There is only one Fourier Transform

Jens V. Fischer *

Microwaves and Radar Institute, German Aerospace Center (DLR), Germany

Abstract
Four Fourier transforms are usually defined, the Integral Fourier transform, the Discrete-Time Fourier transform (DTFT), the Discrete Fourier transform (DFT) and the Integral Fourier transform for periodic functions. However, starting from their definitions, we show that all four Fourier transforms can be reduced to actually only one Fourier transform, the Fourier transform in the distributional sense.

Keywords
Integral Fourier Transform, Discrete-Time Fourier Transform (DTFT), Discrete Fourier Transform (DFT), Integral Fourier Transform for periodic functions, Fourier series, Poisson Summation Formula, periodization trick, interpolation trick

Introduction
The fact that “there is only one Fourier transform” actually needs no proof. It is commonly known and often discussed in the literature, for example in [1-2].

However frequently asked questions are: (1) What is the difference between calculating the Fourier series and the Fourier transform of a continuous function? (2) What is the difference between Discrete-Time Fourier Transform (DTFT) and Discrete Fourier Transform (DFT)? (3) When do we need which Fourier transform? and many others. The default answer today to all these questions is to invite the reader to have a look at the so-called Fourier-Poisson cube, e.g. in [2]. However, this advice may not be very helpful.

In this paper, in contrast to what has been done so far, we introduce two novel symbols, one \( \Box \) for discretization, i.e., to sample functions, and one \( \triangle \) for periodization, i.e., to create periodic functions from non-periodic ones. They will help us understanding the role of Poisson’s Summation Formula in describing transitions from one Fourier transform variant to another. The overall intention of this study is to provide simple means to get all these things sorted and to help the reader to gain more confidence in dealing with finite and infinite Fourier series as well as with different Fourier transform variants. To keep things simple, we only treat the one-dimensional case \( t \in \mathbb{R} \). For n-dimensional cases \( t \in \mathbb{R}^n \), \( n > 1 \) one may refer to [3-5].

Two Important Operations
There are two operations which are being strongly related to four different variants of the Fourier transform, i.e., discretization and periodization. For any real-valued \( T > 0 \) and \( \delta(t) \) being the Dirac impulse, let

\[
(\Box f)(t) := \sum_{k=-\infty}^{\infty} f(kT) \delta(t - kT)
\]

be the function that results from a discretization of \( f(t) \) and let

\[
(\triangle f)(t) := \sum_{k=-\infty}^{\infty} f(t - kT)
\]

be the function that results from a periodization of \( f(t) \), both defined as in [6-7]. Let us furthermore use four different Fourier transform definitions (see e.g. [2]), given in the Appendix. We then claim that starting from these definitions we are able to show that they reduce to

\[
F_{pers}: \quad F\{\triangle f\}(f(t)) = \frac{1}{T} \Box f(t)
\]

and

\[
F_{DFT}: \quad F\{\Box f\}(f(t)) = \triangle f(t)
\]

and

\[
F_{DIFF}: \quad F\{\Box f\}(f(t)) = \frac{1}{N} \Box f(t)
\]

and

\[
F_{DFT^{-1}}: \quad F\{\triangle f\}(f(t)) = \Box f(t)
\]

with \( F \), the Fourier transform in the distributional sense [8].
Fourier Transforms for Periodic Functions $\mathcal{F}_{\text{per}}$

In this subsection we will see that if a periodic function is Fourier transformed via the Fourier transform for periodic functions (9) then the result is a discrete function. Indeed, inserting some $p(t) = (\ldots \sigma \ldots \mathcal{F} (\ldots \sigma \ldots ))(t)$ into (9) yields

$$\hat{p}(m) = \frac{1}{T} \int_{0}^{T} x(t) e^{-2\pi i \frac{m}{T} t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} f(t) e^{-2\pi i \frac{m}{T} t} dt$$

$$= \frac{1}{T} (\mathcal{F} f)(\frac{m}{T})$$

where we used the popular periodization trick \([10-17]\) and Fourier transform definition (7). Inserting these coefficients into (10) we obtain

$$\mathcal{F} (\ldots \sigma \ldots \mathcal{F} (\ldots \sigma \ldots ))(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} (\mathcal{F} f)(\frac{m}{T}) e^{2\pi i \frac{m}{T} t}$$

$$= \frac{1}{T} \mathcal{F}^{-1} \left\{ \sum_{m=-\infty}^{\infty} (\mathcal{F} f)(\frac{m}{T}) e^{2\pi i \frac{m}{T} t} \right\}$$

$$= \frac{1}{T} \mathcal{F}^{-1} \left\{ (\ldots \sigma \ldots ) (\mathcal{F} f))((\sigma) \right\}$$

$$= \frac{1}{T} (\mathcal{F} (\ldots \sigma \ldots \mathcal{F} f))(t)$$

Therefore, (9) inserted into (10) reduces to

$$\mathcal{F} (\ldots \sigma \ldots \mathcal{F} f) = \frac{1}{T} (\ldots \sigma \ldots (\mathcal{F} f))$$

as a function of $\sigma \in \mathbb{R}$. This is formula (3).
Table 1: Discrete Functions vs. Periodic Functions.

