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A NEW EXTENSION OF BETA AND HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The main objective of this paper is to introduce a further extension of extended (p,q)-beta function by considering two Mittag-Leffler function in the kernel. We investigate various properties of this newly defined beta function such as integral representations, summation formulas and Mellin transform. We define extended beta distribution and its mean, variance and moment generating function with the help of extension of beta function. Also, we establish an extension of extended (p,q)-hypergeometric and (p,q)-confluent hypergeometric functions by using the extension of beta function. Various properties of newly defined extended hypergeometric and confluent hypergeometric functions such as integral representations, Mellin transformations, differentiation formulas, transformation and summation formulas are investigated.

1. INTRODUCTION

We start with the classical beta function which is defined by

$$B(\delta_1,\delta_2)=\int_0^\infty t^{\delta_1-1}(1-t)^{\delta_2-1}dt, (\Re(\delta_1)>0,\Re(\delta_2)>0) \tag{1.1}$$

and its relation with well known gamma function

$$\Gamma(\vartheta)=\int_0^\infty t^{\vartheta-1}e^{-t}dt, (\Re(\vartheta)>0) \tag{1.2}$$

is given by

$$B(\delta_1,\delta_2)=\frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1+\delta_2)}, \Re(\delta_1)>0, \Re(\delta_2)>0.$$

The Gauss hypergeometric and confluent hypergeometric functions which are defined (see [16]) as

$${}_2F_1(\delta_1,\delta_2;\delta_3;z)=\sum_{n=0}^\infty \frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}\frac{z^n}{n!}, (|z|<1), \tag{1.3}$$

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$\Big(\delta_1, \delta_2, \delta_3 \in \mathbb{C}$ and $\delta_3 \neq 0, -1, -2, -3, \cdots\Big)$, and

$$ {}_1\Phi_1(\delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_2)_n}{(\delta_3)_n} \frac{z^n}{n!}, (|z| < 1), \tag{1.4}$$

$\Big(\delta_2, \delta_3 \in \mathbb{C}$ and $\delta_3 \neq 0, -1, -2, -3, \cdots\Big)$, respectively.
The integral representation of hypergeometric and confluent hypergeometric functions are respectively defined by

$$ {}_2F_1(\delta_1, \delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1}(1 - zt)^{-\delta_1} dt, \tag{1.5}$$

$\Big(\Re(\delta_3) > \Re(\delta_2) > 0, |\arg(1-z)| < \pi\Big)$, and

$$ {}_1\Phi_1(\delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1} e^{zt} dt, \tag{1.6}$$

$$\Big(\Re(\delta_3) > \Re(\delta_2) > 0\Big).$$

Chaudhry et al. [4] introduced the extended beta function is defined by

$$B(\delta_1, \delta_2; p) = B_p(\delta_1, \delta_2) = \int_0^{\infty} t^{\delta_1 - 1}(1-t)^{\delta_2 - 1} e^{-\frac{p}{t(1-t)}} dt \tag{1.7}$$

(where $\Re(p) > 0, \Re(\delta_1) > 0, \Re(\delta_2) > 0$) respectively. When $p = 0$, then $B(\delta_1, \delta_2; 0) = B(\delta_1, \delta_2)$.
Also, they defined the following extended beta distribution by

$$f(t) = \begin{cases} \frac{1}{B_p(\delta_1,\delta_2)} t^{\delta_1 - 1}(1-t)^{\delta_2 - 1} \exp[-\frac{p}{t(1-t)}], 0 < t < 1 \\ 0, otherwise \end{cases}. \tag{1.8}$$

The extended hypergeometric and confluent hypergeometric functions introduced in [5] by using the definition of extended beta function $B_p(\delta_1, \delta_2)$ as follows:

$$F_p(\delta_1, \delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\delta_2 + n, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} (\delta_1)_n \frac{z^n}{n!}, \tag{1.9}$$

where $p \geq 0$ and $\Re(\delta_3) > \Re(\delta_2) > 0$, $|z| < 1$ and

$$\Phi_p(\delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{B_p(\delta_2 + n, \delta_3 - \delta_2)}{B(\delta_2, \delta_3 - \delta_2)} \frac{z^n}{n!}, \tag{1.10}$$

where $p \geq 0$ and $\Re(\delta_3) > \Re(\delta_2) > 0$.

In the same paper, they defined the following integral representations of extended hypergeometric and confluent hypergeometric functions as

$$F_p(\delta_1, \delta_2; \delta_3; z) = \frac{1}{B(\delta_2, \delta_3 - \delta_2)} \times \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1}(1 - zt)^{-\delta_1} \exp\left(\frac{-p}{t(1-t)}\right) dt, \tag{1.11}$$

$$\Big(p \geq 0, \Re(\delta_3) > \Re(\delta_2) > 0, |\arg(1-z)| < \pi\Big),$$

and

$$\Phi_p(\delta_2;\delta_3;z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\exp\left(zt - \frac{-p}{t(1-t)}\right)dt, \tag{1.12}$$

$$\Big(p \geq 0, \Re(\delta_3) > \Re(\delta_2) > 0\Big).$$

It is clear that when $p = 0$, then the equations (1.9)-(1.12) reduce to the well known hypergeometric and confluent hypergeometric series and their integral representation respectively.

Choi et al. [6] introduced the extended (p, q)-beta function is defined by

$$B(\delta_1,\delta_2;p,q) = B_{p,q}(\delta_1,\delta_2) = \int\limits_0^\infty t^{\delta_1-1}(1-t)^{\delta_2-1}e^{-\frac{p}{t}-\frac{q}{(1-t)}}dt \tag{1.13}$$

(where $\Re(p), \Re(q) > 0, \Re(\delta_1) > 0, \Re(\delta_2) > 0$) respectively. When $p = q$, then $B_{p,q}(\delta_1,\delta_2;0) = B_p(\delta_1,\delta_2)$ and if $p = q = 0$, then $B_{p,q}(\delta_1,\delta_2;0) = B(\delta_1,\delta_2)$.

