find that inflation risk associated with investment could be hedged by investing in inflation-linked bond, with some assumptions of stochastic cash inflows and outflows of the company. Liang and Zhao[5] investigate the efficient frontier and optimal strategies of a family under mean-variance efficiency, which also take into account the inflation risk and consider the problem in the real market instead of the nominal price.

Such kind of bonds are applied in life insurance. Kwak and Lim[6] investigate a continuous time optimal consumption, investment and life insurance decision problem of a family under inflation risk, which explicit solutions are derived by using martingale method. Then Han and Hung[7] solve a similar investment problem of a wage earner before retirement with the method of dynamic programming approach. In order to hedge against inflation risk, an inflation-indexed bond is introduced in both two problems above.

As the investment of a DC pension plan involves quite a long period of time, it seems implausible to ignore inflation risk in the long run. Also, a DC pension plan is a function of the contribution that the pension member paid during his or her working life, in which only the contributions are fixed, and therefore the benefits depend solely on the returns of the fund's portfolio, so it is meaningful to protect inflation risk in such kind of pension scheme. Yao et al.[8] solve a mean-variance problem by

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considering the real wealth process including the influence of inflation. Okoro and Nkeki[9] examine the optimal variational Merton portfolios with inflation protection strategy. Both expected values of pension plan member's terminal wealth and efficient frontier are obtained in their work.

Generally, a pension scheme contains an accumulation (contribution) phase, which is the period before retirement, and a decumulation (distribution) phase, which is the period after retirement. There are some applications of inflation bond in pension plans which concentrate on the optimal management during the accumulation phase. Zhang et al.[10] and Zhang and Ewald[11] investigate an optimal investment problem by investing an indexed bond, and present a way to deal with the optimization problem using the martingale method. In the work of Han and Hung[12], stochastic dynamic programming approach is used to investigate the optimal asset allocation for a DC pension plan with downside protection under stochastic inflation, and the inflation-indexed bond is again included in the asset menu to cope with the inflation risk. According to Chen et al.[13], an optimal investment strategy for a DC plan member who pays close attention to inflation risks and requires a minimum performance at retirement is solved by martingale approach.

As the DC pension scheme also confronted with inflation risk in the decumulation phase, in this paper we apply the inflation bond in this period and consider an optimal control problem, which continuously decides weights of investment in different assets, including a zero coupon bond, an inflation-indexed bond and a riskless asset, in order to maximize the terminal wealth with the consideration of the influence of inflation.

Another motivation of our work is to investigate whether the investment efficiency is improved by the inflation-index bond. The question is whether the optimal utility function is increased with the investment of the index bond. In order to do the comparative study, we follow the definition of the indexed bond price, see, for instance, Nkeki and Nwozo[4] or Han and Hung[7], but find another SDE to describe its price. In our work, the price of an ordinary bond is just a special case of the price of the indexed bond.

The rest of the paper is structured as follows. Section 2 describes the financial market with stochastic interest rate, stochastic price level and three tradable assets which are of interest for our problem. Also the demographic pattern is given by a drifted Brownian motion. Section 3 solves an optimal investment problem with investment in a complete market including an indexed bond, an ordinary bond and a bank account. The closed form solutions of this stochastic control problem are given by solving the related HJB equation. The counterpart, Section 4 solves a similar problem except the indexed bond is excluded. At last, Section 5 compares the results given by Section 3 and 4 and presents some numerical results. The comparative study shows that the investment in inflation-indexed bond do has significant advantage to hedge the inflation risk.

2. Model assumptions and notations

2.1. The financial market

The instantaneously nominal interest rate $R(t)$ is assumed to be stochastic. Here we use the Ornstein-Uhlenbeck process introduced by Vasicek[14], i.e., $R(t)$ satisfies the following differential equation:

$$dR(t) = b\big(a - R(t)\big)dt + \sigma_R\, dZ_1(t) \tag{1}$$

where $Z_1(t)$ is a standard Brownian motion under probability measure P, $R(0) = R_0$ and all parameters are positive constants.

Assume there is a financial market consisting of three tradeable underlying instruments which are traded continuously over time and are perfectly divisible. In addition, we assume that there are no transaction costs or taxes in the context. Borrowing and short-selling is permitted.