<table>
<thead>
<tr>
<th>No</th>
<th>Rule</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$F(\lfloor 1/2 \rfloor f) = T \cdots \cdots f$</td>
<td>Poisson Sum Formula</td>
</tr>
<tr>
<td>2</td>
<td>$F(\lfloor 2/2 \rfloor f) = 1/2 \lfloor 2/2 \rfloor f$</td>
<td>Poisson Sum Formula</td>
</tr>
<tr>
<td>3</td>
<td>$F(\lfloor 3/2 \rfloor f) = \cdots \cdots f$</td>
<td>abbreviated for $T = 1$</td>
</tr>
<tr>
<td>4</td>
<td>$F(\lfloor 4/2 \rfloor f) = \cdots \cdots f$</td>
<td>abbreviated for $T = 1$</td>
</tr>
<tr>
<td>5</td>
<td>$F(\lfloor 5/1 \rfloor f) = \cdots \cdots f$</td>
<td>where $F = \delta$</td>
</tr>
<tr>
<td>6</td>
<td>$F(\lfloor 6/2 \rfloor f) = \cdots \cdots f$</td>
<td>where $F = \delta$</td>
</tr>
<tr>
<td>7</td>
<td>$F(\lfloor 7/1 \rfloor f) = \cdots \cdots f$</td>
<td>Dirac comb invariance</td>
</tr>
<tr>
<td>8</td>
<td>$F(\lfloor 8/2 \rfloor f) = \cdots \cdots f$</td>
<td>Dirac comb invariance</td>
</tr>
<tr>
<td>9</td>
<td>$\lfloor 9/1 \rfloor f = \cdots \cdots f$</td>
<td>Dirac comb identity</td>
</tr>
<tr>
<td>10</td>
<td>$\lfloor 10/1 \rfloor f = \cdots \cdots f$</td>
<td>Dirac comb identity</td>
</tr>
<tr>
<td>11</td>
<td>$F(\lfloor 11/2 \rfloor f) = T \cdots \cdots f$</td>
<td>Dirac comb invariance</td>
</tr>
<tr>
<td>12</td>
<td>$F(\lfloor 12/2 \rfloor f) = \lfloor 12/2 \rfloor f$</td>
<td>Dirac comb invariance</td>
</tr>
<tr>
<td>13</td>
<td>$F(f * g) = Ff * Fg$</td>
<td>multiplication</td>
</tr>
<tr>
<td>14</td>
<td>$Ff * Fg$</td>
<td>convolution</td>
</tr>
<tr>
<td>15</td>
<td>$\lfloor 15/g \rfloor f = \lfloor 15/g \rfloor f$</td>
<td>discretization</td>
</tr>
<tr>
<td>16</td>
<td>$\lfloor 16/0 \rfloor f = \lfloor 16/0 \rfloor f$</td>
<td>periodization</td>
</tr>
<tr>
<td>17</td>
<td>$\lfloor 17/1 \rfloor f = \lfloor 17/1 \rfloor f$</td>
<td>discretization of $f$</td>
</tr>
<tr>
<td>18</td>
<td>$\lfloor 18/2 \rfloor f = \lfloor 18/2 \rfloor f$</td>
<td>periodization of $f$</td>
</tr>
</tbody>
</table>

where $\lfloor \cdot \rfloor$ is a discrete function sampled at integers and $\lfloor \cdot \rfloor f$ is a one-periodic function.

Table 2: Discrete Periodic Functions.