Also, they [6] defined the following extended beta distribution by

$$f(t) = \begin{cases} \frac{1}{B_{p,q}(\delta_1,\delta_2)}t^{\delta_1-1}(1-t)^{\delta_2-1}\exp[-\frac{p}{t} - \frac{q}{(1-t)}], 0 < t < 1 \\ 0, otherwise \end{cases}. \tag{1.14}$$

The extended hypergeometric and confluent hypergeometric functions introduced in [6] by using the definition of extended beta function $B_{p,q}(\delta_1,\delta_2)$ as follows:

$$F_{p,q}(\delta_1,\delta_2;\delta_3;z) = \sum_{n=0}^\infty \frac{B_{p,q}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}(\delta_1)_n\frac{z^n}{n!}, \tag{1.15}$$

where $p, q \geq 0$ and $\Re(\delta_3) > \Re(\delta_2) > 0$, $|z| < 1$ and

$$\Phi_{p,q}(\delta_2;\delta_3;z) = \sum_{n=0}^\infty \frac{B_{p,q}(\delta_2+n,\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}, \tag{1.16}$$

where $p, q \geq 0$ and $\Re(\delta_3) > \Re(\delta_2) > 0$.

In the same paper, they defined the following integral representations of extended hypergeometric and confluent hypergeometric functions as

$$F_{p,q}(\delta_1,\delta_2;\delta_3;z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \times \int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-zt)^{-\delta_1}\exp\left(\frac{-p}{t} - \frac{q}{(1-t)}\right)dt, \tag{1.17}$$

$$\Big(\Re(p), \Re(q) > 0, \Re(\delta_3) > \Re(\delta_2) > 0, |\arg(1-z)| < \pi\Big),$$

and

$$\Phi_{p,q}(\delta_2;\delta_3;z) = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\exp\left(zt - \frac{-p}{t} - \frac{q}{(1-t)}\right)dt, \tag{1.18}$$

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$$\Big(p \geq 0, \Re(\delta_3) > \Re(\delta_2) > 0\Big).$$

It is clear that when $p = q$, then the equations (1.15)-(1.18) reduce to the well known extended hypergeometric and confluent hypergeometric series and their integral representation respectively (see e.g., (1.9)-(1.12)).

Very recently, Pucheta [15] introduced a new and modified extension of gamma and beta function which are respectively defined by

$$\Gamma_b^\lambda(\delta_1) = \int_0^\infty t^{\delta_1 - 1} E_\lambda(-bv), \Re(\delta_1) > 0 \tag{1.19}$$

$$B_p^\lambda(\delta_1, \delta_2) = \int_0^1 t^{\delta_1 - 1}(1-t)^{\delta_2 - 1} E_\lambda\Big(-pt(1-t)\Big)dt, \tag{1.20}$$

where $\Re(\delta_1) > 0$, $\Re(\delta_2) > 0$ and $E_\lambda\Big(.\Big)$ is the Mittag-Leffler function. Obviously, when $\lambda = 1$ and $p = 0$ then $B_p^\lambda(\delta_1, \delta_2) = B(\delta_1, \delta_2)$.
Very recently Shadab et al. [7] introduced a new and modified extension of beta function as:

$$B_p^\alpha(\sigma_1, \sigma_2) = \int_0^1 t^{\sigma_1 - 1}(1-t)^{\sigma_2 - 1} E_\alpha\Big(-\frac{p}{t(1-t)}\Big)dt, \tag{1.21}$$

where $\Re(\sigma_1) > 0$, $\Re(\sigma_2) > 0$ and $E_\alpha\Big(.\Big)$ is Mittag-Leffler function defined by

$$E_\alpha\Big(z\Big) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{1.22}$$

Obviously, when $\alpha = 1$ then $B_p^1(x, y) = B_p(x, y)$ is the extended beta function (see[4]). Similarly, when when $\alpha = 1$ and $p = 0$, then $B_0^1(x, y) = B_0(x, y)$ is the classical beta function.
They [7] also defined extended hypergeometric function and its integral representation

$$\begin{aligned} F_p^\alpha(\sigma_1, \sigma_2; \sigma_3; z) = {}_2F_1\Big(\sigma_1, \sigma_2; \sigma_3; z; p, \alpha\Big) &= \sum_{n=0}^\infty (\sigma_1)_n \frac{B_p^\alpha(\sigma_2 + n, \sigma_3 - \sigma_2)}{B(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!} \\ &= \sum_{n=0}^\infty (\sigma_1)_n \frac{B(\sigma_2 + n, \sigma_3 - \sigma_2; p, \alpha)}{B(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!} \end{aligned} \tag{1.23}$$

where $p, \alpha \geq 0$, $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}$ and $|z| < 1$.

$$F_p^\alpha(\sigma_1, \sigma_2; \sigma_3; z) = \frac{1}{\beta(\sigma_2; \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2 - 1}(1-t)^{\sigma_3 - \sigma_2 - 1}(1 - tz)^{-\sigma_1} E_\alpha\Big(-\frac{p}{t(1-t)}\Big)dt, \tag{1.24}$$

where $\Re(p) > 0$, $\Re(\alpha) > 0$, $\Re(\sigma_3) > \Re(\sigma_2) > 0$. In they same paper defined the following extended confluent hypergeometric function and its integral representation as:

$$\Phi_p^\alpha(\sigma_2; \sigma_3; z) = \Phi\Big(\sigma_1, \sigma_2; \sigma_3; z; p, \alpha\Big) = \sum_{n=0}^\infty \frac{B_p^\alpha(\sigma_2 + n, \sigma_3 - \sigma_2)}{B(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!}$$