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a. A money market account $M(t)$ satisfies the following equation:

$$\frac{dM(t)}{M(t)} = R(t)\, dt \tag{2}$$

with initial value $M(0) = M_0$.

b. A zero coupon nominal bond which pays one monetary unit at expiration time T_0, and its value $B(t, T_0)$ at time t can be written as the conditional expectation under the so-called equivalent martingale measure corresponding to the arbitrage free market, under which $e^{-\int_0^t R(u)du} B(t, T_0)$ is a local martingale.:

$$B(t, T_0) = \mathbb{E}_Q \left\{ e^{-\int_t^{T_0} R(u)du} \middle| \mathscr{F}_t \right\} \tag{3}$$

Considering the martingale property of the discounting process of $B(t, T_0)$ and following the work of Menoncin[15], we get the dynamics of the bond price:

$$\begin{aligned} dB(t, T_0) &= B(t, T_0)\, R(t)\, dt + \frac{\partial B(t, T_0)}{\partial R(t)}\, \sigma_R\, dZ_1^Q(t) \\ B(T_0, T_0) &= 1 \end{aligned} \tag{4}$$

where $Z_1^Q(t)$ is a standard Brownian motion under measure Q. Let λ_R be the market price for the interest rate risk, then by Girsanov's theorem, $dZ_1^Q(t) = \lambda_R dt + dZ_1(t)$ is a Wiener process and the Radon-Nikodym derivative is defined by $\Lambda(T_0) = \frac{dQ}{dP}(Z_1[0, T_0]) = exp\left\{-\lambda_R\, Z_1(T_0) - \frac{1}{2}\, \lambda_R^2\, T_0\right\}$. Consequently, one can get the stochastic differential equation of $B(t, T_0)$ under the original measure P.

$$\begin{aligned} \frac{dB(t, T_0)}{B(t, T_0)} &= \left(R(t) + \nabla_R^B\, \sigma_R\, \lambda_R\right)dt + \nabla_R^B\, \sigma_R\, dZ_1(t) \\ B(T_0, T_0) &= 1 \end{aligned} \tag{5}$$

where $\nabla_R^B = \frac{\partial B(t,T_0)}{\partial R(t)} \frac{1}{B(t,T_0)}$ is the semi-elasticity of the bond price with respect to the interest rate, which is negative because the bond negatively reacts to the shocks on the interest rate.

Remark 1. *Since the bond has a positive premium compared with the riskless asset, λ_R is negative.*

c. With regards to the stochastic price level, we define the inflation rate process as:

$$\frac{dP(t)}{P(t)} = idt + \sigma_P\, dZ_2(t) \tag{6}$$

with $P(0) = P_0 > 0$. The constant i is the expected rate of inflation and σ_P represents its volatility. $Z_2(t)$ is another Brownian motion under the physical measure P, which generates uncertainty in the price level and is independent of $Z_1(t)$. If we set λ_p as the market price of risk with respect to $dZ_2(t)$, then $dZ_2^Q(t) = \lambda_P dt + dZ_2(t)$ is a Wiener process, where $Z_2^Q(t)$ is a Brownian motion under the risk neutral measure Q. $P(t)$ has the explicit form as the following:

$$P(t) = P_0 \exp\left\{(i - \sigma_P\lambda_p - {\sigma_P}^2/2)t + \sigma_P Z_2^Q(t)\right\} \tag{7}$$

As in the work of Han and Hung[12], we define the inflation-indexed zero coupon bond, whose price at time t is denoted by $I(t, T_0)$, from which the investor can get P_{T_0} units of money at maturity time T_0. By the fundamental theorem of asset pricing, it is well known that in an arbitrage free market

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the price of any asset coincides with the expected present value of its future cash flows under the equivalent martingale measure Q:

$$\begin{aligned} I(t,T_0) &= \mathbb{E}_Q\left\{P_{T_0}\, e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t\right\} \\ &= P_0 e^{(i-\sigma_P\lambda_P-\sigma_P{}^2/2)T_0} e^{\sigma_P Z_2^Q(t)}\,\mathbb{E}_Q\left\{e^{\sigma_P\left(Z_2^Q(T_0)-Z_2^Q(t)\right)} e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t\right\} \end{aligned} \tag{8}$$

Lemma 1. *In Eq.(8),*

$$\begin{aligned} &\mathbb{E}_Q\left\{e^{\sigma_P\left(Z_2^Q(T_0)-Z_2^Q(t)\right)} e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t\right\} \\ &= \mathbb{E}_Q\left\{e^{\sigma_P\left(Z_2^Q(T_0)-Z_2^Q(t)\right)}\right\}\mathbb{E}_Q\left\{e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t\right\} \\ &= e^{\sigma_P^2(T_0-t)/2} B(t,T_0) \end{aligned} \tag{9}$$

