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Well-posedness and asymptotic stability to a laminated beam in thermoelasticity of type III

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Abstract

In this paper, we study the well-posedness and asymptotic behaviour of solutions to a laminated beam in thermoelasticity of type III. We first give the well-posedness of the system by using the semigroup method. Then, we show that the system is exponentially stable under the assumption of equal wave speeds. Furthermore, it is proved that the system is lack of exponential stability for case of nonequal wave speeds. In this regard, a polynomial stability result is proved.

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Keywords: laminated beam, thermoelasticity of type III, exponential decay, polynomial decay.

1 Introduction

In this paper, we consider a coupled system of a laminated beam with thermoelasticity of type III, which has the form

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi-\varphi_x)_x = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_{\rho_1}(3\omega-\psi)_{tt} - D(3\omega-\psi)_{xx} - G(\psi-\varphi_x) + \alpha\theta_x = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ I_{\rho_1}\omega_{tt} - D\omega_{xx} + G(\psi-\varphi_x) + \frac{4}{3}\beta_1\omega + \frac{4}{3}\beta_2\omega_t = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\theta_{tt} - \delta\theta_{xx} + \gamma(3\omega-\psi)_{ttx} - k\theta_{txx} = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \varphi(x,0)=\varphi_0(x), \psi(x,0)=\psi_0(x), & x\in[0,1],\\ \omega(x,0)=\omega_0(x), \theta(x,0)=\theta_0(x), & x\in[0,1],\\ \varphi_t(x,0)=\varphi_1(x), \psi_t(x,0)=\psi_1(x), & x\in[0,1],\\ \omega_t(x,0)=\omega_1(x), \theta_t(x,0)=\theta_1(x), & x\in[0,1],\\ \varphi_x(0,t)=\varphi_x(1,t)=\psi(0,t)=\psi(1,t)=0, & t\in[0,+\infty),\\ \omega(0,t)=\omega(1,t)=\theta_x(0,t)=\theta_x(1,t)=0, & t\in[0,+\infty), \end{cases} \tag{1.1}$$

where $\varphi(x,t)$ denotes the transverse displacement, $\psi(x,t)$ represents the rotation angle, $\omega(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable

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x, $\theta(x,t)$ is the differential temperature, respectively, and ρ_1, ρ_2, I_{ρ_1}, G, D, k, α, β_1, β_2, δ, γ are positive constants. Moreover, $\sqrt{\frac{G}{\rho_1}}$ and $\sqrt{\frac{D}{I_{\rho_1}}}$ are two wave speeds.

The asymptotic behaviors of the laminated beam have been investigated by several authors over the past twenty years. Roughly speaking, laminated beam describes that two identical homogeneous beams are allowed between the beams, which were placed on top of each and a slip at the interface. These composite laminates usually have superior structural properties such as adaptability. The design of their piezoelectric materials can be invoked as both actuators and sensors. Based on the Timoshenko system, the model for this structure was first introduced by Hansen [9]. Later on, Hansen and Spies [10] studied the boundary stabilization of laminated beams with structural damping, which is

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3\omega - \psi)_{tt} - D(3\omega - \psi)_{xx} - G(\psi - \varphi_x) = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ 3I_\rho\omega_{tt} - 3D\omega_{xx} + 3G(\psi - \varphi_x) + 4\gamma\omega + 4\beta\omega_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{1.2}$$

where $G, \rho, I_\rho, D, \gamma, \beta > 0$ are the shear stiffness, the density of the beams, mass moment of inertia, flexural rigidity, adhesive stiffness of the beams and the adhesive damping parameter, respectively. In [31], the following boundary feedback controls were proposed to exponentially stabilize system (1.2):

$$u_1(t) = k_1\varphi_t(1,t),\ u_2(t) = -k_2(3\omega_t - \psi_t)(1,t),$$

where k_1 and k_2 were positive constant feedback gains. Then the boundary conditions became

$$\begin{cases} \varphi(0,t) = \psi(0,t) = \omega(0,t) = 0, & t > 0, \\ \psi(1,t) - \varphi_x(1,t) = u_1(t),\ \omega_x(1,t) = 0,\ (3\omega_x - \psi_x)(1,t) = u_2(t), & t \in [0,+\infty), \end{cases}$$

and the close-loop system had both internal damping and boundary controls. They assumed that $r_1 := \dfrac{G}{\rho} \neq \dfrac{D}{I_\rho} =: r_2,\ k_i \neq r_i,\ i = 1,2$, and found out an explicit asymptotic formula for the matrix fundamental solutions. Then they carried out the asymptotic analysis for the eigenpairs by using an invertible matrix function with an eigenvalue parameter and an asymptotic technique for the first order matrix differential equation. Cao et al. [2] considered an exponential stabilization of the system (1.2) in case of non-equal speeds. They obtained that the 'dominant' part of the close loop system is exponentially stable. Tatar [29] proved that the system with condition $\rho G < I_\rho$ could be stabilized in an exponential manner using boundary controls. As the same problem in [31], their results improved the few existing similar works in the literature. Furthermore, Lo and Tatar [19] investigated uniform stability of the system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho(3\omega - \psi)_{tt} - (3\omega - \psi)_{xx} - G(\psi - \varphi_x) & \\ \qquad + \displaystyle\int_0^t h(t-\tau)(3\omega - \psi)_{xx}(\tau)d\tau = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_\rho\omega_{tt} - \omega_{xx} + G(\psi - \varphi_x) + \dfrac{4}{3}\gamma\omega + \displaystyle\int_0^t g(t-\tau)\omega_{xx}(\tau)d\tau = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases}$$

when a viscoelastic damping acted on the effective rotation and in the slip. This extended previous works where boundary controls were used in addition to a frictional damping in the dynamic of the slip. For other asymptotic behavior results to laminated beams, we refer the reader to [1, 15, 18, 20] and the references therein.

It is easy to find that if the slip ω is assumed to be identically zero, then the first two equations of system (1.2) can be reduced exactly to the Timoshenko beam system. For Timoshenko system in thermoelasticity of type III, the theory of which was proposed by Green and Naghdi [8], a large number of interesting decay results depending on the stability number have been established (see [6, 21, 22, 24, 27] and references therein). Messaoudi and Said-Houari [23] considered the following one-dimensional linear thermoelastic system:

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\psi_{tt} - b\psi_{xx} - K(\varphi_x+\psi) + \beta\theta_x = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, & (x,t)\in(0,1)\times(0,+\infty), \end{cases}$$

which modeled the transverse vibration of a thick beam with heat conduction. Under the condition $\frac{k}{\rho_1} = \frac{b}{\rho_2}$, they proved that weak solution decay exponentially by using the energy method. Moreover, Kafini et al. [13] studied the following Timoshenko system of thermoelasticity of type III with delay of the form:

$$\begin{cases} \rho_1\phi_{tt} - K(\phi_x+\psi)_x + \mu_1\phi_t(x,t) + \mu_2\phi_t(x,t-\tau) = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_2\psi_{tt} - b\psi_{xx} + K(\phi_x+\psi) + \beta\theta_{tx} = 0, & (x,t)\in(0,1)\times(0,+\infty),\\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0, & (x,t)\in(0,1)\times(0,+\infty). \end{cases}$$

Under the initial and boundary conditions

$$\begin{cases} \theta(\cdot,0)=\theta_0,\ \theta_t(\cdot,0)=\theta_1, \psi(\cdot,0)=\psi_0, & x\in[0,1],\\ \psi_t(\cdot,0)=\psi_1,\ \phi(\cdot,0)=\phi_0, \phi_t(\cdot,0)=\phi_1, & x\in[0,1],\\ \phi_t(x,t-\tau)=f_0(x,t-\tau), & t\in(0,\tau),\\ \phi(0,t)=\phi(1,t)=\psi(0,t)=\psi(1,t)=\theta_x(0,t)=\theta_x(1,t)=0, & t\in[0,+\infty), \end{cases}$$

the energy of system decays exponentially in the case of equal wave speeds and polynomially in the case of nonequal wave speeds. For other related results, we refer the reader to [3, 4, 5, 11, 12, 16, 17, 26, 28].

Motivated by the above results, in the present work, we study the well-posedness and asymptotic behaviour of solutions to the laminated beam (1.1) in thermoelasticity of type III. By using semigroup method and Lumer-Philips theorem, we prove the existence and uniqueness of the solution. By using the perturbed energy method and construct some Lyapunov functionals, we then obtain the exponential decay result for the case of equal wave speeds, i.e., $\frac{G}{\rho_1} = \frac{D}{I_{\rho_1}}$. When $\frac{G}{\rho_1} \neq \frac{D}{I_{\rho_1}}$, we obtain the lack of exponential stability by using Gearhart-Herbst-Prüss-Huang theorem. For this case, by introducing the extra second-order energy, we prove the polynomial decay result.

The rest of our paper is organized as follows. In the next section, we introduce some preliminaries and state the main results. In Section 3, we establish the well-posedness of the system. In Section 4, we prove that the system is exponentially stable in the case of equal wave speeds. In Section 5, we show that the system is lack of exponential stability with different wave-propagation speeds. The proof of the polynomial decay result is given in Section 6.