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$$= \sum_{n=0}^{\infty} \frac{B(\sigma_2 + n, \sigma_3 - \sigma_2; p, \alpha)}{B(\sigma_2, \sigma_3 - \sigma_2)} \frac{z^n}{n!} \quad (1.25)$$

where $p, \alpha \geq 0,\ , \sigma_2, \sigma_3 \in \mathbb{C}$, and

$$\Phi_p^{\alpha}(\sigma_2; \sigma_3; z) = \frac{1}{B(\sigma_2; \sigma_3 - \sigma_2)} \int_0^1 t^{\sigma_2 - 1}(1-t)^{\sigma_3 - \sigma_2 - 1} \exp(tz) E_{\alpha}\Big(- \frac{p}{t(1-t)}\Big) dt, \quad (1.26)$$

where $\Re(p) > 0$, $\Re(\alpha) > 0$, $\Re(\sigma_3) > \Re(\sigma_2) > 0$.

Obviously when $\alpha = 1$, then (1.23)-(1.26) will reduce to the extended hypergeometric function (1.9)-(1.12) and similarly when $\alpha = 1$ and $p = 0$ then the hypergeometric function (1.23) will reduce to the hypergeometric function (1.3)-(1.6).
The extended beta distribution function is defined in [7] as:

$$f(t) = \begin{cases} \frac{1}{B_p(\delta_1, \delta_2)} t^{\delta_1 - 1}(1-t)^{\delta_2 - 1} E_{\lambda}\Big(- \frac{p}{t(1-t)}\Big), 0 < t < 1 \\ 0, otherwise \end{cases} \quad (1.27)$$

For various extensions and generalizations of beta function and hypergeometric functions the interested readers may refer to the recent work of researchers (see e.g., [1–3, 9, 11–14]).

2. Extension of beta function and its properties

In this section, we define a new extension of beta function and its properties such as Mellin transforms and integral representations.

Definition 2.1. *The extension of extended beta function is defined as:*

$$B_p^{\lambda}(\delta_1, \delta_2) = \int_0^1 z^{\delta_1 - 1}(1-z)^{\delta_2 - 1} E_{\lambda}\left(-\frac{p}{z}\right) E_{\lambda}\left(-\frac{q}{(1-z)}\right) dz, \quad (2.1)$$

where $\Re(\delta_1) > 0\,, \Re(\delta_2) > 0,\ p, q \geq 0,\ \Re(\lambda) > 0$ *and* $E_{\lambda}(.)$ *is Mittag-Leffler function.*

Remark 2.1. *Note that:*
(i) if $\lambda = 1$*, then (2.1) reduces to the well known extended beta function (1.13).*
(ii) if $\lambda = 1$ *and* $p = q$*, then (2.1) reduces reduces to the classical beta function (1.7).*
(iii) if $\lambda = 1$ *and* $p = q = 0$*, then (2.1) reduces reduces to the classical beta function (1.1).*

Theorem 2.1. *The extension of extended beta function have the following Mellin transform relation:*

$$M\{B_{p,q}^{\lambda}(\delta_1, \delta_2); p \to r, q \to s\} = B(\delta_1 + r, \delta_2 + s)\Gamma^{\lambda}(r)\Gamma^{\lambda}(s), \quad (2.2)$$

where $\Re(s) > 0, \Re(\delta_1 + s) > 0$ *and* $\Re(\delta_2 + s) > 0$.

Proof. Applying the Mellin transform on (2.1), we have

$$M\{B_{p,q}^{\lambda}(\delta_1, \delta_2); p \to r, q \to s\} = \int_0^{\infty}\int_0^{\infty} p^{r-1} q^{s-1}$$
$$\times \left(\int_0^1 z^{\delta_1 - 1}(1-z)^{\delta_2 - 1} E_{\lambda}\left(-\frac{p}{z}\right) E_{\lambda}\left(-\frac{q}{(1-z)}\right) dz\right) dpdq$$

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Interchanging the order integrations, we have

$$M\{B_{p,q}^{\lambda}(\delta_1,\delta_2);p\to s\}=\int_0^1 z^{\delta_1-1}(1-z)^{\delta_2-1} \times\Big\{\int_0^\infty p^{r-1}E_\lambda\left(-\frac{p}{z}\right)dp\Big\}\Big\{\int_0^\infty p^{s-1}E_\lambda\left(-\frac{q}{(1-z)}\right)dq\Big\}dz \tag{2.3}$$

Substituting $u=\frac{p}{z}$ and $v=\frac{q}{(1-z)}$ in (2.3), we have

$$M\{B_{p,q}^{\lambda}(\delta_1,\delta_2);p\to s\}=\int_0^1 z^{\delta_1+r-1}(1-z)^{\delta_2+s-1} \times\Big\{\int_0^\infty u^{r-1}E_\lambda(-w)\,du\Big\}\Big\{\int_0^\infty v^{s-1}E_\lambda(-v)\,dv\Big\}dz. \tag{2.4}$$

By applying the definition of $\Gamma^\lambda(.)$ to (2.4) (see [15]), we get the following desired result.

$$M\{B_{p,q}^{\lambda}(\delta_1,\delta_2);p\to s\}=B(\delta_1+r,\delta_2+s)\Gamma^\lambda(r)\Gamma^\lambda(s).$$

□

Corollary 2.1. *The following integral representation holds true*

$$\int_0^\infty B_p^\lambda(\delta_1,\delta_2)dp=B(\delta_1+1,\delta_2+1).$$

Proof. By taking $r=s=1$ and $\lambda=1$ in Theorem 2.1, we get the required result. □

