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Global Existence, Exponential Decay and Blow-Up of Solutions for a Class of Fractional Pseudo-Parabolic Equations with Logarithmic Nonlinearity

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Abstract

In this paper, we study the fractional pseudo-parabolic equations $u_t + (-\triangle)^s u + (-\triangle)^s u_t = u\log|u|$. Firstly, we recall the relationship between the fractional Laplace operator $(-\triangle)^s$ and the fractional Sobolev space H^s and discuss the invariant sets and the vacuum isolating behavior of solutions with the help of a family of potential wells. Then, we derive a threshold result of existence of global weak solution: for the low initial energy $J(u_0) < d$, the solution is global in time with $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$ and blows up $+\infty$ with $I(u_0) < 0$; for the critical initial energy $J(u_0) = d$, the solution is global in time with $I(u_0) \geq 0$ and blows up at $+\infty$ with $I(u_0) < 0$. The decay estimate of the energy functional for the global solution is also given.

2000 Mathematics Subject Classification: 35R11, 35A15, 45K05.
Keywords: blow-up; fractional pseudo-parabolic equations; initial energy; logarithmic nonlinearity.

1 Introduction

In this paper, we consider the following initial-boundary value problem for a class of fractional pseudo-parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t + (-\triangle)^s u + (-\triangle)^s u_t = u\log|u|, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \mathbb{R}^n \setminus \Omega,\ t \geq 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and the operator $(-\triangle)^s$ with $0 < s < 1$ is the fractional Laplacian defined by

$$(-\triangle)^s u(x) = -\frac{C(n,s)}{2}\int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy.$$

The equation in (1.1) is an important physical model, appears in many applications to natural sciences, such as the unidirectional propagation of nonlinear, dispersive, long waves [1], the aggregation of population [20] and the nonstationary processes in crystalline semiconductors [9].

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In the classical case, we have

$$\begin{cases} u_t - \triangle u_t - \triangle u = u \log |u|, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1.2}$$

where $\triangle$ is the standard Laplace operator. It's well known that problem (1.2) has been studied by many authors. A powerful technique for treating problem (1.2) is the so called "potential well method", which was established by Sattinger [22], Payne and Sattinger [21], and then improved by Liu and Zhao [17] by introducing a family of potential wells. Recently, there are some interesting results about the global existence and blow-up of solutions for problem (1.2) in [3], in which Chen and Tian proved global existence, blow-up at $+\infty$, the behavior of vacuum isolation and asymptotic behavior of solutions with initial energy $J(u_0) \leq d$. For other related works, we refer the readers to [2, 11, 6, 15] and the references therein.

In the fractional case, Nezza et al. [19] established the corresponding Sobolev inequality and Poincaré inequality on the cone Sobolev spaces. Then in [5], Fu and Pucci proved the existence theorem of global solutions with exponential decay and showed the blow-up in finite time of solutions to the space-fractional diffusion problem

$$\begin{cases} u_t + (-\Delta)^s u = |u|^{p-1} u, & x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \mathbb{R}^n \backslash \Omega, \ t \geq 0, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $n > 2s$, and p satisfies $1 < p \leq 2_s^* - 1 = \dfrac{n+2s}{n-2s}$. In [4], Cotsiolis and Tavoularis proved the existence of sharp logarithmic Sobolev inequalities with higher fractional derivatives. More works on fractional equations can be found in [7, 18, 23] and the references therein.

In this paper, we aim to use the logarithmic Sobolev inequalities with higher fractional derivatives and the improved potential well theory to prove the invariant sets, the vacuum isolating behavior, the global existence, decay and blow-up at $+\infty$ of solutions for problem (1.1) in fractional Sobolev space. For our purpose, we introduce a family of potential wells and its corresponding sets, and construct the relation between the existence of solution and the initial data $u_0(x)$ via the method of the potential wells. Then, by the usage of Faedo-Galerkin method and properties of a family of potential wells, we derive a threshold result of existence and nonexistence of global weak solution: for the low initial energy case (i.e., $J(u_0) < d$), the solution is global in time with $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$ and blows up at $+\infty$ with $I(u_0) < 0$; for the critical initial energy case (i.e., $J(u_0) = d$), the solution is global in time with $I(u_0) \geq 0$ and blows up at $+\infty$ with $I(u_0) < 0$. The decay estimate of the energy functional for the global solution is given by making use of a differential inequality technique.

The outline of this paper is as follows. In Section 2, we recall the fractional Laplace operator $(-\triangle)^s$, the fractional Sobolev space H^s and the corresponding properties. In Section 3, we give some preliminaries about the family of potential wells, after which we discuss the invariant sets and the vacuum isolating behavior of solutions for problem (1.1). In Section 4, we show the global existence, decay and blow-up at $+\infty$ for problem (1.1) with low initial energy $J(u_0) < d$. In Section

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5, we obtain the global existence, decay and blow-up at $+\infty$ for problem (1.1) with critical initial energy $J(u_0) = d$.

2 Preliminaries

In this section, we recall some preliminary results which are introduced in [5, 19] and will be useful in this paper.

We start by fixing the fractional exponent s in $(0,1)$ and give the definition of the fractional Sobolev spaces and the fractional Laplace operator. For any $p \in [1, +\infty)$, we define $W^{s,p}(\mathbb{R}^n)$ as follows

$$W^{s,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\mathbb{R}^n \times \mathbb{R}^n) \right\};$$

i.e., an intermediary Banach space between $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} := \left(\|u\|^p_{L^p(\mathbb{R}^n)} + \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}, \tag{2.1}$$

where the term

$$[u]_{W^{s,p}(\mathbb{R}^n)} := \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$$

is the so-called *Gagliardo (semi)norm* of u.

We focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ and $W_0^{s,2}(\mathbb{R}^n)$ turn out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^n)$ and $H_0^s(\mathbb{R}^n)$, respectively.

Before giving the definition of the fractional Laplace operator, we consider the Schwartz space $\mathscr{I}$ of rapidly decaying C^∞ functions in $\mathbb{R}^n$. The topology of this space is generated by the seminorms

$$p_N(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \le N} |D^\alpha \varphi(x)|, N = 0, 1, 2, \cdots,$$

where $\phi \in \mathscr{I}(\mathbb{R}^n)$. Let $\mathscr{I}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of $\mathscr{I}(\mathbb{R}^n)$. As usual, for any $\varphi \in \mathscr{I}(\mathbb{R}^n)$, we denote by

$$\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \varphi(x) dx$$

the Fourier transform of φ and we recall that one can extend $\mathcal{F}$ from $\mathscr{I}(\mathbb{R}^n)$ to $\mathscr{I}'(\mathbb{R}^n)$.

For any $u \in \mathscr{I}$ and $s \in (0,1)$, the fractional Laplacian operator $(-\triangle)^s$ is defined as

$$\begin{aligned} (-\triangle)^s u(x) &= C(n,s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \\ &= C(n,s) \lim_{\varepsilon \to 0^+} \int_{\ell B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy. \end{aligned} \tag{2.2}$$

Here, P.V. is a commonly used abbreviation for "in the principal value sense" and $C(n,s)$ is a dimensional constant that depends on n and s, precisely given by

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos\xi_1}{|\xi|^{n+2s}} d\xi \right)^{-1}.$$

The following proposition shows that one may write the singular integral in (2.2) as a weighted second order differential quotient.

Proposition 2.1 *[19, Lemma 3.2] Let $s \in (0,1)$, for any $u \in \mathcal{L}$,*

$$(-\triangle)^s u(x) = -\frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}} dy, \tag{2.3}$$

for all $x \in \mathbb{R}^n$.

Now, we will show that $H^s(\mathbb{R}^n)$ is strictly related to the fractional Laplacian $(-\triangle)^s$. For this purpose, we take into account an alternative definition of the space $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ via the Fourier transform. Precisely, for $s \in (0,1)$, we may define

$$\hat{H}^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\}. \tag{2.4}$$

The following proposition shows that the fractional Laplacian $(-\triangle)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$.

Proposition 2.2 *[19, Proposition 3.3] Let $s \in (0,1)$ and let $(-\triangle)^s : \mathcal{L} \to L^2(\mathbb{R}^n)$ be the fractional Laplacian operator defined by (2.3). Then, for any $u \in \mathcal{L}$,*

$$(-\triangle)^s u = \mathcal{F}^{-1}\left(|\xi|^{2s} (\mathcal{F}u) \right), \qquad \forall \xi \in \mathbb{R}^n.$$

The following proposition gives the equivalence of the space $\hat{H}^s(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$.

Proposition 2.3 *[19, Proposition 3.4] Let $s \in (0,1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^n)$ coincides with $\hat{H}^s(\mathbb{R}^n)$. In particular, for any $u \in H^s(\mathbb{R}^n)$,*

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi.$$

Then, following from Proposition 2.2 and 2.3, we get the relationship between the fractional Laplacian $(-\triangle)^s$ and the fractional Sobolev space H^s.

Proposition 2.4 *[19, Proposition 3.6] Let $s \in (0,1)$ and let $u \in H^s(\mathbb{R}^n)$. Then,*

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(R^n)}.$$

To handle logarithmic nonlinear term $u \log|u|$, we need the following logarithmic Sobolev inequality.

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Proposition 2.5 *[4, Theorem 2.1] Let s be a positive real number. Any function $f \in H^s(\mathbb{R}^n)$ satisfies*

$$\int_{\mathbb{R}^n} |f(x)|^2 \log\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) dx + \left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|f\|_2^2 \leq \frac{\alpha^2}{\pi^s}\|(-\Delta)^{\frac{s}{2}} f\|_2^2 \tag{2.5}$$

where $\alpha > 0$ be any number.

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Proposition 2.6 *[19, Theorem 6.5] Let $s \in (0,1)$ and $p \in [1, +\infty)$ such that $sp < n$. Then there exists a positive constant $C = C(n,p,s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\|f\|_{L^{p^*}(\mathbb{R}^n)}^p \leq C \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy \tag{2.6}$$

where $p^* = p^*(n,s)$ is the so-called "fractional critical exponent" and it is equal to $\dfrac{np}{n-sp}$.

By [23], we have that $[u]_{H^s(\mathbb{R}^n)}$ is also a norm equivalent to the usual one defined in (2.1). In this paper, we consider (1.1) in $X_0(\Omega) = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$. We know

$$\|u\|_{X_0(\Omega)} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\right)^{\frac{1}{2}}$$

is a norm on $X_0(\Omega)$ and $X_0(\Omega) = \left(X_0(\Omega), \|\cdot\|_{X_0(\Omega)}\right)$ is a Hilbert space with inner product

$$(u,v)_{X_0(\Omega)} = \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{n+2s}} dxdy.$$

Since $u \in X_0(\Omega)$, we know that the norm and inner product can be extended to all $\mathbb{R}^n \times \mathbb{R}^n$.