Proof. Denote $\mathscr{F}_t^1 = \sigma\left\{Z_1^Q(s), s\le t\right\}$, $\mathscr{F}_t^2 = \sigma\left\{Z_2^Q(s), s\le t\right\}$, and set $\mathscr{F}_t = \mathscr{F}_t^1 \vee \mathscr{F}_t^2$. Since $e^{-\int_t^{T_0} R(u)du}$ is independent of $e^{\sigma_P\left(Z_2^Q(T_0)-Z_2^Q(t)\right)}$ and $\mathscr{F}_t^2$, and $\mathscr{F}_t^1$ is independent of $\mathscr{F}_t^2$, we have:

$$\begin{aligned} &\mathbb{E}_Q\left\{e^{\sigma_P\left(Z_2^Q(T_0)-Z_2^Q(t)\right)} e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t\right\} \\ &= \mathbb{E}_Q\left\{e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t^1\right\}\mathbb{E}_Q\left\{e^{\sigma_P\left(Z_2^Q(T_0)-Z_2^Q(t)\right)}\Big|\mathscr{F}_t^2\right\}. \end{aligned} \tag{10}$$

Since $R(t)$ is independent of $\mathscr{F}_t^2$, we have $\mathbb{E}_Q\left\{e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t^1\right\} = \mathbb{E}_Q\left\{e^{-\int_t^{T_0} R(u)du}\Big|\mathscr{F}_t\right\}$. $\mathscr{F}_t^2$ disappears in the second part because of the property of independent increment of a Brownian motion, and the last equality of (9) holds by the property of exponential martingale. □

At last, we get a formulation of $I(t,T_0)$ related with $B(t,T_0)$:

$$I(t,T_0) = P_0 e^{(i-\sigma_P\lambda_P)T_0} e^{-\sigma_P{}^2 t/2} e^{\sigma_P Z_2^Q(t)} B(t,T_0) \tag{11}$$

By the chain rule of Itô's formula, the evolution of $I(t,T_0)$ is described by the following stochastic differential equation:

$$dI(t,T_0) = P_0 e^{(i-\sigma_P\lambda_P)T_0}\left[e^{-\sigma_P{}^2 t/2} e^{\sigma_P Z_2^Q(t)} dB(t,T_0) + e^{-\sigma_P{}^2 t/2} B(t,T_0)\sigma_P e^{\sigma_P Z_2^Q(t)} dZ_2^Q(t)\right] \tag{12}$$

Finally we have:

$$\begin{aligned} \frac{dI(t,T_0)}{I(t,T_0)} &= \frac{dB(t,T_0)}{B(t,T_0)} + \sigma_P dZ_2^Q(t) \\ &= (R(t) + \nabla_R^B\sigma_R\lambda_R + \sigma_P\lambda_P)dt + \nabla_R^B\sigma_R dZ_1(t) + \sigma_P dZ_2(t) \end{aligned} \tag{13}$$

Thus we get a correlation between the price of the inflation-indexed bond and that of an ordinary zero coupon bond.

2.2. The demographic pattern

As to the demographic pattern, a meaningful structure is presented in Zimbidis and Pantelous[16]. It is assumed that the dynamic of number of death follows a stochastic differential equation:

$$-dl(t) = \theta(t)dt + \nu(t)dZ_3(t) \tag{14}$$

where $\theta(t)$ and $\nu(t)$ describe the instantaneous drift and volatility, respectively. $Z_3(t)$ is another standard Brownian Motion under the physical measure P , which is independent of $Z_1(t)$ and $Z_2(t)$.

Now set $\mathscr{F}_t = \mathscr{F}_t^1 \vee \mathscr{F}_t^2 \vee \mathscr{F}_t^3$, where $\mathscr{F}_t^3 = \sigma\left\{Z_3^Q(s), s \le t\right\}$. It is not hard to show that the result in Lemma 1 still holds.

3. DC Pension fund management with investment of inflation-indexed bond

We consider a typical DC pension scheme, see, for example, [16], in which a member pays contributions during his or her employment period and beneficiaries of each pensioner (who dies at time t) receive an accumulated amount, as a whole life assurance with a death benefit.

The problem is to find the optimal investment policy for assets over the life of a participant in the plan, from retirement time, i.e., modeling time $t = 0$ to the planning horizon, i.e., terminal time $t = T$, which is equal or smaller than the maturity of bonds T_0. In this case the financial market is complete. The market can be represented as the following matrix form:

$$\begin{bmatrix} \dfrac{dB}{B} \\[2ex] \dfrac{dI}{I} \end{bmatrix} = \begin{bmatrix} R + \nabla_R^B\, \sigma_R\, \lambda_R \\[2ex] R + \nabla_R^B \sigma_R \lambda_R + \sigma_P \lambda_P \end{bmatrix} dt + \underbrace{\begin{bmatrix} \nabla_R^B\, \sigma_R & 0 \\[2ex] \nabla_R^B \sigma_R & \sigma_P \end{bmatrix}}_{\Sigma} \begin{bmatrix} dZ_1 \\[2ex] dZ_2 \end{bmatrix} \tag{15}$$

where the matrix Σ is invertible.