<table>
<thead>
<tr>
<th>No</th>
<th>Rule</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>$F(\lfloor 1/2 \rfloor \cdots \cdots f) = \frac{1}{T} \lfloor 1/2 \rfloor \cdots \cdots f$</td>
<td>Rule 1 + 2</td>
</tr>
<tr>
<td>ii</td>
<td>$F(\lfloor 2/2 \rfloor \cdots \cdots f) = \frac{1}{T} \lfloor 2/2 \rfloor \cdots \cdots f$</td>
<td>Rule 2 + 1</td>
</tr>
<tr>
<td>iii</td>
<td>$F(\lfloor 3/2 \rfloor \cdots \cdots f) = \cdots \cdots f$</td>
<td>identity</td>
</tr>
<tr>
<td>iv</td>
<td>$F(\lfloor 4/2 \rfloor \cdots \cdots f) = \frac{k}{N} \lfloor 4/2 \rfloor \cdots \cdots f$</td>
<td>Rule 3 + 2</td>
</tr>
<tr>
<td>v</td>
<td>$F(\lfloor 5/1 \rfloor \cdots \cdots f) = \frac{k}{N} \lfloor 5/1 \rfloor \cdots \cdots f$</td>
<td>Rule 4 + 1</td>
</tr>
</tbody>
</table>

Fourier Transform for Discrete Functions (DTFT)

We show that if a discrete function is Fourier transformed via the DTFT then the result is a periodic function. As (11) is a Fourier series, it is periodic. The ansatz is therefore to let

$$d(\sigma) = (\cdots \cdots f(\sigma))((\cdots \cdots f(\sigma)) \sigma) \sin (12)$$. 

It yields

$$\int_{-T}^{T} d(\sigma) e^{2\pi i \frac{k}{T} \sigma} d\sigma = \int_{-T}^{T} f(\sigma) e^{2\pi i \frac{k}{T} \sigma} d\sigma = (F^{-1}(Ff))(k/\sigma) = f(k/T)$$.

Inserting these coefficients into (11) we obtain

$$F(\cdots \cdots f(\sigma)) = \frac{1}{T} \sum_{k=\infty}^{\infty} f(k/T) e^{-2\pi i \frac{k}{T} \sigma}$$.

Thus, (12) inserted into (11) reduces to

$$F(\cdots \cdots f) = T \cdots \cdots Ff$$

as a function of $\sigma \in \mathbb{R}$. This is formula (4).

Fourier Transform for Discrete Periodic Functions (DFT)

We show that if a discrete periodic function is Fourier transformed via the DFT, then the result is again a discrete periodic function. First, a simple reasoning tells us that after discretizing a periodic function $\cdots \cdots f(t)$ it can be denoted as $\cdots \cdots f(t) = \cdots \cdots f(t)$ where $N$ is an integer corresponding to its period $T$.

Inserting now its coefficients $\cdots \cdots f$ with $k \in \mathbb{Z}$ into (13) yields

$$\int_{-T}^{T} f(\sigma) e^{2\pi i \frac{k}{T} \sigma} e^{2\pi i \frac{k}{T} \sigma} d\sigma = \frac{1}{N} \sum_{k=\infty}^{\infty} f(k)e^{-2\pi i \frac{k}{N} \sigma}$$.

Thus, (13) inserted into (14) reduces to

$$(\cdots \cdots f(k)) = \frac{1}{N} \sum_{m=\infty}^{\infty} (\cdots \cdots f) \frac{m}{N} e^{2\pi i \frac{k}{N} \sigma}$$.

Using, again, the periodization trick (applied to a discrete function this time), definition (1) and a previous result. Inserting these coefficients into (14) we obtain

$$(\cdots \cdots f(k)) = \frac{1}{N} \sum_{m=\infty}^{\infty} (\cdots \cdots f) \frac{m}{N} e^{2\pi i \frac{k}{N} \sigma}$$.

Thus, (13) inserted into (14) reduces to

$$(\cdots \cdots f(k)) = \frac{1}{N} (\cdots \cdots f)(k) \quad k \in \mathbb{Z}$$.

and with Rule 4 and Fourier transforming both sides it yields

$$F(\cdots \cdots f) = \frac{1}{N} (\cdots \cdots f)$$

as a function of $\sigma \in \mathbb{R}$.

Derivation of the Inverse DFT

As (13) is a Fourier series, it is periodic. Evaluating (14) for $\cdots \cdots f(m) = (\cdots \cdots f)(m)$ where $f = Ff$ yields

$$g(k) = \sum_{m=0}^{\infty} (\cdots \cdots f) \frac{m}{N} e^{2\pi i \frac{k}{N} m}$$.

and inserting

$$\sum_{k=\infty}^{\infty} (\cdots \cdots f) \frac{m}{N} e^{2\pi i \frac{k}{N} m}$$.

$$= (F^{-1}(Ff))(k/\sigma)$$.

and

$$= (F^{-1}f)(\cdots \cdots f)(k/\sigma)$$.

where $\cdots \cdots f$ is the discrete periodic function corresponding to $N$-tuple $[f(0), \ldots, f(N-1)]$. 