Proposition 2.7 *[24, Proposition 9] Denote by*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$$

the distinct eigenvalues and e_k the eigenfunction corresponding to λ_k of the elliptic eigenvalue problem:

$$\begin{cases} (-\triangle)^s u = \lambda u, & x \in \Omega, \\ u(x) = u_0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.7}$$

Then for $k \in n$,

$$\lambda_k = \frac{C(n,s)}{2} \min_{u \in \mathfrak{F}_k\setminus\{0\}} \frac{\|u\|_{X_0(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2},$$

where

$$\mathfrak{F}_1 = X_0(\Omega)$$

and for all $k \geq 2$,

$$\mathfrak{F}_k = \left\{u \in X_0(\Omega) : (u, e_j)_{X_0(\Omega)} = 0 \text{ for all } j = 1, 2, \ldots, k-1\right\}.$$

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Finally, we introduce the following functionals on fractional Sobolev space $H^s(\mathbb{R}^n)$:

$$J(u) = \frac{1}{2}\int_\Omega \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx - \frac{1}{2}\int_\Omega u^2 \log|u|\, dx + \frac{1}{4}\int_\Omega u^2 dx,$$

$$I(u) = \int_\Omega \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx - \int_\Omega u^2 \log|u|\, dx.$$

Since by Proposition 2.4, we have

$$\int_\Omega \left|(-\Delta)^{\frac{s}{2}}u\right|^2 dx = \frac{1}{2}C(n,s)\|u\|^2_{X_0(\Omega)}.$$

Then $J(u)$ and $I(u)$ can be written as

$$J(u) = \frac{1}{4}C(n,s)\|u\|^2_{X_0(\Omega)} - \frac{1}{2}\int_\Omega u^2 \log|u|\, dx + \frac{1}{4}\int_\Omega u^2 dx, \tag{2.8}$$

$$I(u) = \frac{1}{2}C(n,s)\|u\|^2_{X_0(\Omega)} - \int_\Omega u^2 \log|u|\, dx. \tag{2.9}$$

We introduce the potential well

$$W = \{u \in X_0(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\}$$

and the outside sets of the corresponding potential well

$$V = \{u \in X_0(\Omega) | I(u) < 0, J(u) < d\}.$$

We define the potential well depth d as

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda u), u \in X_0(\Omega)\setminus\{0\}, \|u\|_{X_0(\Omega)} \neq 0\right\},$$

and the Nehari manifold

$$\mathcal{N} = \left\{u \in X_0(\Omega) | I(u) = 0, \|u\|_{X_0(\Omega)} \neq 0\right\}.$$

Similar to the results in [25], one has $0 < d = \inf_{u\in\mathcal{N}} J(u)$.

3 Invariant sets and vacuum isolating

In this section, we shall introduce a family of Nehari functionals $I_\delta(u)$ in fractional Sobolev spaces, the family of potential wells sets and give the corresponding lemmas, which will help us to demonstrate the invariant sets and the vacuum isolating behavior of solutions for problem (1.1).

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3.1 Properties of potential wells

In this subsection, we shall introduce a family of potential wells W_δ, its corresponding sets V_δ and give a series of their properties which are useful in the proof of our main results.

Lemma 3.1 *Let $u \in X_0(\Omega) \setminus \{0\}$. Then:*

(1) $\lim_{\lambda \to 0} J(\lambda u) = 0, \ \lim_{\lambda \to +\infty} J(\lambda u) = -\infty.$

(2) *On the interval $0 < \lambda < \infty$, there exists a unique $\lambda^* = \lambda^*(u)$, such that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda=\lambda^*} = 0$.*

(3) *$J(\lambda u)$ is increasing on $0 \le \lambda \le \lambda^*$, decreasing on $\lambda^* \le \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$.*

(4) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$, and $I(\lambda^* u) = 0$.*

Proof. (1) From the definition of $J(u)$, we know

$$J(\lambda u) = \frac{\lambda^2}{4} C(n,s) \|u\|^2_{X_0(\Omega)} - \frac{\lambda^2}{2} \int_\Omega u^2 \log |\lambda u| \, dx + \frac{\lambda^2}{4} \int_\Omega u^2 dx,$$

which gives

$$\lim_{\lambda \to 0} J(\lambda u) = 0$$

and

$$\lim_{\lambda \to +\infty} J(\lambda u) = -\infty.$$

(2) An easy calculation shows that

$$\begin{aligned} \frac{d}{d\lambda} J(\lambda u) &= \frac{\lambda}{2} C(n,s) \|u\|^2_{X_0(\Omega)} - \lambda \int_\Omega u^2 \log |u| \, dx - \lambda \int_\Omega u^2 \log \lambda dx - \frac{\lambda}{2} \int_\Omega u^2 dx + \frac{\lambda}{2} \int_\Omega u^2 dx \\ &= \frac{\lambda}{2} C(n,s) \|u\|^2_{X_0(\Omega)} - \lambda \int_\Omega u^2 \log |u| \, dx - \lambda \log \lambda \int_\Omega u^2 dx. \end{aligned} \tag{3.1}$$

Let $\lambda^* = \exp\left(\frac{\frac{C(n,s)}{2} \|u\|^2_{X_0(\Omega)} - \int_\Omega u^2 \log |u| \, dx}{\|u\|^2_{L^2(\Omega)}} \right)$, then $\left. \frac{dJ(\lambda u)}{d\lambda} \right|_{\lambda=\lambda^*} = 0$.

(3) From

$$\frac{\partial^2 J(\lambda u)}{\partial \lambda^2} = \frac{C(n,s)}{2} \|u\|^2_{X_0(\Omega)} - \int_\Omega u^2 \log |u| \, dx - \log \lambda \int_\Omega u^2 dx - \int_\Omega u^2 dx,$$

we have

$$\left. \frac{\partial^2 J(\lambda u)}{\partial \lambda^2} \right|_{\lambda=\lambda^*} = - \int_\Omega u^2 dx < 0, \qquad \text{as } u \neq 0.$$

So, the conclusion of (3) holds.

(4) The conclusion follows from

$$\begin{aligned} I(\lambda u) &= \frac{\lambda^2}{2} C(n,s)\|u\|^2_{X_0(\Omega)} - \lambda^2 \int_\Omega u^2 \log \lambda\, |u|\, dx \\ &= \frac{\lambda^2}{2} C(n,s)\|u\|^2_{X_0(\Omega)} - \lambda^2 \int_\Omega u^2 \log |u|\, dx - \lambda^2 \log \lambda \int_\Omega u^2 dx \\ &= \lambda \frac{d}{d\lambda} J(\lambda u) \end{aligned}$$

Hence, when $0 < \lambda < \lambda^*$, $I(\lambda u) > 0$; when $\lambda^* < \lambda < \infty$, $I(\lambda u) < 0$; when $\lambda = \lambda^*$, $I(\lambda u) = 0$. □

For $\delta > 0$, we define a set of Nehari functionals in fractional Sobolev spaces.

$$I_\delta(u) = \frac{\delta}{2} C(n,s)\|u\|^2_{X_0(\Omega)} - \int_\Omega u^2 \log |u|\, dx.$$

Lemma 3.2 *Let $u \in X_0(\Omega)$:*

(1) *If $0 < \|u\|_{X_0(\Omega)} \leq \gamma(\delta)$, then $I_\delta(u) \geq 0$.*

(2) *If $I_\delta(u) < 0$, then $\|u\|_{X_0(\Omega)} > \gamma(\delta)$.*

(3) *If $I_\delta(u) = 0$, then $\|u\|_{X_0(\Omega)} \geq \gamma(\delta)$ or $\|u\|_{X_0(\Omega)} = 0$.*

Here $\gamma(\delta) = \lambda_1^{\frac{1}{2}} (2\pi^s \delta)^{\frac{n}{4s}} \sqrt{\frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}} e^{\frac{n}{2}} \left(\frac{2}{C(n,s)}\right)^{\frac{1}{2}}$.

Proof. (1) Using the logarithmic Sobolev inequality (2.5), for any $\alpha > 0$, we have

$$\begin{aligned} I_\delta(u) &= \frac{\delta}{2} C(n,s)\|u\|^2_{X_0(\Omega)} - \int_\Omega u^2 \log |u|\, dx \\ &\geq \frac{2\pi^s\delta}{\alpha^2} \int_\Omega |u|^2 \log\left(\frac{|u|}{\|u\|_{L^2(\Omega)}}\right) dx + \frac{\pi^s \delta}{\alpha^2}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2s}\right)}\right)\|u\|^2_{L^2(\Omega)} \\ &\quad - \int_\Omega |u|^2 \log |u|\, dx. \end{aligned} \tag{3.2}$$

Taking $\alpha = \sqrt{2\pi^s\delta}$ in (3.2), we obtain that

$$I_\delta(u) \geq \left[\frac{1}{2}\left(n + \frac{n}{s}\log\sqrt{2\pi^s\delta} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2s}\right)}\right) - \log \|u\|_{L^2(\Omega)}\right]\|u\|^2_{L^2(\Omega)}. \tag{3.3}$$

From $0 < \|u\|_{X_0(\Omega)} < \gamma(\delta)$, we have

$$\|u\|_{L^2(\Omega)} \leq (2\pi^s\delta)^{\frac{n}{4s}} \sqrt{\frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}} e^{\frac{n}{2}}. \tag{3.4}$$

By (3.3) and (3.4), we have

$$I_\delta(u) > 0.$$

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(2) Notice that $I_\delta(u)<0$, then from (3.3), we have

$$\log\|u\|_{L^2(\Omega)} > \frac{1}{2}\left(n+\frac{n}{s}\log\alpha+\log\frac{s\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2s}\right)}\right),$$

which gives

$$\|u\|_{X_0(\Omega)} = \lambda_1^{\frac{1}{2}}\left(\frac{2}{C(n,s)}\right)^{\frac{1}{2}}\|u\|_{L^2(\Omega)} > \gamma(\delta).$$

(3) If $\|u\|_{X_0(\Omega)}=0$, then

$$I_\delta(u)=0.$$

If $\|u\|_{X_0(\Omega)}\neq 0$ and $I_\delta(u)=0$, then

$$\log\|u\|_{L^2(\Omega)} \geq \frac{1}{2}\left(n+\frac{n}{s}\log\alpha+\log\frac{s\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2s}\right)}\right),$$

which gives

$$\|u\|_{X_0(\Omega)} = \lambda_1^{\frac{1}{2}}\left(\frac{2}{C(n,s)}\right)^{\frac{1}{2}}\|u\|_{L^2(\Omega)} \geq \gamma(\delta).$$

This completes the proof. □

Now, for $\delta>0$, we define the depth of a family of potential wells as follows

$$d(\delta)=\inf_{u\in\mathcal{N}_\delta} J(u),$$

where

$$\mathcal{N}_\delta=\left\{u\in X_0(\Omega)|I_\delta(u)=0,\|u\|_{X_0(\Omega)}\neq 0\right\}. \tag{3.5}$$

Then, the depth $d(\delta)$ and its expression can be estimated. Additionally, we show that how $d(\delta)$ behaves with respect to δ in the following lemma.

