Article

Cross entropy measures of bipolar and interval bipolar neutrosophic sets and their application for multi-attribute decision making

Surapati Pramanik [1,*], Partha Pratim Dey [2], Florentin Smarandache [3] and Jun Ye[4]

[1] Department of Mathematics, Nandalal Ghosh B.T. College, Panpur, P.O.-Narayanpur, District –North 24 Parganas, Pin code-743126,West Bengal, India; sura_pati@yahoo.co.in

[2] Patipukur Pallisree Vidyapith, Patipukur, Kolkata-700048, West Bengal, India; parsur.fuzz@gmail.com

[3] University of New Mexico, Mathematics & Science Department, 705Gurley Ave., Gallup, NM 87301, U.S.A.; smarand@unm.edu

[4] Department of Electrical and Information Engineering, Shaoxing University, 508 Huancheng West Road, Shaoxing, Zhejiang Province 312000, China; yehjun@aliyun.com

* Correspondence: sura_pati@yahoo.co.in; Tel.: +91-947-703-5544

Abstract: Bipolar neutrosophic set is an important extension of bipolar fuzzy set. This set is a hybridization of bipolar fuzzy set and neutrosophic set. Every element of a bipolar neutrosophic set consists of three independent positive membership functions and three independent negative membership functions. In this paper, we develop cross entropy measures of bipolar neutrosophic sets and prove its properties. We also define cross entropy measures of interval bipolar neutrosophic sets and prove its properties. Thereafter, we develop two novel multi-attribute decision making methods based on the proposed cross entropy measures. In the decision making framework, we calculate the weighted cross entropy measures between each alternative and the ideal alternative to rank the alternatives and choose the best one. We solve two illustrative examples of multi-attribute decision making problems and compare the obtained result with the results of other existing methods to show the applicability and effectiveness of the developed method. In the end, the main conclusion and future scope of research are summarized.

Keywords: neutrosophic set; bipolar neutrosophic set; interval bipolar neutrosophic set; multi-attribute decision making; cross entropy measure

---

1. Introduction

According to Shannon and Weaver [1] and Shannon [2], entropy measure is an important decision making apparatus for computing uncertain information. Shannon [2] introduced the concept of cross entropy approach in information theory. In neutrosophic environment [3], Ye [4] proposed single valued neutrosophic cross entropy measures between two single valued neutrosophic sets (SVNSs) [5] by extending the concept of cross entropy and symmetric discrimination information measures between two fuzzy sets [6] due to Shang and Jiang [7]. Şahin [8] proposed two techniques to convert the interval neutrosophic information to single valued neutrosophic information and fuzzy information. In the same study, Şahin [8] defined interval neutrosophic cross entropy measure by utilizing two reduction methods. Tian *et al.* [9] developed a transformation operator to convert interval neutrosophic numbers to single valued neutrosophic

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

numbers and defined cross entropy measures for two SVNSs. In the same study, Tian *et al.* [9] developed a multi-criteria decision making (MCDM) approach based on cross entropy and TOPSIS [10] where the weight of the criterion is incomplete. Ye [11] pointed out that entropy measure of SVNSs defined by Ye [4] has some drawbacks in some situations. Therefore, Ye [11] proposed an improved cross entropy measures of SVNSs in order to overcome the drawbacks discussed in the paper [4] and extended the concept to interval neutrosophic sets (INSs) environment [12] and developed MCDM models using cross entropy measures of SVNSs and INSs.

Bipolar neutrosophic sets (BNSs) was developed by Deli *et al.* [13] by hybridizing the concepts of bipolar fuzzy sets [14, 15] and neutrosophic sets [3]. A BNS has two fully independent parts, which are positive membership degree $T^+ \rightarrow [0, 1]$, $I^+ \rightarrow [0, 1]$, $F^+ \rightarrow [0, 1]$, and negative membership degree $T^- \rightarrow [-1, 0]$, $I^- \rightarrow [-1, 0]$, $F^- \rightarrow [-1, 0]$ where the positive membership degrees T^+, I^+, F^+ represent truth membership degree, indeterminacy membership degree and false membership degree respectively of an element and the negative membership degrees T^-, I^-, F^- represent truth membership degree, indeterminacy membership degree and false membership degree respectively of an element to some implicit counter property corresponding to a BNS. Deli *et al.* [13] defined some operations namely score, accuracy, and certainty functions to compare BNSs and provided some operators in order to aggregate BNSs. Deli and Subas [16] defined correlation coefficient similarity measure for dealing with MCDM problems under single valued neutrosophic setting. Şahin *et al.* [17] proposed Jaccard vector similarity measure for MCDM problems with single valued neutrosophic information. Uluçay *et al.* [18] introduced Dice similarity measure, weighted Dice similarity measure, hybrid vector similarity measure, weighted hybrid vector similarity measure for BNSs and established a MCDM method by using the proposed similarity measures. Dey *et al.* [19] investigated TOPSIS method for solving multi-attribute decision making (MADM) problems with bipolar neutrosophic information where the weights of the attributes are completely unknown to the decision maker. Pramanik *et al.* [20] defined projection, bidirectional projection and hybrid projection measures for BNSs and proved their basic properties. In the same study, Pramanik *et al.* [20], developed three new MADM methods based on the proposed projection, bidirectional projection and hybrid projection measures with bipoar neutrosophic information. Wang *et al.* [21] defined Frank operations of bipolar neutrosophic numbers (BNNs) and proposed Frank bipolar neutrosophic Choquet Bonferroni mean operators by combining Choquet integral operators and Bonferroni mean operators based on Frank operations of BNNs. In the same study, Wang *et al.* [21] established MCDM method based on Frank Choquet Bonferroni operators of BNNs in bipolar neutrosophic environment. Wu [22] defined several cross entropy measures between two multivalued neutrosophic sets and employed the proposed method for selecting the middle-level managers. In 2015, Ezhilmaran and Shankar [23] defined bipolar intuitionistic fuzzy sets, bipolar intuitionistic fuzzy relations, and bipolar intuitionistic fuzzy graphs. The isomorphism of these graphs and several properties of the graphs were also discussed in the same paer [23]. Mahmood *et al.* [24] and Deli *et al.* [25] introduced the hybridized structure called interval bipolar neutrosophic sets (IBNSs) by combining BNSs and INSs and defined some operations and operators for IBNSs.

In this paper, we define a cross entropy and weighted cross entropy measures of BNSs and prove some of their properties and extend the concept to a cross entropy measure of IBNSs. Based on the proposed cross entropy measures, we develop two new MADM methods to rank the alternatives

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

and find the best alternative. Furthermore, two illustrative numerical examples are solved and comparison analysis is given.

The rest of the paper is organized as follows. In section 2, we present some concepts regarding SVNSs, INSs, BNSs, IBNSs. Section 3 proposes cross entropy and weighted cross entropy measures of BNSs and investigates their properties. In the next section, we extend cross entropy measures of BNSs to the cross entropy measures of IBNSs and discuss its basic properties. Two novel MADM methods based on the proposed cross entropy measures under bipolar and interval bipolar neutrosophic setting are devoted in section 5. In section 6, two numerical examples are solved and comparison with other existing methods is given. At the end of the article, conclusions and scope of future work are provided.

2. Preliminary

In this section, we provide some basic definitions regarding SVNSs, INSs, BNSs, IBNSs.

2.1 Single valued neutrosophic sets [5]

A SVNS S in U is characterized by a truth membership function $T_S(x)$, an indeterminate membership function $I_S(x)$ and a falsity membership function $F_S(x)$. A SVNS S over U is defined by

$$S = \{x, \langle T_S(x), I_S(x), F_S(x) \rangle \mid x \in U\}$$

where, $T_S(x)$, $I_S(x)$, $F_S(x) : U \to [0, 1]$ and $0 \le T_S(x) + I_S(x) + F_S(x) \le 3$ for each point $x \in U$.

2.2 Interval neutrosophic set [12]

An interval neutrosophic set P in U is expressed as given below

$$P = \{x, \langle T_P(x), I_P(x), F_P(x) \rangle \mid x \in U\}$$

$$= \{x, [\inf T_P(x), \sup T_P(x)]; [\inf I_P(x), \sup I_P(x)]; [\inf F_P(x), \sup F_P(x)] \mid x \in U\}$$

where $T_P(x)$, $I_P(x)$, $F_P(x)$ are the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively. For each point x in U, $T_P(x)$, $I_P(x)$, $F_P(x) \subseteq [0, 1]$ satisfying the condition $0 \le \sup T_P(x) + \sup I_P(x) + \sup F_P(x) \le 3$.

2.3 Bipolar neutrosophic set [13]

A BNS E in U is presented as given below.

$$E = \{x, \langle T_E^+(x), I_E^+(x), F_E^+(x), T_E^-(x), I_E^-(x), F_E^-(x) \rangle \mid x \in U\}$$

where $T_E^+(x)$, $I_E^+(x)$, $F_E^+(x) : U \to [0, 1]$ and $T_E^-(x)$, $I_E^-(x)$, $F_E^-(x) : U \to [-1, 0]$. Here, $T_E^+(x)$, $I_E^+(x)$, $F_E^+(x)$ denote the truth membership, indeterminate membership, and falsity membership functions of an element $x \in U$ corresponding to a BNS E and $T_E^-(x)$, $I_E^-(x)$, $F_E^-(x)$ denote the truth membership, indeterminate membership, and falsity membership of an element $x \in U$ to some implicit counter property corresponding to E.

Definition 1

Let, $E_1 = \{x, \langle T^+_{E_1}(x), I^+_{E_1}(x), F^+_{E_1}(x), T^-_{E_1}(x), I^-_{E_1}(x), F^-_{E_1}(x) \rangle \mid x \in U\}$ and $E_2 = \{x, \langle T^+_{E_2}(x), I^+_{E_2}(x), F^+_{E_2}(x), T^-_{E_2}(x), I^-_{E_2}(x), F^-_{E_2}(x) \rangle \mid x \in X\}$ be any two BNSs. Then

- $E_1 \subseteq E_2$ if and only if

$T^+_{E_1}(x) \le T^+_{E_2}(x)$, $I^+_{E_1}(x) \le I^+_{E_2}(x)$, $F^+_{E_1}(x) \ge F^+_{E_2}(x)$; $T^-_{E_1}(x) \ge T^-_{E_2}(x)$, $I^-_{E_1}(x) \ge I^-_{E_2}(x)$, $F^-_{E_1}(x) \le F^-_{E_2}(x)$ for all $x \in U$.

- $E_1 = E_2$ if and only if

$T^+_{E_1}(x) = T^+_{E_2}(x)$, $I^+_{E_1}(x) = I^+_{E_2}(x)$, $F^+_{E_1}(x) = F^+_{E_2}(x)$; $T^-_{E_1}(x) = T^-_{E_2}(x)$, $I^-_{E_1}(x) = I^-_{E_2}(x)$, $F^-_{E_1}(x) = F^-_{E_2}(x)$ for all $x \in U$.

- The complement of E is $E^c == \{x, \langle T^+_{E^c}(x), I^+_{E^c}(x), F^+_{E^c}(x), T^-_{E^c}(x), I^-_{E^c}(x), F^-_{E^c}(x) \rangle \mid x \in U\}$

where

$T^+_{E^c}(x) = \{1^+\} - T^+_E(x)$, $I^+_{E^c}(x) = \{1^+\} - I^+_E(x)$, $F^+_{E^c}(x) = \{1^+\} - F^+_E(x)$;

$T^-_{E^c}(x) = \{1^-\} - T^-_E(x)$, $I^-_{E^c}(x) = \{1^-\} - I^-_E(x)$, $F^-_{E^c}(x) = \{1^-\} - F^-_E(x)$.

- The union. $E_1 \cup E_2$ is defined as follows:

$E_1 \cup E_2$ = {Max $(T^+_{E_1}(x), T^+_{E_2}(x))$, Min $(I^+_{E_1}(x), I^+_{E_2}(x))$, Min $(F^+_{E_1}(x), F^+_{E_2}(x))$, Min $(T^-_{E_1}(x), T^-_{E_2}(x))$, Max $(I^-_{E_1}(x), I^-_{E_2}(x))$, Max $(F^-_{E_1}(x), F^-_{E_2}(x))$}, $\forall\ x \in U$.

- The intersection $E_1 \cap E_2$ [20] is defined as follows:

$E_1 \cap E_2$ = {Min $(T^+_{E_1}(x), T^+_{E_2}(x))$, Max $(I^+_{E_1}(x), I^+_{E_2}(x))$, Max $(F^+_{E_1}(x), F^+_{E_2}(x))$, Max $(T^-_{E_1}(x), T^-_{E_2}(x))$, Min $(I^-_{E_1}(x), I^-_{E_2}(x))$, Min $(F^-_{E_1}(x), F^-_{E_2}(x))$}, $\forall\ x \in U$.