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 December 2017
doi:10.20944/preprints201712.0173.v1
this result into (13) we obtain

\[
(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}(\mathcal{F}(f))(m)) = \frac{1}{N} \sum_{k=0}^{N-1} (\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}f)(k) e^{-2\pi i m k/N} = \frac{1}{N} \sum_{k=-\infty}^{\infty} f(k) e^{-2\pi i m k/N} = \frac{1}{N} \mathcal{F}(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}(f))(m).
\]

Thus, (14) inserted into (13) reduces to

\[
(\mathcal{F}(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}f))(m) = N(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}(\mathcal{F}(f))(m), \quad m \in \mathbb{Z}
\]

and with Rule 4 on the left-hand side it yields

\[
\mathcal{F}(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}f) = N(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}(\mathcal{F}(f))\sigma, \quad \sigma \in \mathbb{R}
\]

as a function of \(\sigma\) in \(\mathbb{R}\).

Altogether, we proved that the Discrete Fourier Transform (DFT) together with its inverse follow these two rules

\[
\mathcal{F}(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}f) = \frac{1}{N} \mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}(\mathcal{F}(f)) \quad \text{and} \quad \frac{1}{N} \mathcal{F}(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}f) = \mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle}(\mathcal{F}(f))
\]

as functions of \(\sigma\) in \(\mathbb{R}\), which are (5) and (6), respectively.

**Tricks**

We basically used two tricks in the above derivations. One is well-known. It is the so-called *periodization trick*, explicitly mentioned e.g. in [10-14] and in greater detail described in [15], p.61 and in [16] p.149. It is a standard technique occurring in lecture notes on signal processing (e.g. [17]), also celebrated as the link between Fourier series and Fourier transform. We also applied this trick to *discrete functions* when establishing the link between DTFT and DFT.

The second trick we used can be called *interpolation trick*. It requires generalized functions theory. We compose a loose infinite sequence of sampled values \([\ldots, f(-2), f(-1), f(0), f(1), f(2), \ldots]\) to \([\mathcal{S}_N^{\triangle} f](t)\), i.e., to something that looks like a smooth function depending on variable \(t \in \mathbb{R}\).

In fact, \(\mathcal{S}_N^{\triangle} f(t)\) is a smooth (infinitely differentiable) function in the generalized functions sense. The same trick is furthermore applied to *finite sequences* \([f(0), f(1), \ldots, f(N-1)]\) allowing us to use the DFT as if it were \(\mathcal{F}\) provided the sampling theorem is respected twice while discretizing and periodizing \(f\).

The fundamental idea behind these two tricks which can be considered dual of each other is that periodicity replaces finiteness and discreteness replaces smoothness. Note that the dual of discreteness (discrete functions) is not continuity (continuous functions) as usually taught but smoothness (smooth functions). Further details on these principles can be found in [4]. It remains to mention that both \(\mathcal{S}_N^{\triangle}\) and \(\mathcal{S}_N^{\triangle}\) perform sampling, one in “time domain” and the other in “frequency domain” (Figure 4) which means that the classical sampling theorem must be respected twice (in time and frequency domain) otherwise aliasing effects occur; hence \(f(t)\) and \(\mathcal{S}_N^{\triangle}\mathcal{S}_N^{\triangle} f(t)\) would not represent the same function any more.
Diagrams

Formulas (3)-(6) derived in this paper can moreover be depicted in diagrams such as the one in Figure 1. Additionally, both left-hand side and right-hand side in this diagram are cyclic, see Figure 2. Another most interesting aspect is that Figure 1 forms a Möbius strip if the upper left knot is identified with the lower right and lower left with the upper right. This circumstance can be interpreted in a way that tells us that it actually remains unclear whether the left-hand side is "time domain" or the right-hand side because there is no left-hand side and no right-hand side, just opposite sides.

Having a look at Figure 3 and Figure 4 one may observe furthermore that despite the fact of having only one Fourier transform $\mathcal{F}$, it is actually "captured" either on "fully analog" functions ("analog" in both time and frequency domain), on "fully discrete" functions ("discrete" in both time and frequency domain) or on "half discrete" functions, i.e., functions which are only discrete in either time or frequency domain.

Naming Conventions

The naming of $\mathcal{F}$, $\mathcal{F}_{\text{per}}$, DTFT and DFT in the literature is often not very appropriate and sometimes confusing. By looking at Table 3 and Table 4, they could be called "Fourier transform", "Fourier transform for periodic functions", "Fourier transform for discrete functions" and "Fourier transform for discrete periodic functions". However, they are all the same Fourier transform $\mathcal{F}$ but differ in the kind of functions ($f$, $\triangle \triangle \triangle f$, $\triangle \triangle \triangle \perp \perp f$, $\perp \perp \perp \perp f$) they are applied to.