2 Preliminaries and main results

To exhibit the dissipative nature of the system (1.1), we introduce some new variables

$$\Phi = \varphi_t, \ \ \Psi = \psi_t, \ \ W = \omega_t.$$

Then system (1.1) takes the form:

$$\begin{cases} \rho_1 \Phi_{tt} + G(\Psi - \Phi_x)_x = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_{\rho_1}(3W - \Psi)_{tt} - D(3W - \Psi)_{xx} - G(\Psi - \Phi_x) + \alpha\theta_{tx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ I_{\rho_1} W_{tt} - DW_{xx} + G(\Psi - \Phi_x) + \dfrac{4}{3}\beta_1 W + \dfrac{4}{3}\beta_2 W_t = 0, & (x,t) \in (0,1) \times (0,+\infty), \\ \rho_2 \theta_{tt} - \delta\theta_{xx} + \gamma(3W - \Psi)_{tx} - k\theta_{txx} = 0, & (x,t) \in (0,1) \times (0,+\infty), \end{cases} \tag{2.1}$$

with the initial data and boundary conditions

$$\begin{cases} \Phi(x,0) = \Phi_0(x), \Psi(x,0) = \Psi_0(x), & x \in [0,1], \\ W(x,0) = W_0(x), \theta(x,0) = \theta_0(x), & x \in [0,1], \\ \Phi_t(x,0) = \Phi_1(x), \Psi_t(x,0) = \Psi_1(x), & x \in [0,1], \\ W_t(x,0) = W_1(x), \theta_t(x,0) = \theta_1(x), & x \in [0,1], \\ \Phi_x(0,t) = \Phi_x(1,t) = \Psi(0,t) = \Psi(1,t) = 0, & t \in [0,+\infty), \\ W(0,t) = W(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, & t \in [0,+\infty) \end{cases} \tag{2.2}$$

where

$$\begin{cases} \Phi_0(x) = \varphi_1, \Phi_1(x) = -\dfrac{G}{\rho_1}(\psi_0 - \varphi_{0x})_x, \Psi_0(x) = \psi_1, & x \in [0,1], \\ \Psi_1(x) = -\dfrac{4G}{I_{\rho_1}}(\psi_0 - \phi_{0x}) - \dfrac{D}{I_{\rho_1}}(3\omega_0 - \psi_0)_{xx} + \dfrac{\alpha}{I_{\rho_1}}\theta_{1x} - \dfrac{4\beta_1}{I_{\rho_1}}\omega_0 - \dfrac{4\beta_2}{I_{\rho_1}}\omega_1 + \dfrac{3D}{I_{\rho_1}}\omega_{0xx}, & x \in [0,1], \\ W_0(x) = \omega_1, W_1(x) = -\dfrac{G}{I_{\rho_1}}(\psi_0 - \varphi_{0x}) - \dfrac{4\beta_1}{3I\rho_1}\omega_0 - \dfrac{4\beta_1}{3I_{\rho_1}}\omega_1 + \dfrac{D}{I_{\rho_1}}\omega_{0xx}, & x \in [0,1]. \end{cases}$$

From equations $(2.1)_1$, $(2.1)_3$ and (2.2), we easily verify that

$$\frac{d^2}{dt^2}\int_0^1 \Phi(x,t)dx = 0, \ \ \frac{d^2}{dt^2}\int_0^1 \theta(x,t)dx = 0,$$

and

$$\bar{\Phi}(x,t) := \Phi(x,t) - \int_0^1 \Phi_0(x)dx - t\int_0^1 \Phi_1(x)dx,$$

$$\bar{\theta}(x,t) := \theta(x,t) - \int_0^1 \theta_0(x)dx - t\int_0^1 \theta_1(x)dx.$$

We know that $(\bar{\Phi}, \Psi, W, \bar{\theta})$ satisfies the boundary conditions, and more importantly

$$\int_0^1 \bar{\Phi}(x,t)dx = 0, \quad \int_0^1 \bar{\theta}(x,t)dx = 0.$$

Hence, the use of Poincaré's inequality for $\bar{\Phi}$ and $\bar{\theta}$ is justified. In what follows, we will work with $\bar{\Phi}$ and $\bar{\theta}$. For convenience, we write Φ and θ.

From now on, we let

$$U = (\Phi, 3W - \Psi, W, \theta, \Phi_t, 3W_t - \Psi_t, W_t, \theta_t),$$

then (2.1) and (2.2) can be written as an evolutionary equation

$$\begin{cases} \dfrac{dU(t)}{dt} = \mathcal{A}U(t), \qquad t > 0, \\ U(0) = U_0 = (\Phi_0, 3W_0 - \Psi_0, W_0, \theta_0, \Phi_1, 3W_1 - \Psi_1, W_1, \theta_1), \end{cases} \tag{2.3}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} \Phi_t \\ 3W_t - \Psi_t \\ W_t \\ \theta_t \\ -\dfrac{G}{\rho_1}(\Psi - \Phi_x)_x \\ \dfrac{G}{I_{\rho_1}}(\Psi - \Phi_x) + \dfrac{D}{I_{\rho_1}}(3W - \Psi)_{xx} - \dfrac{\alpha}{I_{\rho_1}}\theta_{xt} \\ -\dfrac{G}{I_{\rho_1}}(\Psi - \Phi_x) - \dfrac{4\beta_1}{3I_{\rho_1}}W - \dfrac{4\beta_2}{3I_{\rho_1}}W_t + \dfrac{D}{I_{\rho_1}}W_{xx} \\ \dfrac{\delta}{\rho_2}\theta_{xx} - \dfrac{\gamma}{\rho_2}(3W - \Psi)_{tx} + \dfrac{k}{\rho_2}\theta_{txx} \end{pmatrix}.$$

We consider the following spaces:

$$\begin{aligned} L^2_*(0,1) &= \left\{ w \in L^2(0,1) : \int_0^1 w(s)ds = 0 \right\}, \\ H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \\ H^2_*(0,1) &= \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0 \right\}, \end{aligned}$$

and the energy space:

$$\mathcal{H} = H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2_*(0,1).$$

The inner product on Hilbert space $\mathcal{H}$ is defined by

$$(U, \bar{U})_{\mathcal{H}} = \gamma\rho_1 \int_0^1 \Phi_t \bar{\Phi}_t dx + \gamma G \int_0^1 (\Psi - \Phi_x)(\bar{\Psi} - \bar{\Phi}_x)dx + \gamma I_{\rho_1} \int_0^1 (3W - \Psi)_t (3\bar{W} - \bar{\Psi})_t dx$$

$$+\gamma\int_0^1 D(3W-\Psi)_x(3\bar{W}-\bar{\Psi})_x dx+3\gamma\int_0^1 I_{\rho_1}W_t\bar{W}_t dx+4\gamma\beta_1\int_0^1 W\bar{W}dx$$
$$+3\gamma D\int_0^1 W_x\bar{W}_x dx+\alpha\rho_2\int_0^1\theta_t\bar{\theta}_t dx+\alpha\delta\int_0^1\theta_x\bar{\theta}_x dx.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A})=\left\{\begin{array}{c}U\in\mathcal{H}\mid\Phi,\theta\in H_*^2(0,1)\cap H_*^1(0,1),\Psi,W\in H^2(0,1)\cap H_0^1(0,1),\\ \Phi_t,\theta_t\in H_*^1(0,1),\Psi_t,W_t\in H_0^1(0,1),\delta\theta+k\theta_t\in H_*^2(0,1)\end{array}\right\}$$

and it is dense in $\mathcal{H}$.

We give the following well-posedness result of problem (2.3) :

Theorem 2.1 *Let $U_0\in\mathcal{H}$, then problem (2.3) exists a unique solution $U\in C(\mathbb{R}^+,\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A})$ then $U\in C(\mathbb{R}^+,D(\mathcal{A}))\cap C^1(\mathbb{R}^+,\mathcal{H})$.*

To state our decay result, we introduce the first energy functional

$$\begin{aligned}E(t)=&\frac{1}{2}\int_0^1\left[\gamma\rho_1\Phi_t^2+\gamma G(\Psi-\Phi_x)^2+\gamma I_{\rho_1}(3W-\Psi)_t^2+\gamma D(3W-\Psi)_x^2\right]dx\\&+\frac{1}{2}\int_0^1\left[3\gamma I_{\rho_1}W_t^2+4\gamma\beta_1W^2+3\gamma DW_x^2+\alpha\rho_2\theta_t^2+\alpha\delta\theta_x^2\right]dx,\end{aligned}\tag{2.4}$$

and the second-order energy functional

$$\begin{aligned}E_2(t)=&\frac{1}{2}\int_0^1\left[\gamma\rho_1\Phi_{tt}^2+\gamma G(\Psi-\Phi_x)_t^2+\gamma I_{\rho_1}(3W-\Psi)_{tt}^2+\gamma D(3W-\Psi)_{xt}^2\right]dx\\&+\frac{1}{2}\int_0^1\left[3\gamma I_{\rho_1}W_{tt}^2+4\gamma\beta_1W_t^2+3\gamma DW_{xt}^2+\alpha\rho_2\theta_{tt}^2+\alpha\delta\theta_{xt}^2\right]dx.\end{aligned}\tag{2.5}$$

Our decay results state as follows.

Theorem 2.2 *Assume that $\dfrac{G}{\rho_1}=\dfrac{D}{I_{\rho_1}}$ and $U_0\in\mathcal{H}$. Then, there exist two positive constants C_0 and s, such that the energy $E(t)$ associated with problem (2.1)-(2.2) satisfies*

$$E(t)\le C_0E(0)e^{-st},\qquad t\ge 0.$$

Theorem 2.3 *Assume that $\dfrac{G}{\rho_1}\neq\dfrac{D}{I_{\rho_1}}$ and $U_0\in\mathcal{H}$. Then the semigroup associated to system (2.1) with boundary conditions (2.2) is not exponentially stable.*

Theorem 2.4 *Assume that $\dfrac{G}{\rho_1}\neq\dfrac{D}{I_{\rho_1}}$ and $U_0\in\mathcal{H}$. Then, there exists a positive constant C_1 such that the energy $E(t)$ associated with problem (2.1)-(2.2) satisfies*

$$E(t)\le\frac{C_1(E(0)+E_2(0))}{t},\ t\ge 0.$$

To study the property of system (2.1), we need the following inequality.