2.4 Interval bipolar neutrosophic sets [24, 25]

An IBNS R = {x, < [inf $T^+_R(x)$, sup $T^+_R(x)$]; [inf $I^+_R(x)$, sup $I^+_R(x)$]; [inf $F^+_R(x)$, sup $F^+_R(x)$]; [inf $T^-_R(x)$, sup $T^-_R(x)$]; [inf $I^-_R(x)$, sup $I^-_R(x)$]; [inf $F^-_R(x)$, sup $F^-_R(x)$] > | $x \in U$} is characterized by positive and negative truth-membership $T^+_R(x)$, $T^-_R(x)$; indeterminacy-membership $I^+_R(x)$, $I^-_R(x)$; falsity-membership $F^+_R(x)$, $F^-_R(x)$ functions respectively. Here, for any $x \in U$, $T^+_R(x)$, $I^+_R(x)$, $F^+_R(x) \subseteq [0,1]$; $T^-_R(x)$, $I^-_R(x)$, $F^-_R(x) \subseteq [-1, 0]$ with the conditions $0 \le \sup T^+_R(x) + \sup I^+_R(x) + \sup F^+_R(x) \le 3$, and $-3 \le \sup T^-_R(x) + \sup I^-_R(x) + \sup F^-_R(x) \le 0$.

 doi:10.20944/preprints201801.0065.v1

Definition 2: Let $R = \{x, <[\inf T_R^+(x), \sup T_R^+(x)]; [\inf I_R^+(x), \sup I_R^+(x)]; [\inf F_R^+(x), \sup F_R^+(x)]; [\inf T_R^-(x), \sup T_R^-(x)]; [\inf I_R^-(x), \sup I_R^-(x)]; [\inf F_R^-(x), \sup F_R^-(x)] > | x \in U\}$ and $S == \{x, <[\inf T_S^+(x), \sup T_S^+(x)]; [\inf I_S^+(x), \sup I_S^+(x)]; [\inf F_S^+(x), \sup F_S^+(x)]; [\inf T_S^-(x), \sup T_S^-(x)]; [\inf I_S^-(x), \sup I_S^-(x)]; [\inf F_S^-(x), \sup F_S^-(x)] > | x \in U\}$ be two IBNSs in U. Then

- $R \subseteq S$ if and only if

$\inf T_R^+(x) \le \inf T_S^+(x)$, $\sup T_R^+(x) \le \sup T_S^+(x)$,

$\inf I_R^+(x) \ge \inf I_S^+(x)$, $\sup I_R^+(x) \ge \sup I_S^+(x)$,

$\inf F_R^+(x) \ge \inf F_S^+(x)$, $\sup F_R^+(x) \ge \sup F_S^+(x)$,

$\inf T_R^-(x) \ge \inf T_S^-(x)$, $\sup T_R^-(x) \ge \sup T_S^-(x)$,

$\inf I_R^-(x) \le \inf I_S^-(x)$, $\sup I_R^-(x) \le \sup I_S^-(x)$,

$\inf F_R^-(x) \le \inf F_S^-(x)$, $\sup F_R^-(x) \le \sup F_S^-(x)$,

for all $x \in U$.

- $R = S$ if and only if

$\inf T_R^+(x) = \inf T_S^+(x)$, $\sup T_R^+(x) = \sup T_S^+(x)$, $\inf I_R^+(x) = \inf I_S^+(x)$, $\sup I_R^+(x) = \sup I_S^+(x)$,

$\inf F_R^+(x) = \inf F_S^+(x)$, $\sup F_R^+(x) = \sup F_S^+(x)$, $\inf T_R^-(x) = \inf T_S^-(x)$, $\sup T_R^-(x) = \sup T_S^-(x)$,

$\inf I_R^-(x) = \inf I_S^-(x)$, $\sup I_R^-(x) = \sup I_S^-(x)$, $\inf F_R^-(x) = \inf F_S^-(x)$, $\sup F_R^-(x) = \sup F_S^-(x)$,

for all $x \in U$.

- The complement of R is defined as $R^C = \{x, <[\inf T_{R^c}^+(x), \sup T_{R^c}^+(x)]; [\inf I_{R^c}^+(x), \sup I_{R^c}^+(x)]; [\inf F_{R^c}^+(x), \sup F_{R^c}^+(x)]; [\inf T_{R^c}^-(x), \sup T_{R^c}^-(x)]; [\inf I_{R^c}^-(x), \sup I_{R^c}^-(x)]; [\inf F_{R^c}^-(x), \sup F_{R^c}^-(x)] > | x \in U\}$ where

$\inf T_{R^c}^+(x) = 1 - \inf T_R^+(x)$, $\sup T_{R^c}^+(x) = 1 - \sup T_R^+(x)$

$\inf I_{R^c}^+(x) = 1 - \inf I_R^+(x)$, $\sup I_{R^c}^+(x) = 1 - \sup I_R^+(x)$

$\inf F_{R^c}^+(x) = 1 - \inf F_R^+(x)$, $\sup F_{R^c}^+(x) = 1 - \sup F_R^+(x)$

$\inf T_{R^c}^-(x) = -1 - \inf T_R^-(x)$, $\sup T_{R^c}^-(x) = -1 - \sup T_R^-(x)$

$$\inf I^-_{R^c}(x) = -1 - \inf I^-_R(x),\ \sup I^-_{R^c}(x) = -1 - \sup I^-_R(x)$$

$$\inf F^-_{R^c}(x) = -1 - \inf F^-_R(x),\ \sup F^-_{R^c}(x) = -1 - \sup F^-_R(x)$$

for all $x \in U$. A conceptual representation of evolution of BNS and IBNS are shown in Figure 1 as given below.

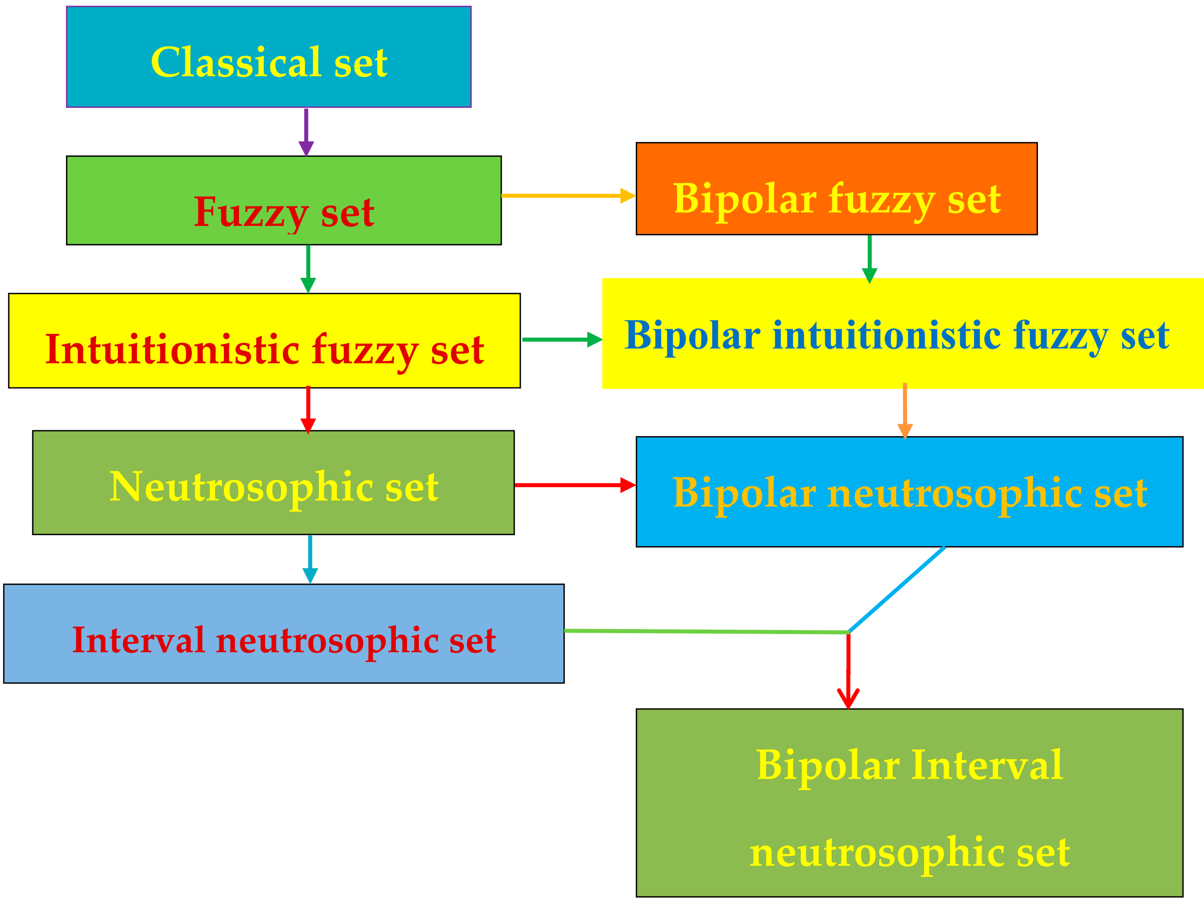

Figure1. Representation of evolution of BNS and IBNS

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

3. Cross entropy measures of bipolar neutrosophic set

In this section we define cross entropy measure between two BNSs and establish some of its basic properties.

Definition 3: For any two BNSs *M* and *N* in *U*, the cross entropy measure can be defined as follows:

$C_B(M, N) =$

$$\sum_{i=1}^{n}\left[\begin{array}{l}\sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}}-\left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right)+\sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}}-\left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right)+\\ \sqrt{\frac{[1-I_M^+(x_i)]+[1-I_N^+(x_i)]}{2}}-\left(\frac{\sqrt{1-I_M^+(x_i)}+\sqrt{[1-I_N^+(x_i)]}}{2}\right)+\sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}}-\left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right)+\\ \sqrt{\frac{-(T_M^-(x_i)+T_N^+(x_i))}{2}}-\left(\frac{\sqrt{(-T_M^-(x_i))}+\sqrt{(-T_N^-(x_i))}}{2}\right)+\sqrt{\frac{-(I_M^-(x_i)+I_N^-(x_i))}{2}}-\left(\frac{\sqrt{(-I_M^-(x_i))}+\sqrt{(-I_N^-(x_i))}}{2}\right)+\\ \sqrt{\frac{[1+I_M^-(x_i)]+[1+I_N^-(x_i)]}{2}}-\left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{[1+I_N^-(x_i)]}}{2}\right)+\sqrt{\frac{-(F_M^-(x_i)+F_N^-(x_i))}{2}}-\left(\frac{\sqrt{(-F_M^-(x_i))}+\sqrt{(-F_N^-(x_i))}}{2}\right)\end{array}\right] \quad (1)$$

Theorem 1. If $M = \langle T_M^+(x_i), I_M^+(x_i), F_M^+(x_i), T_M^-(x_i), I_M^-(x_i), F_M^-(x_i)\rangle$ and N $\langle T_N^+(x_i), I_N^+(x_i), F_N^+(x_i), T_N^-(x_i), I_N^-(x_i), F_N^-(x_i)\rangle$ be two BNSs in *U*, then the cross entropy measure C_B (M, N) satisfies the following properties:

(1) $C_B(M, N) \geq 0$

(2) $C_B(M, N) = 0$ if and only if $T_M(x_i) = T_N(x_i), I_M(x_i) = I_N(x_i), F_M(x_i) = F_N(x_i), \forall\ x \in U$.

(3) $C_B(M, N) = C_B(M, N)$.

Proof

(1) We have the inequality $\left(\frac{a+b}{2}\right)^{1/2} \geq \frac{a^{1/2}+b^{1/2}}{2}$ for all positive numbers *a* and *b*. From the inequality we can easily obtain $C_B(M, N) \geq 0$.

(2) The inequality $\left(\frac{a+b}{2}\right)^{1/2} \geq \frac{a^{1/2}+b^{1/2}}{2}$ becomes the equality $\left(\frac{a+b}{2}\right)^{1/2} = \frac{a^{1/2}+b^{1/2}}{2}$ if and only if $a = b$ and therefore C_B $(M, N) = 0$ if and only if $M = N$, i.e., $T_M^+(x_i) = T_N^+(x_i)$, $I_M^+(x_i) = I_N^+(x_i)$, $F_M^+(x_i) = F_N^+(x_i)$, $T_M^-(x_i) = T_N^-(x_i)$, $I_M^-(x_i) = I_N^-(x_i)$, $F_M^-(x_i) = F_N^-(x_i)$ $\forall\ x \in U$.

(3) $C_B(M, N) =$

$$\sum_{i=1}^{n}\left[\begin{array}{l}\sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}}-\left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right)+\sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}}-\left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right)+\\ \sqrt{\frac{(1-I_M^+(x_i))+(1-I_N^+(x_i))}{2}}-\left(\frac{\sqrt{(1-I_M^+(x_i))}+\sqrt{(1-I_N^+(x_i))}}{2}\right)+\sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}}-\left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right)+\\ \sqrt{\frac{-(T_M^-(x_i)+T_N^+(x_i))}{2}}-\left(\frac{\sqrt{(-T_M^-(x_i))}+\sqrt{(-T_N^-(x_i))}}{2}\right)+\sqrt{\frac{-(I_M^-(x_i)+I_N^-(x_i))}{2}}-\left(\frac{\sqrt{(-I_M^-(x_i))}+\sqrt{(-I_N^-(x_i))}}{2}\right)+\\ \sqrt{\frac{(1+I_M^-(x_i))+(1+I_N^-(x_i))}{2}}-\left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right)+\sqrt{\frac{-(F_M^-(x_i)+F_N^-(x_i))}{2}}-\left(\frac{\sqrt{(-F_M^-(x_i))}+\sqrt{(-F_N^-(x_i))}}{2}\right)\end{array}\right]$$

 doi:10.20944/preprints201801.0065.v1

$$= \sum_{i=}^{n}\left[\begin{array}{l}\sqrt{\frac{T_N^+(x_i)+T_M^+(x_i)}{2}} - \left(\frac{\sqrt{T_N^+(x_i)}+\sqrt{T_M^+(x_i)}}{2}\right) + \sqrt{\frac{I_N^+(x_i)+I_M^+(x_i)}{2}} - \left(\frac{\sqrt{I_N^+(x_i)}+\sqrt{I_M^+(x_i)}}{2}\right) + \\ \sqrt{\frac{(1-I_N^+(x_i))+(1-I_M^+(x_i))}{2}} - \left(\frac{\sqrt{(1-I_N^+(x_i))}+\sqrt{(1-I_M^+(x_i))}}{2}\right) + \sqrt{\frac{F_N^+(x_i)+F_M^+(x_i)}{2}} - \left(\frac{\sqrt{F_N^+(x_i)}+\sqrt{F_M^+(x_i)}}{2}\right) + \\ \sqrt{\frac{-(T_N^-(x_i)+T_M^-(x_i))}{2}} - \left(\frac{\sqrt{(-T_N^-(x_i))}+\sqrt{(-T_M^-(x_i))}}{2}\right) + \sqrt{\frac{-(I_N^-(x_i)+I_M^-(x_i))}{2}} - \left(\frac{\sqrt{(-I_N^-(x_i))}+\sqrt{(-I_M^-(x_i))}}{2}\right) + \\ \sqrt{\frac{(1+I_N^-(x_i))+(1+I_M^-(x_i))}{2}} - \left(\frac{\sqrt{1+I_N^-(x_i)}+\sqrt{1+I_M^-(x_i)}}{2}\right) + \sqrt{\frac{-(F_N^-(x_i)+F_M^-(x_i))}{2}} - \left(\frac{\sqrt{(-F_N^-(x_i))}+\sqrt{(-F_M^-(x_i))}}{2}\right)\end{array}\right]$$

$= C_B(N, M)$.