Theorem 2.2. *The following integral representations holds true*

$$B_{p,q}^{\lambda}(\delta_1,\delta_2)=2\int_0^{\frac{\pi}{2}}\cos^{2\delta_1-1}\theta\sin^{2\delta_2-1}\theta E_\lambda\Big(-\frac{p}{\cos^2\theta}\Big)E_\lambda\Big(-\frac{q}{\sin^2\theta}\Big)d\theta, \tag{2.5}$$

$$B_{p,q}^{\lambda}(\delta_1,\delta_2)=\int_0^\infty\frac{u^{\delta_1-1}}{(1+u)^{\delta_1+\delta_2}}E_\lambda\Big(-\frac{p(1+u)}{u}\Big)E_\lambda\Big(-q(1+u)\Big)du, \tag{2.6}$$

$$B_p^\lambda(\delta_1,\delta_2)=2^{1-\delta_1-\delta_2}\int_{-1}^1(1+u)^{\delta_1-1}(1-u)^{\delta_2-1}E_\lambda\Big(-\frac{2p}{1+u}\Big)E_\lambda\Big(-\frac{2q}{1-u}\Big)du \tag{2.7}$$

and

$$B_p^\lambda(x,y)=(c-a)^{1-\delta_1-\delta_2}\int_a^c(u-a)^{\delta_1-1}(c-u)^{\delta_2-1}E_\lambda\Big(-\frac{p(c-a)}{(u-a)}\Big)E_\lambda\Big(-\frac{q(c-a)}{(c-u)}\Big)du \tag{2.8}$$

$$\Big(p,q\geq 0\,,\lambda>0,\,\Re(p)>0,\Re(q)>0,\Re(\lambda)>0,\Re(\delta_1)>0,\Re(\delta_2)>0\Big).$$

Proof. Equations (2.5)-(2.8) can be easily obtained by taking the transformation $t=\cos^2\theta$, $t=\frac{u}{1+u}$, $t=\frac{1+u}{2}$ and $t=\frac{u-a}{c-a}$ in (2.1), respectively. □

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3. PROPERTIES OF EXTENDED BETA FUNCTION

In this section, we investigate various properties of the extended beta function $B_p^{\lambda}(\delta_1, \delta_2)$.

Theorem 3.1. *The extension of beta function satisfies the following integral representation*

$$B_{p,q}^{\lambda}(\delta_1+1, \delta_2) + B_p^{\lambda}(\delta_1, \delta_2+1) = B_{p,q}^{\lambda}(\delta_1, \delta_2). \tag{3.1}$$

Proof. Consider the left hand side of (3.1), we have

$$\begin{aligned} B_{p,q}^{\lambda}(\delta_1+1, \delta_2) \quad &+ \quad B_{p,q}^{\lambda}(\delta_1, \delta_2+1) \\ &= \int_0^1 \Big[t_1^{\delta}(1-t)^{\delta_2-1} + t^{\delta_1-1}(1-t)_2^{\delta}\Big] E_{\lambda}\Big(-\frac{p}{t}\Big) E_{\lambda}\Big(-\frac{q}{(1-t)}\Big) dt, \end{aligned}$$

which proves the desired result. □

Corollary 3.1. *The following result holds true*

$$B_{p,q}(\delta_1+1, \delta_2) + B_{p,q}(\delta_1, \delta_2+1) = B_{p,q}(\delta_1, \delta_2). \tag{3.2}$$

Proof. Setting $\lambda = 1$ in Theorem 3.1, we get the desired result. □

Corollary 3.2. *The following integral representation holds true*

$$B_p(\delta_1+1, \delta_2) + B_p(\delta_1, \delta_2+1) = B_p(\delta_1, \delta_2). \tag{3.3}$$

Proof. Setting $\lambda = 1$ and $p = q$ in Theorem 3.1, we get the required result. □

Corollary 3.3. *The following integral representation holds true*

$$B(\delta_1+1, \delta_2) + B(\delta_1, \delta_2+1) = B(\delta_1, \delta_2). \tag{3.4}$$

Proof. Setting $\lambda = 1$ and $p = q = 0$ in Theorem 3.1, we get the required result. □

Theorem 3.2. *The extension of beta function satisfies the following summation formulas*

$$B_{p,q}^{\lambda}(\delta_1, 1-\delta_2) = \sum_{n=0}^{\infty} \frac{(\delta_2)_n}{n!} B_{p,q}^{\lambda}(\delta_1+n, 1), (\Re(p) > 0, \lambda > 0). \tag{3.5}$$

Proof. Consider the generalized binomial theorem

$$(1-t)^{-\delta_2} = \sum_{n=0}^{\infty} (\delta_2)_n \frac{t^n}{n!} \ (|t| < 1). \tag{3.6}$$

Applying (3.6) to the definition (2.1) of extended beta function

$$B_p^{\lambda}(\delta_1, 1-\delta_2) = \int_0^1 \sum_{n=0}^{\infty} \frac{(\delta_2)_n t^{\delta_1+n-1}}{n!} E_{\lambda}\Big(-\frac{p}{t}\Big) E_{\lambda}\Big(-\frac{q}{(1-t)}\Big) dt$$

Now, interchanging the order of summation and integration in above equation and using (2.1) proves the desired result. □

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Theorem 3.3. *The extension of beta function satisfies the following infinite summation formulas*

$$B^{\lambda}_{p,q}(\delta_1,\delta_2)=\sum_{n=0}^{\infty}B^{\lambda}_{p,q}(\delta_1+n,\delta_2+1),(\Re(p),\Re(p)>0,\lambda>0). \tag{3.7}$$