Lemma 3.3 *$d(\delta)$ satisfies the following properties:*

(1) $d(\delta)\geq\left(\frac{1}{4}-\frac{\delta}{4}\right)C(n,s)\gamma^2(\delta)+\frac{1}{4}\left(2\pi^s\delta\right)^{\frac{n}{2s}}e^n\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}$ *for* $0<\delta\leq 1$. *In particular,*

$$d\geq\frac{1}{4}\left(2\pi^s\right)^{\frac{n}{2s}}e^n\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}=M;$$

(2) *There exists a unique* b, $b\in\left(1,1+\frac{1}{2\lambda_1}\right]$ *such that* $d(b)=0$, *and* $d(\delta)>0$ *for* $1\leq\delta<b$;

(3) $d(\delta)$ *is increasing on* $0<\delta\leq 1$, *decreasing on* $1\leq\delta\leq b$ *and takes the maximum* $d=d(1)$ *at* $\delta=1$.

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Proof. (1) Let $u \in \mathcal{N}_\delta$, then $I_\delta(u) = 0$. By Lemma 3.2 (3), we have $\|u\|_{X_0(\Omega)} \geq \gamma(\delta)$. Hence from

$$\begin{aligned} J(u) &= \frac{1}{4}C(n,s)\|u\|^2_{X_0(\Omega)} - \frac{1}{2}\int_\Omega u^2 \log|u|\,dx + \frac{1}{4}\int_\Omega u^2 dx, \\ &= \left(\frac{1}{4} - \frac{\delta}{4}\right) C(n,s)\|u\|^2_{X_0(\Omega)} + \frac{1}{4}\int_\Omega u^2 dx \\ &\geq \left(\frac{1}{4} - \frac{\delta}{4}\right) C(n,s)\gamma^2(\delta) + \frac{1}{4}\left(2\pi^s\delta\right)^{\frac{n}{2s}} e^n \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}, \end{aligned}$$

we have

$$d(\delta) = \inf_{u\in\delta} J(u) \geq \left(\frac{1}{4} - \frac{\delta}{4}\right) C(n,s)\gamma^2(\delta) + \frac{1}{4}\left(2\pi^s\delta\right)^{\frac{n}{2s}} e^n \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}$$

for $0 < \delta \leq 1$.

(2) For any $u \in X_0(\Omega)$, $\|u\|_{L^2(\Omega)} \neq 0$, and for any $\delta > 0$, we can define a unique

$$\lambda = \lambda(\delta) = \exp\left(\frac{\frac{\delta C(n,s)}{2}\|u\|^2_{X_0(\Omega)} - \int_\Omega u^2\log|u|\,dx}{\|u\|^2_{L^2(\Omega)}}\right) \tag{3.6}$$

such that $I_\delta(\lambda u) = 0$. Then $\lambda u \in \mathcal{N}_\delta$, and

$$d(\delta) \leq J(\lambda u) = \lambda^2\left[\left(\frac{1}{4} - \frac{\delta}{4}\right) C(n,s)\|u\|^2_{X_0(\Omega)} + \frac{1}{4}\int_\Omega u^2 dx\right].$$

Therefore $d\left(1 + \frac{1}{2\lambda_1}\right) \leq 0$. On the other hand, $d(1) = d \geq \frac{1}{4}\left(2\pi^s\right)^{\frac{n}{2s}} e^n \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}$, $d(\delta)$ is continuous with δ, so there exists a unique $b \in \left(1, 1 + \frac{1}{2\lambda_1}\right]$ such that $d(b) = 0$, and $d(\delta) > 0$ for $1 \leq \delta < b$.

(3) It is enough to prove that for any $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < b$ and $u \in \mathcal{N}_{\delta''}$, there exist $v \in \mathcal{N}_{\delta'}$ and a constant $\varepsilon(\delta', \delta'')$ such that $J(v) < J(u) - \varepsilon(\delta', \delta'')$. In fact, for above u we also define $\lambda(\delta)$, then $I_\delta(\lambda(\delta)u) = 0$ and $\lambda(\delta'') = 1$. Let $g(\lambda) = J(\lambda u)$, we get

$$\begin{aligned} \frac{d}{d\lambda}g(\lambda) &= \frac{1}{\lambda}\left[(1-\delta)\frac{C(n,s)}{2}\|\lambda u\|^2_{X_0(\Omega)} + I_\delta(\lambda u)\right] \\ &= \lambda(1-\delta)\frac{C(n,s)}{2}\|\lambda u\|^2_{X_0(\Omega)}. \end{aligned}$$

Take $v = \lambda(\delta')$, then $v \in \mathcal{N}_{\delta'}$. For $0 < \delta' < \delta'' < 1$, we have

$$\begin{aligned} J(u) - J(v) &= g(1) - g(\lambda(\delta')) = g(\lambda(\delta'')) - g(\lambda\delta') \\ &= \int_{\lambda(\delta')}^{\lambda(\delta'')} \frac{d}{d\lambda}(g(\lambda))\,d\lambda \\ &= \int_{\lambda(\delta')}^{\lambda(\delta'')} (1-\delta)\lambda\frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} d\lambda \\ &\geq (1-\delta'')\frac{C(n,s)}{2}\gamma^2(\delta'')(\lambda(\delta'') - \lambda(\delta'))\lambda(\delta') \\ &= \varepsilon(\delta', \delta'') > 0. \end{aligned}$$

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For $1 < \delta'' < \delta' < b$, then

$$\begin{aligned} J(u) - J(v) &= \int_{\lambda(\delta')}^{\lambda(\delta'')} (1-\delta)\lambda \frac{C(n,s)}{2} \|u\|^2_{X_0(\Omega)} d\lambda \\ &> (1-\delta'') \frac{C(n,s)}{2} \gamma^2(\delta'')(\lambda(\delta'') - \lambda(\delta'))\lambda(\delta') \\ &= \varepsilon(\delta', \delta'') > 0. \end{aligned}$$

Therefore, the conclusion of (3) is proved. □

Lemma 3.4 *Let $u_0 \in X_0(\Omega)$ and $0 < \delta < 1$. Assume that $J(u) \le d(\delta)$:*

(1) *If $I_\delta(u) > 0$, then $\|u\|^2_{X_0(\Omega)} < \dfrac{d(\delta)}{a(\delta)}$, where $a(\delta) = \left(\dfrac{1}{4} - \dfrac{\delta}{4}\right) C(n,s)$.*

(2) *If $\|u\|^2_{X_0(\Omega)} > \dfrac{d(\delta)}{a(\delta)}$, then $I_\delta(u) < 0$.*

(3) *If $I_\delta(u) = 0$, then $\|u\|^2_{X_0(\Omega)} \le \dfrac{d(\delta)}{a(\delta)}$.*

Proof. (1) For $0 < \delta < 1$, we have

$$\left(\frac{1}{4} - \frac{\delta}{4}\right) C(n,s) \|u\|^2_{X_0(\Omega)} + \frac{1}{4} \int_\Omega u^2 dx + \frac{1}{2} I_\delta(u) = J(u) \le d(\delta). \tag{3.7}$$

Then $a(\delta)\|u\|^2_{X_0(\Omega)} < d(\delta)$, i.e., $\|u\|^2_{X_0(\Omega)} < \dfrac{d(\delta)}{a(\delta)}$.

(2) If $\|u\|^2_{X_0(\Omega)} > \dfrac{d(\delta)}{a(\delta)}$, then from (3.7), we get $I_\delta(u) < 0$.

(3) If $I_\delta(u) = 0$, then from (3.7), we have $\|u\|^2_{X_0(\Omega)} \le \dfrac{d(\delta)}{a(\delta)}$. □

Now, we can define

$$d_0 = \lim_{\delta \to 0^+} d(\delta). \tag{3.8}$$

Then by Lemma 3.3, $d_0 \ge 0$.

Lemma 3.5 *Assume $d_0 < J(u) < d$ for some $u \in X_0(\Omega)$, and $\delta_1 < \delta_2$ are two roots of equation $d(\delta) = J(u)$. Then the sign of $I_\delta(u)$ doesn't change for $\delta_1 < \delta < \delta_2$.*

Proof. $J(u) > d_0$ implies $\|u\|_{X_0(\Omega)} \ne 0$. If the sign of $I_\delta(u)$ is changeable for $\delta_1 < \delta < \delta_2$, then we can choose $\bar{\delta} \in (\delta_1, \delta_2)$ and $I_{\bar{\delta}}(u) = 0$. Therefore, we can have $J(u) \ge d(\bar{\delta})$. From Lemma 3.3 (3), we have $J(u) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$, which is contradict with $J(u) \ge d(\bar{\delta})$. □

Now, we are in a position to introduce a series of potential wells. For $0 < \delta < b$, we define

$$\begin{aligned} W_\delta &= \{u \in X_0(\Omega) \mid I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \\ V_\delta &= \{u \in X_0(\Omega) \mid I_\delta(u) < 0, J(u) < d(\delta)\}. \end{aligned}$$

From the definition of W_δ, V_δ and Lemma 3.3, we can obtain the following lemmas:

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Lemma 3.6 (1) *If* $0 < \delta' < \delta'' \leq 1$, *then* $W_{\delta'} \subset W_{\delta''}$.

(2) *If* $1 \leq \delta'' < \delta' < b$, *then* $V_{\delta'} \subset V_{\delta''}$.