The proof is completed.

Example 1. Suppose that M = <0.7, 0.3, 0.4, -0.3, -0.5, -0.1> and N = <0.5, 0.2, 0.5, -0.3, -0.3, -0.2> be two BNSs, then the cross entropy between M and N is calculated as follows:

$C_B(M, N)$ =

$$\left[\begin{array}{l}\sqrt{\frac{0.7+0.5}{2}} - \left(\frac{\sqrt{0.7}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{0.3+0.2}{2}} - \left(\frac{\sqrt{0.3}+\sqrt{0.2}}{2}\right) + \sqrt{\frac{(1-0.3)+(1-0.2)}{2}} - \left(\frac{\sqrt{1-0.3}+\sqrt{1-0.2}}{2}\right) \\ + \sqrt{\frac{0.4+0.5}{2}} - \left(\frac{\sqrt{0.4}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{-(-0.3-0.3)}{2}} - \left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.3)}}{2}\right) + \sqrt{\frac{-(-0.5-0.3)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.3)}}{2}\right) \\ + \sqrt{\frac{(1-0.3)+(1-0.3)}{2}} - \left(\frac{\sqrt{1-0.3}+\sqrt{[1-0.3]}}{2}\right) + \sqrt{\frac{-(-0.1-0.2)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.2)}}{2}\right)\end{array}\right] = 0.01738474.$$

Definition 4: Suppose w_i be the weight of each element x_i, i = 1, 2, ..., n, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, then the weighted cross entropy measure between any two BNSs M and N in U can be defined as follows:

$C_B(M, N)_w$ =

$$\sum_{i=1}^{n} w_i \left[\begin{array}{l}\sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{(1-I_M^+(x_i))+(1-I_N^+(x_i))}{2}} - \left(\frac{\sqrt{1-I_M^+(x_i)}+\sqrt{1-I_N^+(x_i)}}{2}\right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{-(T_M^-(x_i)+T_N^-(x_i))}{2}} - \left(\frac{\sqrt{(-T_M^-(x_i))}+\sqrt{(-T_N^-(x_i))}}{2}\right) + \sqrt{\frac{-(I_M^-(x_i)+I_N^-(x_i))}{2}} - \left(\frac{\sqrt{(-I_M^-(x_i))}+\sqrt{(-I_N^-(x_i))}}{2}\right) + \\ \sqrt{\frac{(1+I_M^-(x_i))+(1+I_N^-(x_i))}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right) + \sqrt{\frac{-(F_M^-(x_i)+F_N^-(x_i))}{2}} - \left(\frac{\sqrt{(-F_M^-(x_i))}+\sqrt{(-F_N^-(x_i))}}{2}\right)\end{array}\right] \quad (2)$$

Theorem 2. If M = < $T_M^+(x_i)$, $I_M^+(x_i)$, $F_M^+(x_i)$, $T_M^-(x_i)$, $I_M^-(x_i)$, $F_M^-(x_i)$ > and N < $T_N^+(x_i)$, $I_N^+(x_i)$, $F_N^+(x_i)$, $T_N^-(x_i)$, $I_N^-(x_i)$, $F_N^-(x_i)$ > be two BNSs in U, then the weighted cross entropy measure $C_B(M, N)_w$ satisfies the following properties:

(1) $C_B(M, N)_w \geq 0$

(2) $C_B(M, N)_w = 0$ if and only if $T_M(x_i) = T_N(x_i)$, $I_M(x_i) = I_N(x_i)$, $F_M(x_i) = F_N(x_i)$, $\forall x \in U$.

(3) $C_B(M, N)_w = C_B(N, M)_w$.

Proof is given in appendix.

Example 2. Suppose that M = <0.7, 0.3, 0.4, -0.3, -0.5, -0.1> and N = <0.5, 0.2, 0.5, -0.3, -0.3, -0.2> be two BNSs and w = 0.4, then the weighted cross entropy between M and N is calculated as given below.

$C_B(M, N)_w =$

$$0.4\times\left[\begin{array}{l}\sqrt{\frac{0.7+0.5}{2}}-\left(\frac{\sqrt{0.7}+\sqrt{0.5}}{2}\right)+\sqrt{\frac{0.3+0.2}{2}}-\left(\frac{\sqrt{0.3}+\sqrt{0.2}}{2}\right)+\sqrt{\frac{(1-0.3)+(1-0.2)}{2}}-\left(\frac{\sqrt{1-0.3}+\sqrt{1-0.2}}{2}\right)\\ +\sqrt{\frac{0.4+0.5}{2}}-\left(\frac{\sqrt{0.4}+\sqrt{0.5}}{2}\right)+\sqrt{\frac{-(-0.3-0.3)}{2}}-\left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.3)}}{2}\right)+\sqrt{\frac{-(-0.5-0.3)}{2}}-\left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.3)}}{2}\right)\\ +\sqrt{\frac{(1-0.3+(1-0.3))}{2}}-\left(\frac{\sqrt{1-0.3}+\sqrt{1-0.3}}{2}\right)+\sqrt{\frac{-(-0.1-0.2)}{2}}-\left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.2)}}{2}\right)\end{array}\right]$$

$= 0.006953896.$

4. Cross entropy measures of IBNSs

This section extends the concepts of cross entropy and weighted cross entropy measures of BNSs to IBNSs.

Definition 5: The cross entropy measure between any two IBNSs R = < $[\inf T_R^+(x_i), \sup T_R^+(x_i)], [\inf I_R^+(x_i), \sup I_R^+(x_i)], [\inf F_R^+(x_i), \sup F_R^+(x_i)], [\inf T_R^-(x_i), \sup T_R^-(x_i)], [\inf I_R^-(x_i), \sup I_R^-(x_i)], [\inf F_R^-(x_i), \sup F_R^-(x_i)]$ > and S = < $[\inf T_S^+(x_i), \sup T_S^+(x_i)], [\inf I_S^+(x_i), \sup I_S^+(x_i)], [\inf F_S^+(x_i), \sup F_S^+(x_i)], [\inf T_S^-(x_i), \sup T_S^-(x_i)], [\inf I_S^-(x_i), \sup I_S^-(x_i)], [\inf F_S^-(x_i), \sup F_S^-(x_i)]$ > in U can be defined as follows:

$$C_{IB}(R, S) = \frac{1}{2}\sum_{i=1}^{n}\left\langle\left[\begin{array}{l}\sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right)+\\ \sqrt{\frac{\inf I_R^+(x_i)+\inf I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf I_R^+(x_i)}+\sqrt{\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup I_R^+(x_i)+\sup I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup I_R^+(x_i)}+\sqrt{\sup I_S^+(x_i)}}{2}\right)+\\ \sqrt{\frac{(1-\inf I_R^+(x_i))+(1-\inf I_S^+(x_i))}{2}}-\left(\frac{\sqrt{1-\inf I_R^+(x_i)}+\sqrt{[1-\inf I_S^+(x_i)]}}{2}\right)+\sqrt{\frac{(1-\sup I_R^+(x_i))+(1-\sup I_S^+(x_i))}{2}}-\\ \left(\frac{\sqrt{1-\sup I_R^+(x_i)}+\sqrt{1-\sup I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right)+\\ \sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right)+\sqrt{\frac{-(\inf T_R^-(x_i)+\inf T_S^+(x_i))}{2}}-\\ \left(\frac{\sqrt{(-\inf T_R^-(x_i))}+\sqrt{(-\inf T_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup T_R^-(x_i)+\sup T_S^+(x_i))}{2}}-\left(\frac{\sqrt{(-\sup T_R^-(x_i))}+\sqrt{(-\sup T_S^-(x_i))}}{2}\right)+\\ \sqrt{\frac{-(\inf I_R^-(x_i)+\inf I_S^-(x_i))}{2}}-\left(\frac{\sqrt{(-\inf I_R^-(x_i))}+\sqrt{(-\inf I_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup I_R^-(x_i)+\sup I_S^-(x_i))}{2}}-\\ \left(\frac{\sqrt{(-\sup I_R^-(x_i))}+\sqrt{(-\sup I_S^-(x_i))}}{2}\right)+\sqrt{\frac{(1+\inf I_R^-(x_i))+(1+\inf I_S^-(x_i))}{2}}-\left(\frac{\sqrt{1+\inf I_R^-(x_i)}+\sqrt{1+\inf I_S^-(x_i)}}{2}\right)+\\ \sqrt{\frac{(1+\sup I_R^-(x_i))+(1+\sup I_S^-(x_i))}{2}}-\left(\frac{\sqrt{1+\sup I_R^-(x_i)}+\sqrt{1+\sup I_S^-(x_i)}}{2}\right)+\sqrt{\frac{-(\inf F_R^-(x_i)+\inf F_S^-(x_i))}{2}}-\\ \left(\frac{\sqrt{(-\inf F_R^-(x_i))}+\sqrt{(-\inf F_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup F_R^-(x_i)+\sup F_S^-(x_i))}{2}}-\left(\frac{\sqrt{(-\sup F_R^-(x_i))}+\sqrt{(-\sup F_S^-(x_i))}}{2}\right)\end{array}\right]\right\rangle \quad (3)$$

Theorem 3. If R = < $[\inf T_R^+(x_i), \sup T_R^+(x_i)], [\inf I_R^+(x_i), \sup I_R^+(x_i)], [\inf F_R^+(x_i), \sup F_R^+(x_i)], [\inf T_R^-(x_i), \sup T_R^-(x_i)], [\inf I_R^-(x_i), \sup I_R^-(x_i)], [\inf F_R^-(x_i), \sup F_R^-(x_i)]$ > and S = < $[\inf T_S^+(x_i), \sup T_S^+(x_i)], [\inf I_S^+(x_i), \sup I_S^+(x_i)], [\inf F_S^+(x_i), \sup F_S^+(x_i)], [\inf T_S^-(x_i), \sup T_S^-(x_i)], [\inf I_S^-(x_i), \sup I_S^-(x_i)], [\inf F_S^-(x_i), \sup F_S^-(x_i)]$ > be two BNSs in U, then the cross entropy measure $C_{IB}(R, S)$ satisfies the following properties:

(1) $C_{IB}(R, S) \geq 0$

(2) C_{IB} $(R, S) = 0$ for $R = S$ i.e., $\inf T_R^+(x_i) = \inf T_S^+(x_i)$, $\sup T_R^+(x_i) = \sup T_S^+(x_i)$, $\inf I_R^+(x_i) = \inf I_S^+(x_i)$, $\sup I_R^+(x_i) = \sup I_S^+(x_i)$, $\inf F_R^+(x_i) = \inf F_S^+(x_i)$, $\sup F_R^+(x_i) = \sup F_S^+(x_i)$, $\inf T_R^-(x_i) = \inf T_S^-(x_i)$, $\sup T_R^-(x_i) = \sup T_S^-(x_i)$, $\inf I_R^-(x_i) = \inf I_S^-(x_i)$, $\sup I_R^-(x_i) = \sup I_S^-(x_i)$, $\inf F_R^-(x_i) = \inf F_S^-(x_i)$, $\sup F_R^-(x_i) = \sup F_S^-(x_i)$ $\forall$ $x \in U$.

(3) C_{IB} $(R, S) = C_{IB}$ (S, R).

Proof

(1) From the inequality stated in theorem 1, we can easily get C_{IB} $(R, S) \geq 0$.

(2) Since $\inf T_R^+(x_i) = \inf T_S^+(x_i)$, $\sup T_R^+(x_i) = \sup T_S^+(x_i)$, $\inf I_R^+(x_i) = \inf I_S^+(x_i)$, $\sup I_R^+(x_i) = \sup I_S^+(x_i)$, $\inf F_R^+(x_i) = \inf F_S^+(x_i)$, $\sup F_R^+(x_i) = \sup F_S^+(x_i)$, $\inf T_R^-(x_i) = \inf T_S^-(x_i)$, $\sup T_R^-(x_i) = \sup T_S^-(x_i)$, $\inf I_R^-(x_i) = \inf I_S^-(x_i)$, $\sup I_R^-(x_i) = \sup I_S^-(x_i)$, $\inf F_R^-(x_i) = \inf F_S^-(x_i)$, $\sup F_R^-(x_i) = \sup F_S^-(x_i)$ $\forall$ $x \in U$, we have C_{IB} $(R, S) = 0$.