Proof. Replacing the following series representation in (2.1)

$$(1-t)^{\delta_2-1}=(1-t)^{\delta_2}\sum_{n=0}^{\infty}t^n,$$

we obtain

$$B^{\lambda}_{p,q}(\delta_1,\delta_2)=\int_0^1(1-t)^{\delta_2}\sum_{n=0}^{\infty}t^{\delta_1+n-1}E_{\lambda}\Big(-\frac{p}{t}\Big)E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dt.$$

By interchanging the order of integration and summation in above equation and using (2.1), we get the desired result. □

Theorem 3.4. *The following relation holds true*

$$B^{\lambda}_{p,q}(a,-a-n)=\sum_{k=0}^{n}\binom{n}{k}B^{\lambda}_{p,q}(a+k,-a-k),(n\in\mathbb{N}_0). \tag{3.8}$$

Proof. The fundamental relation gives

$$B^{\lambda}_{p,q}(u+1,v)+B^{\lambda}_{p,q}(u,v+1)=B^{\lambda}_{p,q}(x,y).$$

Taking $u=a$ and $v=-a-n$ in above relation, we have

$$B^{\lambda}_{p,q}(a,-a-n)=B^{\lambda}_{p,q}(a,-a-n+1)+B^{\lambda}_{p,q}(a+1,-a-n).$$

Starting with $n=1,2,3\cdots$, we have

$$\begin{aligned}
B^{\lambda}_{p,q}(a,-a-1) &= B^{\lambda}_{p,q}(a,-a)+B^{\lambda}_{p,q}(a+1,-a-1),\\
B^{\lambda}_{p,q}(a,-a-2) &= B^{\lambda}_{p,q}(a,-a)+2B^{\lambda}_{p,q}(a+1,-a-1)+B^{\lambda}_{p,q}(a+2,-a-2),\\
B^{\lambda}_{p,q}(a,-a-3) &= B^{\lambda}_{p,q}(a,-a)+3B^{\lambda}_{p,q}(a+1,-a-1)+3B^{\lambda}_{p,q}(a+2,-a-2)\\
&+ B^{\lambda}_{p,q}(a+3,-a-3),
\end{aligned}$$

and so on. The above series behaves like as finite binomials series does. Thus, we can finally obtain the desired relation (3.8). □

Note that, we can also prove the desired inequality by applying induction on n.

Theorem 3.5. *For extension of beta function, we have the following Mellin transformation formula:*

$$B^{\lambda}_{p,q}(\delta_1,\delta_2)=\frac{1}{(2\pi\iota)^2}\int_{\gamma_1-\iota\infty}^{\gamma_1+\iota\infty}\int_{\gamma_2-\iota\infty}^{\gamma_2+\iota\infty}\frac{\Gamma^{\lambda}(r)\Gamma^{\lambda}(s)\Gamma(\delta_1+r)\Gamma(\delta_2+s)}{\Gamma(\delta_1+\delta_2+r+s)}p^{-r}q^{-s}drds \tag{3.9}$$

$$\Big(\Re(p)>0,\Re(\lambda)>0,\gamma_1,\gamma_2>0\Big).$$

Proof. Applying the inverse Mellin transform on both sides of (2.2), we get the desired result. □

4. The Extended beta distribution

Like extended beta function $B_p(x,y)$ and $B_{p,q}(x,y)$ there will be many application of further extension of beta function $B^{\lambda}_{p,q}(x,y)$. One application of this newly extension of beta function $B^{\lambda}_{p,q}(x,y)$ is to define beta distribution to a variables δ_1 and δ_2 with an infinite range, which is an extension of extended beta distribution defined by Chaudhary et al. and Choi et al. (see [4, 6]).

We define the extension of beta distribution by

$$f(t) = \begin{cases} \frac{1}{B^{\lambda}_{p,q}(\delta_1,\delta_2)} t^{\delta_1-1}(1-t)^{\delta_2-1} E_{\lambda}\Big(-\frac{p}{t}\Big) E_{\lambda}\Big(-\frac{q}{(1-t)}\Big), 0 < t < 1 \\ 0, otherwise \end{cases} \tag{4.1}$$

It is clear that the beta distribution (4.1) is the extension of beta distribution defined by Chaudhary et al. and Choi et al. (see [4, 6]).

If v is any real number, then the mean of extended beta distribution (4.1) is defined as;

$$E(X^v) = \frac{B^{\lambda}_{p,q}(\delta_1+v,\delta_2)}{B^{\lambda}_{p,q}(\delta_1,\delta_2)}$$

$$\Big(p,q>0, \lambda>0, -\infty<\delta_1,\delta_2<\infty\Big).$$

When $v=1$, then we get the mean of the distribution

$$\mu = E(X) = \frac{B^{\lambda}_{p,q}(\delta_1+1,\delta_2)}{B^{\lambda}_{p,q}(\delta_1,\delta_2)}.$$

The various of the distribution can defined by

$$\begin{aligned} \sigma^2 &= E(X^2) - \{E(X)\}^2 \\ &= \frac{B^{\lambda}_{p,q}(\delta_1,\delta_2)B^{\lambda}_{p,q}(\delta_1+2,\delta_2) - \{B^{\lambda}_{p,q}(\delta_1+1,\delta_2)\}^2}{\{} B^{\lambda}_{p,q}(\delta_1,\delta_2)\}^2. \end{aligned}$$

The moment generating function of the distribution is defined by

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(X^n) = \frac{1}{B^{\lambda}_{p,q}(\delta_1,\delta_2)} \sum_{n=0}^{\infty} B^{\lambda}_{p,q}(\delta_1+n,\delta_2)\frac{t^n}{n!}. \tag{4.2}$$

The cumulative distribution of (4.1) can be defined as

$$F(x) = \frac{B^{\lambda}_{x,p,q}(\delta_1,\delta_2)}{B^{\lambda}_{p,q}(\delta_1,\delta_2)} \tag{4.3}$$

where

$$B^{\lambda}_{x,p,q}(\delta_1,\delta_2) = \int_0^x t^{\delta_1-1}(1-t)^{\delta_2-1} E_{\lambda}\Big(-\frac{p}{t}\Big) E_{\lambda}\Big(-\frac{q}{(1-t)}\Big) dt, \tag{4.4}$$

$$\Big(p,q>0, \lambda>0, -\infty<\delta_1,\delta_2<\infty\Big)$$

is an extension of incomplete beta function.

