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Cognitive Development Is a Reconstruction Process That May Follow Different Pathways: The Case of Number

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Abstract

Some cognitive functions, shared by humans and certain animals, were acquired early in the course of phylogeny and, in humans, are operational in their primitive form shortly after birth. This is the case for the quantification of discrete objects. The further phylogenetic evolution of the human brain allows such functions to be reconstructed in a much more sophisticated way during child development. Certain functional characteristics of the brain (plasticity, multiple cognitive processes involved in the same response, interactions and substitution relationships between those processes) provide degrees of freedom that open up the possibility of different pathways of reconstruction. The within- and between-individual variability of these developmental pathways offers an original window on the dynamics of development. Here, I will illustrate this theoretical approach to cognitive development — which can be called "reconstructivist" and "pluralistic" — using children's construction of number as an example.

1. Introduction

The objective of every science is to uncover the invariants that underlie the variability of observable phenomena occurring in its domain. Each science nevertheless makes epistemological choices that are specific to it. In doing so, it makes the distinction, among the various forms of variability to which it is confronted, between those it deems relevant to its object of study and those it sees as bothersome. The latter, which it decides to ignore or control in one way or another, by averaging for example, are often the ones that cannot be interpreted in the framework of current theoretical paradigms (Lautrey, Mazoyer, van Geert, 2002).

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In developmental psychology, the variations that are judged relevant are age-related. On the other hand, between-individual variations and within-individual variations have often been neutralized in attempts to extract the general laws of development. Long neglected in general psychology, whether experimental or developmental, variations between individuals nonetheless abide by general laws. These laws were first brought to the fore in differential psychology, a subdiscipline that focuses specifically on the study of between-individual variations. For quite some time, the two approaches (general and differential) ignored each other, each one developing its own conceptual framework. Some examples are the developmental approach proposed by Piaget and the differential approach proposed by Spearman and Thurstone. These two approaches to intelligence gave rise to totally different conceptual frameworks with no connection to each other: the operatory theory on one side and the factor theory on the other. It wasn't until the sixties that research on the relationships between these two frameworks began to emerge (for a review, see Lautrey, 2002). Following Reuchlin (1969), a pioneer of differential psychology in France, the point of view I will adopt here is that it is necessary to account for between- and within-individual variability in any general theory of psychology. The reciprocal of this requirement is that any general theory that cannot account for these phenomena must be modified accordingly.

Admittedly, the major paradigms that dominated developmental psychology in the past — Piagetian constructivism starting in the fifties, neo-nativism starting in the sixties — did not lend themselves to explaining variability phenomena. In constructivism, as theorized by Piaget, cognitive development relies solely on the equilibration of cognitive structures, and this process is seen as the driving force behind the construction of each new cognitive structure, brought about by the linking up of as-yet-unrelated schemes. The order in which these structures are built is fixed and each one spans all knowledge domains. This set of constraints defines a single developmental pathway, that of a theoretical subject — the so-called "epistemic" subject — corresponding to what all children have in common. In this view, the only possible variations between individuals are differences in speed along this one-dimensional pathway. Piaget did, however, take some of the observed within-individual variations into account (those corresponding to fluctuations in the responses of a given child right before the onset of a new stage). He ascribed them to alternated centering on two currently unlinked schemes that are on their way to being coordinated into the structure under construction. Neo-nativism — which had a substantial impact on developmental psychology, notably following Chomsky's theory of language development — also reacted strongly against Piaget's constructivist approach. On this

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point, see the debate between Chomsky and Piaget that took place at the Royaumont Conference (Piattelli-Palmarini, 1979). In positing that deep cognitive structures are both innate and domain-specific, Chomsky's theory differs fundamentally from Piaget's, granted, but it resembles it in its quest for universality. Both are aimed at uncovering the deep cognitive structures that characterize the human species and are invariant across eras, individuals, and cultures. In the end, it is the structuralist approach that leads, in both cases, to the search for what is common to all individuals and considers between- and within-individual variabilities to be irrelevant to this object of study.

2. Conceptual Framework

The goal of this article is to propose a conceptual framework capable of integrating both the general and the variable into cognitive development. The main concepts underlying this approach are *reconstruction*, *plurality*, *interaction*, and *substitution*. Each of these concepts is explicated below and will be illustrated in a concrete way using the example of the development of numerical quantification in children.