(3) $$C_{IB}(R, S) = \frac{1}{2}\sum_{i=1}^{n}\left\langle\left[\begin{array}{l}
\sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\inf I_R^+(x_i)+\inf I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf I_R^+(x_i)}+\sqrt{\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup I_R^+(x_i)+\sup I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup I_R^+(x_i)}+\sqrt{\sup I_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{([1-\inf I_R^+(x_i))+(1-\inf I_S^+(x_i))}{2}}-\left(\frac{\sqrt{1-\inf I_R^+(x_i)}+\sqrt{1-\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{(1-\sup I_R^+(x_i))+(1-\sup I_S^+(x_i))}{2}}-\\
\left(\frac{\sqrt{1-\sup I_R^+(x_i)}+\sqrt{1-\sup I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right)+\\
\sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right)+\sqrt{\frac{-(\inf T_R^-(x_i)+\inf T_S^+(x_i))}{2}}-\\
\left(\frac{\sqrt{(-\inf T_R^-(x_i))}+\sqrt{(-\inf T_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup T_R^-(x_i)+\sup T_S^+(x_i))}{2}}-\left(\frac{\sqrt{(-\sup T_R^-(x_i))}+\sqrt{(-\sup T_S^-(x_i))}}{2}\right)+\\
\sqrt{\frac{-(\inf I_R^-(x_i)+\inf I_S^-(x_i))}{2}}-\left(\frac{\sqrt{(-\inf I_R^-(x_i))}+\sqrt{(-\inf I_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup I_R^-(x_i)+\sup I_S^-(x_i))}{2}}-\\
\left(\frac{\sqrt{(-\sup I_R^-(x_i))}+\sqrt{(-\sup I_S^-(x_i))}}{2}\right)+\sqrt{\frac{(1+\inf I_R^-(x_i))+(1+\inf I_S^-(x_i))}{2}}-\left(\frac{\sqrt{1+\inf I_R^-(x_i)}+\sqrt{1+\inf I_S^-(x_i)}}{2}\right)+\\
\sqrt{\frac{(1+\sup I_R^-(x_i))+(1+\sup I_S^-(x_i))}{2}}-\left(\frac{\sqrt{1+\sup I_R^-(x_i)}+\sqrt{1+\sup I_S^-(x_i)}}{2}\right)+\sqrt{\frac{-(\inf F_R^-(x_i)+\inf F_S^-(x_i))}{2}}-\\
\left(\frac{\sqrt{(-\inf F_R^-(x_i))}+\sqrt{(-\inf F_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup F_R^-(x_i)+\sup F_S^-(x_i))}{2}}-\left(\frac{\sqrt{(-\sup F_R^-(x_i))}+\sqrt{(-\sup F_S^-(x_i))}}{2}\right)
\end{array}\right]\right\rangle$$

$$
=\frac{1}{2}\sum_{i=1}^{n}\left\langle\left[\begin{array}{l}
\sqrt{\frac{\inf T_S^{+}(x_i)+\inf T_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\inf T_S^{+}(x_i)}+\sqrt{\inf T_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\sup T_S^{+}(x_i)+\sup T_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\sup T_S^{+}(x_i)}+\sqrt{\sup T_R^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\inf I_S^{+}(x_i)+\inf I_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\inf I_S^{+}(x_i)}+\sqrt{\inf I_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\sup I_S^{+}(x_i)+\sup I_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\sup I_S^{+}(x_i)}+\sqrt{\sup I_R^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{(1-\inf I_S^{+}(x_i))+(1-\inf I_R^{+}(x_i))}{2}}-\left(\frac{\sqrt{1-\inf I_S^{+}(x_i)}+\sqrt{1-\inf I_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{(1-\sup I_S^{+}(x_i))+(1-\sup I_R^{+}(x_i))}{2}}-\\
\left(\frac{\sqrt{1-\sup I_S^{+}(x_i)}+\sqrt{1-\sup I_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{\inf F_S^{+}(x_i)+\inf F_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\inf F_S^{+}(x_i)}+\sqrt{\inf F_R^{+}(x_i)}}{2}\right)+\\
\sqrt{\frac{\sup F_S^{+}(x_i)+\sup F_R^{+}(x_i)}{2}}-\left(\frac{\sqrt{\sup F_S^{+}(x_i)}+\sqrt{\sup F_R^{+}(x_i)}}{2}\right)+\sqrt{\frac{-(\inf T_S^{-}(x_i)+\inf T_R^{+}(x_i))}{2}}-\\
\left(\frac{\sqrt{(-\inf T_S^{-}(x_i))}+\sqrt{(-\inf T_R^{-}(x_i))}}{2}\right)+\sqrt{\frac{-(\sup T_S^{-}(x_i)+\sup T_R^{+}(x_i))}{2}}-\left(\frac{\sqrt{(-\sup T_S^{-}(x_i))}+\sqrt{(-\sup T_R^{-}(x_i))}}{2}\right)+\\
\sqrt{\frac{-(\inf I_S^{-}(x_i)+\inf I_R^{-}(x_i))}{2}}-\left(\frac{\sqrt{(-\inf I_S^{-}(x_i))}+\sqrt{(-\inf I_R^{-}(x_i))}}{2}\right)+\sqrt{\frac{-(\sup I_S^{-}(x_i)+\sup I_R^{-}(x_i))}{2}}-\\
\left(\frac{\sqrt{(-\sup I_S^{-}(x_i))}+\sqrt{(-\sup I_R^{-}(x_i))}}{2}\right)+\sqrt{\frac{(1+\inf I_S^{-}(x_i))+(1+\inf I_R^{-}(x_i))}{2}}-\left(\frac{\sqrt{1+\inf I_S^{-}(x_i)}+\sqrt{1+\inf I_R^{-}(x_i)}}{2}\right)+\\
\sqrt{\frac{(1+\sup I_S^{-}(x_i))+(1+\sup I_R^{-}(x_i))}{2}}-\left(\frac{\sqrt{1+\sup I_S^{-}(x_i)}+\sqrt{[1+\sup I_R^{-}(x_i)]}}{2}\right)+\sqrt{\frac{-(\inf F_S^{-}(x_i)+\inf F_R^{-}(x_i))}{2}}-\\
\left(\frac{\sqrt{(-\inf F_S^{-}(x_i))}+\sqrt{(-\inf F_R^{-}(x_i))}}{2}\right)+\sqrt{\frac{-(\sup F_S^{-}(x_i)+\sup F_R^{-}(x_i))}{2}}-\left(\frac{\sqrt{(-\sup F_S^{-}(x_i))}+\sqrt{(-\sup F_R^{-}(x_i))}}{2}\right)
\end{array}\right]\right\rangle
$$

$= C_{IB}(S, R)$.

Example 3. Suppose that R = < [0.5, 0.8], [0.4, 0.6], [0.2, 0.6], [-0.3, -0.1], [-0.5, -0.1], [-05, -0.2] > and S = < [0.5, 0.9], [0.4, 0.5], [0.1, 0.4], [-0.5, -0.3], [-0.7, -0.3], [-0.6, -0.3] > be two IBNSs, the cross entropy between R and S is computed as follows:

$C_{IB}(R, S)$ =

$$
\frac{1}{2}\left[\begin{array}{l}
\sqrt{\frac{0.5+0.5}{2}}-\left(\frac{\sqrt{0.5}+\sqrt{0.5}}{2}\right)+\sqrt{\frac{0.8+0.9}{2}}-\left(\frac{\sqrt{0.8}+\sqrt{0.9}}{2}\right)+\sqrt{\frac{0.4+0.4}{2}}-\left(\frac{\sqrt{0.4}+\sqrt{0.4}}{2}\right)+\sqrt{\frac{0.6+0.5}{2}}-\left(\frac{\sqrt{0.6}+\sqrt{0.5}}{2}\right)+\\
\sqrt{\frac{[1-0.4]+[1-0.4]}{2}}-\left(\frac{\sqrt{1-0.4}+\sqrt{[1-0.4]}}{2}\right)+\sqrt{\frac{[1-0.6]+[1-0.5]}{2}}-\left(\frac{\sqrt{1-0.6}+\sqrt{[1-0.5]}}{2}\right)+\sqrt{\frac{0.2+0.1}{2}}-\left(\frac{\sqrt{0.2}+\sqrt{0.1}}{2}\right)+\\
\sqrt{\frac{0.6+0.4}{2}}-\left(\frac{\sqrt{0.6}+\sqrt{0.4}}{2}\right)+\sqrt{\frac{-(-0.3-0.5)}{2}}-\left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.5)}}{2}\right)+\sqrt{\frac{-(-0.1-0.3)}{2}}-\left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right)+\\
\sqrt{\frac{-(-0.5-0.7)}{2}}-\left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.7)}}{2}\right)+\sqrt{\frac{-(-0.1-0.3)}{2}}-\left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right)+\sqrt{\frac{[1-0.5]+[1-0.7]}{2}}-\\
\left(\frac{\sqrt{1-0.5}+\sqrt{[1-0.7]}}{2}\right)+\sqrt{\frac{[1-0.1]+[1-0.3]}{2}}-\left(\frac{\sqrt{1-0.1}+\sqrt{[1-0.3]}}{2}\right)+\sqrt{\frac{-(-0.5-0.6)}{2}}-\left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.6)}}{2}\right)+\\
\sqrt{\frac{-(-0.2-0.3)}{2}}-\left(\frac{\sqrt{-(-0.2)}+\sqrt{-(-0.3)}}{2}\right)
\end{array}\right]
$$

= 0.02984616.

Definition 6: Let w_i be the weight of each element x_i, i = 1, 2, ..., n, and $w_i \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$, then the weighted cross entropy measure between any two IBNSs R = < $[\inf T_R^{+}(x_i), \sup T_R^{+}(x_i)]$, $[\inf I_R^{+}(x_i), \sup I_R^{+}(x_i)]$, $[\inf F_R^{+}(x_i), \sup F_R^{+}(x_i)]$, $[\inf T_R^{-}(x_i), \sup T_R^{-}(x_i)]$, $[\inf I_R^{-}(x_i), \sup I_R^{-}(x_i)]$, $[\inf F_R^{-}(x_i), \sup F_R^{-}(x_i)]$ > and S = < $[\inf T_S^{+}(x_i), \sup T_S^{+}(x_i)]$, $[\inf I_S^{+}(x_i), \sup I_S^{+}(x_i)]$, $[\inf F_S^{+}(x_i), \sup F_S^{+}(x_i)]$, $[\inf T_S^{-}(x_i), \sup T_S^{-}(x_i)]$, $[\inf I_S^{-}(x_i), \sup I_S^{-}(x_i)]$, $[\inf F_S^{-}(x_i), \sup F_S^{-}(x_i)]$ > in U can be defined as follows:

$C_{IB}(R, S)_w$

$$= \frac{1}{2}\sum_{i=1}^{n} w_i \left\langle \left[\begin{array}{l}
\sqrt{\frac{\inf T_R^+(x_i) + \inf T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf T_R^+(x_i)} + \sqrt{\inf T_S^+(x_i)}}{2} \right) + \sqrt{\frac{\sup T_R^+(x_i) + \sup T_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup T_R^+(x_i)} + \sqrt{\sup T_S^+(x_i)}}{2} \right) + \\
\sqrt{\frac{\inf I_R^+(x_i) + \inf I_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf I_R^+(x_i)} + \sqrt{\inf I_S^+(x_i)}}{2} \right) + \sqrt{\frac{\sup I_R^+(x_i) + \sup I_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup I_R^+(x_i)} + \sqrt{\sup I_S^+(x_i)}}{2} \right) + \\
\sqrt{\frac{(1 - \inf I_R^+(x_i)) + (1 - \inf I_S^+(x_i))}{2}} - \left(\frac{\sqrt{1 - \inf I_R^+(x_i)} + \sqrt{1 - \inf I_S^+(x_i)}}{2} \right) + \sqrt{\frac{(1 - \sup I_R^+(x_i)) + (1 - \sup I_S^+(x_i))}{2}} - \\
\left(\frac{\sqrt{1 - \sup I_R^+(x_i)} + \sqrt{(1 - \sup I_S^+(x_i))}}{2} \right) + \sqrt{\frac{\inf F_R^+(x_i) + \inf F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\inf F_R^+(x_i)} + \sqrt{\inf F_S^+(x_i)}}{2} \right) + \\
\sqrt{\frac{\sup F_R^+(x_i) + \sup F_S^+(x_i)}{2}} - \left(\frac{\sqrt{\sup F_R^+(x_i)} + \sqrt{\sup F_S^+(x_i)}}{2} \right) + \sqrt{\frac{-(\inf T_R^-(x_i) + \inf T_S^+(x_i))}{2}} - \\
\left(\frac{\sqrt{(-\inf T_R^-(x_i))} + \sqrt{(-\inf T_S^-(x_i))}}{2} \right) + \sqrt{\frac{-(\sup T_R^-(x_i) + \sup T_S^+(x_i))}{2}} - \left(\frac{\sqrt{(-\sup T_R^-(x_i))} + \sqrt{(-\sup T_S^-(x_i))}}{2} \right) + \\
\sqrt{\frac{-(\inf I_R^-(x_i) + \inf I_S^-(x_i))}{2}} - \left(\frac{\sqrt{(-\inf I_R^-(x_i))} + \sqrt{(-\inf I_S^-(x_i))}}{2} \right) + \sqrt{\frac{-(\sup I_R^-(x_i) + \sup I_S^-(x_i))}{2}} - \\
\left(\frac{\sqrt{(-\sup I_R^-(x_i))} + \sqrt{(-\sup I_S^-(x_i))}}{2} \right) + \sqrt{\frac{(1 + \inf I_R^-(x_i)) + (1 + \inf I_S^-(x_i))}{2}} - \left(\frac{\sqrt{1 + \inf I_R^-(x_i)} + \sqrt{1 + \inf I_S^-(x_i)}}{2} \right) + \\
\sqrt{\frac{(1 + \sup I_R^-(x_i)) + (1 + \sup I_S^-(x_i))}{2}} - \left(\frac{\sqrt{1 + \sup I_R^-(x_i)} + \sqrt{[1 + \sup I_S^-(x_i)]}}{2} \right) + \sqrt{\frac{-(\inf F_R^-(x_i) + \inf F_S^-(x_i))}{2}} - \\
\left(\frac{\sqrt{(-\inf F_R^-(x_i))} + \sqrt{(-\inf F_S^-(x_i))}}{2} \right) + \sqrt{\frac{-(\sup F_R^-(x_i) + \sup F_S^-(x_i))}{2}} - \left(\frac{\sqrt{(-\sup F_R^-(x_i))} + \sqrt{(-\sup F_S^-(x_i))}}{2} \right)
\end{array} \right] \right\rangle \quad (4)$$

Theorem 4. For any two IBNSs $R = <[\inf T_R^+(x_i), \sup T_R^+(x_i)], [\inf I_R^+(x_i), \sup I_R^+(x_i)], [\inf F_R^+(x_i), \sup F_R^+(x_i)], [\inf T_R^-(x_i), \sup T_R^-(x_i)], [\inf I_R^-(x_i), \sup I_R^-(x_i)], [\inf F_R^-(x_i), \sup F_R^-(x_i)]>$ and $S = <[\inf T_S^+(x_i), \sup T_S^+(x_i)], [\inf I_S^+(x_i), \sup I_S^+(x_i)], [\inf F_S^+(x_i), \sup F_S^+(x_i)], [\inf T_S^-(x_i), \sup T_S^-(x_i)], [\inf I_S^-(x_i), \sup I_S^-(x_i)], [\inf F_S^-(x_i), \sup F_S^-(x_i)]>$ be in *U*, the cross entropy measure $C_{IB}(R, S)_w$ also satisfies the following properties:

(1) $C_{IB}(R, S)_w \geq 0$

(2) $C_{IB}(R, S)_w = 0$ if and only if $R = S$ i.e., $\inf T_R^+(x_i) = \inf T_S^+(x_i)$, $\sup T_R^+(x_i) = \sup T_S^+(x_i)$, $\inf I_R^+(x_i) = \inf I_S^+(x_i)$, $\sup I_R^+(x_i) = \sup I_S^+(x_i)$, $\inf F_R^+(x_i) = \inf F_S^+(x_i)$, $\sup F_R^+(x_i) = \sup F_S^+(x_i)$, $\inf T_R^-(x_i) = \inf T_S^-(x_i)$, $\sup T_R^-(x_i) = \sup T_S^-(x_i)$, $\inf I_R^-(x_i) = \inf I_S^-(x_i)$, $\sup I_R^-(x_i) = \sup I_S^-(x_i)$, $\inf F_R^-(x_i) = \inf F_S^-(x_i)$, $\sup F_R^-(x_i) = \sup F_S^-(x_i)$ $\forall$ $x \in U$.