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5. Extension of hypergeometric functions

In this section, we introduce further extension of hypergeometric and confluent hypergeometric functions by using the extension of beta function (2.1).

Definition 5.1. *The extension of extended hypergeometric function is defined as;*

$$F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\sum_{n=0}^{\infty}(\delta_1)_n\frac{B^{\lambda}_{p,q}(\delta_2+n;\delta_3-\delta_2)}{B(\delta_2;\delta_3-\delta_2)}\frac{z^n}{n!},\tag{5.1}$$

where $p,q\geq 0\,,\lambda>0\,,\Re(\delta_3)>\Re(\delta_2)>0,\ |z|<1$ *and* $E_\lambda(.)$ *is the Mittag-Leffler function.*

Definition 5.2. *The extension of extended confluent hypergeometric function is defined as;*

$$\Phi^{\lambda}_{p,q}\Big(\delta_2;\delta_3;z\Big)=\sum_{n=0}^{\infty}\frac{B^{\lambda}_{p,q}(\delta_2+n;\delta_3-\delta_2)}{B(\delta_2;\delta_3-\delta_2)}\frac{z^n}{n!},\tag{5.2}$$

where $p,q\geq 0\,,\lambda>0\,,\Re(\delta_3)>\Re(\delta_2)>0$.

Remark 5.1. *It is clear that*
(i) if we letting $\lambda=1$*, then (5.1) and (5.2) reduce to the* (p,q)*-extended hypergeometric and confluent hypergeometric functions (1.15) and (1.16) respectively.*
(ii) if we letting $\lambda=1$ *and* $p=q$*, then (5.1) and (5.2) reduce to the extended hypergeometric and confluent hypergeometric functions (1.9) and (1.10) respectively.*
(iii) if we letting $\lambda=1$ *and* $p=q=0$*, then (5.1) and (5.2) reduce to the classical hypergeometric and confluent hypergeometric functions (1.3) and (1.4) respectively.*

6. Integral representations of extended hypergeometric functions

In this section, we define the integral representations of extended hypergeometric and confluent hypergeometric functions (5.1) and (5.2).

Theorem 6.1. *The extended hypergeometric has the following integral representation;*

$$F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\times\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-tz)^{-\delta_1}E_\lambda\Big(-\frac{p}{t}\Big)E_\lambda\Big(-\frac{q}{(1-t)}\Big)dt\tag{6.1}$$

where $p,q,\lambda>0,\ p,q=0,\ \Re(\delta_3)>\Re(\delta_2)>0$ *and* $|\arg(1-z)|<\pi$.

Proof. By using (2.1) in (5.1), we have

$$F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\times\int_0^1 t^{\delta_2-1}(1-t)\delta_3-\delta_2-1E_\lambda\Big(-\frac{p}{t}\Big)E_\lambda\Big(-\frac{q}{(1-t)}\Big)\sum_{n=0}^{\infty}(\delta_1)_n\frac{(zt)^n}{n!}dt.\tag{6.2}$$

Thus by using $\sum_{n=0}^{\infty}(\delta_1)_n\frac{(zt)^n}{n!}=(1-tz)^{-\delta_1}$ in (6.2), we get the desired result. □

Theorem 6.2. *The following integral representations holds true:*

$$F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\times\int_0^1 u^{\delta_2-1}(1+u)^{\delta_1-\delta_3}(1-u(1+z))^{-\delta_1}E_{\lambda}\Big(-\frac{p(1+u)}{u}\Big)E_{\lambda}\Big(-q(1+u)\Big)dt, \tag{6.3}$$

$$F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\frac{2}{B(\delta_2,\delta_3-\delta_2)}\times\int_0^{\frac{\pi}{2}}\frac{(\sin\theta)^{2\delta_2-1}(\cos\theta)^{2\delta_3-2\delta_2-1}}{(1-z\sin^2\theta)^{\delta_1}}E_{\lambda}\Big(-p\csc^2\theta\Big)E_{\lambda}\Big(-q\sec^2\theta\Big)d\theta \tag{6.4}$$

and

$$F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\frac{2}{B(\delta_2,\delta_3-\delta_2)}\times\int_0^{\infty}\frac{(\sinh\theta)^{2\delta_2-1}(\cosh\theta)^{2\delta_1-2\delta_3-1}}{(cosh^2\theta-z\sinh^2\theta)^{\delta_1}}E_{\lambda}\Big(-p\coth^2\theta\Big)E_{\lambda}\Big(-q\cosh^2\theta\Big)d\theta. \tag{6.5}$$

Proof. By substituting $t=\frac{u}{1+u}$, $t=\sin^2\theta$ and $t=\tanh^2\theta$ in (5.1) respectively, we get the desired representations (6.3)-(6.5) □

Next, we prove integral representations of extended confluent hypergeometric function.