It is easily to get the following equation which describes the evolution of the fund:

$$\begin{aligned} dX(t) &= \left(1 - u_1^{BI}(t) - u_2^{BI}(t)\right) X(t)\, \frac{dM(t)}{M(t)} + u_1^{BI}(t)\, X(t)\, \frac{dB(t,T_0)}{B(t,T_0)} \\ &\quad + u_2^{BI}(t)\, X(t)\, \frac{dI(t,T_0)}{I(t,T_0)} - X(t)c(t)\big(-dl(t)\big) \\ &= X(t)\Big[R(t) + u_1^{BI}(t)\nabla_R^B\sigma_R\lambda_R + u_2^{BI}(t)(\nabla_R^B\sigma_R\lambda_R + \sigma_P\lambda_P) - c(t)\theta(t)\Big]dt \\ &\quad + X(t)\left(u_1^{BI}(t) + u_2^{BI}(t)\right)\nabla_R^B\sigma_R dZ_1(t) + X(t)u_2^{BI}(t)\sigma_P dZ_2(t) - X(t)c(t)\nu(t)dZ_3(t) \\ X(0) &= X_0 \end{aligned} \tag{16}$$

where $c(t)$ is the weight of the benefit received immediately by the beneficiaries, i.e., the benefit associated with the pension fund is assumed to be a deterministic proportion $c(t)$ of the pention wealth. $u_1^{BI}(t)$ and $u_2^{BI}(t)$ are weights invested into the zero coupon bond and the inflation-index bond, respectively. Denote $u^{BI}(t) = \left(u_1^{BI}(t), u_2^{BI}(t)\right)$. It is called admissible if it satisfies the following conditions, and we denote the set of all admissible controls by Π.

1. $u^{BI}(t)$ is progressively measurable with respect to $\{\mathscr{F}_t\}_{t\ge 0}$;
2. $\mathbb{E}\left\{\int_0^T \left[\left(X(t)\left(u_1^{BI}(t) + u_2^{BI}(t)\right)\nabla_R^B\, \sigma_R\right)^2 + \left(X(t)u_2^{BI}(t)\sigma_P\right)^2 + \left(X(t)c(t)\nu(t)\right)^2\right]dt\right\} < \infty$;
3. Eq.(16) has a unique strong solution for the initial value $(t_0, R_0, X_0) \in [0,T] \times (0,\infty)^2$.

Usually, the period T for the pension fund is very long, hence the effect of inflation becomes noticeable for the pension manager. Denote the real wealth process including the impact of inflation by $\overline{X}(t)$, i.e., $\overline{X}(t) = X(t)/P(t)$. By the chain rule, we have that $\overline{X}(t)$ follows:

$$\begin{aligned}\frac{d\overline{X}(t)}{\overline{X}(t)} &= \Big[R(t) + u_1^{BI}(t)\nabla_R^B\sigma_R\lambda_R + u_2^{BI}(t)(\nabla_R^B\sigma_R\lambda_R + \sigma_P\lambda_P - \sigma_P^2) - c(t)\theta(t) + \sigma_P^2 - i\Big]dt \\ &\quad + (u_1^{BI}(t) + u_2^{BI}(t))\nabla_R^B\sigma_R dZ_1(t) + (u_2^{BI}(t)\sigma_P - \sigma_P)dZ_2(t) - c(t)\nu(t)dZ_3(t) \qquad (17)\\ \overline{X}(0) &= X_0/P_0\end{aligned}$$

The pension sponsor would like to maximize the expected utility of terminal real fund $\overline{X}(t)$. Our optimal problem can be written as:

$$\max_{\{u^{BI}\in\Pi\}} \mathbb{E}\Big[U(\overline{X}(T))\Big]. \qquad (18)$$

where $U(x)$ is the utility function which describes the preference over wealth. Here we consider the typical CRRA utility function, for which we can derive the explicit form of the solution, as follows:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0 \ \ and \ \ \gamma \neq 1, \qquad (19)$$

where γ is the relative risk aversion.