(3) $C_{IB}(R, S)_w = C_{IB}(S, R)_w$.

Proof is given in appendix.

Example 4. Consider R = < [0.5, 0.8], [0.4, 0.6], [0.2, 0.6], [-0.3, -0.1], [-0.5, -0.1], [-05, -0.2] > and S = < [0.5, 0.9], [0.4, 0.5], [0.1, 0.4], [-0.5, -0.3], [-0.7, -0.3], [-0.6, -0.3] > be two IBNSs, and w = 0.3, then the weighted cross entropy between *R* and *S* is calculated as follows:

$$C_{IB}(R, S) = \frac{1}{2} \times 0.3 \times \left[\begin{array}{l}
\sqrt{\frac{0.5+0.5}{2}} - \left(\frac{\sqrt{0.5}+\sqrt{0.5}}{2}\right) + \sqrt{\frac{0.8+0.9}{2}} - \left(\frac{\sqrt{0.8}+\sqrt{0.9}}{2}\right) + \sqrt{\frac{0.4+0.4}{2}} - \left(\frac{\sqrt{0.4}+\sqrt{0.4}}{2}\right) + \sqrt{\frac{0.6+0.5}{2}} - \left(\frac{\sqrt{0.6}+\sqrt{0.5}}{2}\right) + \\
\sqrt{\frac{[1-0.4]+[1-0.4]}{2}} - \left(\frac{\sqrt{1-0.4}+\sqrt{[1-0.4]}}{2}\right) + \sqrt{\frac{[1-0.6]+[1-0.5]}{2}} - \left(\frac{\sqrt{1-0.6}+\sqrt{[1-0.5]}}{2}\right) + \sqrt{\frac{0.2+0.1}{2}} - \left(\frac{\sqrt{0.2}+\sqrt{0.1}}{2}\right) + \\
\sqrt{\frac{0.6+0.4}{2}} - \left(\frac{\sqrt{0.6}+\sqrt{0.4}}{2}\right) + \sqrt{\frac{-(-0.3-0.5)}{2}} - \left(\frac{\sqrt{-(-0.3)}+\sqrt{-(-0.5)}}{2}\right) + \sqrt{\frac{-(-0.1-0.3)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right) + \\
\sqrt{\frac{-(-0.5-0.7)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.7)}}{2}\right) + \sqrt{\frac{-(-0.1-0.3)}{2}} - \left(\frac{\sqrt{-(-0.1)}+\sqrt{-(-0.3)}}{2}\right) + \sqrt{\frac{[1-0.5]+[1-0.7]}{2}} - \\
\left(\frac{\sqrt{1-0.5}+\sqrt{[1-0.7]}}{2}\right) + \sqrt{\frac{[1-0.1]+[1-0.3]}{2}} - \left(\frac{\sqrt{1-0.1}+\sqrt{[1-0.3]}}{2}\right) + \sqrt{\frac{-(-0.5-0.6)}{2}} - \left(\frac{\sqrt{-(-0.5)}+\sqrt{-(-0.6)}}{2}\right) + \\
\sqrt{\frac{-(-0.2-0.3)}{2}} - \left(\frac{\sqrt{-(-0.2)}+\sqrt{-(-0.3)}}{2}\right)
\end{array}\right]$$

$= 0.00895385.$

5. MADM methods based on cross entropy measures

In this section, we propose two new MADM methods based on cross entropy measures under bipolar neutrosophic and interval bipolar neutrosophic environments. Consider $B = \{B_1, B_2, \ldots, B_m\}$, $(m \geq 2)$ be a discrete set of m feasible alternatives which are to be evaluated based on n attributes $C = \{C_1, C_2, \ldots, C_n\}$, $(n \geq 2)$ and w_j be the weight vector of the attributes such that $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$.

5.1 MADM method based on cross entropy measures of BNS

The procedure for solving MADM problems under bipolar neutrosophic environment is given by the following steps

Step 1. The rating of performance value of alternative B_i (i = 1, 2, …, m) with respect to the predefined attribute C_j (j = 1, 2, …, n) can be expressed in terms of bipolar neutrosophic information as follows:

$$B_i = \{C_j, < T^{+}_{B_i}(C_j), I^{+}_{B_i}(C_j), F^{+}_{B_i}(C_j), \ T^{-}_{B_i}(C_j), I^{-}_{B_i}(C_j), F^{-}_{B_i}(C_j) > \mid C_j \in C_j, \ j = 1, 2, \ldots, n\}$$

where $0 \leq T^{+}_{B_i}(C_j) + I^{+}_{B_i}(C_j) + F^{+}_{B_i}(C_j) \leq 3$, and $-3 \leq T^{-}_{B_i}(C_j) + I^{-}_{B_i}(C_j) + F^{-}_{B_i}(C_j) \leq 0$, i = 1, 2, …, m; j = 1, 2, …, n. Let $\tilde{d}_{ij} = < T^{+}_{ij}, I^{+}_{ij}, F^{+}_{ij}, \ T^{-}_{ij}, I^{-}_{ij}, F^{-}_{ij} >$ be the bipolar neutrosophic decision matrix whose entries are the rating values of the alternatives with respect to the attributes provided by the expert or decision maker. The bipolar neutrosophic decision matrix $[\tilde{d}_{ij}]_{m \times n}$ can be expressed as follows:

$$[\tilde{d}_{ij}]_{m \times n} = \begin{array}{c} \\ B_1 \\ B_2 \\ . \\ . \\ B_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \left(\begin{array}{cccc} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{array}\right) \end{array}$$

Step 2. The positive ideal solution (PIS) $<p^{*} = (d_1^{*}, d_2^{*}, \ldots, d_n^{*})>$ of bipolar neutrosophic information is obtained as follows:

$p_j^* = \left\langle T_j^{*+}, I_j^{*+}, F_j^{*+}, T_j^{*-}, I_j^{*-}, F_j^{*-} \right\rangle$ = <[{ $\underset{i}{\mathrm{Max}}(T_{ij}^{+})$ | j∈ H1}; { $\underset{i}{\mathrm{Min}}(T_{ij}^{+})$ | j∈ H2}], [{ $\underset{i}{\mathrm{Min}}(I_{ij}^{+})$ | j∈ H1}; { $\underset{i}{\mathrm{Max}}(I_{ij}^{+})$ | j∈ H2}], [{ $\underset{i}{\mathrm{Min}}(F_{ij}^{+})$ | j∈ H1}; { $\underset{i}{\mathrm{Max}}(F_{ij}^{+})$ | j∈ H2}], [{ $\underset{i}{\mathrm{Min}}(T_{ij}^{-})$ | j∈ H1}; { $\underset{i}{\mathrm{Max}}(T_{ij}^{-})$ | j∈ H2}], [{ $\underset{i}{\mathrm{Max}}(I_{ij}^{-})$ | j∈ H1}; { $\underset{i}{\mathrm{Min}}(I_{ij}^{-})$ | j∈ H2}],[{ $\underset{i}{\mathrm{Max}}(F_{ij}^{-})$ | j∈ H1}; { $\underset{i}{\mathrm{Min}}(F_{ij}^{-})$ | j∈ H2}]>, j = 1, 2, …, *n*;

where H1 and H2 represent benefit and cost type attributes respectively.

Step 3. The weighted cross entropy between an alternative B_i, i = 1, 2, …, *m*, and the ideal alternative p^* is determined by

$$C_B(B_i, p^*)_w = \sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{T_{ij}^{+}+T_j^{*+}}{2}} - \left(\frac{\sqrt{T_{ij}^{+}}+\sqrt{T_j^{*+}}}{2}\right) + \sqrt{\frac{I_{ij}^{+}+I_j^{*+}}{2}} - \left(\frac{\sqrt{I_{ij}^{+}}+\sqrt{I_j^{*+}}}{2}\right) + \sqrt{\frac{[1-I_{ij}^{+}]+[1-I_j^{*+}]}{2}} - \\ \left(\frac{\sqrt{1-I_j^{+}}+\sqrt{[1-I_j^{*+}]}}{2}\right) + \sqrt{\frac{F_{ij}^{+}+F_j^{*+}}{2}} - \left(\frac{\sqrt{F_{ij}^{+}}+\sqrt{F_j^{*+}}}{2}\right) + \sqrt{\frac{-(T_{ij}^{+}+T_j^{*+})}{2}} - \\ \left(\frac{\sqrt{(-T_{ij}^{+})}+\sqrt{(-T_j^{*+})}}{2}\right) + \sqrt{\frac{-(I_{ij}^{+}+I_j^{*+})}{2}} - \left(\frac{\sqrt{(-I_{ij}^{+})}+\sqrt{(-I_j^{*+})}}{2}\right) + \sqrt{\frac{[1+I_{ij}^{+}]+[1+I_j^{*+}]}{2}} \\ - \left(\frac{\sqrt{1+I_{ij}^{+}}+\sqrt{[1+I_j^{*+}]}}{2}\right) + \sqrt{\frac{-(F_{ij}^{+}+F_j^{*+})}{2}} - \left(\frac{\sqrt{(-F_{ij}^{+})}+\sqrt{(-F_j^{*+})}}{2}\right) \end{array} \right] \quad (5)$$

Step 4. The smaller value of $C_B(B_i, p^*)_w$, i = 1, 2, ..., *m* represents that an alternative B_i, i = 1, 2, …, *m* is closer to the PIS p^*. Therefore, an alternative with the smallest weighted cross entropy measure means the best alternative.

5.2 MADM method based on cross entropy measures of IBNSs

The steps for solving MADM problems with interval bipolar neutrosophic information are provided as follows.

Step 1. In interval bipolar neutrosophic environment, the rating of performance value of alternative B_i (i = 1, 2, …, *m*) with respect to the predefined attribute C_j (j = 1, 2, …, *n*) can be represented as follows:

B_i = {C_j, < [$\inf T_{B_i}^{+}(C_j)$, $\sup T_{B_i}^{+}(C_j)$], [$\inf I_{B_i}^{+}(C_j)$, $\sup I_{B_i}^{+}(C_j)$], [$\inf F_{B_i}^{+}(C_j)$, $\sup F_{B_i}^{+}(C_j)$], [$\inf T_{B_i}^{-}(C_j)$, $\sup T_{B_i}^{-}(C_j)$], [$\inf I_{B_i}^{-}(C_j)$, $\sup I_{B_i}^{-}(C_j)$], [$\inf F_{B_i}^{-}(C_j)$, $\sup F_{B_i}^{-}(C_j)$] > | $C_j \in C_j$, j = 1, 2, …, *n*},

where $0 \le \sup T_{B_i}^{+}(C_j) + \sup I_{B_i}^{+}(C_j) + \sup F_{B_i}^{+}(C_j) \le 3$, and $-3 \le \sup T_{B_i}^{-}(C_j) + \sup I_{B_i}^{-}(C_j) + \sup F_{B_i}^{-}(C_j) \le 0$.; j = 1, 2, …, *n*. Consider $\tilde{g}_{ij}$ = <[${}^{L}T_{ij}^{+}, {}^{U}T_{ij}^{+}$], [${}^{L}I_{ij}^{+}, {}^{U}I_{ij}^{+}$], [${}^{L}F_{ij}^{+}, {}^{U}F_{ij}^{+}$], [${}^{L}T_{ij}^{-}, {}^{U}T_{ij}^{-}$], [${}^{L}I_{ij}^{-}, {}^{U}I_{ij}^{-}$], [${}^{L}F_{ij}^{-}, {}^{U}F_{ij}^{-}$]> be the bipolar neutrosophic decision matrix whose entries are the rating values of the alternatives with respect to the attributes provided by the expert or decision maker. The interval bipolar neutrosophic decision matrix $[\tilde{g}_{ij}]_{m\times n}$ can be presented as follows:

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

$$
[\tilde{g}_{ij}]_{m\times n} = \begin{matrix} \\ B_1 \\ B_2 \\ . \\ . \\ B_m \end{matrix} \begin{matrix} C_1 & C_2 & \cdots & C_n \\ \end{matrix} \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{pmatrix}
$$