Lemma 2.1 *[30, Lemma 2.2] Let $f\in\{H^1[0,1]\mid f(0)=0\}$, then it holds that*

$$\int_0^1 f^2(x)dx\le\frac{1}{2}\int_0^1\left(f'(x)\right)^2dx.$$

3 Well-posedness

In this section, we give the proof of the well-posedness of problem (2.1)-(2.2) by making use of Lumer-Philips theorem [7, 25].

Proof of Theorem 2.1. To prove the well-posedness result, it suffices to show that $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ is a maximal monotone operator, which means $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. First, an easy computation implies that

$$\begin{aligned}(\mathcal{A}U, U)_{\mathcal{H}} =&\gamma\rho_1 \int_0^1 \Phi_{tt}\Phi_t dx + \gamma G \int_0^1 (\Psi - \Phi_x)_t(\Psi - \Phi_x)dx + \gamma I_{\rho_1} \int_0^1 (3W - \Psi)_{tt}(3W - \Psi)_t dx \\ &+ \gamma \int_0^1 D(3W - \Psi)_{tx}(3W - \Psi)_x dx + 3\gamma \int_0^1 I_{\rho_1} W_{tt}W_t dx + 4\gamma\beta_1 \int_0^1 W_t W dx \\ &+ 3\gamma D \int_0^1 W_{tx}W_x dx + \alpha\rho_2 \int_0^1 \theta_{tt}\theta_t dx + \alpha\delta \int_0^1 \theta_{tx}\theta_x dx.\end{aligned}$$

For any $U \in D(\mathcal{A})$, we have

$$(\mathcal{A}U, U)_{\mathcal{H}} = -4\gamma \int_0^1 \beta_2 W_t^2 dx - \alpha \int_0^1 k\theta_{tx}^2 dx \leq 0.$$

Consequently, $\mathcal{A}$ is a dissipative operator.

Next, let $F = (f_1, \cdots, f_8)^T \in \mathcal{H}$, we seek $V = (v_1, \cdots, v_8)^T \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})V = F, \tag{3.1}$$

that is

$$\begin{cases} v_1 - v_5 = f_1, \\ v_2 - v_6 = f_2, \\ v_3 - v_7 = f_3, \\ v_4 - v_8 = f_4, \\ \rho_1 v_5 - G\partial_{xx}v_1 - G\partial_x v_2 + 3G\partial_x v_3 = \rho_1 f_5, \\ I_{\rho_1} v_6 + G\partial_x v_1 + Gv_2 - 3Gv_3 - D\partial_{xx}v_2 + \alpha\partial_x v_8 = I_{\rho_1} f_6, \\ I_{\rho_1} v_7 - Gv_2 + 3Gv_3 - G\partial_x v_1 + \dfrac{4}{3}\beta_1 v_3 + \dfrac{4}{3}\beta_2 v_7 - D\partial_{xx}v_3 = I_{\rho_1} f_7, \\ \rho_2 v_8 - \delta\partial_{xx}v_4 + \gamma\partial_x v_6 - k\partial_{xx}v_8 = \rho_2 f_8. \end{cases} \tag{3.2}$$

From $(3.2)_1$-$(3.2)_4$, we have

$$\begin{cases} v_5 = v_1 - f_1, \\ v_6 = v_2 - f_2, \\ v_7 = v_3 - f_3, \\ v_8 = v_4 - f_4. \end{cases} \tag{3.3}$$

By combining (3.2) and (3.3), it can be shown that v_1, v_2, v_3, v_4 satisfy

$$\begin{cases} \rho_1 v_1 - G\partial_{xx} v_1 - G\partial_x v_2 + 3G\partial_x v_3 = \rho_1 \left(f_1 + f_5\right), \\ I_{\rho_1} v_2 + G\partial_x v_1 + Gv_2 - 3Gv_3 - D\partial_{xx} v_2 + \alpha\partial_x v_4 = I_{\rho_1} \left(f_2 + f_6\right) + \alpha\partial_x f_4, \\ I_{\rho_1} v_3 - Gv_2 + 3Gv_3 - G\partial_x v_1 + \frac{4}{3}\beta_1 v_3 + \frac{4}{3}\beta_2 v_3 - D\partial_{xx} v_3 = I_{\rho_1} \left(f_3 + f_7\right) + \frac{4}{3}\beta_2 f_3, \\ \rho_2 v_4 - \delta\partial_{xx} v_4 + \gamma\partial_x v_2 - k\partial_{xx} v_4 = \rho_2 \left(f_4 + f_8\right) + \gamma\partial_x f_2 - k\partial_{xx} f_4. \end{cases} \tag{3.4}$$

Multiplying $(3.4)_1$-$(3.4)_4$ by $\gamma\bar{v}_1$, $\gamma\bar{v}_2$, $3\gamma\bar{v}_3$ and $\alpha\bar{v}_4$ and integrating over $(0,1)$, we arrive at

$$\begin{cases} \gamma\int_0^1 \rho_1 v_1\bar{v}_1 dx - \gamma\int_0^1 G\partial_{xx} v_1\bar{v}_1 dx - \gamma\int_0^1 G\partial_x v_2\bar{v}_1 dx + 3\gamma\int_0^1 G\partial_x v_3\bar{v}_1 dx \\ \quad = \gamma\int_0^1 \rho_1 \left(f_1 + f_5\right)\bar{v}_1 dx, \\ \gamma\int_0^1 I_{\rho_1} v_2\bar{v}_2 dx + \gamma\int_0^1 G\partial_x v_1\bar{v}_2 dx + \gamma\int_0^1 Gv_2\bar{v}_2 dx - 3\gamma\int_0^1 Gv_3\bar{v}_2 dx - \gamma\int_0^1 D\partial_{xx} v_2\bar{v}_2 dx \\ \qquad +\gamma\int_0^1 \alpha\partial_x v_4\bar{v}_2 dx = \gamma\int_0^1 I_{\rho_1}\left(f_2 + f_6\right)\bar{v}_2 dx + \gamma\int_0^1 \alpha\partial_x f_4\bar{v}_2 dx, \\ 3\gamma\int_0^1 I_{\rho_1} v_3\bar{v}_3 dx - 3\gamma\int_0^1 Gv_2\bar{v}_3 dx + 9\gamma\int_0^1 Gv_3\bar{v}_3 dx - 3\gamma\int_0^1 G\partial_x v_1\bar{v}_3 dx + 4\gamma\int_0^1 \beta_1 v_3\bar{v}_3 dx \\ \qquad +4\gamma\int_0^1 \beta_2 v_3\bar{v}_3 dx - 3\gamma\int_0^1 D\partial_{xx} v_3\bar{v}_3 dx = 3\gamma\int_0^1 I_{\rho_1}\left(f_3 + f_7\right)\bar{v}_3 dx + 4\gamma\int_0^1 \beta_2 f_3\bar{v}_3 dx, \\ \alpha\int_0^1 \rho_2 v_4\bar{v}_4 dx - \alpha\int_0^1 \delta\partial_{xx} v_4\bar{v}_4 dx + \alpha\int_0^1 \gamma\partial_x v_2\bar{v}_4 dx - \alpha\int_0^1 k\partial_{xx} v_4\bar{v}_4 dx \\ \quad = \alpha\int_0^1 \rho_2\left(f_4 + f_8\right)\bar{v}_4 dx + \alpha\int_0^1 \gamma\partial_x f_2\bar{v}_4 dx - \alpha\int_0^1 k\partial_{xx} f_4\bar{v}_4 dx. \end{cases} \tag{3.5}$$

The sum of these equations in (3.5) gives the following variational formulation:

$$a\left((v_1, v_2, v_3, v_4)^T, (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)^T\right) = \bar{a}\left((\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)^T\right), \tag{3.6}$$
$$\forall(\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)^T \in H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_*^1(0,1),$$

where

$$\begin{aligned} & a\left((v_1, v_2, v_3, v_4)^T, (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)^T\right) \\ =& \gamma\int_0^1 G(-\partial_x v_1 - v_2 + 3v_3)(-\partial_x\bar{v}_1 - \bar{v}_2 + 3\bar{v}_3)dx + \gamma\int_0^1 \rho_1 v_1\bar{v}_1 dx \\ & + \gamma\int_0^1 I_{\rho_1} v_2\bar{v}_2 dx + (3\gamma I_{\rho_1} + 4\gamma\beta_1 + 4\gamma\beta_2)\int_0^1 v_3\bar{v}_3 dx \\ & + \alpha\int_0^1 \rho_2 v_4\bar{v}_4 dx + \gamma\int_0^1 D\partial_x v_2\partial_x\bar{v}_2 dx + 3\gamma\int_0^1 D\partial_x v_3\partial_x\bar{v}_3 dx \\ & + \alpha\int_0^1 \delta\partial_x v_4\partial_x\bar{v}_4 dx + \alpha\int_0^1 k\partial_x v_4\partial_x\bar{v}_4 dx \\ & + \gamma\alpha\int_0^1 \partial_x v_4\bar{v}_2 dx + \alpha\gamma\int_0^1 \partial_x v_2\bar{v}_4 dx \end{aligned} \tag{3.7}$$

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and

$$\begin{aligned}\bar{a}\left((\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4)^T\right)=&\gamma\int_0^1\rho_1(f_1+f_5)\bar{v}_1dx+\gamma\int_0^1 I_{\rho_1}(f_2+f_6)\bar{v}_2dx\\&+3\gamma\int_0^1 I_{\rho_1}(f_3+f_7)\bar{v}_3dx+4\gamma\int_0^1\beta_2 f_3\bar{v}_3dx+\alpha\int_0^1\rho_2(f_4+f_8)\bar{v}_4dx\\&+\alpha\gamma\int_0^1\partial_x f_2\bar{v}_4dx+\alpha k\int_0^1\partial_x f_4\partial_x\bar{v}_4dx+\alpha\gamma\int_0^1\partial_x f_4\bar{v}_2dx.\end{aligned}\tag{3.8}$$