In addition, we define

$$\begin{aligned} B_\delta &= \left\{ u \in X_0(\Omega) \mid \|u\|_{X_0(\Omega)} < \gamma(\delta) \right\}, \\ \bar{B}_\delta &= B_\delta \cup \partial B_\delta = \left\{ u \in X_0(\Omega) \mid \|u\|_{X_0(\Omega)} \leq \gamma(\delta) \right\}, \\ B_\delta^c &= \left\{ u \in X_0(\Omega) \mid \|u\|_{X_0(\Omega)} > \gamma(\delta) \right\}. \end{aligned}$$

Lemma 3.7 *Let* $0 < \delta < b$. *Then*

$$B_{\gamma_1(\delta)} \subset W_\delta \subset B_{\gamma_2(\delta)}, V_\delta \subset B_\delta^c$$

where

$$\begin{aligned} B_{\gamma_1(\delta)} &= \left\{ u \in X_0(\Omega) | \|u\|^2_{X_0(\Omega)} < \min\left\{\gamma^2(\delta), \gamma_0^2(\delta)\right\} \right\}, \\ B_{\gamma_2(\delta)} &= \left\{ u \in X_0(\Omega) | \|u\|^2_{X_0(\Omega)} < \frac{d(\delta)}{a(\delta)} \right\}, \end{aligned}$$

where $\gamma_0(\delta)$ *is the unique real root of equation*

$$\frac{C(n,s)}{4}\gamma^2 + \frac{1}{4}\left(2\pi^s\delta\right)^{\frac{n}{2s}} e^n \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} = d(\delta).$$

Proof. Firstly, $\|u\|_{X_0(\Omega)} < \gamma(\delta)$ gives $\|u\|_{X_0(\Omega)} = 0$ or $I_\delta(u) > 0$ and

$$\frac{1}{4}\int_\Omega u^2 dx < \frac{1}{4}\left(2\pi^s\delta\right)^{\frac{n}{2s}} e^n \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}.$$

On the other hand,

$$J(u) \leq \frac{1}{4}C(n,s)\|u\|^2_{X_0(\Omega)} + \frac{1}{4}\int_\Omega u^2 dx$$

and $\|u\|^2_{X_0(\Omega)} < \gamma_0^2(\delta)$ yield $J(u) < d(\delta)$. Hence we have $B_{\gamma_1(\delta)} \subset W_\delta$. The remainder of this lemma follows from Lemma 3.2 and Lemma 3.4. □

3.2 Invariant sets and Vacuum isolating

In this subsection, we prove the invariance of some sets under the flow of (1.1) and the vacuum isolating behavior of problem (1.1).

Definition 1 (Maximal existence time) *Let* $u(t)$ *be a weak solution of problem* (1.1). *We define the maximal existence time* T_{max} *of* $u(t)$ *as follows:*

(1) *If* $u(t)$ *exists for* $0 \leq t < \infty$, *then* $T_{max} = +\infty$.

(2) *If there exists a* $t_0 \in (0, \infty)$ *such that* $u(t)$ *exists for* $0 \leq t < t_0$, *but doesn't exist at* $t = t_0$, *then* $T_{max} = t_0$.

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Definition 2 (Blow-up at $+\infty$) *Let $u(t)$ be a weak solution of problem* (1.1). *We say $u(x,t)$ blows up at $+\infty$ if $T_{max} = +\infty$ and*

$$\lim_{t\to+\infty}\left(\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)}\right) = +\infty.$$

Definition 3 (Weak solution) *Function $u = u(x,t)$ is called a weak solution of problem* (1.1) *on $\Omega\times[0,T_{max})$, with $0 < T_{max} \le +\infty$ being the maximal existence time, if $u \in L^\infty(0,T_{max};X_0(\Omega))$ with $u_t \in L^2(0,T_{max};X_0(\Omega))$ and satisfies problem* (1.1) *in the distribution sense, i.e.,*

(1) $\forall\, v \in X_0(\Omega)$, $t \in [0,T_{max})$,

$$(u_t,v)_{L^2(\Omega} + \left((-\Delta)^{\frac{s}{2}} u_t, (-\Delta)^{\frac{s}{2}} v\right)_{L^2(\Omega} + \left((-\Delta)^{\frac{s}{2}} u, (-\Delta)^{\frac{s}{2}} v\right)_{L^2(\Omega} = (u\log|u|\,,v)_{L^2(\Omega}. \tag{3.9}$$

(2) $u(x,0) = u_0(x)$ *in* $X_0(\Omega)$.

(3) *For* $0 \le t < T_{max}$,

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)}d\tau + \int_0^t \frac{C(n,s)}{2}\|u_\tau\|^2_{X_0(\Omega)}d\tau + J(u(t)) \le J(u_0). \tag{3.10}$$

Now, we discuss the invariance of some sets corresponding to problem (1.1) inspired by the ideas in [16].

Theorem 3.1 *Let $u_0 \in X_0(\Omega)$, $0 < e < d$, $\bar\delta \in (1,b)$ be the root of equation $d(\delta) = e$. Then:*

(1) *All weak solutions u of problem* (1.1) *with $0 < J(u_0) \le e$ belong to W_δ for $1 \le \delta < \bar\delta$, $0 \le t < T_{max}$, provided $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$.*

(2) *All weak solutions u of problem* (1.1) *with $0 < J(u_0) \le e$ belong to V_δ for $1 \le \delta < \bar\delta$, $0 \le t < T_{max}$, provided $I(u_0) < 0$,*

where T_{max} is the maximal existence time of $u(t)$.

Proof. (1) Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) \le e$, $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$. T_{max} is the existence time of $u(t)$. If $\|u_0\|_{X_0(\Omega)} = 0$, then $u_0(x) \in W_\delta$. If $I(u_0) > 0$, then from Lemma 3.5, it follows $I_\delta(u_0) > 0$ and $J(u_0) < d(\delta)$. Then $u_0(x) \in W_\delta$ for $1 \le \delta < \bar\delta$.

Next, we should prove $u(t) \in W_\delta$ for $1 \le \delta < \bar\delta$ and $0 < t < T_{max}$. Arguing by contradiction, by the continuity of $I(u)$ we suppose that there must exist a $\delta_0 \in (1,\bar\delta)$ and $t_0 \in (0,T_{max})$ such that $u(t_0) \in \partial W_{\delta_0}$, and $I_{\delta_0}(u(t_0)) = 0$, $\|u_0\|_{X_0(\Omega)} \ne 0$ or $J(u(t_0)) = d(\delta_0)$. From

$$\begin{aligned}&\int_0^t \|u_\tau\|^2_{L^2(\Omega)}d\tau + \int_0^t \frac{C(n,s)}{2}\|u_\tau\|^2_{X_0(\Omega)}d\tau + J(u(t))\\ \le& J(u_0) < d(\delta), \qquad 1 \le \delta < \bar\delta, \qquad 0 \le t < T_{max},\end{aligned} \tag{3.11}$$

we can see that $J(u(t_0)) \ne d(\delta_0)$. If $I_{\delta_0}(u(t_0)) = 0$, $\|u(t_0)\|_{X_0(\Omega)} \ne 0$, then by the definition of $d(\delta)$, we have $J(u(t_0)) \ge d(\delta_0)$, which contradicts (3.11).

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(2) Let $u(t)$ be a weak solution of problem (1.1) with $0 < J(u_0) \le e < d$, $I(u_0) < 0$. From $J(u_0) \le e$, $I(u_0) < 0$ and Lemma 3.5, it follows $I_\delta(u_0) < 0$ and $J(u_0) < d(\delta)$. Then $u_0(x) \in V_\delta$ for $1 \le \delta < \bar{\delta}$.

We prove $u(t) \in V_\delta$ for $1 \le \delta < \bar{\delta}$ and $0 < t < T_{max}$. Arguing by contradiction, by time continuity of $I(u)$ we suppose that there must exist a $\delta_0 \in (1, \bar{\delta})$ and $t_0 \in (0, T_{max})$ such that $u(t_0) \in \partial V_{\delta_0}$, and $I_{\delta_0}(u(t_0)) = 0$ or $J(u(t_0)) = d(\delta_0)$. By (3.11) we can see that $J(u(t_0)) \neq d(\delta_0)$. Assume $I_{\delta_0}(u(t_0)) = 0$ and t_0 is the first time such that $I_{\delta_0}(u(t)) = 0$, then $I_{\delta_0}(u(t)) < 0$ for $0 \le t < t_0$. By Lemma 3.2 (2) we have $\|u_0\|_{X_0(\Omega)} > \gamma(\delta_0)$ for $0 \le t < t_0$. Hence $\|u(t_0)\|_{X_0(\Omega)} \ge \gamma(\delta_0)$, then $\|u(t_0)\|_{X_0(\Omega)} \neq 0$. From $u(t_0) \in \mathcal{N}_{\delta_0}$ and $J(u(t_0)) \neq d(\delta_0)$, we have $J(u(t_0)) > d(\delta_0)$, which contradicts to (3.11). □

Corollary 3.1 *Let $u_0 \in X_0(\Omega)$, $d_0 < e < d$, $\delta_1 < \delta_2$ be the two roots of equation $d(\delta) = e$. Then:*

(1) *All weak solutions u of problem* (1.1) *with $0 < J(u_0) \le e$ belong to W_δ for $\delta_1 < \delta < \delta_2$, $0 \le t < T_{max}$, provided $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$.*

(2) *All weak solutions u of problem* (1.1) *with $0 < J(u_0) \le e$ belong to V_δ for $\delta_1 < \delta < \delta_2$, $0 \le t < T_{max}$, provided $I(u_0) < 0$,*

where T_{max} is the maximal existence time of $u(t)$.

Proof. The proof is similar to Theorem 3.1, then we omit it. □

Remark 3.1 *If $u_0 \in X_0(\Omega)$, $J(u_0) \le 0$ and $I(u_0) < 0$, $u(x,t)$ is a weak solution of problem* (1.1), *then $I(u(t)) < 0$ for all $0 \le t < T$. This can be deduced from Theorem 3.3 and* (3.10).

To deal with the critical case, we have the following proposition.

Proposition 3.1 *If $u_0 \in X_0(\Omega)$, $J(u_0) = d$, $u(x,t)$ is a weak solution of problem* (1.1). *Then*

(1) *$I(u(t)) > 0$ for all $0 \le t < T_{max}$, provided $I(u_0) > 0$,*

(2) *$I(u(t)) < 0$ for all $0 \le t < T_{max}$, provided $I(u_0) < 0$.*

Here T_{max} is the maximal existence time of $u(x,t)$.

Proof. (1) If the result is false, then there exists $t_1 \in (0, T_{max})$ such that

$$I(u(t_1)) = 0, \quad I(u(t)) > 0, \quad \text{for all } 0 < t < t_1.$$

Then combining the fact $I(u) = -(u, u_t) - \left((-\Delta)^{\frac{s}{2}} u_t, (-\Delta)^{\frac{s}{2}} u\right) \neq 0$, one has $\|u_t\|_{L^2(\Omega)} > 0$ or $\|u_t\|_{X_0(\Omega)} > 0$ for $0 < t < t_1$. Hence, we have

$$\int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{C(n,s)}{2} \|u_\tau\|^2_{X_0(\Omega)} d\tau > 0, \text{ for all } 0 < t \le t_1$$

and

$$J(u(t)) \le J(u_0) - \int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau - \int_0^t \frac{C(n,s)}{2} \|u_\tau\|^2_{X_0(\Omega)} d\tau < d, \text{ for all } 0 < t \le t_1. \tag{3.12}$$

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Also $I(u(t_1))) = 0$ and $I(u(t)) > 0$ for all $0 < t < t_1$ imply that $\|u(t_1)\|_{X_0(\Omega)} \geq \gamma(1) \neq 0$. Then by the definition of d, we have $J(u(t_1)) \geq d$, which is contradictive with (3.12).