The problem can be solved by the dynamic programming method. Denote $V(t,R,x) = \mathbb{E}\left\{U(\overline{X}(T))|\overline{X}(t) = x, R(t) = R\right\}$ as the value function. In stochastic optimal control theory, the HJB equation accomplishes the connection between the value function and the optimal control, see [18] or [17]. We have the associated HJB equation for the above problem as follows:

$$\sup_{\{u^{BI}\in\Pi\}} \Psi(u_1^{BI}, u_2^{BI}) = 0 \qquad (20)$$

where

$$\begin{aligned}\Psi(u_1^{BI}, u_2^{BI}) &= V_t + V_x x\big[R + u_1^{BI}\nabla_R^B\sigma_R\lambda_R + u_2^{BI}(\nabla_R^B\,\sigma_R\lambda_R + \sigma_P\lambda_P - \sigma_P^2) - c\theta + \sigma_P^2 - i\big] \\ &\quad + \frac{1}{2}V_{xx}\,x^2\big[(u_1^{BI} + u_2^{BI})^2\,{\nabla_R^B}^2\,{\sigma_R}^2 + (u_2^{BI}\sigma_P - \sigma_P)^2 + c^2\nu^2\big] \qquad (21)\\ &\quad + V_R\,b(a-R) + \frac{1}{2}V_{RR}\,{\sigma_R}^2 + V_{Rx}\,x\,(u_1^{BI} + u_2^{BI})\,\nabla_R^B\,{\sigma_R}^2 = 0\end{aligned}$$

where V_t, V_x, V_R, V_{xx}, V_{RR} and V_{Rx} denote the first and second order partial derivatives of V with respect to t, x and R, respectively.

The maximization of $\Psi(u_1^{BI}, u_2^{BI})$ can be obtained by the optimal functional u_1^{BI*} and u_2^{BI*}, which satisfy the following necessary conditions:

$$\begin{aligned}&\Psi(u_1^{BI*}, u_2^{BI*}) = 0 \\ &\frac{d\Psi}{du_1^{BI}}(u_1^{BI*}) = 0 \qquad (22)\\ &\frac{d\Psi}{du_2^{BI}}(u_2^{BI*}) = 0\end{aligned}$$

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The first order conditions expressed as feedback formulas in term of derivatives of the value function are:

$$
\begin{aligned}
u_1^{BI*} &= -\frac{V_x}{V_{xx}\,x}\frac{\lambda_R}{\nabla_R^B\,\sigma_R} - \frac{V_{Rx}}{V_{xx}\,x}\frac{1}{\nabla_R^B} - \frac{V_x}{V_{xx}\,x}\frac{\sigma_P-\lambda_P}{\sigma_P} - 1 \\
u_2^{BI*} &= \frac{V_x}{V_{xx}\,x}\frac{\sigma_P-\lambda_P}{\sigma_P} + 1
\end{aligned}
\tag{23}
$$

Substituting the above two Eqs.(23) into the HJB Eq.(20), we can finally get the explicit forms of the value function and the optimal investment strategies. They are given in the following theorem.

Theorem 1. *The optimal utility and optimal investment strategies satisfy the following equations:*

$$
\begin{aligned}
V(t,R,x) &= \frac{x^{1-\gamma}}{1-\gamma}\,e^{A_1^{BI}(t)+A_2^{BI}(t)R} \\
u_1^{BI}(t)^* &= \frac{1}{\gamma}\frac{\lambda_R}{\nabla_R^B\,\sigma_R} + \frac{1}{\gamma}\frac{1}{\nabla_R^B}A_2^{BI}(t) + \frac{1}{\gamma}\frac{\sigma_P-\lambda_P}{\sigma_P} - 1 \\
u_2^{BI}(t)^* &= \frac{1}{\gamma}\frac{\sigma_P-\lambda_P}{\sigma_P} + 1
\end{aligned}
\tag{24}
$$

where

$$
A_2^{BI}(t) = -\frac{1-\gamma}{b}\left[e^{-b(T-t)} - 1\right] \tag{25}
$$

$$
A_1^{BI}(t) = \int_t^T \alpha^{BI}(s)ds \tag{26}
$$

$$
\begin{aligned}
\alpha^{BI}(t) = &\frac{1}{2}\frac{1-\gamma}{\gamma}(\lambda_R^2 + (\sigma_P-\lambda_P)^2) + (1-\gamma)(\sigma_P\lambda_P - c\theta - i) - \frac{1}{2}\gamma(1-\gamma)c^2\nu^2 \\
&+A_2^{BI}(t)ba + \frac{1}{2}A_2^{BI}(t)^2\sigma_R^2 + \frac{1-\gamma}{\gamma}A_2^{BI}(t)\lambda_R\sigma_R + \frac{1}{2}\frac{1-\gamma}{\gamma}A_2^{BI}(t)^2\sigma_R^2
\end{aligned}
\tag{27}
$$

Proof. See Appendix. □

In order to do some comparison with Section 4, we denote $V(t,R,x) \triangleq V^{BI}(t,R,x)$.