Theorem 6.3.

$$\Phi_{p,q}^{\lambda}\Big(\delta_2;\delta_3;z\Big)=\frac{1}{B(\delta_2;\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}\exp(zt)E_{\lambda}\Big(-\frac{p}{t}\Big)E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dt \tag{6.6}$$

and

$$\Phi_{p,q}^{\lambda}\Big(\delta_2;\delta_3;z\Big)=\frac{\exp(z)}{B(\delta_2;\delta_3-\delta_2)}\times\int_0^1(1-t)^{\delta_2-1}t^{\delta_3-\delta_2-1}\exp(-zt)E_{\lambda}\Big(-\frac{p}{t}\Big)E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dt \tag{6.7}$$

Proof. By using definition of extended beta function (2.1) in (5.2), we have

$$\Phi_{p,q}^{\lambda}\Big(\delta_2;\delta_3;z\Big)=\frac{1}{B(\delta_2;\delta_3-\delta_2)}\times\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}E_{\lambda}\Big(-\frac{p}{t}\Big)E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)\Big(\sum_{n=0}^{\infty}\frac{(zt)^n}{n!}\Big)dt. \tag{6.8}$$

Using

$$\sum_{n=0}^{\infty}\frac{(zt)^n}{n!}=\exp(zt),$$

in (6.8), we get the proof of (6.6). To prove (6.7), substituting $t=1-t$ in (6.6). □

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7. Differentiation formulas for the extended hypergeometric functions

In this section, we derive differentiations formulas for the extended hypergeometric and confluent hypergeometric functions.

Theorem 7.1. *The following formula hold true:*

$$\frac{d^n}{dz^n}\left\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)\right\}=\frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}F^{\lambda}_{p,q}\Big(\delta_1+n,\delta_2+n;\delta_3+n;z\Big) \tag{7.1}$$

Proof. Differentiating (5.1) with respect to z, we have

$$\begin{aligned}\frac{d}{dz}\left\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)\right\}&=\frac{d}{dz}\sum_{n=0}^{\infty}(\delta_1)_n\frac{B^{\lambda}_{p,q}(\delta_2+n;\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}\\&=\sum_{n=1}^{\infty}(\delta_1)_n\frac{B^{\lambda}_{p,q}(\delta_2+n;\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^{n-1}}{(n-1)!}.\end{aligned} \tag{7.2}$$

Changing n to $n+1$ in (7.2), we have

$$\frac{d}{dz}\left\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)\right\}=\sum_{n=0}^{\infty}(\delta_1)_{n+1}\frac{B^{\lambda}_{p,q}(\delta_2+n+1;\delta_3-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\frac{z^n}{n!}. \tag{7.3}$$

Since

$$B(b,c-b)=\frac{c}{b}B(b+1,c-b) \tag{7.4}$$

Applying (7.4) to (7.3), we get

$$\begin{aligned}\frac{d}{dz}\left\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)\right\}&=\frac{\delta_1\delta_2}{\delta_3}\sum_{n=0}^{\infty}(\delta_1+1)_n\frac{B^{\lambda}_{p,q}(\delta_2+n+1;\delta_3-\delta_2)}{B(\delta_2+1,\delta_3-\delta_2)}\frac{z^n}{n!}\\&=\frac{\delta_1\delta_2}{\delta_3}F^{\lambda}_{p,q}\Big(\delta_1+1,\delta_2+1;\delta_3+1;z\Big).\end{aligned} \tag{7.5}$$

Again differentiating (7.5) with respect to z, we obtain

$$\frac{d^2}{dz^2}\left\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big)\right\}=\frac{\delta_1(\delta_1+1)\delta_2(\delta_2+1)}{\delta_3(\delta_3+1)}F^{\lambda}_{p,q}\Big(\delta_1+2,\delta_2+2;\delta_3+2;z\Big). \tag{7.6}$$

Continuing up to n times, we get the required result. □

Theorem 7.2. *The following formula hold true:*

$$\frac{d^n}{dz^n}\left\{\Phi^{\lambda}_{p,q}\Big(\delta_2;\delta_3;z\Big)\right\}=\frac{(\delta_2)_n}{(\delta_3)_n}\Phi^{\lambda}_{p,q}\Big(\delta_2+n;\delta_3+n;z\Big). \tag{7.7}$$

Proof. Applying the similar procedure used in Theorem 7.1, we get the desired result. □

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8. Mellin transformation of extended hypergeometric functions

In this section, we derive the Mellin transformation of extended hypergeometric and confluent hypergeometric functions (5.1) and (5.2).

Theorem 8.1. *The extended hypergeometric function has the following Mellin transform;*

$$M\Big\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big);p\to s\Big\} = \frac{\Gamma^{\lambda}(r)\Gamma^{\lambda}(s)B(\delta_2+r,\delta_3+s-\delta_2)}{B(\delta_2,\delta_3-\delta_2)} \times F\Big(\delta_1,\delta_2+r;\delta_3+r+s;z\Big), \tag{8.1}$$

where $\Re(\delta_2+r)>0$*, and* $\Re(\delta_3+s)>0$.

Proof. Applying Mellin transform on both sides of (6.1), we have

$$M\Big\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big);p\to r,q\to s\Big\} = \frac{1}{B(\delta_2,\delta_3-\delta_2)} \times\int_0^{\infty}\int_0^{\infty}p^{r-1}q^{s-1}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-tz)^{-\delta_1}E_{\lambda}\Big(-\frac{p}{t}\Big)E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dtdpdq$$

Interchanging the order of integrations in above equation, we have

$$M\Big\{F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big);p\to r,q\to s\Big\} = \frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_2-1}(1-tz)^{-\delta_1} \times\Big\{\int_0^{\infty}p^{r-1}E_{\lambda}\Big(-\frac{p}{t}\Big)dp\Big\}\Big\{\int_0^{\infty}q^{s-1}E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dq\Big\}dt \tag{8.2}$$

Using the following definition of $\Gamma^{\lambda}(.)$ (see [15]) in above equation, we get the desired result.