(2) If the result is false, then there exists $t_1 \in (0, T_{max})$ such that

$$I(u(t_1)) = 0, \quad I(u(t)) < 0, \quad \text{for all } 0 < t < t_1.$$

Similar to the proof of (i), we have

$$J(u(t)) \leq J(u_0) - \int_0^t \|u_\tau\|^2_{L^2(\Omega)} d\tau - \int_0^t \frac{C(n,s)}{2} \|u_\tau\|^2_{X_0(\Omega)} d\tau < d, \text{ for all } 0 < t \leq t_1. \tag{3.13}$$

Also from Lemma 3.2 and $I(u(t)) < 0$ for all $0 \leq t < t_1$, then $\|u(t_1)\|_{X_0(\Omega)} \geq \gamma(1) \neq 0$. By the definition of d, $J(u(t_1)) \geq d$, which is contradictive with (3.13). □

For the invariant of the solutions with negative level energy, we also have the following results.

Proposition 3.2 *All nontrivial solutions of problem* (1.1) *with* $J(u_0) = 0$ *belong to*

$$B^c_{\gamma_0} = \left\{ u \in X_0(\Omega) | \|u\|_{X_0(\Omega)} \geq \gamma_0 \right\},$$

where

$$\gamma_0 = \lambda_1^{\frac{1}{2}} (2\pi^s)^{\frac{n}{4s}} \sqrt{\frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}} e^{\frac{n+1}{2}} \left(\frac{2}{C(n,s)} \right)^{\frac{1}{2}} \tag{3.14}$$

Proof. Let $u(t)$ be any solution of problem (1.1) with $J(u_0) = 0$, T_{max} be the maximal existence time of $u(t)$. From (3.10), we get $J(u) \leq 0$ for $0 \leq t < T_{max}$. Hence by

$$\frac{1}{4} C(n,s) \|u\|^2_{X_0(\Omega)} + \frac{1}{4} \int_\Omega u^2 dx \leq \frac{1}{2} \int_\Omega u^2 \log |u| \, dx$$

and (2.5), we have

$$\frac{1}{4} C(n,s) \|u\|^2_{X_0(\Omega)} + \frac{1}{4} \int_\Omega u^2 dx \leq \frac{\alpha^2}{\pi^s} \frac{1}{8} C(n,s) \|u\|^2_{X_0(\Omega)} + \frac{1}{2} \int_\Omega u^2 \log \|u\|_{L^2(\Omega)} dx$$
$$- \frac{1}{4} \left(n + \frac{n}{s} \log \alpha + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} \right) \|u\|^2_{L^2(\Omega)} \tag{3.15}$$

Taking $\alpha = \sqrt{2\pi^s}$ in (3.15), we obtain that

$$\frac{1}{2} \left(1 + n + \frac{n}{s} \log \sqrt{2\pi^s} + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} \right) \|u\|^2_{L^2(\Omega)} \leq \|u\|^2_{L^2(\Omega)} \log \|u\|_{L^2(\Omega)}, \tag{3.16}$$

which gives

$$\|u\|_{L^2(\Omega)} \geq (2\pi^s)^{\frac{n}{4s}} \sqrt{\frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}} e^{\frac{n+1}{2}}.$$

Therefore, the Proposition is proved. □

Theorem 3.2 *Let* $u_0 \in X_0(\Omega)$. *Assume that* $J(u_0) < 0$ *or* $J(u_0) = 0$ *and* $\|u_0\|_{X_0(\Omega)} \neq 0$. *Then all solutions of problem* (1.1) *belong to* V_δ *for* $0 < \delta < b$.

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Proof. Let $u(t)$ be any solution of problem (1.1) with $J(u_0) < 0$ or $J(u_0) = 0$ and $\|u_0\|_{X_0(\Omega)} \neq 0$, T_{max} be the maximal existence time of $u(t)$. The energy inequality gives

$$\left(\frac{1}{4} - \frac{\delta}{4}\right) C(n,s)\|u\|^2_{X_0(\Omega)} + \frac{1}{4}\int_\Omega u^2dx + \frac{1}{2}I_\delta(u) = J(u) \leq J(u_0), \qquad 0 < \delta < b. \tag{3.17}$$

From (3.17) it follows that if $J(u_0) < 0$, then $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $0 < \delta < b$; if $J(u_0) = 0$ and $\|u_0\|_{X_0(\Omega)} \neq 0$, then by Proposition 3.2 we have $\|u_0\|_{X_0(\Omega)} \geq \gamma_0$ for $0 \leq t < T_{max}$. Again by (3.17) we get $I_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for $0 < \delta < b$. Hence for above two cases we always have $u(t) \in V_\delta$ for $0 < \delta < b$, $0 \leq t < T_{max}$. □

Corollary 3.2 *Let $u_0 \in X_0(\Omega)$. Assume that $J(u_0) < 0$ or $J(u_0) = 0$ and $\|u_0\|_{X_0(\Omega)} \neq 0$. Then all weak solutions of problem* (1.1) *belong to $\bar{B}^c_b$.*

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) < 0$ or $J(u_0) = 0$ and $\|u_0\|_{X_0(\Omega)} \neq 0$, T_{max} be the maximal existence time of $u(t)$. Then Theorem 3.2 gives

$$u(t) \in V_\delta \text{ for } 0 < \delta < b,\ 0 \leq t < T_{max}.$$

From this and Lemma 3.2 we get $\|u_0\|_{X_0(\Omega)} > \gamma(\delta)$ for $0 < \delta < b$, $0 \leq t < T_{max}$. Letting $\delta \to b$, we obtain $\|u_0\|_{X_0(\Omega)} \geq \gamma(b)$ for $0 \leq t < T_{max}$. □

Now, we discuss the vacuum isolating to problem (1.1) with $J(u_0) < d$.

Theorem 3.3 *Let $e \in (d_0, d)$. Suppose δ_1, δ_2 are the two roots of $d(\delta) = e$. Then for all weak solutions of problem* (1.1) *with $J(u_0) \leq e$, there is a vacuum region*

$$U_e = \left\{u \in X_0(\Omega) | I_\delta(u) = 0, \|u_0\|_{X_0(\Omega)} \neq 0, \delta_1 < \delta < \delta_2\right\}$$

such that there is no any weak solution of problem (1.1) *in U_e.*

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) \leq e$, T_{max} be the maximal existence time of $u(t)$. We only need to prove that if $\|u_0\|_{X_0(\Omega)} \neq 0$ and $J(u_0) \leq e$, then for all $\delta \in (\delta_1, \delta_2)$, $u(t) \notin N_\delta$, i.e. $I_\delta(u(t)) \neq 0$, for all $t \in [0, T_{max})$.

At first, it is clear that $I_\delta(u_0) \neq 0$. Since if $I_\delta(u_0) = 0$, then $J(u_0) \geq d(\delta) > d(\delta_1) = d(\delta_2)$, which contradicts with $J(u_0) \leq e$.

Suppose there is $t_1 > 0$ s.t. $u(t_1) \in U_e$. Namely, there must exist a $\delta_0 \in (\delta_1, \delta_2)$ such that $u(t_1) \in N_{\delta_0}$. From (3.10), we get $J(u_0) \geq J(u(t_1)) \geq d(\delta) > J(u_0)$, which leads to a contradiction. □

Remark 3.2 *The vacuum region U_e becomes bigger and bigger with decreasing of e. As the limit case we obtain*

$$U_0 = \left\{u \in X_0(\Omega) | I_\delta(u) = 0, \|u_0\|_{X_0(\Omega)} \neq 0, 0 < \delta < \delta_3\right\}$$

where $\delta_3 > 1$ and satisfies $d_0 = d(\delta_3)$.

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4 Low initial energy $J(u_0) < d$

In this section, we prove a threshold result of global existence and nonexistence of solutions for problem (1.1) with the low initial energy $J(u_0) < d$.

4.1 Global existence with exponential decay

In this subsection, we establish the global existence of weak solutions for problem (1.1) when $J(u_0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$ by using Galerkin approximation technique and potential well theory. Meanwhile, we obtain the asymptotic stability of global solutions. The following lemma will be used to prove the asymptotic stability.

Lemma 4.1 *Let $y(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function. Assume that there is a constant $A > 0$ such that*

$$\int_s^{+\infty} y(t)dt \le Ay(s), \quad 0 \le s < +\infty.$$

Then $y(t) \le y(0)e^{1-\frac{t}{A}}$, for all $t > 0$.

Theorem 4.1 (Global existence and decay for $J(u_0) < d$) *Let $u_0 \in X_0(\Omega)$. Assume that $J(u_0) < d$ and $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$. Then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, \infty; X_0(\Omega))$ with $u_t \in L^2(0, \infty; X_0(\Omega))$. Moreover $u(t) \in W$ for $0 \le t < \infty$ and*

(1) *if $J(u_0) < M$, we have*

$$\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} \le \left(\|u_0\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u_0\|^2_{X_0(\Omega)}\right) e^{1-4Dt},$$

where

$$D = \min\left\{\frac{1}{2} - \frac{\alpha^2}{4\pi^s}, \frac{1}{4}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \frac{1}{4}\log 8J(u_0)\right\},$$

for any $\left[\dfrac{8J(u_0)\Gamma(\frac{n}{2s})}{e^n s\Gamma(\frac{n}{2})}\right]^{\frac{s}{n}} < \alpha < \sqrt{2\pi^s}.$

(2) *if $J(u_0) = M$, we have*

$$\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} \le \left(\|u_0\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u_0\|^2_{X_0(\Omega)}\right) e^{1-4D_1t},$$

where

$$D_1 = \min\left\{\frac{1}{2} - \frac{\alpha^2}{4\pi^s}, \frac{1}{4}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \frac{1}{4}\log 8(M-\beta)\right\},$$

for any $\left[\dfrac{8(M-\beta)\Gamma(\frac{n}{2s})}{e^n s\Gamma(\frac{n}{2})}\right]^{\frac{s}{n}} < \alpha < \sqrt{2\pi^s}.$

Proof. We divide the proof into two steps.

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Step 1 Proof of global existence.