2.1 The Notion of Reconstruction

Cognitive development is not seen as a product of innate structures, nor as a *de novo* construction process, but as a process by way of which primitive functions are reconstructed. The neo-nativist trend gave rise to studies on the cognitive abilities of infants and contributed to a complete renewal of knowledge in this field. Many studies using methods suited to the capacities of infants showed that certain behaviors regarded by Piaget as indicators of the acquisition of new structures had been observed much earlier in development. It was sometimes concluded that early infant abilities evidenced in this way are underlain by the same cognitive structures as those found in older children, and hence, that these structures are innate. My own hypothesis about the resemblance between infants' abilities and those of older children is that it does not originate in the fact that they have the same underlying cognitive structure, but the fact that they perform the same function at both ages. The major cognitive functions that enable living beings to adapt to their environment — for example, categorizing, quantifying, orienting oneself in space, and communicating — were selected in the course of phylogenesis and were gradually integrated into the genetic heritage of certain animal species, including the human species. In this primitive form, they are operational soon after birth. But the cognitive structures in which they are rooted are different in nature from those that will be reconstructed during development. In humans, the further phylogenetic evolution of the brain endowed the species with other capacities, such as symbolic representation and cognitive control. One can assume

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that in situations where the primitive function is elicited, the underlying neural structure acts as an attractor around which will aggregate groups of neurons that are likely to perform the function in a more reliable and efficient way using other means.

2.2 The Notion of Plurality

As the reconstruction process takes place, a system is constituted that aggregates all cognitive processes capable of performing a given function. In the end, it is a plurality of processes that get activated to fulfill one and the same function, but not all of these processes will necessarily treat the same information. Some will be more suited to treating certain situations; some will be preferred by certain individuals. As we shall see below, processing plurality gives degrees of freedom to cognitive functioning and provides several possible pathways toward the reconstruction of the function (Lautrey, 1990, 1993, 2003).

2.3 The Notion of Interaction

If the different processes activated to fulfill a given function interact, then — insofar as they do not all process the same information — each one can transmit to each of the others, either directly or via a common interface, information that is not available to the other processes. When this occurs, the conditions are satisfied for the joint generation of a system in which the functioning of each process affects the functioning of each of the others. Models of such dynamic systems have shown that a system with these characteristics can be a source of self-organization. In the approach to development based on dynamic-system modeling (Thelen & Smith, 1994; van der Maas and Molenaar, 1992; van Geert, 2004), this kind of self-organization is seen as one of the potential sources of developmental change. Several types of interaction are possible. The type considered here is an interaction of mutual support or reciprocal causality that is particularly conducive to initiating an improving self-organization process. For an example, see van der Maas et al. (2006), who simulated the self-development of a system with five initially unrelated but mutually supportive components. There are, of course, other sources of development, such as the myelination of neural structures, but interactions between processes capable of carrying out a given function can be another source of change.

2.4 The Notion of Substitution[1]

[1] In his model, Reuchlin (1978) used the French word "*vicariance*" to refer to this type of relationship between processes. For lack of an exact equivalent in English, I use the expression "substitution" to translate this term.

Whenever several cognitive processes are capable of fulfilling the same function, a certain amount of redundancy is generated in the system they form, and this offers some functional degrees of freedom. Consequently, if one of the processes in the system is damaged, another can compensate for its absence — either partially or fully — by performing the shared function. In this case, we speak of compensatory relationships between processes. Substitution possibilities may also exist when none of the co-functional processes are damaged but have different probabilities of activation, depending on the individual and the situation. Reuchlin (1978) proposed a probabilistic model of substitution relationships between cognitive processes. It postulates the existence of an activation-probability hierarchy that ranks the processes that perform the same function. The hierarchy is not necessarily the same for all individuals confronted with the same situation. This is a source of *between-individuals variability* in the nature of the cognitive processes activated to accomplish the task. Symmetrically, the hierarchy may not be the same, for a given individual, in different situations or at different moments in time. It will depend on the degree of situational affordance of each process. This is a source of *within-individual variability*. Substitution relationships between processes give plasticity to the cognitive system and contribute to its reliability. They also help account for — and this is our key point of interest here — between- and within-individual variations in a general model of development (Lautrey, 1993; 2003).

The conceptual framework presented above is theoretical and largely hypothetical. In what follows, I will try to demonstrate its validity using the quantification of sets of discrete objects as an example to illustrate and concretize the conceptual framework proposed.