Step 2. The PIS $\langle q^* = (g_1^*, g_2^*, \ldots, g_n^*)\rangle$ of interval bipolar neutrosophic information is obtained as follows:

$$
q_j^* = \langle [{}^LT_{ij}^{*+}, {}^UT_{ij}^{*+}], [{}^LI_{ij}^{*+}, {}^UI_{ij}^{*+}], [{}^LF_{ij}^{*+}, {}^UF_{ij}^{*+}], [{}^LT_{ij}^{*-}, {}^UT_{ij}^{*-}], [{}^LI_{ij}^{*-}, {}^UI_{ij}^{*-}], [{}^LF_{ij}^{*-}, {}^UF_{ij}^{*-}] \rangle =
$$

$\langle[\{\operatorname{Max}_i({}^L\mathrm{T}_{ij}^+) \mid j\in H_1\}; \{\operatorname{Min}_i({}^L\mathrm{T}_{ij}^+) \mid j\in H_2\}, \{\operatorname{Max}_i({}^U\mathrm{T}_{ij}^+) \mid j\in H_1\}; \{\operatorname{Min}_i({}^U\mathrm{T}_{ij}^+) \mid j\in H_2\}], [\{\operatorname{Min}_i({}^L\mathrm{I}_{ij}^+) \mid j\in H_1\}; \{\operatorname{Max}_i({}^L\mathrm{I}_{ij}^+) \mid j\in H_2\}, \{\operatorname{Min}_i({}^U\mathrm{I}_{ij}^+) \mid j\in H_1\}; \{\operatorname{Max}_i({}^U\mathrm{I}_{ij}^+) \mid j\in H_2\}], [\{\operatorname{Min}_i({}^L\mathrm{F}_{ij}^+) \mid j\in H_1\}; \{\operatorname{Max}_i({}^L\mathrm{F}_{ij}^+) \mid j\in H_2\}, \{\operatorname{Min}_i({}^U\mathrm{F}_{ij}^+) \mid j\in H_1\}; \{\operatorname{Max}_i({}^U\mathrm{F}_{ij}^+) \mid j\in H_2\}], [\{\operatorname{Min}_i({}^L\mathrm{T}_{ij}^-) \mid j\in H_1\}; \{\operatorname{Max}_i({}^L\mathrm{T}_{ij}^-) \mid j\in H_2\}, \{\operatorname{Min}_i({}^U\mathrm{T}_{ij}^-) \mid j\in H_1\}; \{\operatorname{Max}_i({}^U\mathrm{T}_{ij}^-) \mid j\in H_2\}], [\{\operatorname{Max}_i({}^L\mathrm{I}_{ij}^-) \mid j\in H_1\}; \{\operatorname{Min}_i({}^L\mathrm{I}_{ij}^-) \mid j\in H_2\}, \{\operatorname{Max}_i({}^U\mathrm{I}_{ij}^-) \mid j\in H_1\}; \{\operatorname{Min}_i({}^U\mathrm{I}_{ij}^-) \mid j\in H_2\}], [\{\operatorname{Max}_i({}^L\mathrm{F}_{ij}^-) \mid j\in H_1\}; \{\operatorname{Min}_i({}^L\mathrm{F}_{ij}^-)\},$ $\{\operatorname{Max}_i({}^U\mathrm{F}_{ij}^-) \mid j\in H_1\}; \{\operatorname{Min}_i({}^U\mathrm{F}_{ij}^-)\ j\in H_2\}]\rangle$, $j = 1, 2, \ldots, n$;

where H_1 and H_2 represent benefit and cost type attributes respectively.

Step 3. The weighted cross entropy between an alternative B_i, i = 1, 2, …, m, and the ideal alternative q^* under interval bipolar neutrosophic setting is computed as follows:

$$
C_{IB}(B_i, q^*)_w = \frac{1}{2} \times \sum_{i=1}^{n} w_i =
$$

$$
\left[\begin{array}{l}
\sqrt{\frac{{}^LT_{ij}^+ + {}^LT_j^{*+}}{2}} - \left(\frac{\sqrt{{}^LT_{ij}^+} + \sqrt{{}^LT_j^{*+}}}{2}\right) + \sqrt{\frac{{}^UT_{ij}^+ + {}^UT_j^{*+}}{2}} - \left(\frac{\sqrt{{}^UT_{ij}^+} + \sqrt{{}^UT_j^{*+}}}{2}\right) + \sqrt{\frac{{}^LI_{ij}^+ + {}^LI_j^{*+}}{2}} - \left(\frac{\sqrt{{}^LI_{ij}^+} + \sqrt{{}^LI_j^{*+}}}{2}\right) + \\
\sqrt{\frac{[1-{}^LI_{ij}^+] + [1-{}^LI_j^{*+}]}{2}} - \left(\frac{\sqrt{1-{}^LI_j^+} + \sqrt{[1-{}^LI_j^{*+}]}}{2}\right) + \sqrt{\frac{{}^UI_{ij}^+ + {}^UI_j^{*+}}{2}} - \left(\frac{\sqrt{{}^UI_{ij}^+} + \sqrt{{}^UI_j^{*+}}}{2}\right) + \sqrt{\frac{[1-{}^UI_{ij}^+] + [1-{}^UI_j^{*+}]}{2}} - \\
\left(\frac{\sqrt{1-{}^UI_j^+} + \sqrt{[1-{}^UI_j^{*+}]}}{2}\right) + \sqrt{\frac{{}^LF_{ij}^+ + {}^LF_j^{*+}}{2}} - \left(\frac{\sqrt{{}^LF_{ij}^+} + \sqrt{{}^LF_j^{*+}}}{2}\right) + \sqrt{\frac{{}^UF_{ij}^+ + {}^UF_j^{*+}}{2}} - \left(\frac{\sqrt{{}^UF_{ij}^+} + \sqrt{{}^UF_j^{*+}}}{2}\right) + \sqrt{\frac{-({}^LT_{ij}^+ + {}^LT_j^{*+})}{2}} - \\
\left(\frac{\sqrt{(-{}^LT_{ij}^+)} + \sqrt{(-{}^LT_j^{*+})}}{2}\right) + \sqrt{\frac{-({}^UT_{ij}^+ + {}^UT_j^{*+})}{2}} - \left(\frac{\sqrt{(-{}^UT_{ij}^+)} + \sqrt{(-{}^UT_j^{*+})}}{2}\right) + \sqrt{\frac{-({}^LI_{ij}^+ + {}^LI_j^{*+})}{2}} - \left(\frac{\sqrt{(-{}^LI_{ij}^+)} + \sqrt{(-{}^LI_j^{*+})}}{2}\right) + \\
\sqrt{\frac{-({}^UI_{ij}^+ + {}^UI_j^{*+})}{2}} - \left(\frac{\sqrt{(-{}^UI_{ij}^+)} + \sqrt{(-{}^UI_j^{*+})}}{2}\right) + \sqrt{\frac{[1+{}^LI_{ij}^+] + [1+{}^LI_j^{*+}]}{2}} - \left(\frac{\sqrt{1+{}^LI_{ij}^+} + \sqrt{[1+{}^LI_j^{*+}]}}{2}\right) + \sqrt{\frac{[1+{}^UI_{ij}^+] + [1+{}^UI_j^{*+}]}{2}} - \\
\left(\frac{\sqrt{1+{}^UI_{ij}^+} + \sqrt{[1+{}^UI_j^{*+}]}}{2}\right) + \sqrt{\frac{-({}^LF_{ij}^+ + {}^LF_j^{*+})}{2}} - \left(\frac{\sqrt{(-{}^LF_{ij}^+)} + \sqrt{(-{}^LF_j^{*+})}}{2}\right) + \sqrt{\frac{-({}^UF_{ij}^+ + {}^UF_j^{*+})}{2}} - \left(\frac{\sqrt{(-{}^UF_{ij}^+)} + \sqrt{(-{}^UF_j^{*+})}}{2}\right)
\end{array}\right] \quad (6)
$$

Step 4. The smaller value of C_{IB} (B_i, p^*)$_w$, i = 1, 2, ..., m indicates that an alternative B_i, i = 1, 2, …, m is closer to the PIS q^*. Hence, an alternative with the smallest weighted cross entropy measure will be identified as the best alternative.

A conceptual model of the proposed approach is shown in Figure 2.

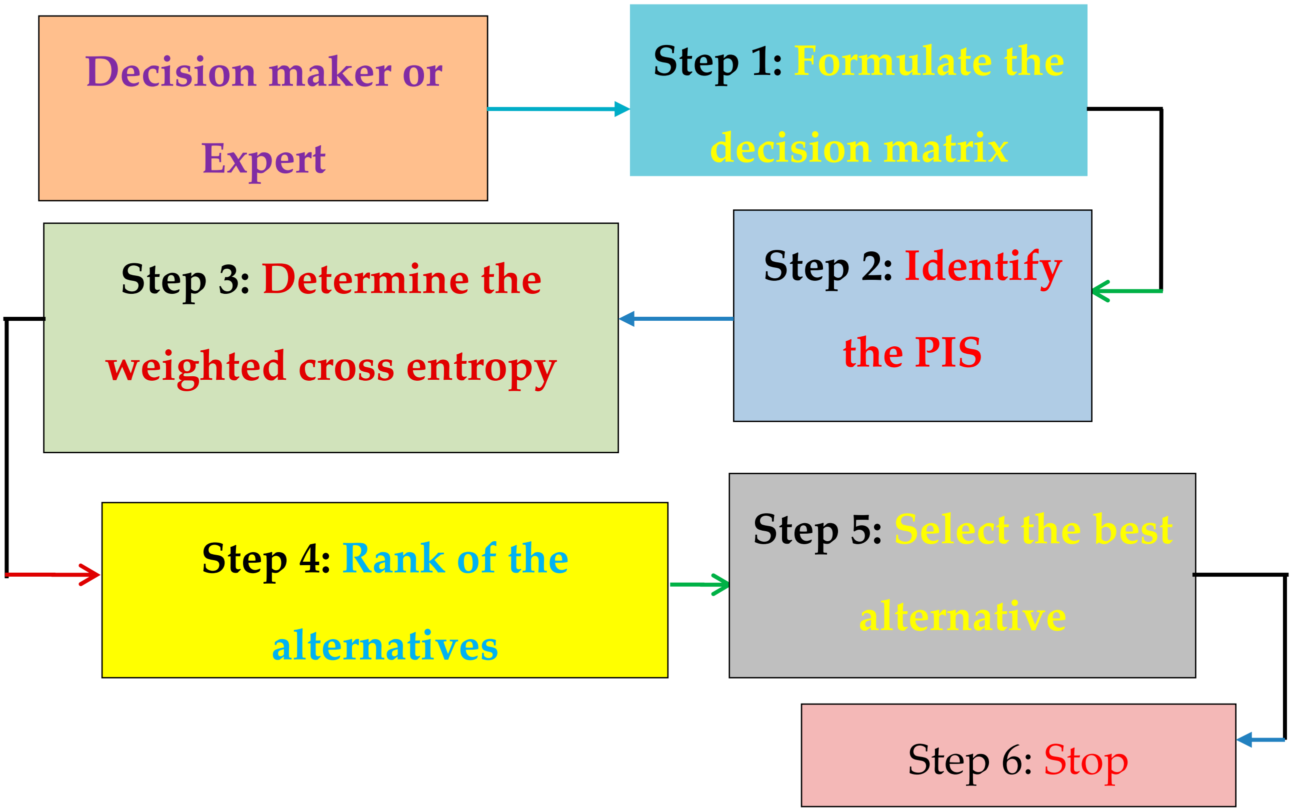

Figure2. Conceptual model of the proposed approach

6. Illustrative example

In this section we solve two numerical MADM problems and comparison with other existing methods is presented to verify the applicability and effectiveness of the proposed approach under bipolar neutrosophic and interval bipolar neutrosophic environments.

6.1 Car selection problem with bipolar neutrosophic information

Consider the problem discussed in [13, 18, 19, 20] where a buyer wants to purchase a car based on some predefined attributes. Suppose that there are four types cars (alternatives) B_i, (i = 1, 2, 3, 4) are available in the market. Four attributes are taken into consideration in the decision making environment namely Fuel economy (C_1), Aerod (C_2), Comfort (C_3), Safety (C_4) to select the most desirable car. Assume the weight vector for the four attributes are known and given by w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125). Therefore the bipolar neutrosophic decision matrix $\langle d_{ij} \rangle_{4\times4}$ can be obtained as given below.

The bipolar neutrosophic decision matrix $[\tilde{d}_{ij}]_{4\times4}$ =

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

	C_1	C_2	C_3	C_4
B_1	<0.5, 0.7, 0.2, -0.7, -0.3, -0.6>	<0.4, 0.4, 0.5, -0.7, -0.8, -0.4>	<0.7, 0.7, 0.5, -0.8, -0.7, -0.6>	<0.1, 0.5, 0.7, -0.5, -0.2, -0.8>
B_2	<0.9, 0.7, 0.5, -0.7, -0.7, -0.1>	<0.7, 0.6, 0.8, -0.7, -0.5, -0.1>	<0.9, 0.4, 0.6, -0.1, -0.7, -0.5>	<0.5, 0.2, 0.7, -0.5, -0.1, -0.9>
B_3	<0.3, 0.4, 0.2, -0.6, -0.3, -0.7>	<0.2, 0.2, 0.2, -0.4, -0.7, -0.4>	<0.9, 0.5, 0.5, -0.6, -0.5, -0.2>	<0.7, 0.5, 0.3, -0.4, -0.2, -0.2>
B_4	<0.9, 0.7, 0.2, -0.8, -0.6, -0.1>	<0.3, 0.5, 0.2, -0.5, -0.5, -0.2>	<0.5, 0.4, 0.5, -0.1, -0.7, -0.2>	<0.2, 0.4, 0.8, -0.5, -0.5, -0.6>

The positive ideal bipolar neutrosophic solutions are computed from $[\tilde{d}_{ij}]_{4\times4}$ as follows:

p^* = [<0.9, 0.4, 0.2, -0.8, -0.3, -0.1>, <0.7, 0.2, 0.2, -0.7, -0.5, -0.1>, <0.9, 0.4, 0.5, -0.8, -0.5, -0.2>, <0.7, 0.2, 0.3, -0.5, -0.1, -0.2>].