Let $\{\omega_j(x)\}$ be a system of base functions in $X_0(\Omega)$. Now we construct the following approximate solutions $u_m(t,x)$ of problem (1.1):

$$u_m(t,x) = \sum_{j=1}^{m} g_{jm}(t)\omega_j(x), \qquad m = 1, 2, \cdots,$$

which satisfies

$$(u_{mt}, \omega_s)_2 + \left((-\Delta)^{\frac{s}{2}} u_m, (-\Delta)^{\frac{s}{2}} \omega_s\right)_2 + \left((-\Delta)^{\frac{s}{2}} u_{mt}, (-\Delta)^{\frac{s}{2}} \omega_s\right)_2 = (u_m \log |u_m|, \omega_s)_2, \quad (4.1)$$

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm}\omega_j(x) \to u_0(x) \text{ in } X_0(\Omega). \quad (4.2)$$

Multiplying (4.1) by $g'_{sm}(t)$, summing for s, and integrating with respect to t from 0 to t, we have

$$\int_0^t \|u_{m\tau}\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{C(n,s)}{2} \|u_{m\tau}\|^2_{X_0(\Omega)} d\tau + J(u_m) \le J(u_m(0)), \qquad 0 \le t < \infty.$$

By (4.2) we can get $J(u_m(0)) \to J(u_0)$, then for sufficiently large m, we have

$$\int_0^t \|u_{m\tau}\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{C(n,s)}{2} \|u_{m\tau}\|^2_{X_0(\Omega)} d\tau + J(u_m) < d, \qquad 0 \le t < \infty. \quad (4.3)$$

From (4.3) and the proof of Theorem 3.1, we can get $u_m(t) \in W$ for $0 \le t < \infty$ and sufficiently large m. Hence, by (4.3) and

$$J(u_m) = \frac{1}{4}\int_\Omega u_m^2 dx + \frac{1}{2} I(u_m),$$

we obtain

$$\int_0^t \|u_{m\tau}\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{C(n,s)}{2} \|u_{m\tau}\|^2_{X_0(\Omega)} d\tau + \frac{1}{4}\int_\Omega u_m^2 dx < d, \qquad 0 \le t < \infty, \quad (4.4)$$

for sufficiently large m, which yields

$$\int_\Omega u_m^2 dx < 4d, \qquad 0 \le t < \infty. \quad (4.5)$$

Taking $\alpha = \sqrt{\pi^s}$ in (2.5), we have

$$\begin{aligned} &\frac{C(n,s)}{2} \|u_m\|^2_{X_0(\Omega)} \quad (4.6)\\ &= 2I(u_m) + 2\int_\Omega u_m^2 \log |u_m| \, dx - \frac{C(n,s)}{2} \|u_m\|^2_{X_0(\Omega)} \\ &\le 2I(u_m) + 2\int_\Omega u_m^2 \log |u_m| \, dx - 2\int_\Omega u_m^2 \log |u_m| \, dx + 2\int_\Omega u_m^2 \log \|u_m\|_{L^2(\Omega)} dx \\ &\quad - \left(n + n \log \sqrt{\pi} + \log \frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) \|u_m\|^2_{L^2(\Omega)} \end{aligned}$$

$$= 4J(u_m) + \left(2\log\|u_m\|_{L^2(\Omega)} - 1 - \left(n + n\log\sqrt{\pi} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\right)\|u_m\|^2_{L^2(\Omega)}$$
$$\leq C_d. \tag{4.7}$$

Also, (4.3) implies

$$\int_0^t \|u_{m\tau}\|^2_{L^2(\Omega)}d\tau + \int_0^t \frac{C(n,s)}{2}\|u_{m\tau}\|^2_{X_0(\Omega)}d\tau < d, \qquad 0 \leq t < \infty.$$

On the other hand, by a direct calculation, we have

$$\int_\Omega (u_m \log|u_m|)^2\,dx = \int_{\{x\in\Omega;u_m(x)\leq 1\}} (u_m\log|u_m|)^2\,dx + \int_{\{x\in\Omega;u_m(x)>1\}} (u_m\log|u_m|)^2\,dx$$
$$\leq e^{-2}|\Omega| + \left(\frac{n-2s}{2s}\right)^2 \int_{\{x\in\Omega;u_m(x)>1\}} u_m^{\frac{2n}{n-2s}}\,dx$$
$$\leq e^{-2}|\Omega| + \left(\frac{n-2s}{2s}\right)^2 \|u_m\|^{\frac{2n}{n-2s}}_{L^{\frac{2n}{n-2s}}(\Omega)} \tag{4.8}$$

By (4.8) and Proposition 2.6, we have

$$\int_\Omega (u_m\log|u_m|)^2\,dx \leq e^{-2}|\Omega| + \left(\frac{n-2s}{2s}\right)^2 C^{2^*}\|u_m\|^{2^*}_{X_0(\Omega)}$$
$$\leq C_d$$

Therefore, there exist a u and a subsequence $\{u_v\}$ such that

$$u_v \to u \text{ in } L^\infty(0,\infty;X_0(\Omega)) \text{ weakly star and a.e. in } \Omega\times[0,\infty),$$
$$u_{vt} \to u_t \text{ in } L^2(0,\infty;X_0(\Omega)) \text{ weakly star},$$
$$u_v\log|u_v| \to u\log|u| \text{ in } L^\infty\left(0,\infty;L^2(\Omega)\right) \text{ weakly star and a.e. in } \Omega\times[0,\infty).$$

In (4.1), we fixed s, letting $m = v \to \infty$. Then, we get

$$(u_t,\omega_s)_{L^2(\Omega)} + \left((-\Delta)^{\frac{s}{2}}u,(-\Delta)^{\frac{s}{2}}\omega_s\right)_{L^2(\Omega} + \left((-\Delta)^{\frac{s}{2}}u_t,(-\Delta)^{\frac{s}{2}}\omega_s\right)_{L^2(\Omega} = (u\log|u|,\omega_s)_{L^2(\Omega}$$

and

$$(u_t,v)_{L^2(\Omega} + \left((-\Delta)^{\frac{s}{2}}u,(-\Delta)^{\frac{s}{2}}v\right)_{L^2(\Omega} + \left((-\Delta)^{\frac{s}{2}}u_t,(-\Delta)^{\frac{s}{2}}v\right)_{L^2(\Omega} = (u\log|u|,v)_{L^2(\Omega},$$

for all $v \in X_0(\Omega), t \in (0,T_{max})$. From (4.2) we obtain $u(x,0) = u_0(x)$ in $X_0(\Omega)$, $t \in (0,T_{max})$. By density we obtain $u \in L^\infty(0,\infty;X_0(\Omega))$ with $u_t \in L^2(0,\infty;X_0(\Omega))$ is a global weak solution of problem (1.1). It is obvious that $u(t) \in W$ for $0 \leq t < \infty$.

Step 2 Proof of decay.

Taking $v = u$ in (3.9), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)}\right) = -\frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} + \int_\Omega u^2\log u\,dx = -I(u).$$

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Then we have

$$\int_t^T I(u) = \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{4}\|u(t)\|^2_{X_0(\Omega)} - \left(\frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{4}\|u(T)\|^2_{X_0(\Omega)}\right)$$
$$\leq \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{4}\|u(t)\|^2_{X_0(\Omega)} \tag{4.9}$$

By the definition of $I(u)$, we have

$$\begin{aligned} I(u) &= \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} - \int_\Omega u^2 \log u dx \\ &\geq \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} - \frac{\alpha^2}{2\pi^s}\frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)} - \frac{1}{2}\|u\|^2_{L^2(\Omega)}\log\|u\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{2}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|u\|^2_{L^2(\Omega)} \\ &= \frac{C(n,s)}{2}\left(1 - \frac{\alpha^2}{2\pi^s}\right)\|u\|^2_{X_0(\Omega)} + \left[\frac{1}{2}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \log\|u\|_{L^2(\Omega)}\right]\|u\|^2_{L^2(\Omega)} \\ &\geq \frac{C(n,s)}{2}\left(1 - \frac{\alpha^2}{2\pi^s}\right)\|u\|^2_{X_0(\Omega)} + \left[\frac{1}{2}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \frac{1}{2}\log 4J(u_0)\right]\|u\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.10}$$

If $J(u_0) < M$, then for $\left[\frac{4J(u_0)\Gamma(\frac{n}{2s})}{e^n s\Gamma(\frac{n}{2})}\right]^{\frac{s}{n}} < \alpha < \sqrt{2\pi^s}$, we have

$$I(u) \geq D\left(\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)}\right), \tag{4.11}$$

where

$$D = \min\left\{1 - \frac{\alpha^2}{2\pi^s}, \frac{1}{2}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \frac{1}{2}\log 4J(u_0)\right\}.$$

Then (4.9) and (4.11) imply

$$\int_t^T \left(\|u(\tau)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(\tau)\|^2_{X_0(\Omega)}\right) dx \leq \frac{1}{2D}\left(\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)}\right) \tag{4.12}$$

Let $T \to +\infty$ and from Lemma 4.1, one has

$$\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)} \leq \left(\|u_0\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u_0\|^2_{X_0(\Omega)}\right)e^{1-2Dt}, \text{ for all } t \geq 0.$$

If $J(u_0) = M$, then we have

$$\frac{1}{2}\left(n + \frac{n}{s}\log\sqrt{2\pi^s} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right) - \frac{1}{2}\log 4J(u_0) = 0.$$

From Theorem 3.1 and $J(u_0) = M$, we have

$$I(u) = -(u, u_t) - \left((-\Delta)^{\frac{s}{2}}u_t, (-\Delta)^{\frac{s}{2}}u\right) > 0.$$

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This implies that

$$\int_0^t \left(\|u_\tau\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2} \|u_\tau\|^2_{X_0(\Omega)} \right) d\tau$$

is strictly positive for $0 < t < \infty$. On the other hand, for any given sufficiently small positive number β, there exists $t_\beta > 0$ such that

$$M - \beta = J(u(t_\beta)) = J(u_0) - \int_0^{t_\beta} \left(\|u_\tau\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2} \|u_\tau\|^2_{X_0(\Omega)} \right) d\tau.$$

Then by the energy inequality we get

$$0 < J(u) \le M - \beta < M \le d \text{ for all } t_\beta \le t < +\infty.$$

If we take $t = t_\beta$ as the initial time, then similar to $J(u_0) < M$, we have

$$\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2} \|u(t)\|^2_{X_0(\Omega)} \le \left(\|u_0\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2} \|u_0\|^2_{X_0(\Omega)} \right) e^{1-2D_1 t}, \text{ for all } t \ge 0.$$

where

$$D_1 = \min\left\{ 1 - \frac{\alpha^2}{2\pi^s}, \frac{1}{2}\left(n + \frac{n}{s}\log\alpha + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})} \right) - \frac{1}{2}\log 4(M-\beta) \right\},$$

for any $\left[\frac{4(M-\beta)\Gamma(\frac{n}{2s})}{e^n s\Gamma(\frac{n}{2})} \right]^{\frac{s}{n}} < \alpha < \sqrt{2\pi^s}$.