3. Initial State of Numerical Cognition Development

Two non-verbal, non-symbolic systems via which infants quantify sets of discrete objects have been brought to the fore. The first, called the "Approximate Number System" (ANS), provides an approximate, noisy estimate of the numerosity of large sets of objects. The second, called the "Parallel Individuation System" (PIS), gives an exact quantification of small sets of no more than three or four objects. Only the main characteristics of these two systems will be presented here. More detailed descriptions can be found elsewhere (for reviews, see Feigenson, Dehaene, & Spelke, 2004; Piazza, 2010).

3.1 The Approximate Number System (ANS)

The function of the ANS system of numerical information processing is to provide an approximate estimate of the numerosity of a set of perceived objects. In the experiments

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designed to study its properties, the stimuli are sets of points or objects. The number of items in the set is varied while controlling the continuous variables likely to co-vary with that number (total area occupied, total perimeter, density). The task consists of discriminating or comparing the number of points or objects presented. With infants, the experimental paradigm used is habituation (e.g., Xu & Spelke, 2000). With older children or adults, the task usually consists of having the participant point to the greater of the two numerosities presented. In this case, the experimenter ensures that the display time is too short to allow the participant to count (e.g., Halberda, Mazzocco, & Feigenson, 2008). With primates, this same task is presented using conditioning methods (e.g., Nieder, 2013).

In all cases, whether with humans (infants or adults) or apes, the participants prove capable of approximately estimating the number of items to discriminate or compare, provided the difference in number is large enough. The larger the to-be-distinguished numerosities, the noisier the estimates and hence, the farther away from each other the numerosities must be to be perceived. The ratio between this distance and the set size nevertheless remains constant. More specifically, the approximate estimation of numerosity obeys Weber's law: the estimates are a logarithmic function of the real numerosity. This logarithmic function is considered to be the signature of the ANS, and Weber's fraction[2] tells us about the acuity of the ANS, i.e., the smallest difference in numerosity that an individual is capable of detecting.

The acuity of the ANS increases with the child's age, rapidly at first and then more slowly. In 6-month-old infants, a reaction to novelty does not occur unless the ratio between the two numerosities to be discriminated is about 1:2 (on average). For example, an infant habituated to the numerosity of sets of 8 points exhibits a novelty reaction during the test phase only for numerosities of at least 16 points. This ratio is about 2:3 at 9 months (8 can thus be discriminated from 12), and continues to decrease exponentially until it stabilizes at about 7:8 in adulthood. Note, however, that at any given age, stable individual differences in ANS acuity exist, even at the early age of 6 months (Libertus & Brannon, 2010). Among 14-year-olds, for example, the dispersion of this ratio ranges from 2:3 to 9:10 (Halberda, Mazzocco, & Feigenson, 2008). The search for correlations between ANS acuity and mathematical ability — a question we will address below — is based on these individual differences.

[2] Weber's fraction w, which is constant across a range of numerosities, is the difference between the two closest discriminable numerosities, normalized by their size. The same information is given by the coefficient of variation (standard deviation/mean).

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The signature of the ANS on number discrimination and comparison tasks has been found in indigenous populations that do not have a numerical system at their disposal (Piazza et al., 2013). It has also been found in certain animals, especially primates (Nieder, 2013). In the neurophysiology research, it has been shown, using tasks that call upon the ANS, that homologous brain areas are activated in man and primates, in particular the intraparietal sulcus. In this cerebral region, there is a population of neurons that are specifically activated by the perception of a particular numerosity. Each of these "number neurons" exhibits maximal activation when the number of objects it codes is perceived, and has a weaker and weaker activation level as the number of elements perceived moves farther and farther away from that number. Also, the dispersion of a number neuron's activation level increases as the number it codes rises. These two characteristics are such that the function relating the amplitude of neural activation to the number of perceived objects looks just like the logarithmic response function described above. It may therefore be its origin (Dehaene 2003; Nieder, Dehaene 2009 for review).

In sum, the behavioral signature of the ANS in numerosity quantification tasks is the same in man and some animals. This numerical information processing system also activates homologous brain regions in man and primates. It is operational soon after birth in humans (at three months in Izard, Dehaene-Lambertz, and Dehaene's, 2008 experiment). These three characteristics suggest that the ANS was integrated into the genetic heritage of our species relatively early in the course of its phylogenesis and can therefore be seen as the initial state in the development of the numerical quantification function of human beings (Dehaene, 2011).