Using Eq. (5), the weighted cross entropy measure $C_B(B_i, p^*)_w$ is obtained as follows:

$C_B(B_1, p^*)_w = 0.0734$, $C_B(B_2, p^*)_w = 0.0688$, $C_B(B_3, p^*)_w = 0.0642$, $C_B(B_4, p^*)_w = 0.0516$.

According to weighted cross entropy measure $C_B(B_i, p^*)_w$ the order of the four alternatives is $B_4 \prec B_3 \prec B_2 \prec B_1$ and therefore B_4 is the best car.

We compare our obtained result with the results of other existing methods (see Table 1), where the known weight of the attribute is given by $w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125)$. It is to be noted that the ranking results obtained from the other existing methods are different from the result of the proposed method in some cases. The reason is that the different authors adopted different decision making methods and thereby obtained different ranking results. However, the proposed method is simple and straightforward and can effectively solve decision making problems with bipolar neutrosophic information.

Table 1. The results of car selection problem obtained from different methods

Methods	Ranking results	Best option
The proposed weighted cross entropy measure	$B_4 \prec B_3 \prec B_2 \prec B_1$	B_4
TOPSIS method [19]	$B_1 \prec B_3 \prec B_2 \prec B_4$	B_4
Deli et al.'s method [13]	$B_1 \prec B_2 \prec B_4 \prec B_3$	B_3
Projection measure [20]	$B_3 \prec B_4 \prec B_1 \prec B_2$	B_2
Bidirectional projection measure [20]	$B_2 \prec B_1 \prec B_4 \prec B_3$	B_3
Hybrid projection measure [20] with $\rho = 0.25$	$B_2 \prec B_1 \prec B_3 \prec B_4$	B_4
Hybrid projection measure [20] with $\rho = 0.50$	$B_3 \prec B_2 \prec B_1 \prec B_4$	B_4
Hybrid projection measure [20] with $\rho = 0.75$	$B_1 \prec B_3 \prec B_4 \prec B_2$	B_2
Hybrid projection measure [20] with $\rho = 0.90$	$B_3 \prec B_4 \prec B_2 \prec B_1$	B_1
Hybrid similarity measure [18] with $\rho = 0.25$	$B_2 \prec B_4 \prec B_1 \prec B_3$	B_3
Hybrid similarity measure [18] with $\rho = 0.30$	$B_2 \prec B_4 \prec B_1 \prec B_3$	B_3
Hybrid similarity measure [18] with $\rho = 0.60$	$B_2 \prec B_4 \prec B_1 \prec B_3$	B_3

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

Hybrid similarity measure [18] with $\rho = 0.90$	$B_2 \prec B_4 \prec B_3 \prec B_1$	B_1

6.2 Interval bipolar neutrosophic MADM investment problem

Consider an interval bipolar neutrosophic MADM problem studied in [24] with four possible alternatives to invest a sum of money in the best choice. The four alternatives are:

> a food company (B_1),
> a car company (B_2),
> a arm company (B_3), and
> a car computer (B_4).

The investment company selects the best option based on the three predefined attributes namely growth analysis (C_1), risk analysis (C_2), and environment analysis (C_3). We consider C_1, C_2 are benefit type attributes and C_3 is cost type attribute based on Ye [26]. Assume that the weight vector [24] of C_1, C_2, and C_3 is given by $w = (w_1, w_2, w_3) = (0.35, 0.25, 0.4)$. The interval bipolar neutrosophic decision matrix $[\tilde{g}_{ij}]_{4\times3}$ presented by the decision maker or expert

Interval bipolar neutrosophic decision matrix $[\tilde{g}_{ij}]_{4\times3} =$

C_1

$$\begin{pmatrix} B_1 & [[0.4,0.5],[0.2,0.3],[0.3,0.4],[-0.3,-0.2],[-0.4,-0.3],[-0.5,-0.4]] \\ B_2 & [[0.6,0.7],[0.1,0.2],[0.2,0.3],[-0.2,-0.1],[-0.3,-0.2],[-0.7,-0.6]] \\ B_3 & [[0.3,0.6],[0.2,0.3],[0.3,0.4],[-0.3,-0.2],[-0.4,-0.3],[-0.6,-0.3]] \\ B_4 & [[0.7,0.8],[0.0,0.1],[0.1,0.2],[-0.1,-0.0],[-0.2,-0.1],[-0.8,-0.7]] \end{pmatrix}$$

C_2

$$\begin{pmatrix} B_1 & [[0.4,0.6],[0.1,0.3],[0.2,0.4],[-0.3,-0.1],[-0.4,-0.2],[-0.6,-0.4]] \\ B_2 & [[0.6,0.7],[0.1,0.2],[0.2,0.3],[-0.2,-0.1],[-0.3,-0.2],[-0.7,-0.6]] \\ B_3 & [[0.5,0.6],[0.2,0.3],[0.3,0.4],[-0.3,-0.2],[-0.4,-0.3],[-0.6,-0.5]] \\ B_4 & [[0.6,0.7],[0.1,0.2],[0.1,0.3],[-0.2-0.1],[-0.3,-0.1],[-0.7,-0.6]] \end{pmatrix}$$

C_3

$$\begin{pmatrix} B_1 & [[0.7,0.9],[0.2,0.3],[0.4,0.5],[-0.3,-0.2],[-0.5,-0.4],[-0.9,-0.7]] \\ B_2 & [[0.3,0.6],[0.3,0.5],[0.8,0.9],[-0.5,-0.3],[-0.9,-0.8],[-0.6,-0.3]] \\ B_3 & [[0.4,0.5],[0.2,0.4],[0.7,0.9],[-0.4,-0.2],[-0.9,-0.7],[-0.5,-0.4]] \\ B_4 & [[0.6,0.7],[0.3,0.4],[0.8,0.9],[-0.4,-0.3],[-0.9,-0.8],[-0.7,-0.6]] \end{pmatrix}$$

From the matrix $[\tilde{g}_{ij}]_{4\times3}$ we determine positive ideal interval bipolar neutrosophic solution (q^*) by using Eq. (6) as follows:

q^*= < [0.7, 0.8], [0.0, 0.1], [0.1, 0.2], [-0.3, -0.2], [-0.2, -0.1], [-0.5, -0.3]; < [0.6, 0.7], [0.1, 0.2], [0.1, 0.3], [-0.3, -0.2], [-0.3, -0.1], [-0.6, -0.4]; < [0.3, 0.5], [0.3, 0.5], [0.8, 0.9], [-0.3, -0.2], [-0.9, -0.8], [-0.9, -0.7].

The weighted cross entropy between an alternative B_i, i = 1, 2, …, m, and the ideal alternative q^* can be obtained as given below.

$C_{IB}(B_1, q^*)_w = 0.0606$, $C_{IB}(B_2, q^*)_w = 0.0286$, $C_{IB}(B_3, q^*)_w = 0.0426$, $C_{IB}(B_4, q^*)_w = 0.0423$.

On the basis of weighted cross entropy measure $C_{IB}(B_i, q^*)_w$ the order of the four alternatives is $B_2 \prec B_4 \prec B_3 \prec B_1$ and therefore B_2 is the best choice.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 January 2018 doi:10.20944/preprints201801.0065.v1

Next, the comparison of the results obtained from different methods is presented in Table 2 where the weight vector of the attribute is given by $w = (w_1, w_2, w_3) = (0.35, 0.25, 0.4)$. We observe that B_2 is the best option obtained by the proposed method and B_4 is the best option obtained by the method of Mahmood *et al.* [24]. The reason may be that we use interval bipolar neutrosophic cross entropy method whereas Mahmood *et al.* [24] derived the most desirable alternative based on weighted arithmetic average operator under interval bipolar neutrosophic setting.

Table 2. The results of the investment problem obtained from different methods

Methods	Ranking results	Best option
The proposed weighted cross entropy measure	$B_2 \prec B_4 \prec B_3 \prec B_1$	B_2
Mahmood *et al.*'s method [24]	$B_2 \prec B_3 \prec B_1 \prec B_4$	B_4

7. Conclusion

In the paper we define cross entropy and weighted cross entropy measures of bipolar neutrosophic sets and prove their basic properties. We also extend the proposed concept to interval bipolar neutrosophic environment and prove their basic properties. The proposed cross entropy measures are then employed to develop two new multi-attribute decision making models. Two illustrative numerical examples are solved and finally comparisons with existing methods are provided to demonstrate the simplicity and efficiency of the proposed approach. We hope that the proposed cross entropy measures can be effective in dealing with group decision making, weaver selection, data analysis, medical diagnosis problems. In the future, the cross entropy measures can be extended to the other environments, such as bipolar neutrosophic soft expert sets, bipolar neutrosophic refined sets.

Author Contributions: "S. Pramanik and P.P. Dey conceived and designed the experiments; S. Pramanik performed the experiments; J. Ye and F. Smarandache analyzed the data; S. Pramanik and P.P. Dey wrote the paper."

Conflicts of Interest: "The authors declare no conflict of interest."

Appendix A

Proof of theorem 2

(1) From the inequality stated in theorem 1, we can easily obtain $C_B(M, N)_w \geq 0$.

(2) $C_B(M, N)_w = 0$ if and only if $M = N$,

i.e., $T_M^+(x_i) = T_N^+(x_i)$, $I_M^+(x_i) = I_N^+(x_i)$, $F_M^+(x_i) = F_N^+(x_i)$, $T_M^-(x_i) = T_N^-(x_i)$, $I_M^-(x_i) = I_N^-(x_i)$, $F_M^-(x_i) = F_N^-(x_i)$

$\forall x \in U$.

(3) $C_B(M, N)_w =$

$$\sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{T_M^+(x_i)+T_N^+(x_i)}{2}} - \left(\frac{\sqrt{T_M^+(x_i)}+\sqrt{T_N^+(x_i)}}{2}\right) + \sqrt{\frac{I_M^+(x_i)+I_N^+(x_i)}{2}} - \left(\frac{\sqrt{I_M^+(x_i)}+\sqrt{I_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{(1-I_M^+(x_i))+(1-I_N^+(x_i))}{2}} - \left(\frac{\sqrt{1-I_M^+(x_i)}+\sqrt{1-I_N^+(x_i)}}{2}\right) + \sqrt{\frac{F_M^+(x_i)+F_N^+(x_i)}{2}} - \left(\frac{\sqrt{F_M^+(x_i)}+\sqrt{F_N^+(x_i)}}{2}\right) + \\ \sqrt{\frac{-(T_M^-(x_i)+T_N^-(x_i))}{2}} - \left(\frac{\sqrt{(-T_M^-(x_i))}+\sqrt{(-T_N^-(x_i))}}{2}\right) + \sqrt{\frac{-(I_M^-(x_i)+I_N^-(x_i))}{2}} - \left(\frac{\sqrt{(-I_M^-(x_i))}+\sqrt{(-I_N^-(x_i))}}{2}\right) + \\ \sqrt{\frac{(1+I_M^-(x_i))+(1+I_N^-(x_i))}{2}} - \left(\frac{\sqrt{1+I_M^-(x_i)}+\sqrt{1+I_N^-(x_i)}}{2}\right) + \sqrt{\frac{-(F_M^-(x_i)+F_N^-(x_i))}{2}} - \left(\frac{\sqrt{(-F_M^-(x_i))}+\sqrt{(-F_N^-(x_i))}}{2}\right) \end{array} \right]$$

$$= \sum_{i=1}^{n} w_i \left[\begin{array}{l} \sqrt{\frac{T_N^+(x_i)+T_M^+(x_i)}{2}} - \left(\frac{\sqrt{T_N^+(x_i)}+\sqrt{T_M^+(x_i)}}{2}\right) + \sqrt{\frac{I_N^+(x_i)+I_M^+(x_i)}{2}} - \left(\frac{\sqrt{I_N^+(x_i)}+\sqrt{I_M^+(x_i)}}{2}\right) + \\ \sqrt{\frac{(1-I_N^+(x_i))+(1-I_M^+(x_i))}{2}} - \left(\frac{\sqrt{1-I_N^+(x_i)}+\sqrt{1-I_M^+(x_i)}}{2}\right) + \sqrt{\frac{F_N^+(x_i)+F_M^+(x_i)}{2}} - \left(\frac{\sqrt{F_N^+(x_i)}+\sqrt{F_M^+(x_i)}}{2}\right) + \\ \sqrt{\frac{-(T_N^-(x_i)+T_M^+(x_i))}{2}} - \left(\frac{\sqrt{(-T_N^-(x_i))}+\sqrt{(-T_M^-(x_i))}}{2}\right) + \sqrt{\frac{-(I_N^-(x_i)+I_M^-(x_i))}{2}} - \left(\frac{\sqrt{(-I_N^-(x_i))}+\sqrt{(-I_M^-(x_i))}}{2}\right) + \\ \sqrt{\frac{(1+I_N^-(x_i))+(1+I_M^-(x_i))}{2}} - \left(\frac{\sqrt{1+I_N^-(x_i)}+\sqrt{1+I_M^-(x_i)}}{2}\right) + \sqrt{\frac{-(F_N^-(x_i)+F_M^-(x_i))}{2}} - \left(\frac{\sqrt{(-F_N^-(x_i))}+\sqrt{(-F_M^-(x_i))}}{2}\right) \end{array} \right]$$

$= C_B(N, M)_w$.

Appendix B

Proof of Theorem 4

(1) Obviously, we can easily get $C_{IB}(R, S)_w \geq 0$.