The proof is complete. □

Corollary 4.1 *In Theorem 4.1, if the assumptions "$J(u_0) < d$, $I(u_0) > 0$" is replaced by "$0 < J(u_0) < d$, $I_{\delta_2}(u_0) > 0$", where (δ_1, δ_2) is the maximal interval including $\delta = 1$ such that $d_\delta > J(u_0)$ for $\delta \in (\delta_1, \delta_2)$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, \infty; X_0(\Omega))$ with $u_t \in L^2(0, \infty; X_0(\Omega))$ and $u(t) \in W_\delta$, for $0 \le t < \infty$.*

Proof. Making use of Lemma 3.5, we obtain from $0 < J(u_0) < d$, $I_{\delta_2}(u_0) > 0$ that $I_\delta(u_0) > 0$ for all $\delta \in (\delta_1, \delta_2)$. Repeating the arguments of Theorem 4.1 for $\delta_1 < \delta < \delta_2$, then the conclusion of Corollary 4.1 holds. □

Corollary 4.2 *In Corollary 4.1, if the assumptions "$I_{\delta_2}(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$" is replaced by $\|u_0\|_{X_0(\Omega)} < \gamma(\delta)$, then problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, \infty; X_0(\Omega))$ with $u_t \in L^2(0, \infty; X_0(\Omega))$ and satisfies*

$$\|u_0\|^2_{X_0(\Omega)} < \frac{d(\delta)}{a(\delta)}, \qquad \text{for } \delta_1 < \delta < \min\{\delta_2, b\},\ 0 \le t < \infty. \tag{4.13}$$

In particular, we have

$$\|u_0\|^2_{X_0(\Omega)} \le \frac{d(\delta_1)}{a(\delta_1)} \qquad \text{for } 0 \le t < \infty. \tag{4.14}$$

Proof. $\|u_0\|_{X_0(\Omega)} < \gamma(\delta)$ gives $I_{\delta_2}(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$. Hence, from Corollary 4.1, it follows that problem (1.1) admits a global weak solution $u(t) \in L^\infty(0, \infty; X_0(\Omega))$ with $u_t \in L^2(0, \infty; X_0(\Omega))$ and $u(t) \in W_\delta$, for $0 \le t < \infty$. Moreover, from Lemma 3.4, we can deduce that (4.13) holds. Letting $\delta \to \delta_1$, the conclusion (4.14) is also obtained. □

4.2 Blow-up at $+\infty$ of solution

In this subsection, we establish blow-up at $+\infty$ of solution for problem (1.1) when $J(u_0) < d$ and $I(u_0) < 0$ by using properties of a family of potential wells.

Theorem 4.2 *Let $u_0 \in X_0(\Omega)$. Suppose that $J(u_0) < d$, $I(u_0) < 0$, and $0 < s < 1$ satisfies*

$$\frac{\Gamma\left(\frac{n}{2s}\right)}{\Gamma\left(\frac{n}{2}\right)} \leq s(2\pi^s)^{\frac{n}{2s}} e^n. \tag{4.15}$$

Then the weak solution of problem (1.1) blows up at $+\infty$, i.e., the maximal existence time $T_{max} = +\infty$ and

$$\lim_{t\to+\infty}\left(\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)}\right) = +\infty. \tag{4.16}$$

Proof. We divide the proof into two steps.

Step 1 We prove that the weak solution of problem (1.1) can't blow up in finite time.

Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) < d$ and $I(u_0) < 0$. Assume by contradiction that $T_{max} < +\infty$ and

$$\lim_{t\to T_{max}^-}\left(\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)}\right) = +\infty.$$

Then we can choose a time t_0 small than but close to T_{max} such that

$$\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)} \geq 1,$$

for any $t \in [t_0, T_{max})$. We define

$$L(t) = \int_{t_0}^{t}\|u(\tau)\|^2_{L^2(\Omega)}d\tau + \int_{t_0}^{t}\frac{C(n,s)}{2}\|u(\tau)\|^2_{X_0(\Omega)}d\tau, \tag{4.17}$$

for $t \in [t_0, T_{max})$. Furthermore, for $t \in [t_0, T_{max})$,

$$L'(t) = \|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)}. \tag{4.18}$$

And, consequently,

$$L''(t) = 2\int_\Omega u(t)u_t(t)dx + 2\int_\Omega(-\Delta)^{\frac{s}{2}}u(t)(-\Delta)^{\frac{s}{2}}u_t(t)dx,$$

for $t \in [t_0, T_{max})$. Then taking $\alpha = \sqrt{2\pi^s}$ in (2.5), for $t \in [t_0, T_{max})$, we have

$$\begin{aligned}
&L'(t)\log L'(t) - L''(t)\\
=&\left(\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)}\right)\log\left(\|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)}\right)\\
&-2\int_\Omega u(t)u_t(t)dx - 2\int_\Omega(-\Delta)^{\frac{s}{2}}u(t)(-\Delta)^{\frac{s}{2}}u_t(t)dx
\end{aligned}$$

$$
\begin{aligned}
&\geq 2\left[\|u(t)\|_{L^2(\Omega)}^2 \log \|u(t)\|_{L^2(\Omega)} + I(u)\right] \\
&\geq \left(n + \frac{n}{s}\log\sqrt{2\pi^s} + \log\frac{s\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2s}\right)}\right)\|u(t)\|_{L^2(\Omega)}^2. \qquad (4.19)
\end{aligned}
$$

Combining (4.15), we have

$$
L'(t)\log L'(t) - L''(t) \geq 0. \qquad (4.20)
$$

Then we have

$$
(\log L'(t))' \leq \log L'(t)
$$

and

$$
\log L'(t) \leq e^{t-t_0}\log L'(t_0) = e^{t-t_0}\log\left(\|u(t_0)\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u(t_0)\|_{X_0(\Omega)}^2\right),
$$

for any $t \in [t_0, T_{max})$. This implies

$$
\|u(t)\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u(t)\|_{X_0(\Omega)}^2 \leq A_0\left(\|u(t_0)\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u(t_0)\|_{X_0(\Omega)}^2\right)^{e^t}, \qquad (4.21)
$$

for all $t \in [t_0, T_{max})$, where $A_0 = \left(\|u(t_0)\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u(t_0)\|_{X_0(\Omega)}^2\right)^{e^{-t_0}}$. This contradicts to

$$
\lim_{t\to T_{max}^-}\left(\|u\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u\|_{X_0(\Omega)}^2\right) = +\infty.
$$

Step 2 We prove that the weak solution of problem (1.1) blows up at $+\infty$.

For any $T > 0$ and for all $t \in [0, T]$, we define

$$
M(t) = \int_0^t \|u(\tau)\|_{L^2(\Omega)}^2 d\tau + \int_0^t \frac{C(n,s)}{2}\|u(\tau)\|_{X_0(\Omega)}^2 d\tau + (T-t)\left(\|u_0\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u_0\|_{X_0(\Omega)}^2\right) \qquad (4.22)
$$

where b and T_0 are positive constants which will be specified later. Furthermore,

$$
\begin{aligned}
M'(t) &= \|u(t)\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u(t)\|_{X_0(\Omega)}^2 - \left(\|u_0\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u_0\|_{X_0(\Omega)}^2\right) \\
&= 2\int_0^t (u(\tau), u_\tau(\tau))_2\, d\tau + 2\int_0^t \left((-\Delta)^{\frac{s}{2}}u(\tau), (-\Delta)^{\frac{s}{2}}u_\tau(\tau)\right)_2 d\tau \qquad (4.23)
\end{aligned}
$$

and, consequently,

$$
M''(t) = 2\int_\Omega u(t)u_t(t)dx + 2\int_\Omega (-\Delta)^{\frac{s}{2}}u(t)(-\Delta)^{\frac{s}{2}}u_t(t)dx,
$$

Therefore, we get

$$
M(t)M''(t) - M'(t)^2
$$

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$$
\begin{aligned}
=&M(t)M''(t)+4\Bigg[\eta(t)-\left(M(t)-(T-t)\left(\|u_0\|^2_{L^2(\Omega)}+\frac{C(n,s)}{2}\|u_0\|^2_{X_0(\Omega)}\right)\right)\\
&\times\left(\int_0^t\|u_\tau(\tau)\|^2_{L^2(\Omega)}+\int_0^t\frac{C(n,s)}{2}\|u_\tau(\tau)\|^2_{X_0(\Omega)}d\tau\right)\Bigg]
\end{aligned}
$$

where $\eta(t):[0,T]\to\mathbb{R}_+$ is the functional defined by

$$
\begin{aligned}
\eta(t)=&\left(\int_0^t\|u(\tau)\|^2_{L^2(\Omega)}d\tau+\int_0^t\frac{C(n,s)}{2}\|u(\tau)\|^2_{X_0(\Omega)}d\tau\right)\\
&\times\left(\int_0^t\|u_\tau(\tau)\|^2_{L^2(\Omega)}d\tau+\int_0^t\frac{C(n,s)}{2}\|u_\tau(\tau)\|^2_{X_0(\Omega)}d\tau\right)\\
&-\left(\int_0^t(u(\tau),u_\tau(\tau))_2\,d\tau+\int_0^t\left((-\Delta)^{\frac{s}{2}}u(\tau),(-\Delta)^{\frac{s}{2}}u_\tau(\tau)\right)_2d\tau\right)^2\geq 0.
\end{aligned}
$$

As a consequence, we read the differential inequality

$$
\begin{aligned}
&M(t)M''(t)-M'(t)^2\\
\geq&M(t)M''(t)-4M(t)\left(\int_0^t\|u(\tau)\|^2_{L^2(\Omega)}d\tau+\int_0^t\frac{C(n,s)}{2}\|u(\tau)\|^2_{X_0(\Omega)}d\tau\right)\\
=&M(t)\xi(t),
\end{aligned}\tag{4.24}
$$

for almost every $t\in[0,T]$, where $\xi:[0,T]\to\mathbb{R}$ is the map defined by

$$
\begin{aligned}
\xi(t)=&2\int_\Omega u(t)u_t(t)dx+2\int_\Omega(-\Delta)^{\frac{s}{2}}u(t)(-\Delta)^{\frac{s}{2}}u_t(t)dx-4\int_0^t\|u_\tau(\tau)\|^2_{L^2(\Omega)}d\tau\\
&-4\int_0^t\frac{C(n,s)}{2}\|u_\tau(\tau)\|^2_{X_0(\Omega)}d\tau.
\end{aligned}\tag{4.25}
$$