To study the variational formulation, we introduce the Hilbert space $V=H^1_*(0,1)\times H^1_0(0,1)\times H^1_0(0,1)\times H^1_*(0,1)$ equipped with the norm

$$\|(v_1,v_2,v_3,v_4)\|_V^2=\|-\partial_x v_1-v_2+3v_3\|_2^2+\|v_1\|_2^2+\|\partial_x v_2\|_2^2+\|\partial_x v_3\|_2^2+\|v_4\|_2^2+\|\partial_x v_4\|_2^2.$$

It is clear that $a(\cdot,\cdot)$ and $\bar{a}(\cdot)$ are bounded. Furthermore, we obtain

$$\begin{aligned}a\left((v_1,v_2,v_3,v_4)^T,(v_1,v_2,v_3,v_4)^T\right)=&\gamma\int_0^1 G(-\partial_x v_1-v_2+3v_3)^2dx+\gamma\int_0^1\rho_1 v_1^2dx+\gamma\int_0^1 I_{\rho_1}v_2^2dx\\&+(3\gamma I_{\rho_1}+4\gamma\beta_1+4\gamma\beta_2)\int_0^1 v_3^2dx+\alpha\int_0^1\rho_2 v_4^2dx\\&+\gamma\int_0^1 D(\partial_x v_2)^2dx+3\gamma\int_0^1 D(\partial_x v_3)^2dx\\&+\alpha\int_0^1\delta(\partial_x v_4)^2dx+\alpha\int_0^1 k(\partial_x v_4)^2dx\\\leq& C\|(v_1,v_2,v_3,v_4)\|_V^2,\end{aligned}$$

which implies that $a(\cdot,\cdot)$ is coercive. Consequently, applying Lax-Milgram theorem, we obtain that (3.5) has a unique solution $(v_1,v_2,v_3,v_4)^T\in V$.

Substituting v_1, v_2, v_3 and v_4 into (3.3), we obtain

$$(v_5,v_6,v_7,v_8)\in H^1_*(0,1)\times H^1_0(0,1)\times H^1_0(0,1)\times H^1_*(0,1).$$

It remains to show that

$$\begin{aligned}(v_1,v_2,v_3,v_4)\in&\left(H^2_*(0,1)\cap H^1_*(0,1)\right)\times\left(H^2(0,1)\cap H^1_0(0,1)\right)\times\left(H^2(0,1)\cap H^1_0(0,1)\right)\\&\times\left(H^2_*(0,1)\cap H^1_*(0,1)\right).\end{aligned}$$

Taking $(\bar{v}_2,\bar{v}_3,\bar{v}_4)=(0,0,0)\in H^1_0(0,1)\times H^1_0(0,1)\times H^1_*(0,1)$ in (3.6), we get

$$\begin{aligned}a\left((v_1,v_2,v_3,v_4)^T,(\bar{v}_1,0,0,0)^T\right)&=\gamma\int_0^1\rho_1 v_1\bar{v}_1dx+\gamma\int_0^1 G(-\partial_{xx}v_1\bar{v}_1-\partial_x v_2\bar{v}_1+3\partial_x v_3\bar{v}_1)dx\\&=\gamma\int_0^1\rho_1(f_1+f_5)\bar{v}_1dx,\end{aligned}\tag{3.9}$$

for all $\bar{v}_1\in H^1_*(0,1)$, which implies

$$G\partial_{xx}v_1=\rho_1 v_1-G\partial_x v_2+3G\partial_x v_3-\rho_1(f_1+f_5)\in L^2_*(0,1).$$

Thus, by L^2 theory for the linear elliptic equations, we obtain that

$$v_1 \in H^2(0,1) \cap H^1_*(0,1).$$

Moreover, (3.9) is also true for any $\phi \in C^1[0,1] \subset H^1_*(0,1)$ $(\phi(0) = 0)$. Hence, we get

$$\int_0^1 G\partial_x v_1 \partial_x \phi dx + \int_0^1 \rho_1 v_1 \phi dx - \int_0^1 G\partial_x v_2 \phi dx + 3\int_0^1 G\partial_x v_3 \phi dx = \int_0^1 \rho_1 (f_1 + f_5)\phi dx.$$

By using the integration by parts, we have

$$\partial_x v_1(1)\phi(1) - \partial_x v_1(0)\phi(0) = 0 \quad \forall \phi \in C^1[0,1], \phi(0) = 0.$$

Therefore, $\partial_x v_1(1)\phi(1) = 0$ and we deduce that $v_1 \in H^2_*(0,1) \cap H^1_*(0,1)$. In the same way, taking $(\bar{v}_1, \bar{v}_3, \bar{v}_4) = (0,0,0) \in H^1_*(0,1) \times H^1_0(0,1) \times H^1_*(0,1)$ in (3.7), one has

$$\begin{aligned} a\left((v_1,v_2,v_3,v_4)^T, (0,\bar{v}_2,0,0)^T\right) =& \gamma \int_0^1 G(\partial_x v_1 \bar{v}_2 + v_2\bar{v}_2 - 3v_3\bar{v}_2)dx + \gamma \int_0^1 I_{\rho_1} v_2 \bar{v}_2 dx \\ &+ \gamma \int_0^1 D\partial_x v_2 \partial_x \bar{v}_2 dx + \gamma\alpha \int_0^1 \partial_x v_4 \bar{v}_2 dx \\ =& \gamma \int_0^1 I_{\rho_1}(f_2 + f_6)\bar{v}_2 dx + \alpha\gamma \int_0^1 \partial_x f_4 \bar{v}_2 dx. \end{aligned} \tag{3.10}$$

Recalling $(3.2)_2$ and $(3.2)_4$, we arrive at

$$\begin{aligned} &\int_0^1 G(\partial_x v_1 \bar{v}_2 + v_2\bar{v}_2 - 3v_3\bar{v}_2)dx + \int_0^1 I_{\rho_1} v_2\bar{v}_2 dx + \int_0^1 D\partial_x v_2 \partial_x \bar{v}_2 dx + \alpha \int_0^1 \partial_x v_4 \bar{v}_2 dx \\ =& \int_0^1 I_{\rho_1}(v_2 - v_6 + f_6)\bar{v}_2 dx + \alpha \int_0^1 \partial_x (v_4 - v_8)\bar{v}_2 dx. \end{aligned}$$

Hence, one has

$$\int_0^1 D\partial_x v_2 \partial_x \bar{v}_2 dx = \int_0^1 \left[I_{\rho_1} f_6 - G\left(\partial_x v_1 + v_2 - 3v_3\right) - \alpha\partial_x v_8 - I_{\rho_1} v_6\right]\bar{v}_2 dx, \tag{3.11}$$

for all $\bar{v}_2 \in H^1_0(0,1)$, which implies $I_{\rho_1} f_6 - G\left(\partial_x v_1 + v_2 - 3v_3\right) - \alpha\partial_x v_8 - I_{\rho_1} v_6 \in L^2(0,1)$. Consequently, (3.11) takes the form

$$\int_0^1 \left(-D\partial_{xx} v_2 + G\partial_x v_1 + Gv_2 - 3Gv_3 + \alpha\partial_x v_8 + I_{\rho_1} v_6 - I_{\rho_1} f_6\right)\bar{v}_2 dx = 0.$$

We obtain

$$-D\partial_{xx} v_2 + G(\partial_x v_1 + v_2 - 3v_3) + \alpha\partial_x v_8 + I_{\rho_1} v_6 = I_{\rho_1} f_6,$$

and $v_2 \in H^2(0,1) \cap H^1_0(0,1)$, which gives $(3.2)_6$. Similarly, we get

$$v_3 \in H^2(0,1) \cap H^1_0(0,1).$$

However, if we take $(\bar{v}_1, \bar{v}_2, \bar{v}_3) = (0,0,0) \in H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1)$ in (3.7), thanks to $(3.2)_2$ and $(3.2)_4$, we get

$$\delta\partial_{xx} v_4 + k\partial_{xx} v_8 = \rho_2 f_8 - \gamma\partial_x v_6 - \rho_2 v_8 \in L^2_*(0,1),$$

and conclude that

$$\delta v_4 + k v_8 \in H^2(0,1).$$

Furthermore, it is obvious from

$$\delta \partial_x v_4 + k \partial_x v_8 = \rho_2 \int_0^x f_8 dx - \gamma v_6 - \rho_2 \int_0^x v_8 dx,$$

that

$$(\delta \partial_x v_4 + k \partial_x v_8)(0) = (\delta \partial_x v_4 + k \partial_x v_8)(1) = 0,$$

then, we get

$$\delta v_4 + k v_8 \in H^2_*(0,1).$$

Hence, there exists a unique $V \in D(\mathcal{A})$ such that (3.1) is satisfied, the operator $Id - \mathcal{A}$ is surjective. Moreover, it is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Consequently, the well-posedness result follows from Lumer-Philips theorem. □

4 Exponential decay for $\frac{G}{\rho_1} = \frac{D}{I_{\rho_1}}$

In this section, we prove the exponential decay result in Theorem 2.2. It will be accomplished by a Lyapunov functional $\mathcal{L}(t)$ which equivalent to $E(t)$. To construct the Lyapunov functional $\mathcal{L}(t)$, we modify some classical multipliers used in [13] and prove several lemmas.