$$\Big\{\int_0^{\infty}p^{r-1}E_{\lambda}\Big(-\frac{p}{t}\Big)dp\Big\}\Big\{\int_0^{\infty}q^{-s}E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dq\Big\} = t^r(1-t)^s\Gamma^{\lambda}(r)\Gamma^{\lambda}(s),$$

by substituting $u=\frac{p}{t(1-t)}$ and $v=\frac{q}{(1-t)}$. □

Theorem 8.2. *The following result holds true;*

$$F^{\lambda}_{p,q}\Big(\delta_1,\delta_2;\delta_3;z\Big) = \frac{1}{(2\pi\iota)^2B(\delta_2,\delta_3-\delta_2)}\int_{\gamma_1-\iota\infty}^{\gamma_1+\iota\infty}\int_{\gamma_2-\iota\infty}^{\gamma_2+\iota\infty}\frac{\Gamma^{\lambda}(r)\Gamma^{\lambda}(s)\Gamma(\delta_2+r)\Gamma(\delta_3+s-\delta_2)}{\Gamma(\delta_3+r+s)} \times F\Big(\delta_1,\delta_2+r;\delta_3+r+s;z)p^{-r}q^{-s}drds, (\gamma_1,\gamma_2>0). \tag{8.3}$$

Proof. Taking the inverse Mellin transform of both sides on (8.1), we get the required result. □

In similar way, we can prove the following theorems for extended confluent hypergeometric functions.

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Theorem 8.3. *The extended confluent hypergeometric function has the following Mellin transform;*

$$M\left\{\Phi_{p,q}^{\lambda}\Big(\delta_2;\delta_3;z\Big);p\to r,q\to s\right\}=\frac{\Gamma^{\lambda}(r)\Gamma^{\lambda}(s)B(\delta_2+r,\delta_3+r-\delta_2)}{B(\delta_2,\delta_3-\delta_2)}\times\Phi\Big(\delta_2+s;\delta_3+r+s;z\Big),\tag{8.4}$$

where $\Re(\delta_2+s)>0,$ *and* $\Re(\delta_3+s)>0.$

Theorem 8.4. *The following result holds true;*

$$\Phi_{p,q}^{\lambda}\Big(\delta_2;\delta_3;z\Big)=\frac{1}{(2\pi\iota)^2B(\delta_2,\delta_3-\delta_2)}\int_{\gamma_1-\iota\infty}^{\gamma_1+\iota\infty}\int_{\gamma_2-\iota\infty}^{\gamma_2+\iota\infty}\frac{\Gamma^{\lambda}(s)\Gamma^{\lambda}(r)\Gamma(\delta_2+r)\Gamma(\delta_3+s-\delta_2)}{\Gamma(\delta_3+r+s)}\times\Phi\Big(\delta_2+r;\delta_3+r+s;z\Big)p^{-r}q^{-s}drds,(\gamma_1,\gamma_2>0).\tag{8.5}$$

9. Transformation and summation formulas

In this section, we obtain transformation and summation formulas for the extended hypergeometric and confluent hypergeometric functions as follows:

Theorem 9.1. *The following transformation for extended hypergeometric function holds true for* $p,\lambda>0$*:*

$$F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;z\Big)=(1-z)^{-\delta_1}F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;-\frac{z}{1-z}\Big),\tag{9.1}$$

where $|\arg(1-z)|<\pi.$

Proof. Replacing t by $(1-t)$ in (6.1), we get the desired result. □

Theorem 9.2. *The following transformation for extended confluent hypergeometric function holds true for* $p,\lambda>0$*:*

$$\Phi_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\exp(z)\Phi_{p,q}^{\lambda}\Big(\delta_3-\delta_2;\delta_3;-z\Big),\tag{9.2}$$

where $|\arg(1-z)|<\pi.$

Proof. From (6.6) and (6.7), we can easily establish the required result. □

Theorem 9.3. *The following summation formula holds true*

$$F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;z\Big)=\frac{B_{p,q}^{\lambda}(\delta_2,\delta_3-\delta_1-\delta_2)}{B(\delta_2,\delta_3-\delta_2)},\tag{9.3}$$

where $p,q\geq 0,$ $\lambda>0$ *and* $\Re(\delta_3-\delta_1-\delta_2)>0.$

Proof. Taking $z=1$ in (6.1), we have

$$F_{p,q}^{\lambda}\Big(\delta_1,\delta_2;\delta_3;1\Big)=\frac{1}{B(\delta_2,\delta_3-\delta_2)}\int_0^1 t^{\delta_2-1}(1-t)^{\delta_3-\delta_1-\delta_2-1}E_{\lambda}\Big(-\frac{p}{t}\Big)E_{\lambda}\Big(-\frac{q}{(1-t)}\Big)dt$$

By applying definition (2.1) to the above equation, we get the desired result. □

10. Concluding remarks

In this paper, the authors established the extension of extended beta function. They defined several properties, integral representations and extension of beta distribution. Also, they defined further extension of extended hypergeometric and confluent hypergeometric functions with the help of newly defined beta function and established various properties, integral representations and differentiation formulas of extended hypergeometric and confluent hypergeometric functions. The authors conclude that if we letting $\lambda = 1$ throughout in the paper then all the results will be reduced to the work of Chaudhry et al. (see [6]). Also, if we letting $\lambda = 1$ and $p = q$ throughout in the paper then all the results will be reduced to results of extended beta function, extended beta distribution, extended Gauss hypergeometric and extended confluent hypergeometric functions (see [4, 5]). In similar way, if we letting $\lambda = 1$ and $p = q = 0$ throughout in the paper then all the results will be reduced to the classical results of beta function, beta distribution, Gauss hypergeometric and confluent hypergeometric functions (see [8], [16]).

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