By (1.1) and (3.10), we have

$$
\begin{aligned}
\xi(t)&=2\int_\Omega\left(u^2\log|u|-\left|(-\Delta)^{\frac{s}{2}}u\right|^2\right)dx-4\int_0^t\|u_\tau(\tau)\|^2_{L^2(\Omega)}d\tau-4\int_0^t\frac{C(n,s)}{2}\|u_\tau(\tau)\|^2_{X_0(\Omega)}d\tau\\
&\geq 2\int_\Omega\left(u^2\log|u|-\left|(-\Delta)^{\frac{s}{2}}u\right|^2\right)dx-4J(u_0)+4J(u(t))\\
&>\int_\Omega u^2dx-4d.
\end{aligned}
$$

By the definition of d and Lemma 3.1 (3), we have $d\leq\frac{\lambda_*^2}{4}\int_\Omega u^2dx$, where

$$
\lambda_*=\exp\left(\frac{\frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)}-\int_\Omega u^2\log|u|\,dx}{\|u\|^2_{L^2(\Omega)}}\right).
$$

From Theorem 3.1 and Remark 3.1, we have $I(u)<0$, for $t>0$. This implies $\lambda_*<1$, then we can get $d<\frac{1}{4}\int_\Omega u^2dx$. So we have

$$
\xi(t)>0,\qquad t\in[0,T],
$$

which implies

$$M(t)M''(t) - M'(t)^2 > 0, \qquad t \in [0, T]. \tag{4.26}$$

Thus, by directly calculation, we can see that

$$(\log M(t))' = \frac{M'(t)}{M(t)}, \qquad (\log M(t))'' = \frac{M(t)M''(t) - M'(t)^2}{M^2(t)}.$$

Then we have

$$(\log M(t))' = (\log M(t_2))' + \int_{t_2}^{t} \frac{M(\tau)M''(\tau) - M'(\tau)^2}{M^2(\tau)} d\tau,$$

and

$$\log M(t) - \log M(t_2) = \int_{t_2}^{t} (\log M(\tau))' \, d\tau \geq \frac{M'(t_2)}{M(t_2)} (t - t_2),$$

where $0 \leq t_2 \leq t \leq T$. Then

$$M(t) \geq M(t_2) e^{\frac{M'(t_2)}{M(t_2)}(t-t_2)}, \qquad t \in [t_2, T]$$

Then for $t \in [t_2, T]$, we have

$$\begin{aligned} & \|u(t)\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u(t)\|_{X_0(\Omega)}^2 \\ \geq & \frac{M'(t_2)}{M(t_2)} M(t) + \left(\|u_0\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u_0\|_{X_0(\Omega)}^2 \right) \\ \geq & M'(t_2) e^{\frac{M'(t_2)}{M(t_2)}(t-t_2)} + \left(\|u_0\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u_0\|_{X_0(\Omega)}^2 \right). \end{aligned} \tag{4.27}$$

Then from (4.21) and (4.27), $\|u\|_{L^2(\Omega)}^2 + \frac{C(n,s)}{2}\|u\|_{X_0(\Omega)}^2$ will blow up at $+\infty$.

The proof is complete. □

Remark 4.1 *If $s \to 1$, (4.15) holds naturally. When $n > 2$, by the inequality in [8, Theorem 1] for the gamma function, we can simplify (4.15) as*

$$n^{\frac{n}{2}(\frac{1}{s}-1)} e^{-\frac{n}{2}(1+\frac{1}{s})} s^{-\frac{1}{2}(1+\frac{n}{s})} 2^{-\frac{n}{2}(\frac{2}{s}-1)} \leq \pi^{\frac{n}{2}},$$

in which the gamma function is removed.

Remark 4.2 *We note from*

$$\frac{1}{4}\|u_0\|_{L^2(\Omega)}^2 + \frac{1}{2} I(u_0) = J(u_0)$$

and

$$\frac{C(n,s)}{2}\|u_0\|_{L^2(X_0)(\Omega)}^2 = \lambda_1 \|u_0\|_{L^2(\Omega)}^2$$

that, if $J(u_0) < 0$, then $I(u_0) \geq 0$ is impossible. If $J(u_0) = 0$, then either $I(u_0) > 0$ or $I(u_0) = 0$ with $\|u_0\|_{X_0(\Omega)} \neq 0$ is impossible. If $0 < J(u_0) < d$, it follows from the definition of d that $I(u_0) = 0$ with $\|u_0\|_{X_0(\Omega)} \neq 0$ is impossible. Thus, all possible cases have been considered in Theorems 4.1 and 4.2.

From the discussion above, a threshold result of global existence and nonexistence of solutions for problem (1.1) has been obtained as follows.

Corollary 4.3 *Assume that $u_0 \in X_0(\Omega)$ and $J(u_0) < d$. Then problem (1.1) admits a global weak solution provided $I(u_0) > 0$ or $\|u_0\|_{X_0(\Omega)} = 0$; problem (1.1) dose not admit any global solution provided $I(u_0) < 0$.*

5 Critical initial energy $J(u_0) = d$

In this section, we prove the global existence and blow-up at $+\infty$ of solutions for problem (1.1) with the critical initial condition $J(u_0) = d$.

Theorem 5.1 (Global existence for $J(u_0) = d$) *Let $u_0 \in X_0(\Omega)$. Suppose $J(u_0) = d$ and $I(u_0) \geq 0$. Then problem (1.1) has a global weak solution $u \in L^\infty(0, \infty; X_0(\Omega))$ with $u_t \in L^2(0, \infty; X_0(\Omega))$, and $u(t) \in \bar{W} = W \cup \partial W$ for $0 \leq t < \infty$.*

Proof. Let $\mu_m = 1 - \dfrac{1}{m}$ and $u_{0m} = \mu_m u_0$, $m = 2, 3, \cdots$. We consider the following problem

$$\begin{cases} u_t + (-\Delta)^s u_t + (-\Delta)^s u = u \log|u|, & x \in \Omega,\ t > 0, \\ u(x, 0) = u_{0m}(x), & x \in \Omega, \\ u(x, t) = 0, & x \in \Omega,\ t \geq 0. \end{cases} \tag{5.1}$$

From $I(u_0) \geq 0$ and Lemma 3.1, we have $\lambda^* = \lambda^*(u_0) \geq 1$. Thus, we get $I(u_{0m}) = I(\mu_m u_0) > 0$ and $J(u_{0m}) = J(\mu_m u_0) < J(u_0) < d$. So it follows from Theorem 4.1 that, for each m, problem (5.1) admits a global weak solution $u_m(t) \in L^\infty(0, \infty; X_0(\Omega))$ with $u_{mt} \in L^2(0, \infty; X_0(\Omega))$ and $u_{mt} \in W$ for $0 \leq t < \infty$, satisfying

$$(u_{mt}, v)_2 + \left((-\Delta)^{\frac{s}{2}} u_m, (-\Delta)^{\frac{s}{2}} v\right)_2 + \left((-\Delta)^{\frac{s}{2}} u_{mt}, (-\Delta)^{\frac{s}{2}} v\right)_2 = (u \log|u|, v)_2 \text{ for any } v \in X_0(\Omega)$$

and

$$\int_0^t \|u_{m\tau}\|^2_{L^2(\Omega)} d\tau + \int_0^t \frac{C(n,s)}{2} \|u_{m\tau}\|^2_{X_0(\Omega)} d\tau + J(u_m) \leq J(u_{0m}) < J(u_0) = d \text{ for } t \in (0, \infty). \tag{5.2}$$

By similar ways in Theorem 4.1 we can deduce (4.6) and (4.8) for each m. Hence there exists a u and a subsequence still denoted as $\{u_m\}$, such that, as $m \to \infty$,

$u_m \to u$ in $L^\infty(0, \infty; X_0(\Omega))$ weakly star and a.e. in $\Omega \times [0, \infty)$,

$u_{mt} \to u_t$ in $L^2(0, \infty; X_0(\Omega))$ weakly star,

$u_v \log|u_v| \to u \log|u|$ in $L^\infty\left(0, \infty; L^2(\Omega)\right)$ weakly star and a.e. in $\Omega \times [0, \infty)$.

The proof of global existence for the solution is the same as that in the first part of the Theorem 4.1. □

Theorem 5.2 (Blow-up at $+\infty$ for $J(u_0) = d$) *Let $u_0 \in X_0(\Omega)$. Suppose $J(u_0) = d$, $I(u_0) < 0$ and $0 < s < 1$, where s satisfies (4.15). Then the weak solution of problem (1.1) blows up at $+\infty$, i.e., the maximal existence time $T_{max} = +\infty$ and*

$$\lim_{t\to+\infty}\left(\|u\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u\|^2_{X_0(\Omega)}\right) = +\infty. \tag{5.3}$$

Proof. Let $u(t)$ be any weak solution of problem (1.1) with $J(u_0) = d$ and $I(u_0) < 0$. We define

$$L(t) = \int_0^t \|u(\tau)\|^2_{L^2(\Omega)}d\tau + \int_0^t \frac{C(n,s)}{2}\|u(\tau)\|^2_{X_0(\Omega)}d\tau. \tag{5.4}$$

Furthermore,

$$L'(t) = \|u(t)\|^2_{L^2(\Omega)} + \frac{C(n,s)}{2}\|u(t)\|^2_{X_0(\Omega)} \tag{5.5}$$

and, consequently,

$$L''(t) = 2\int_\Omega u(t)u_t(t)dx + 2\int_\Omega (-\Delta)^{\frac{s}{2}}u(t)(-\Delta)^{\frac{s}{2}}u_t(t)dx.$$

To complete the proof of Theorem 5.2, by the proof of Theorem 4.2, we only need to prove that there is a time $t_0 > 0$ such that $J(u(t_0)) < d$. By (3.10), we have

$$\int_0^{t_0} \|u_\tau\|^2_{L^2(\Omega)}d\tau + \int_0^{t_0} \frac{C(n,s)}{2}\|u_\tau\|^2_{X_0(\Omega)}d\tau + J(u(t_0)) \le J(u_0) = d. \tag{5.6}$$

By Proposition 3.1, we have $I(u(t)) < 0$ for all $0 \le t < T_{max}$. This implies $\|u_t\|_{L^2(\Omega)} > 0$ or $\|u_t\|_{X_0(\Omega)} > 0$ for $0 \le t < T_{max}$. So we have

$$\int_0^{t_0} \|u_\tau\|^2_{L^2(\Omega)}d\tau + \int_0^{t_0} \frac{C(n,s)}{2}\|u_\tau\|^2_{X_0(\Omega)}d\tau > 0. \tag{5.7}$$

By (5.6) and (5.7), we have $J(u(t_0)) < d$, which contradicts to $J(u(t_0)) \ge d$. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nov. 11301277), the Natural Science Foundation of Jiangsu Province (Grant No. BK20151523), the Six Talent Peaks Project in Jiangsu Province (Grant No. 2015-XCL-020), and the Qing Lan Project of Jiangsu Province.

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