The following lemma shows that the associated energy is non-increasing in time.

Lemma 4.1 *The energy functional $E(t)$ defined by* (2.4) *satisfies*

$$\frac{d}{dt}E(t) = -4\gamma \int_0^1 \beta_2 W_t^2 dx - \alpha \int_0^1 k\theta_{tx}^2 dx \leq 0.$$

Proof. Mulitiplying equation $(2.2)_1$ by $\gamma\Phi_t$, $(2.2)_2$ by $\gamma(3W-\Psi)_t$, $(2.2)_3$ by $3\gamma W_t$ and $(2.2)_4$ by $\alpha\theta_t$ respectively, then, integrating over (0,1) and summing up, we get

$$\frac{d}{dt}\frac{\gamma}{2}\int_0^1 \left[\rho_1\Phi_t^2 + G(\Psi-\Phi_x)^2 + I_{\rho_1}(3W_t-\Psi_t)^2 + D(3W_x-\Psi_x)^2 + 3I_{\rho_1}W_t^2 + 4\beta_1 W^2 + 3DW_x^2\right] dx$$
$$+\frac{d}{dt}\frac{\alpha}{2}\int_0^1 \left(\rho_2\theta_t^2 + \delta\theta_x^2\right) dx + 4\gamma\int_0^1 \beta_2 W_t^2 dx + \alpha\int_0^1 k(\theta_{tx})^2 dx - G\int_0^1 (\Psi-\Phi_x)\Psi_t dx$$
$$-\gamma\int_0^1 G(\Psi-\Phi_x)(3W-\Psi)_t dx + 3\gamma\int_0^1 G(\Psi-\Phi_x)W_t dx$$
$$=0,$$

we can conclude that

$$\frac{d}{dt}E(t) = -4\gamma\int_0^1 \beta_2 W_t^2 dx - \alpha\int_0^1 k(\theta_{tx})^2 dx.$$

This completes the proof. □

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Lemma 4.2 *Let* (Φ, Ψ, W, θ) *be the solution of problem* (2.1)-(2.2). *The functional*

$$I_1(t) := -\rho_1 \int_0^1 \Phi\Phi_t dx + I_{\rho_1} \int_0^1 WW_t dx$$

satisfies

$$\begin{aligned} I_1'(t) \leq &- \rho_1 \int_0^1 \Phi_t^2 dx - \frac{2\gamma\beta}{3} \int_0^1 W^2 dx - \frac{D}{2} \int_0^1 W_x^2 dx + C \int_0^1 W_t^2 dx \\ &+ C \int_0^1 (\Psi - \Phi_x)^2 dx + C \int_0^1 (3W_x - \Psi_x)^2 dx. \end{aligned} \tag{4.1}$$

Proof. By differentiating I_1 and using $(2.1)_1$, we conclude that

$$\begin{aligned} I_1(t) =& - \rho_1 \int_0^1 \Phi_t^2 dx - \rho_1 \int_0^1 \Phi\Phi_{tt} dx + I_{\rho_1} \int_0^1 W_t^2 dx + I_{\rho_1} \int_0^1 WW_{tt} dx \\ =& - \rho_1 \int_0^1 \Phi_t^2 dx + I_{\rho_1} \int_0^1 W_t^2 dx - G \int_0^1 \Phi_x(\Psi - \Phi_x) dx - G \int_0^1 W(\Psi - \Phi_x) dx \\ &- \frac{4\beta_1}{3} \int_0^1 W^2 dx - \frac{4\beta_2}{3} \int_0^1 WW_t dx - D \int_0^1 W_x^2 dx. \end{aligned}$$

Making use of Young's inequality, we get

$$\begin{aligned} I_1'(t) \leq& - \rho_1 \int_0^1 \Phi_t^2 dx - \frac{4\beta_1}{3} \int_0^1 W^2 dx - D \int_0^1 W_x^2 dx + I_{\rho_1} \int_0^1 W_t^2 dx + G \int_0^1 (\Psi - \Phi_x)^2 dx \\ &+ \frac{D}{36G} \int_0^1 \Psi^2 dx + \frac{9G^2}{D} \int_0^1 (\Psi - \Phi_x)^2 dx + \frac{\beta_1}{3} \int_0^1 W^2 dx + \frac{3G^2}{4\beta_1} \int_0^1 (\Psi - \Phi_x)^2 dx \\ &+ \frac{\beta_1}{3} \int_0^1 W^2 dx + \frac{4\beta_2^2}{3\beta_1} \int_0^1 W_t^2 dx. \end{aligned}$$

Using Lemma 2.1 and Young's inequality again, we obtain (4.1). □

Lemma 4.3 *Let* (Φ, Ψ, W, θ) *be the solution of problem* (2.1)-(2.2). *The functional*

$$I_2(t) := I_{\rho_1} \int_0^1 (3W - \Psi)(3W - \Psi)_t dx$$

satisfies

$$I_2'(t) \leq -\frac{D}{2} \int_0^1 (3W_x - \Psi_x)^2 dx + I_{\rho_1} \int_0^1 (3W_t - \Psi_t)^2 dx + C \int_0^1 (\Psi - \Phi_x)^2 dx + C \int_0^1 \theta_t^2 dx.$$

Proof. By differentiating I_2 and using $(2.1)_2$, we conclude that

$$\begin{aligned} I_2'(t) =& I_{\rho_1} \int_0^1 (3W_t - \Psi_t)^2 dx + \int_0^1 (3W - \Psi)[G(\Psi - \Phi_x) + D(3W - \Psi)_{xx} - \alpha\theta_{xt}] dx \\ =& I_{\rho_1} \int_0^1 (3W_t - \Psi_t)^2 dx + \int_0^1 (3W - \Psi) G(\Psi - \Phi_x) dx \\ &- D \int_0^1 (3W_x - \Psi_x)^2 dx + \alpha \int_0^1 (3W_x - \Psi_x)\theta_t dx. \end{aligned}$$

Using Young's inequality and Lemma 2.1, we obtain the result. □

Lemma 4.4 *Let* (Φ,Ψ,W,θ) *be the solution of problem* (2.1)-(2.2). *The functional*

$$I_3(t) := I_{\rho_1}\rho_2\int_0^1 (3W-\Psi)_t\int_0^x \theta_t(y,t)dydx - \delta I_{\rho_1}\int_0^1 \theta_x(3W-\Psi)dx$$

satisfies

$$\begin{aligned}I_3'(t) \le & -\frac{\gamma I_\rho}{2}\int_0^1 (3W_t-\Psi_t)^2dx + \alpha I_{\rho_1}\int_0^1 \theta_t^2dx + \varepsilon_3 I_{\rho_1}\int_0^1 G(\Psi-\Phi_x)^2dx\\ & + c(\varepsilon_3)\int_0^1 \theta_t^2dx + \varepsilon_3 I_{\rho_!}D\int_0^1 (3W_x-\Psi_x)^2dx.\end{aligned}$$

Proof. By differentiating I_3 and using $(2.1)_4$, we conclude that

$$\begin{aligned}I_3'(t) =& \int_0^1 I_{\rho_1}(3W-\Psi)_t\int_0^x [\delta\theta_{xx} - \gamma(3W-\Psi)_{tx} + k\theta_{txx}]dydx\\ & + \int_0^1 \rho_2[G(\Psi-\Phi_x) + D(3W-\Psi)_{xx} - \alpha\theta_{xt}]\int_0^x \theta_t(y,t)dydx\\ & - \delta I_{\rho_1}\int_0^1 \theta_{tx}(3W-\Psi)dx - \delta I_{\rho_1}\int_0^1 \theta_x(3W-\Psi)_tdx\\ =& \int_0^1 I_{\rho_!}(3W-\Psi)_t[\delta\theta_x - \gamma(3W-\Psi)_t + k\theta_{tx}]dx + \rho_2\int_0^1 G\Psi\int_0^x \theta_t(y,t)dydx\\ & + \int_0^1 \rho_2[-G\Phi_x + D(3W-\Psi)_{xx} - \alpha\theta_{xt}]\int_0^x \theta_t(y,t)dydx\\ & - \delta I_{\rho_1}\int_0^1 \theta_{tx}(3W-\Psi)dx - \delta I_{\rho_1}\int_0^1 \theta_x(3W-\Psi)_tdx\\ =& -\gamma I_{\rho_1}\int_0^1 (3W-\Psi)_t^2dx + I_{\rho_1}k\int_0^1 (3W-\Psi)_t\theta_{tx}dx + \rho_2\int_0^1 G(\Psi-\Phi_x)\int_0^x \theta_t(y,t)dydx\\ & - \delta I_{\rho_1}\int_0^1 \theta_{tx}(3W-\Psi)dx - \rho_2\int_0^1 D(3W-\Psi)_x\theta_tdx + \alpha\rho_2\int_0^1 \theta_t^2dx\\ & + \left[\rho_2(-G\Phi + D(3W-\Psi)_x - \alpha\theta_t)\int_0^x \theta_t(y,t)dy\right]\Big|_{x=0}^{x=1}.\end{aligned}$$

Note that

$$\int_0^1 \theta_t(y,t)dy = \frac{d}{dt}\int_0^1 \theta(y,t)dy = 0,$$

then, by Young's inequality, we obtain the result. □

Lemma 4.5 *Let* (Φ,Ψ,W,θ) *be the solution of problem* (2.1)-(2.2). *The functional*

$$I_4(t) := \int_0^1 \left[\rho_2\theta_t\theta + \frac{k}{2}\theta_x^2 + \gamma(3W-\Psi)_x\theta\right]dx$$

satisfies

$$I_4'(t) \le -\delta\int_0^1 \theta_x^2dx + \left(\frac{\gamma^2}{4\varepsilon_4} + \rho_2\right)\int_0^1 \theta_t^2dx + \varepsilon_4\int_0^1 (3W_x-\Psi_x)^2dx. \tag{4.2}$$