(2) If $C_{IB}(R, S)_w = 0$ then $R = S$ and if $\inf T_R^+(x_i) = \inf T_S^+(x_i)$, $\sup T_R^+(x_i) = \sup T_S^+(x_i)$, $\inf I_R^+(x_i) = \inf I_S^+(x_i)$, $\sup I_R^+(x_i) = \sup I_S^+(x_i)$, $\inf F_R^+(x_i) = \inf F_S^+(x_i)$, $\sup F_R^+(x_i) = \sup F_S^+(x_i)$, $\inf T_R^-(x_i) = \inf T_S^-(x_i)$, $\sup T_R^-(x_i) = \sup T_S^-(x_i)$, $\inf I_R^-(x_i) = \inf I_S^-(x_i)$, $\sup I_R^-(x_i) = \sup I_S^-(x_i)$, $\inf F_R^-(x_i) = \inf F_S^-(x_i)$, $\sup F_R^-(x_i) = \sup F_S^-(x_i)$ $\forall x \in U$, then we have obtain $C_{IB}(R, S) = 0$.

(3) $C_{IB}(R, S)$

$$
=\frac{1}{2}\sum_{i=1}^{n} w_i \left\langle \left[\begin{aligned}
&\sqrt{\frac{\inf T_R^+(x_i)+\inf T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf T_R^+(x_i)}+\sqrt{\inf T_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup T_R^+(x_i)+\sup T_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup T_R^+(x_i)}+\sqrt{\sup T_S^+(x_i)}}{2}\right)+\\
&\sqrt{\frac{\inf I_R^+(x_i)+\inf I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf I_R^+(x_i)}+\sqrt{\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\sup I_R^+(x_i)+\sup I_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup I_R^+(x_i)}+\sqrt{\sup I_S^+(x_i)}}{2}\right)+\\
&\sqrt{\frac{(1-\inf I_R^+(x_i))+(1-\inf I_S^+(x_i))}{2}}-\left(\frac{\sqrt{1-\inf I_R^+(x_i)}+\sqrt{1-\inf I_S^+(x_i)}}{2}\right)+\sqrt{\frac{(1-\sup I_R^+(x_i))+(1-\sup I_S^+(x_i))}{2}}-\\
&\left(\frac{\sqrt{1-\sup I_R^+(x_i)}+\sqrt{1-\sup I_S^+(x_i)}}{2}\right)+\sqrt{\frac{\inf F_R^+(x_i)+\inf F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\inf F_R^+(x_i)}+\sqrt{\inf F_S^+(x_i)}}{2}\right)+\\
&\sqrt{\frac{\sup F_R^+(x_i)+\sup F_S^+(x_i)}{2}}-\left(\frac{\sqrt{\sup F_R^+(x_i)}+\sqrt{\sup F_S^+(x_i)}}{2}\right)+\sqrt{\frac{-(\inf T_R^-(x_i)+\inf T_S^+(x_i))}{2}}-\\
&\left(\frac{\sqrt{(-\inf T_R^-(x_i))}+\sqrt{(-\inf T_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup T_R^-(x_i)+\sup T_S^+(x_i))}{2}}-\left(\frac{\sqrt{(-\sup T_R^-(x_i))}+\sqrt{(-\sup T_S^-(x_i))}}{2}\right)+\\
&\sqrt{\frac{-(\inf I_R^-(x_i)+\inf I_S^-(x_i))}{2}}-\left(\frac{\sqrt{(-\inf I_R^-(x_i))}+\sqrt{(-\inf I_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup I_R^-(x_i)+\sup I_S^-(x_i))}{2}}-\\
&\left(\frac{\sqrt{(-\sup I_R^-(x_i))}+\sqrt{(-\sup I_S^-(x_i))}}{2}\right)+\sqrt{\frac{(1+\inf I_R^-(x_i))+(1+\inf I_S^-(x_i))}{2}}-\left(\frac{\sqrt{1+\inf I_R^-(x_i)}+\sqrt{1+\inf I_S^-(x_i)}}{2}\right)+\\
&\sqrt{\frac{(1+\sup I_R^-(x_i))+(1+\sup I_S^-(x_i))}{2}}-\left(\frac{\sqrt{1+\sup I_R^-(x_i)}+\sqrt{1+\sup I_S^-(x_i)}}{2}\right)+\sqrt{\frac{-(\inf F_R^-(x_i)+\inf F_S^-(x_i))}{2}}-\\
&\left(\frac{\sqrt{(-\inf F_R^-(x_i))}+\sqrt{(-\inf F_S^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup F_R^-(x_i)+\sup F_S^-(x_i))}{2}}-\left(\frac{\sqrt{(-\sup F_R^-(x_i))}+\sqrt{(-\sup F_S^-(x_i))}}{2}\right)
\end{aligned} \right] \right\rangle
$$

$$
=\frac{1}{2}\sum_{i=1}^{n} w_i \left\langle \left[\begin{aligned}
&\sqrt{\frac{\inf T_S^+(x_i)+\inf T_R^+(x_i)}{2}}-\left(\frac{\sqrt{\inf T_S^+(x_i)}+\sqrt{\inf T_R^+(x_i)}}{2}\right)+\sqrt{\frac{\sup T_S^+(x_i)+\sup T_R^+(x_i)}{2}}-\left(\frac{\sqrt{\sup T_S^+(x_i)}+\sqrt{\sup T_R^+(x_i)}}{2}\right)+\\
&\sqrt{\frac{\inf I_S^+(x_i)+\inf I_R^+(x_i)}{2}}-\left(\frac{\sqrt{\inf I_S^+(x_i)}+\sqrt{\inf I_R^+(x_i)}}{2}\right)+\sqrt{\frac{\sup I_S^+(x_i)+\sup I_R^+(x_i)}{2}}-\left(\frac{\sqrt{\sup I_S^+(x_i)}+\sqrt{\sup I_R^+(x_i)}}{2}\right)+\\
&\sqrt{\frac{(1-\inf I_S^+(x_i))+(1-\inf I_R^+(x_i))}{2}}-\left(\frac{\sqrt{1-\inf I_S^+(x_i)}+\sqrt{1-\inf I_R^+(x_i)}}{2}\right)+\sqrt{\frac{(1-\sup I_S^+(x_i))+(1-\sup I_R^+(x_i))}{2}}-\\
&\left(\frac{\sqrt{1-\sup I_S^+(x_i)}+\sqrt{[1-\sup I_R^+(x_i)]}}{2}\right)+\sqrt{\frac{\inf F_S^+(x_i)+\inf F_R^+(x_i)}{2}}-\left(\frac{\sqrt{\inf F_S^+(x_i)}+\sqrt{\inf F_R^+(x_i)}}{2}\right)+\\
&\sqrt{\frac{\sup F_S^+(x_i)+\sup F_R^+(x_i)}{2}}-\left(\frac{\sqrt{\sup F_S^+(x_i)}+\sqrt{\sup F_R^+(x_i)}}{2}\right)+\sqrt{\frac{-(\inf T_S^-(x_i)+\inf T_R^+(x_i))}{2}}-\\
&\left(\frac{\sqrt{(-\inf T_S^-(x_i))}+\sqrt{(-\inf T_R^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup T_S^-(x_i)+\sup T_R^+(x_i))}{2}}-\left(\frac{\sqrt{(-\sup T_S^-(x_i))}+\sqrt{(-\sup T_R^-(x_i))}}{2}\right)+\\
&\sqrt{\frac{-(\inf I_S^-(x_i)+\inf I_R^-(x_i))}{2}}-\left(\frac{\sqrt{(-\inf I_S^-(x_i))}+\sqrt{(-\inf I_R^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup I_S^-(x_i)+\sup I_R^-(x_i))}{2}}-\\
&\left(\frac{\sqrt{(-\sup I_S^-(x_i))}+\sqrt{(-\sup I_R^-(x_i))}}{2}\right)+\sqrt{\frac{(1+\inf I_S^-(x_i))+(1+\inf I_R^-(x_i))}{2}}-\left(\frac{\sqrt{1+\inf I_S^-(x_i)}+\sqrt{1+\inf I_R^-(x_i)}}{2}\right)+\\
&\sqrt{\frac{(1+\sup I_S^-(x_i))+(1+\sup I_R^-(x_i))}{2}}-\left(\frac{\sqrt{1+\sup I_S^-(x_i)}+\sqrt{[1+\sup I_R^-(x_i)]}}{2}\right)+\sqrt{\frac{-(\inf F_S^-(x_i)+\inf F_R^-(x_i))}{2}}-\\
&\left(\frac{\sqrt{(-\inf F_S^-(x_i))}+\sqrt{(-\inf F_R^-(x_i))}}{2}\right)+\sqrt{\frac{-(\sup F_S^-(x_i)+\sup F_R^-(x_i))}{2}}-\left(\frac{\sqrt{(-\sup F_S^-(x_i))}+\sqrt{(-\sup F_R^-(x_i))}}{2}\right)
\end{aligned} \right] \right\rangle
$$

$= C_{IB}(R, S)$.

This completes the proof.

References

1. Shannon, C.E.; Weaver, W. *The mathematical theory of communications*; The University of Illinois Press: Urbana, 1949.
2. Shannon, C.E. A mathematical theory of communications, *Bell. Syst. Tech J* .**1948**, *27*, 379-423.
3. Smarandache, F. *A unifying field of logics. Neutrosophy: Neutrosophic probability, set and logic*; American Research Press: Rehoboth, 1998.
4. Ye, J. Single valued neutrosophic cross-entropy for multicriteria decision making problems. *Appl. Math. Model*. **2014**, *38(3)*, 1170-1175, DOI:10.1016/j.apm.2013.07.020.
5. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. Single valued neutrosophic sets. *Multi-spa. Multi-Struc*. **2010**, *4*, 410-413.
6. Zadeh, LA. Fuzzy sets. *Inform. Cont*. **1965**, *8(3)*, 338-353, DOI:10.1016/S0019-9958(65)90241-X.

7. Shang, X.G.; Jiang, W.S. A note on information measures. *Patt. Recog. Lett*. **1997**, *18(5)*, 425-432, DOI:10.1016/S0167-8655(97)00028-7.
8. Şahin, M. Cross entropy measure on interval neutrosophic sets and its application in multi criteria decision making. *Neur. Comput. Appl*. **2017**, *28(5)*, 1177-1187, DOI: 10.1007/s00521-015-2131-5.
9. Tian, Z.; Zhang, H.; Wang, J.; Wang, J.; Chen, X. Multi-criteria decision making method based on a cross-entropy with interval neutrosophic sets. *Int. J. of Syst. Sci*. **2016**, *47(15)*, 3598-3608, DOI:10.1080/00207721.2015.110235.
10. Hwang, C.L.; Yoon, K.. *Multiple attribute decision making: methods and applications*; Springer: New York. 1981.
11. Ye, J. Improved cross entropy measures of single valued neutrosophic sets and their multi criteria decision making methods. *Cyber. Infor. Tech*, **2015**, *15(4)*, 13-26, DOI:10.1515/cait-2015-0051.
12. Wang, H.; Smarandache, F.; Zhang, Y.Q; Sunderraman, R. *Interval neutrosophic sets and logic: theory and applications in computing*, Hexis: Arizona, 2005.
13. Deli, I, Ali, M, Smarandache, F. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. In Proceedings of the 2015 International conference on advanced mechatronic systems (ICAMechS), Beijing, China, August 2015, 249-254.
14. Zhang, W.R. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. In Proceedings of IEEE Conference, 1994, 305-309, DOI: 10.1109/IJCF.1994.375115.
15. Zhang, W.R. Bipolar fuzzy sets. In Proceedings of the IEEE World Congress on Computational Science (FuzzIEEE), Anchorage, Alaska, 1998, 835-840, DOI: 10.1109/FUZZY.1998.687599.
16. Deli, I.; Subas Y.A. Multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure. In International Conference on mathematics and mathematics education (ICMME-2016), Frat University, Elazg, Turkey, 12-14 May 2016.
17. Şahin, M.; Deli, I.; Uluçay, V. Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making. In International Conference on Natural Science and Engineering (ICNASE'16), Killis, 19-20 March 2016.
18. Uluçay, V.; Deli, I.; Şahin, M. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neur. Comput. Appl*. **2016**, DOI:10.1007/s00521-016-2479-1.
19. Dey, P.P.; Pramanik, S.; Giri, B.C. TOPSIS for solving multi-attribute decision making problems under bi-polar neutrosophic environment. In *New Trends in Neutrosophic Theory and Applications*; Smarandache, F.; Pramanik, S.; Eds., Pons asbl: Brussells, 2016, 65-77.
20. Pramanik, S.; Dey, P.P.; Giri, B.C.; Smarandache, F. Bipolar neutrosophic projection based models for solving multi-attribute decision making problems. *Neut. Sets Systs*. **2017**, *15*, 74-83, DOI:10.5281/zenedo.570936.
21. Wang, L.; Zhang, H.; Wang, J. Frank Choquet Bonferroni operators of bipolar neutrosophic sets and their applications to multi-criteria decision-making problems. *Int. J. of Fuzzy Systs*. **2017**, DOI: 10.1007/s40815-017-0373-3.
22. Wu, X. Cross entropy measures of multivalued neutrosophic sets and its application in selecting middle-level manager. *Int. J. Uncert. Quant*. **2017**, 155-176, DOI: 101665/Int.J. UncertaintyQuantification2017019444.
23. Ezhilmaran, D.; Shankar, K. Morphism of bipolar intuitionistic fuzzy graphs. *J. Discrete Math. Sci. Crypt*. **2015**, *18(5)*, 605-621, DOI: 10.1080/09720529.2015.1013673.

 doi:10.20944/preprints201801.0065.v1

24. Mahmood, T.; Ye, J.; Khan, Q. Bipolar interval neutrosophic set and its application in multicriteria decision making. https://archive.org/details/BipolarIntervalNeutrosophicSet. Retrieved on October 9, 2017.
25. Deli, I.; Şubaş, Y.; Smarandache, F.; Ali, M. Interval valued bipolar neutrosophic sets and their application in pattern recognition. https://www.researchgate.net/publication/289587637. Retrieved on October 9, 2017.
26. Ye, J. Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. *J. Intell. Fuzzy Syst*. **2014**, *26(1)*, 165-172, DOI:10.3233/IFS-120724.