4. DC Pension fund management without the investment of inflation-indexed bond

In order to investigate the role of the inflation-linked bond in pension management, we consider another optimal portfolio selection problem with the indexed bond excluded in this section. Here we abuse the notation and set the wealth process again denoted by $X(t)$, but the portfolio is only consist of a bank account and an ordinary T-bond. In this case the financial market is incomplete: there is no enough assets for hedging against the inflation risk, i.e., the stochasticity dZ_2. The financial assets on the market can be summarized in the following matrix form:

$$
\frac{dB}{B} = \left[R(t) + \nabla_R^B\,\sigma_R\,\lambda_R\right]dt + \underbrace{\left[\nabla_R^B\,\sigma_R \quad 0\right]}_{\Sigma}\begin{bmatrix} dZ_1 \\ dZ_2 \end{bmatrix} \tag{28}
$$

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in which case the matrix Σ is not invertible. Actually, there does not exist any linear combination of assets to replicate the inflation risk.

Similarly, the corresponding wealth process can be defined as:

$$\begin{aligned} dX(t) &= \left(1 - u_1^B(t)\right)\, X(t)\, \frac{dM(t)}{M(t)} + u_1^B(t)\, X(t)\, \frac{dB(t,T_0)}{B(t,T_0)} - X(t)c(t)\big(-dI(t)\big) \\ &= X(t)\Big[R(t) + u_1^B(t)\nabla_R^B \sigma_R \lambda_R - c(t)\theta(t)\Big]dt + X(t)u_1^B(t)\nabla_R^B \sigma_R dZ_1(t) \\ &\quad - X(t)c(t)\nu(t)dZ_3(t) \\ X(0) &= X_0 \end{aligned} \tag{29}$$

where $u_1^B(t)$ describes the weight allocated to the zero coupon bond.

Again denote the real wealth process including the impact of inflation by $\overline{X}(t)$, and by the chain rule, the dynamic is:

$$\begin{aligned} \frac{d\overline{X}(t)}{\overline{X}(t)} &= \Big[R(t) + u_1^B(t)\nabla_R^B \sigma_R \lambda_R - c(t)\theta(t) + \sigma_P^2 - i\Big]dt + u_1^B(t)\nabla_R^B \sigma_R dZ_1(t) \\ &\quad - \sigma_P dZ_2(t) - c(t)\nu(t)dZ_3(t) \\ \overline{X}(0) &= X_0/P_0 \end{aligned} \tag{30}$$

We have exactly the same objective function and the same optimization problem in Section 3 and 4, except that the investment of an inflation-indexed bond has been removed. Analogously, the HJB equation is:

$$\begin{aligned} \sup_{\{u^B \in \Pi\}} \quad & V_t + V_x\, x\big(R + u_1^B \nabla_R^B\, \sigma_R\, \lambda_R - c\theta + \sigma_P^2 - i\big) + \frac{1}{2} V_{xx} x^2 \Big[{u_1^B}^2\, {\nabla_R^B}^2\, \sigma_R^2 + {\sigma_P}^2 + c^2\nu^2\Big] \\ & + V_R\, b(a - R) + \frac{1}{2} V_{RR}\, {\sigma_R}^2 + V_{Rx}\, x\, u_1^B\, \nabla_R^B\, {\sigma_R}^2 = 0 \end{aligned} \tag{31}$$

where $V(t, R, x)$ is again the value function corresponding the optimal problem.

By differentiating with respect to u_1^B, the optimal weight can be expressed in the form of the value function as:

$$u_1^{B*} = -\frac{V_x}{V_{xx}\, x} \frac{\lambda_R}{\nabla_R^B\, \sigma_R} - \frac{V_{Rx}}{V_{xx}\, x} \frac{1}{\nabla_R^B} \tag{32}$$

Similarly, we can finally get the explicit forms of the value function and the optimal investment strategies in the same way. They are given in the following theorem.

Theorem 2. *The the corresponding value function and the optimal weight for the T-bond satisfy:*

$$\begin{aligned} V(t, R, x) \triangleq V^B(t, R, x) &= \frac{x^{1-\gamma}}{1-\gamma}\, e^{A_1^B(t) + A_2^B(t) R} \\ u_1^B(t)^* &= \frac{1}{\gamma} \frac{\lambda_R}{\nabla_R^B\, \sigma_R} + \frac{1}{\gamma} \frac{1}{\nabla_R^B} A_2^B(t) \end{aligned} \tag{33}$$

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where $A_2^B(t) = A_2^{BI}(t)$ *and*

$$A_1^B(t) = \int_t^T \alpha^B(s)ds \tag{34}$$

$$\begin{aligned}\alpha^B(t) = &\frac{1}{2}\frac{1-\gamma}{\gamma}\lambda_R^2 + (1-\gamma)(-c\theta - i + \sigma_P{}^2) - \frac{1}{2}\gamma(1-\gamma)(\sigma_P{}^2 + c^2v^2) \\ &+ A_2^B(t)ba + \frac{1}{2}A_2^B(t)^2\sigma_R^2 + \frac{1-\gamma}{\gamma}A_2^B(t)\lambda_R\sigma_R + \frac{1}{2}\frac{1-\gamma}{\gamma}A_2^B(t)^2\sigma_R^2\end{aligned} \tag{35}$$

Now we denote that $A_2^B(t) = A_2^{BI}(t) \triangleq A_2(t)$.