Proof. By differentiating I_4 and using $(2.1)_4$, we conclude that

$$I_4'(t) = \int_0^1 \rho_2\theta_{tt}\theta dx + \int_0^1 \rho_2\theta_t^2dx + \int_0^1 \frac{k}{2}(\theta_{xt}\theta_x + \theta_x\theta_{xt})dx$$

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$$
\begin{aligned}
&+\int_0^1 \gamma(3W-\Psi)_{xt}\theta dx+\int_0^1 \gamma(3W-\Psi)_x\theta_t dx \\
&=\int_0^1 [\delta\theta_{xx}-\gamma(3W-\Psi)_{xt}+k\theta_{txx}]\theta dx+\int_0^1 \rho_2\theta_t^2 dx-k\int_0^1 \theta_{xxt}\theta dx \\
&+\int_0^1 \gamma(3W-\Psi)_{xt}\theta dx+\int_0^1 \gamma(3W-\Psi)_x\theta_t dx \\
&=\int_0^1 \delta\theta_{xx}\theta dx+\int_0^1 \rho_2\theta_t^2 dx+\int_0^1 \gamma(3W-\Psi)_x\theta_t dx.
\end{aligned}
$$

Thanks to young's inequality, (4.2) is established. □

Lemma 4.6 *Let* (Φ,Ψ,W,θ) *be the solution of problem* (2.1)-(2.2). *The functional*

$$
I_5(t):=I_{\rho_1}\int_0^1 (3W-\Psi)_t(\Phi_x-\Psi)dx+\frac{D}{G}\rho_1\int_0^1 (3W-\Psi)_x\Phi_t dx
$$

satisfies

$$
\begin{aligned}
I_5'(t)\le &-\frac{G}{2}\int_0^1 (\Psi-\Phi_x)^2dx+C\int_0^t \theta_{tx}^2dx+I_{\rho_1}\int_0^1 (3W-\Psi)_t^2dx \\
&+\varepsilon_5\int_0^1 (3W-\Psi)_t^2dx+C(\varepsilon_5)\int_0^1 W_t^2dx+\left(\frac{D}{G}\rho_1-I_{\rho_1}\right)\int_0^1 (3W-\Psi)_{xt}\Phi_t dx. \qquad (4.3)
\end{aligned}
$$

Proof. By differentiating I_5 and using $(2.1)_2$, we conclude that

$$
\begin{aligned}
I_5'(t)=&I_{\rho_1}\int_0^1 (3W-\Psi)_{tt}(\Phi_x-\Psi)dx+I_{\rho_1}\int_0^1 (3W-\Psi)_t(\Phi_x-\Psi)_t dx \\
&+\frac{D}{G}\rho_1\int_0^1 (3W-\Psi)_{xt}\Phi_t dx+\frac{D}{G}\rho_1\int_0^1 (3W-\Psi)_x\Phi_{tt}dx \\
=&-\int_0^1 G(\Psi-\Phi_x)^2dx+\int_0^1 D(3W-\Psi)_{xx}(\Phi_x-\Psi)dx-\int_0^1 \alpha\theta_{tx}(\Phi_x-\Psi)dx \\
&-I_{\rho_1}\int_0^1 (3W-\Psi)_t(\Psi-\Phi_x)_t dx+\frac{D}{G}\rho_1\int_0^1 (3W-\Psi)_{xt}\Phi_t dx-D\int_0^1 (3W-\Psi)_x(\Psi-\Phi_x)_x dx \\
=&-\int_0^1 G(\Psi-\Phi_x)^2dx+\int_0^1 \alpha\theta_{tx}(\Psi-\Phi_x)dx+I_{\rho_1}\int_0^1 (3W-\Psi)_t\Psi_t dx \\
&+\left(\frac{D}{G}\rho_1-I_{\rho_1}\right)\int_0^1 (3W-\Psi)_{xt}\Phi_t dx. \qquad (4.4)
\end{aligned}
$$

Similarly, using young's inequality, (4.3) is established. □ **Proof.** of Theorem 2.2: To finalize the proof, we assume $\frac{G}{\rho_1}=\frac{D}{I_{\rho_1}}$ and define a Lyapunov functional $\mathcal{L}$ as follows

$$
\mathcal{L}(t):=NE(t)+I_1(t)+N_2I_2(t)+N_3I_3(t)+I_4(t)+N_5I_5(t),
$$

where N, N_2, N_3,N_5 are positive constants to be chosen properly later. Using Cauchy-Schwarz inequality and the Poincare inequality, one can easily see that all $I_i(t)$, $i=1,2,3,4,5$ are bounded by an expression with the existing terms in the energy $E(t)$. This leads to the equivalence of

$\mathcal{L}(t)$ and $E(t)$. Gathering the estimates in the previous lemmas and using $\int_0^1 \theta_t^2 dx \leq \int_0^1 \theta_{tx}^2 dx$, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -(4N\gamma\beta_2 - C - N_5 C(\varepsilon_5)) \int_0^1 W_t^2 dx \\ & - (N\alpha G - N_3 C(\varepsilon_3) - N_5 C - N_2 C - I_{\rho_1}\alpha N_3 - N_3 C(\varepsilon_3) - C(\varepsilon_4) - \rho_2) \int_0^1 \theta_{xt}^2 dx \\ & - \rho_1 \int_0^1 \Phi_t^2 dx - \frac{2\beta_1}{3} \int_0^1 W^2 dx - \frac{D}{2} \int_0^1 W_x^2 dx - \delta \int_0^1 \theta_x^2 dx \\ & - \left(\frac{G}{2} N_5 - C - CN_2 - \varepsilon_3 N_3\right) \int_0^1 (\Psi - \Phi_x)^2 dx \\ & - \left(\frac{D}{2} N_2 - C - \varepsilon_4\right) \int_0^1 (3W_x - \Psi_x)^2 dx \\ & - \left(\frac{\gamma I_{\rho_1}}{2} N_3 - I_{\rho_1} N_2 - I_{\rho_1} N_5 - N_5 \varepsilon_5\right) \int_0^1 (3W_t - \Psi_t)^2 dx. \end{aligned} \tag{4.5}$$

At this point, we need to choose our constants carefully. First, we take N_2 large enough and ε_4 small, such that $\frac{D}{2} N_2 - C - \varepsilon_4 > 0$. Then, we choose N_5 large enough, so that $\frac{G}{2} N_5 - C - CN_2 > 0$. Next, we pick ε_5 small and choose N_3 large enough such that $\frac{\gamma}{2} I_{\rho_1} N_3 - I_{\rho_1} N_2 - I_{\rho_1} N_5 - N_5 \varepsilon_5 > 0$. We then select ε_3 so small that $\frac{D}{2} N_2 - C - \varepsilon_4 - N_3 \varepsilon_3 > 0$ and $\frac{G}{2} N_5 - C - CN_2 - \varepsilon_3 N_3 > 0$. Finally, we choose N so large such that $4N\gamma\beta_2 - C - N_5 C(\varepsilon_5) > 0$ and $N\alpha G - N_3 C(\varepsilon_3) - N_5 C - N_2 C - I_{\rho_1}\alpha N_3 - N_3 C(\varepsilon_3) - C(\varepsilon_4) - \rho_2 > 0$. From the above, we deduce that for some positive constants γ_1, γ_2, one has $\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t)$. Therefore, (4.5) becomes

$$\mathcal{L}'(t) \leq -cE(t).$$

For $s = \frac{c}{\gamma_2}$, we get

$$\mathcal{L}'(t) \leq -s\mathcal{L}(t), \ \forall t \geq 0. \tag{4.6}$$

A simple integration of (4.6) over (0,t) leads to

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-st}, \ \forall t \geq 0.$$

It gives the desired result Theorem 2.2 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. $\square$

5 Lack of exponential stability

In this section, by using Gearhart-Herbst-Prüss-Huang theorem [14], we give the proof of Theorem 2.3 which concludes the lack of exponential decay result. We consider that there exists a sequence of imaginary number λ_μ and functions

$$F_\mu = (f^1, f^2, f^3, f^4, f^5, f^6, f^7, f^8)^T \in \mathcal{H},$$

with $\|F_\mu\|_{\mathcal{H}} \leq 1$ such that $\|(\lambda_\mu I - \mathcal{A})^{-1} F_\mu\|_{\mathcal{H}} \to \infty$ where

$$\lambda_\mu U_\mu - \mathcal{A} U_\mu = F_\mu \tag{5.1}$$

with $U_\mu = (\Phi, 3W - \Psi, W, \theta, \Phi_t, 3W_t - \Psi_t, W_t, \theta_t)$ not bounded. Rewrite the spectral equation (5.1) in term of its components, for $\lambda_\mu = \lambda$, we have

$$\begin{cases} \lambda\Phi - \Phi_t = f^1, \\ \lambda(3W - \Psi) - (3W - \Psi)_t = f^2, \\ \lambda W - W_t = f^3, \\ \lambda\theta - \theta_t = f^4, \\ \rho_1 \lambda \Phi_t - G\partial_{xx}\Phi - G\partial_x(3W - \Psi) + 3G\partial_x W = \rho_1 f^5, \\ I_{\rho_1}\lambda(3W - \Psi)_t + G\partial_x\Phi + G(3W - \Psi) - 3GW - D\partial_{xx}(3W - \Psi) + \alpha\partial_x\theta_t = I_{\rho_1} f^6, \\ I_{\rho_1}\lambda W_t - G(3W - \Psi) + 3GW - G\partial_x\Phi + \frac{4}{3}\beta_1 W + \frac{4}{3}\beta_2 W_t - D\partial_{xx}W = I_{\rho_1} f^7, \\ \rho_2\lambda\theta_t - \delta\partial_{xx}\theta + \gamma\partial_x(3W - \Psi)_t - k\partial_{xx}\theta_t = \rho_2 f^8. \end{cases}$$