An introduction of two transforms $\mathcal{F}_{\text{per}}$ and DTFT is also not advisable because they are, apart from having an inverse sign (which indicates that they are inverse to each other), the same transform (Figure 3). Most appropriately, it could be called the "Fourier Series Transform" as it switches between two Fourier series representations, i.e., between discrete functions (Fourier series coefficients) and periodic functions (Fourier series). Its two formulas should be called "Fourier Series Analysis" and "Fourier Series Synthesis" formula, as already done in many textbooks.

Applications

A main application of the above derivations lies of course in the fact that they can well be used for educational purposes, i.e., for teaching the interrelationships between Fourier series and Fourier transform and between different Fourier transform variants. Furthermore, the rules above can be fed into symbolic calculation environments such as Wolfram Mathematica [18] and/or Python SymPy [19] and by that become an indispensable toolbox for algorithm design. Recall that Poisson’s summation formula allows switching from badly converging algorithms to rapidly converging ones [6].

Conclusions

It is shown that instead of four Fourier transforms, we only have one Fourier transform, the Fourier transform in the distributional sense. We found that Poisson’s Summation Formula is the actual link in-between these four transform definitions and, in contrast to the Fourier-Poisson cube, we were able to explicitly write down these links in connecting formulas which do moreover rigorously hold in the tempered distributions sense.

APPENDIX – Fourier Transforms

There are mainly three ways of how to deal with the factor $2\pi$ in Fourier transform definitions. Here, we use the so-called "unitary, ordinary frequency" Fourier transform [20] as it can be found for example in [2]. It uses 1-periodic exponential functions $e^{2\pi i t}$ rather than $2\pi$-periodic ones $e^{\sigma}$ and thereby yields the most symmetric results in Fourier transform pairs, e.g. $F\delta = 1$ and $\mathcal{F}1 = \delta$. As a result of this, "time domain" and "frequency domain" become fully equivalent.

Now, let $f(t)$ be a suitable function such that it can be Fourier transformed and let $f(\sigma)$ be its Fourier transform. Then the following four Fourier transform variants are usually defined.

Fourier Transform

The Integral Fourier Transform (for non-discrete non-periodic functions) is defined by

$$\hat{f}(\sigma) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t \sigma} \, dt \quad \text{Analysis} \ (7)$$

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\sigma) e^{2\pi i t \sigma} \, d\sigma \quad \text{Synthesis} \ (8)$$

for suitable $f(t)$.

Fourier Transform (for periodic functions)

The Integral Fourier Transform for (non-discrete) periodic functions, used for Fourier series analysis, is defined by

$$\hat{f}(m) = \frac{1}{T} \int_{0}^{T} f(t) e^{-2\pi i m \frac{t}{T}} \, dt \quad \text{Analysis} \ (9)$$

$$f(t) = \sum_{m=-\infty}^{\infty} \hat{f}(m) e^{2\pi i \frac{t}{T} m} \quad \text{Synthesis} \ (10)$$

where (10) is the Fourier series of $f(t)$ and (9) determines its coefficients. The coefficients are discrete.

Fourier Transform (for discrete functions)

The Fourier Transform for discrete (non-periodic) functions, also called Discrete-Time Fourier Transform (DTFT), used for Fourier series synthesis, is defined by

$$\hat{f}(\sigma) = \frac{1}{T} \sum_{k=-\infty}^{\infty} f(k) e^{-2\pi i \frac{\sigma}{T} k} \quad \text{Analysis} \ (11)$$

$$f(k) = \int_{0}^{T} \hat{f}(\sigma) e^{2\pi i \frac{\sigma}{T} k} \, d\sigma \quad \text{Synthesis} \ (12)$$

where (11) is a Fourier series. Hence, it is periodic but $f(t)$ itself is discrete, its samples are determined by (12).

Fourier Transform (for discrete periodic functions)

The Fourier Transform for discrete periodic functions, called Discrete Fourier Transform (DFT), is defined by

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-2\pi i \frac{m}{N} k} \quad \text{Analysis} \ (13)$$

$$f(k) = \sum_{m=0}^{N-1} \hat{f}(m) e^{2\pi i \frac{k}{N} m} \quad \text{Synthesis} \ (14)$$

where both, (13) and (14), are (finite-sum) Fourier series. Thus, they are periodic and discrete, simultaneously.
References