Proof. The proof is similar to that of Theorem 1 so we omit it here. □

5. Comparison and Conclusion

A comparison between Section 3 and Section 4 is shown in the following theorem:

Theorem 3. *When* $\lambda_P > \sigma_P$, *we have* $V^{BI}(t, R, x) > V^B(t, R, x)$, *which means that the maximum expected utility of the terminal wealth by investing in an inflation-indexed bond is higher than that of a portfolio consist of only a zero coupon bond and a money market account.*

Proof. When $\lambda_P > \sigma_P$, we have $\alpha^{BI}(t) > \alpha^B(t)$, thus $A_1^{BI}(t) > A_1^B(t)$, and the result follows. □

Remark 2. *Explanation of the condition* $\lambda_P > \sigma_P$*: Since* $\lambda_P = \frac{i-\sigma_P}{\sigma_P}$, $\lambda_P > \sigma_P$ *is equivalent to* $i > \sigma_P{}^2 + \sigma_P$, *i.e., when the expected inflation rate is significantly low, there is no remarkable advantage to invest in the inflation-indexed bond, or probably we can conclude that the hedging is not significant.*

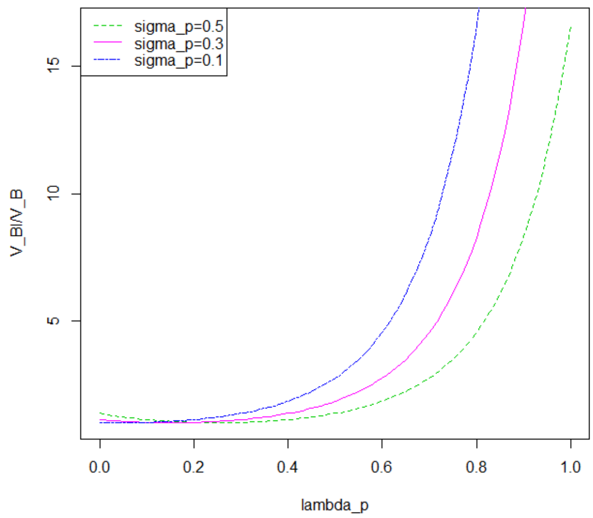

Figure 1. The Impact of λ_p

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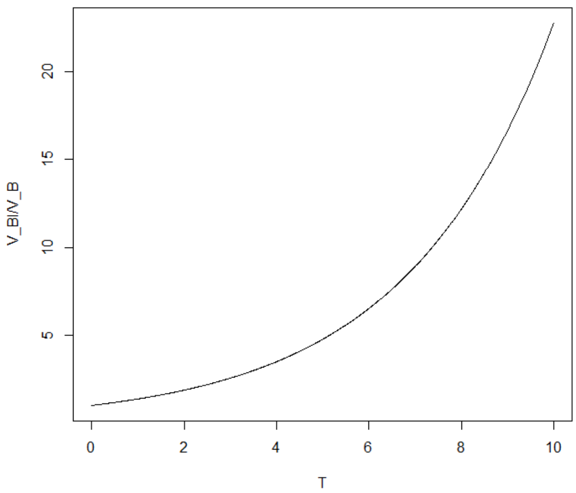

Figure 2. The Impact of T

Now we investigate the influence of the difference between λ_p and σ_p to our value functions. If we set $\gamma = 0.5$, $t = 0$, $T = 10$, and $\sigma_p = 0.5$, respectively, it would be clear that in Figure 1, when λ_p is greater than σ_p, i.e., $\lambda_p > 0.5$, the ratio between the value function corresponding the optimal problem with inflation-indexed bond and the value function corresponding the optimal problem without the indexed bond is almost increasing exponentially with the increasing of $\lambda_p - \sigma_p$. When we change the value of σ_p to 0.3 and 0.1, it is easy to conclude that the curve increasing faster with σ_p becomes smaller.