Taking $f^1 = f^2 = f^3 = f^4 = 0$, we arrive at

$$\begin{cases} \rho_1\lambda^2\Phi - G\partial_{xx}\Phi - G\partial_x(3W - \Psi) + 3G\partial_x W = \rho_1 f^5, \\ I_{\rho_1}\lambda^2(3W - \Psi) + G\partial_x\Phi + G(3W - \Psi) - 3GW - D\partial_{xx}(3W - \Psi) + \lambda\alpha\partial_x\theta = I_{\rho_1} f^6, \\ I_{\rho_1}\lambda^2 W - G(3W - \Psi) + 3GW - G\partial_x\Phi + \frac{4}{3}\beta_1 W + \frac{4}{3}\lambda\beta_2 W - D\partial_{xx}W = I_{\rho_1} f^7, \\ \rho_2\lambda^2\theta - \delta\partial_{xx}\theta + \lambda\gamma\partial_x(3W - \Psi) - k\lambda\partial_{xx}\theta = \rho_2 f^8. \end{cases}$$

Because of the boundary conditions given by (2.2), we assume that

$$\Phi = A\cos(\mu\pi x),\ (3W - \Psi) = B\sin(\mu\pi x),\ W = C\sin(\mu\pi x),\ \theta = E\cos(\mu\pi x).$$

Now, choosing

$$f^5 = b_1\cos(\mu\pi x),\ f^6 = b_2\sin(\mu\pi x),\ f^7 = b_3\sin(\mu\pi x),\ f^8 = b_4\cos(\mu\pi x),$$

we arrive at

$$\begin{cases} \rho_1\lambda^2 A\cos(\mu\pi x) + G(\mu\pi)^2 A\cos(\mu\pi x) - G\mu\pi B\cos(\mu\pi x) + 3G\mu\pi C\cos(\mu\pi x) \\ \quad = \rho_1 b_1\cos(\mu\pi x), \\ I_{\rho_1}\lambda^2 B\sin(\mu\pi x) - G\mu\pi A\sin(\mu\pi x) + GB\sin(\mu\pi x) - 3GC\sin(\mu\pi x) \\ +D(\mu\pi)^2 B\sin(\mu\pi x) - \lambda\alpha\mu\pi E\sin(\mu\pi x) \\ \quad = I_{\rho_1} b_2\sin(\mu\pi x), \\ I_{\rho_1}\lambda^2 C\sin(\mu\pi x) - GB\sin(\mu\pi x) + 3GC\sin(\mu\pi x) + G\mu\pi A\sin(\mu\pi x) \\ +\frac{4}{3}\beta_1 C\sin(\mu\pi x) + \frac{4}{3}\lambda\beta_2 C\sin(\mu\pi x) + D(\mu\pi)^2 C\sin(\mu\pi x) \\ \quad = I_{\rho_1} b_3\sin(\mu\pi x), \\ \rho_2\lambda^2 E\cos(\mu\pi x) + \delta(\mu\pi)^2 E\cos(\mu\pi x) + \lambda\gamma\mu\pi B\cos(\mu\pi x) + k\lambda(\mu\pi)^2 E\cos(\mu\pi x) \\ \quad = \rho_2 b_4\cos(\mu\pi x). \end{cases}$$

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Choosing $b_1 = \dfrac{1}{\rho_1}$, $b_2 = b_3 = b_4 = 0$, we have

$$\begin{cases} \rho_1\lambda^2 A + G(\mu\pi)^2 A - GB\mu\pi + 3G\mu\pi C = 1, \\ I_{\rho_1}\lambda^2 B - AG\mu\pi + GB - 3GC + D(\mu\pi)^2 B - \lambda\alpha\mu\pi E = 0, \\ I_{\rho_1}\lambda^2 C - GB + 3GC + G\mu\pi A + \dfrac{4}{3}\beta_1 C + \dfrac{4}{3}\lambda\beta_2 C + D(\mu\pi)^2 C = 0, \\ \rho_2\lambda^2 E + \delta(\mu\pi)^2 E + \lambda\gamma\mu\pi B + k\lambda(\mu\pi)^2 E = 0. \end{cases}$$

Now, we take $\lambda = \lambda_\mu$, such that

$$\rho_1\lambda^2 + G(\mu\pi)^2 = 0.$$

Therefore, the above system can be written as

$$\begin{cases} -GB\mu\pi + 3G\mu\pi C = 1, \\ -AG\mu\pi + \left[\left(D - \dfrac{I_{\rho_1}G}{\rho_1}\right)(\mu\pi)^2 + G\right]B - 3GC - \sqrt{\dfrac{G}{\rho_1}}\alpha(\mu\pi)^2 iE = 0, \\ -GB + G\mu\pi A + \left[\left(D - \dfrac{I_{\rho_1}G}{\rho_1}\right)(\mu\pi)^2 + \dfrac{4}{3}\beta_1 + 3G + \dfrac{4}{3}\sqrt{\dfrac{G}{\rho_1}}\beta_2\mu\pi i\right]C = 0, \\ \sqrt{\dfrac{G}{\rho_1}}\gamma(\mu\pi)^2 iB + \left(-\dfrac{\rho_2 G}{\rho_1} + \delta + k\sqrt{\dfrac{G}{\rho_1}}\mu\pi i\right)(\mu\pi)^2 E = 0. \end{cases} \tag{5.2}$$

Add $(5.2)_2$ to $(5.2)_3$, we conclude that

$$\left(D - \frac{I_{\rho_1}G}{\rho_1}\right)(\mu\pi)^2 B + \left[\left(D - \frac{I_{\rho_1}G}{\rho_1}\right)(\mu\pi)^2 + \frac{4}{3}\beta_1 + \frac{4}{3}\sqrt{\frac{G}{\rho_1}}\beta_2\mu\pi i\right]C - \sqrt{\frac{G}{\rho_1}}\alpha(\mu\pi)^2 iE = 0. \tag{5.3}$$

It follows from $(5.2)_4$, we get

$$E = \frac{\gamma\sqrt{\frac{G}{\rho_1}}i}{\frac{\rho_2 G}{\rho_1} - \delta - k\sqrt{\frac{G}{\rho_1}}(\mu\pi)i}B. \tag{5.4}$$

Combining (5.4) and (5.3) yields

$$C = \frac{N}{M}B,$$

where

$$N = \frac{\alpha\pi\gamma G}{\mu\rho_1}\left(\frac{\rho_2 G}{\rho_1} - \delta - k\sqrt{\frac{G}{\rho_1}}(\mu\pi)i\right)^{-1} - \left(D - \frac{I_{\rho_1}G}{\rho_1}\right)(\mu\pi)^2,$$

$$M = \left(D - \frac{I_{\rho_1}G}{\rho_1}\right)(\mu\pi)^2 + \frac{4}{3}\beta_1 + \frac{4}{3}\sqrt{\frac{G}{\rho_1}}\beta_2(\mu\pi)i.$$

Subtracting C into $(5.2)_3$ and $(5.2)_1$, we get

$$A = \frac{G - (M + 3G)\frac{N}{M}}{G(\mu\pi)}B, \quad B = \frac{M}{(3N - M)G(\mu\pi)}.$$

When $\mu \to \infty$, we get $B(\mu\pi) \to -\frac{1}{4G}$. Then, substituting B into A, C, E, we obtain

$$A \to -\frac{1}{4},\ C \to o\left(\frac{1}{\mu}\right),\ E \to o\left(\frac{1}{\mu}\right).$$

Thus

$$\begin{aligned}\|U_\mu\|_{\mathcal{H}}^2 \geq & G\int_0^1 (\Psi - \Phi_x)^2 dx = G\int_0^1 (-B + 3C + \mu\pi A)^2 \sin^2(\mu\pi x)\, dx \\ & = GA^2(\mu\pi)^2 \int_0^1 \sin^2(\mu\pi x)\, dx = \frac{1}{2}GA^2(\mu\pi)^2 - \frac{1}{4}GA^2(\mu\pi)\sin(2\mu\pi x)\,|_{x=0}^{x=1} \\ & \to \infty \text{ as } \mu \to \infty.\end{aligned}$$

Therefore, this completes the proof.