Remark 3. *Investigate the influence of terminal time T: In Figure 2, the ratio between two value functions again increasing exponentially with the terminal time. A rational explanation of this phenomenon is that our investment becomes more risky with a longer time interval, and the hedging becomes more necessary with an inflation-indexed bond.*

We make a conclusion as follows: This paper analyses, by means of dynamic programming approach, the optimal investment strategy for the decumulative phase in DC type pension schemes under inflation environment. The objective is to determine the investment strategy, maximizing the expected CRRA utility of the terminal wealth in a complete market consisting of an indexed bond, a zero coupon bond and a riskless asset. The explicit solution of the optimal problem is derived from the corresponding HJB equation. In order to investigate the role of an indexed bond, we also solve another optimal investment problem in an incomplete market with the indexed bond excluded. We find that under some rational assumptions, the value function in the complete market is higher than the value function in the incomplete market, thus we may conclude that an inflation-indexed bond do has significant advantage to hedge inflation risk.

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Abbreviations

The following abbreviations are used in this manuscript:

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DC	Defined contribution
HJB	Hamilton-Jacobi-Bellman
CRRA	Constant relative risk aversion

Appendix

The proof of Theorem 1.

Substituting Eq.(23) into the HJB equation (20), we get:

$$\begin{aligned} & V_t + V_x\, x\, R - \frac{1}{2}\frac{{V_x}^2}{V_{xx}}({\lambda_R}^2 + (\sigma_P - \lambda_P)^2) + V_x\, x(\sigma_P\lambda_P - c\theta - i) + \frac{1}{2}V_{xx}\, x^2c^2\nu^2 \\ & + V_R\, \mu + \frac{1}{2}V_{RR}\, \sigma_R^2 - \frac{V_xV_{Rx}}{V_{xx}}\lambda_R\sigma_R - \frac{1}{2}\frac{V_{Rx}^2}{V_{xx}}\sigma_R^2 = 0 \end{aligned} \tag{A1}$$

We may try that the solution of $V(t,R,x)$ has a form as follows:

$$V(t,R,x) = \frac{x^{1-\gamma}}{1-\gamma}\, e^{A_1^{BI}(t)+A_2^{BI}(t)R} \tag{A2}$$

Differentiating it, we get:

$$\begin{aligned} & V_t = \left(A_1^{BI'}(t) + A_2^{BI'}(t)R\right) V \qquad V_x = \frac{1-\gamma}{x}\, V \qquad V_{xx} = -\gamma(1-\gamma)x^{-2}\, V \\ & V_R = A_2^{BI}(t)\, V \qquad V_{RR} = A_2^{BI}(t)^2\, V \qquad V_{Rx} = \frac{1-\gamma}{x}\, A_2^{BI}(t)\, V \end{aligned} \tag{A3}$$

Substituting Eqs.(A3) into Eq.(A1), and arranging it by order of R, we have:

$$\begin{aligned} & R\left[A_2^{BI'}(t) - bA_2^{BI}(t) + (1-\gamma)\right] \\ & + A_1^{BI'}(t) + \frac{1}{2}\frac{1-\gamma}{\gamma}(\lambda_R^2 + (\sigma_P - \lambda_P)^2) + (1-\gamma)(\sigma_P\lambda_P - c\theta - i) - \frac{1}{2}\gamma(1-\gamma)c^2\nu^2 \\ & + A_2^{BI}(t)\, b\, a + \frac{1}{2}A_2^{BI}(t)^2\sigma_R^2 + \frac{1-\gamma}{\gamma}A_2^{BI}(t)\lambda_R\sigma_R + \frac{1}{2}\frac{1-\gamma}{\gamma}A_2^{BI}(t)^2\sigma_R^2 = 0 \end{aligned} \tag{A4}$$

with terminal conditions $A_1^{BI}(T) = A_2^{BI}(T) = 0$.

The above equation satisfies for every R, so it is equivalent to the following two equation systems:

$$\begin{cases} A_2^{BI'}(t) - bA_2^{BI}(t) + (1-\gamma) = 0 \\ A_2^{L}(T) = 0 \end{cases} \tag{A5}$$

$$\begin{cases} A_1^{BI'}(t) + \frac{1}{2}\frac{1-\gamma}{\gamma}(\lambda_R^2 + (\sigma_P - \lambda_P)^2) + (1-\gamma)(\sigma_P\lambda_P - c\theta - i) - \frac{1}{2}\gamma(1-\gamma)c^2\nu^2 \\ + A_2^{BI}(t)\, b\, a + \frac{1}{2}A_2^{BI}(t)^2\sigma_R^2 + \frac{1-\gamma}{\gamma}A_2^{BI}(t)\lambda_R\sigma_R + \frac{1}{2}\frac{1-\gamma}{\gamma}A_2^{BI}(t)^2\sigma_R^2 = 0 \\ A_1^{BI}(T) = 0 \end{cases} \tag{A6}$$

The results hold by solving the above ordinary differential equations. By substituting the value function into the first order conditions in Eqs.(23), the optimal investment weights are thus obtained.
□

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