6 Polynomial decay for $\frac{G}{\rho_1} \neq \frac{D}{I_{\rho_1}}$

In this section, we prove the polynomial decay result for the case $\frac{G}{\rho_1} \neq \frac{D}{I_{\rho_1}}$. Similar as Lemma 4.1, a simple calculation about the second-order energy implies

$$\frac{d}{dt}E_2(t) = -4\gamma \int_0^1 \beta_2 W_{tt}^2 dx - \alpha \int_0^1 k\theta_{xtt}^2 dx \leq 0.$$

Now, we prove Theorem 2.4. Let $\frac{G}{\rho_1} - \frac{D}{I_{\rho_1}} \neq 0$, the last term in (4.4) can be handled as follows. Using $(2.1)_4$ and Young's inequality, we obtain

$$\begin{aligned}\int_0^1 (3W - \Psi)_{xt}\Phi_t dx = & \frac{1}{\gamma}\int_0^1 (k\theta_{xxt} + \delta\theta_{xx} - \rho_2\theta_{tt})\Phi_t dx \\ = & -\frac{\rho_2}{\gamma}\int_0^1 \theta_{tt}\Phi_t dx + \frac{\delta}{\gamma}\int_0^1 \theta_{xx}\Phi_t dx + \frac{k}{\gamma}\int_0^1 \theta_{xxt}\Phi_t dx \\ = & -\frac{\rho_2}{\gamma}\int_0^1 \theta_{tt}\Phi_t dx + \frac{\delta}{\gamma}\int_0^1 \theta_{xt}\Phi_x dx + \frac{k}{\gamma}\int_0^1 \theta_{xtt}\Phi_x dx \\ & -\frac{d}{dt}\int_0^1 \left(\frac{\delta}{\gamma}\theta_x\Phi_x + \frac{k}{\gamma}\theta_{xt}\Phi_x\right) dx.\end{aligned}$$

Multiplying by $\frac{D}{G}\rho_1 - I_{\rho_1}$, we get

$$\begin{aligned}\left(\frac{D}{G}\rho_1 - I_\rho\right)\int_0^1 (3W - \Psi)_{xt}\Phi_t dx \leq & -\frac{d}{dt}\left(\frac{D}{G}\rho_1 - I_{\rho_1}\right)\int_0^1 \left(\frac{\delta}{\gamma}\theta_x\Phi_x + \frac{k}{\gamma}\theta_{xt}\Phi_x\right) dx \\ & + \varepsilon_6 \int_0^1 (\Phi_t^2 + \Phi_x^2) dx + C(\varepsilon_6)\int_0^1 (\theta_{xt}^2 + \theta_{xtt}^2) dx,\end{aligned}$$

where we have used that

$$\begin{aligned}\Phi_x^2 = (\Phi_x - \Psi + \Psi)^2 \leq & 2(\Phi_x - \Psi)^2 + 2(\Psi - 3W + 3W)^2 \\ \leq & 2(\Phi_x - \Psi)^2 + 4(\Psi - 3W)^2 + 36W^2.\end{aligned}$$

We then define

$$L(t) = \bar{L}(t) + \left(\frac{D}{G}\rho_1 - I_{\rho_1} \right) \int_0^1 \left(\frac{\delta}{\gamma}\theta_x \Phi_x + \frac{k}{\gamma}\theta_{xt}\Phi_x \right) dx,$$

where

$$\bar{L}(t) = N\left(E(t) + E_2(t)\right) + I_1(t) + N_2 I_2(t) + N_3 I_3(t) + I_4(t) + N_5 I_5(t).$$

From (4.5) and (2.4), we get

$$\begin{aligned} L'(t) \leq & - cE(t) + \varepsilon_6 \int_0^1 \left[\Phi_t^2 + 2(\Phi_x - \Psi)^2 + 4(\Psi - 3W)^2 + 36W^2 \right] dx \\ & - (N - C(\varepsilon_6)) \int_0^1 \theta_{xtt}^2 dx - (N - C(\varepsilon_6)) \int_0^1 \theta_{xt}^2 dx. \end{aligned} \tag{6.1}$$

Then we choose ε_6 small enough and take N large enough so that L is positive and

$$N\alpha G - N_3 C(\varepsilon_3) - N_5 C - N_2 C - I_\rho \alpha N_3 - N_3 C(\varepsilon_3) - C(\varepsilon_4) - \rho_2 - C(\varepsilon_6) > 0.$$

Depending on above constants, we deduce that

$$L'(t) \leq -\frac{c}{2}E(t).$$

Integrating over $(0,t)$, we have

$$tE(t) \leq \int_0^t E(s)ds \leq \frac{2}{c}(L'(0) - L'(t)) \leq \frac{2L'(0)}{c}.$$

Consequently,

$$E(t) \leq \frac{2L(0)}{ct} \leq \frac{C_1(E(0) + E_2(0))}{t} \quad \forall\, t > 0.$$

This completes the proof.

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References

[1] T. A. Apalara, Uniform stability of a laminated beam with structural damping and second sound, Z. Angew. Math. Phys. **68** (2017), no. 2, 68:41.

[2] X.-G. Cao, D.-Y. Liu and G.-Q. Xu, Easy test for stability of laminated beams with structural damping and boundary feedback controls, J. Dyn. Control Syst. **13** (2007), no. 3, 313–336.

[3] M. M. Cavalcanti et al., Uniform decay rates for the energy of Timoshenko system with the arbitrary speeds of propagation and localized nonlinear damping, Z. Angew. Math. Phys. **65** (2014), no. 6, 1189–1206.

[4] M. M. Chen, W. J. Liu and W. C. Zhou, Existence and general stabilization of the Timoshenko system of thermo-viscoelasticity of type III with frictional damping and delay terms, Adv. Nonlinear Anal., in press. doi:10.1515/anona-2016-0085

[5] S. Drabla, S. A. Messaoudi and F. Boulanouar, A general decay result for a multi-dimensional weakly damped thermoelastic system with second sound, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), no. 4, 1329–1339.

[6] A. Fareh and S. A. Messaoudi, Stabilization of a type III thermoelastic Timoshenko system in the presence of a time-distributed delay, Math. Nachr. **290** (2017), no. 7, 1017–1032.

[7] J. A. Goldstein, *Semigroups of linear operators and applications*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1985.

[8] A. E. Green and P. M. Naghdi, A re-examination of the basic postulates of thermomechanics, Proc. Roy. Soc. London Ser. A **432** (1991), no. 1885, 171–194.

[9] S. W. Hansen. A model for a two-layered plate with interfacial slip. Control and estimation of distributed parameter systems: nonlinear phenomena (Vorau, 1993), 143–170, Internat. Ser. Numer. Math. vol. 118, Birkhauser, Basel, 1994.

[10] S. W. Hansen and R. Spies, Structural damping in laminated beams due to interfacial slip, J. Sound Vibration. **204** (1997), no. 2, 183–202.

[11] J. Hao and P. Wang, Asymptotical stability for memory-type porous thermoelastic system of type III with constant time delay, Math. Methods Appl. Sci. **39** (2016), no. 13, 3855–3865.

[12] J. Hao and P. Wang, Exponential decay of solution to the viscoelastic porous-thermoelastic system of type III with boundary time-varying delay, Math. Methods Appl. Sci. **39** (2016), no. 13, 3659–3668.

[13] M. Kafini et al., Well-posedness and stability results in a Timoshenko-type system of thermoelasticity of type III with delay, Z. Angew. Math. Phys. **66** (2015), no. 4, 1499–1517.

[14] A. A. Keddi, T. A. Apalaras and S. A. Messaoudi, Exponential and polynomial decay in a thermoelastic-Bresse system with second sound, Appl. Math. Optim., in press. DOI:10.1007/s00245-016-9376-y

[15] G. Li, X. Y. Kong and W. J. Liu, General decay for a laminated beam with structural damping and memory: the case of non-equal wave speeds, J. Integral Equations Appl., in press.

[16] W. J. Liu, K. W. Chen and J. Yu, Existence and general decay for the full von Kármán beam with a thermo-viscoelastic damping, frictional dampings and a delay term, IMA J. Math. Control Inform. **34** (2017), no. 2, 521–542.

[17] W. J. Liu, K. W. Chen and J. Yu, Asymptotic stability for a non-autonomous full von Kármán beam with thermo-viscoelastic damping, Appl. Anal., in press. doi: 10.1080/00036811.2016.1268688

[18] W. J. Liu and W. F. Zhao, Stabilization of a thermoelastic laminated beam with past history, Appl. Math. Optim., in press. doi: 10.1007/s00245-017-9460-y

[19] A. Lo and N. Tatar, Uniform Stability of a Laminated Beam with Structural Memory, Qual. Theory Dyn. Syst. **15** (2016), no. 2, 517–540.

[20] A. Lo and N. Tatar, Stabilization of laminated beams with interfacial slip, Electron. J. Differential Equations **2015** (2015), No. 129, 14 pp.

[21] S. A. Messaoudi and T. A. Apalara, General stability result in a memory-type porous thermoelasticity system of type III, Arab J. Math. Sci. **20** (2014), no. 2, 213–232.

[22] S. A. Messaoudi and A. Fareh, Energy decay in a Timoshenko-type system of thermoelasticity of type III with different wave-propagation speeds, Arab. J. Math. (Springer) **2** (2013), no. 2, 199–207.

[23] S. A. Messaoudi and B. Said-Houari, Energy decay in a Timoshenko-type system of thermoelasticity of type III, J. Math. Anal. Appl. **348** (2008), no. 1, 298–307.

[24] M. I. Mustafa, On the decay rates for thermoviscoelastic systems of type III, Appl. Math. Comput. **239** (2014), 29–37.

[25] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer, New York, 1983.

[26] Y. Qin, X.-G. Yang and Z. Ma, Global existence of solutions for the thermoelastic Bresse system, Commun. Pure Appl. Anal. **13** (2014), no. 4, 1395–1406.

[27] B. Said-Houari and T. Hamadouche, The Cauchy problem of the Bresse system in thermoelasticity of type III, Appl. Anal. **95** (2016), no. 11, 2323–2338.

[28] F. Tahamtani, On energy decay of an n-dimensional thermoelasticity system with a nonlinear weak damping, Iran. J. Sci. Technol. Trans. A Sci. **32** (2008), no. 1, 45–51, 83.

[29] N.-E. Tatar, Stabilization of a laminated beam with interfacial slip by boundary controls, Bound. Value Probl. **2015**, 2015:169, 11 pp.

[30] Z. Tian and G.-Q. Xu. Exponential stability analysis of Timoshenko beam system with boundary delays. Appl. Anal. (2016), 1–29.

[31] J.-M. Wang, G.-Q. Xu and S.-P. Yung, Exponential stabilization of laminated beams with structural damping and boundary feedback controls, SIAM J. Control Optim. **44** (2005), no. 5, 1575–1597.