3.2 The Parallel Individuation System (PIS)

When the number of objects is less than or equal to three, quantification behavior differs from that described above for larger numerosities. Discrimination accuracy no longer depends on the ratio of the two perceived quantities; it is the same for 1 vs 3, 1 vs 2, and 2 vs 3. This also holds true for comparison tasks. Infants capable of choosing the recipient containing the greatest number of food items when shown 1 vs 2, 2 vs 3, or 1 vs 3, fail as soon as there are more than three items (e.g., 2 vs 4 or 1 vs 4). At a more general level, neither accuracy nor response time depends on the ratio of the two quantities compared. This time, we are dealing with a form of exact representation of small quantities. The system relies on parallel processing in which each object perceived is individualized and represented in short-term memory (STM) by its own symbol, a kind of place holder. The STM representation incorporates the characteristics (shape, area, etc.) that enable the perceiver to track each object through time and space. Unlike with

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ANS, these continuous variables cannot be dissociated from the number in this system. It is sometimes called the Object Tracking System (Piazza, 2010) and sometimes the Parallel Individuation System (Le Corre, Carey, 2007).

The information contained in the PIS-based representation of objects is not itself numerical, but number is implicitly taken into account in the term-by-term comparison between the objects perceived and their symbols already stored in short-term memory. Accordingly, the infants in Wynn's (1992) study showed surprise when one of the objects was secretly added to or taken away from the set initially presented. For up to three objects, this system thus gives an exact representation of quantity and furnishes a non-numerical equivalent of adding or subtracting one object.

The cortical network supporting the PIS is different from that of the ANS. This time, rather than the inferior intraparietal sulcus, it is the occipital-temporal sites that are activated (Hyde & Spelke, 2012). Like ANS, PIS is operational shortly after birth and develops rapidly. The number of objects an infant can simultaneously take into account goes from one at about one month, to two or three at about twelve months (Piazza, 2010). PIS is also present in the repertoire of primates (Hauser & Carey, 2003) and like ANS, gives rise to substantial individual differences (Vogel & Machizawa, 2004).

The above findings suggest that the quantification of small numerosities (from 1 to 3) is based on a system different from the ANS, but one that, like ANS, became part of the genetic heritage of humanity relatively early in phylogenesis. Some authors call the ANS and PIS "core systems", defined as domain-specific representational systems that constrain the cultural acquisition of new representations (Feigenson et al., 2004; Spelke, Kinzler, 2007).

These core systems are very different from what will later become the cognitive structures underlying the concept of number, so they are neither innate nor constructed *de novo* by a general process of equilibration. The resemblance between the quantification behavior of infants and older children lies in the function common to the two kinds of cognitive structures implicated in these behaviors. The development of numerical cognition in children, then, should be regarded as a process that reconstructs the primitive function of discrete-object quantification. The reconstruction process relies on its evolutionary precursors (ANS and PIS), on cognitive abilities that have emerged more recently in the phylogenesis of the human species, and on the knowledge that these abilities have allowed us to construct and transmit. It follows from all this that the reconstruction of the quantification of discrete objects is in fact grounded in a plurality of cognitive processes that treat different kinds of information about number.

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4. Plurality of Processes Supporting Reconstruction

For Piaget, the construction of number in children relied on logical operations, or more specifically, on the synthesis of the operations needed to understand its two great properties: seriation operations, which enable the child to grasp the order relations that structure the sequence of numbers, and class inclusion operations, which enable the child to understand inclusion relationships between sets whose cardinals correspond to consecutive numbers (Piaget & Szeminska, 1941). Later studies painted a much more complicated picture by providing evidence of several other processes that play a substantial role in the construction process. Below is a brief description of what appear to be the most important processes. They can be divided into two main categories on the basis of whether they can be called "analogical"[3] or "symbolic". The former quantify on the basis of an analogical relationship between the size of a set of objects and the representation of that size. The latter are propositional and rely on arbitrary symbols (number words or Arabic numerals) to represent numbers.

4.1 Analogical Processing

Two of these processing systems, the ANS and the PIS, were presented above. The analogical nature of ANS lies in the (logarithmic) relation between numerosity and its representation; for PIS, it lies in the correspondence between the number of objects perceived and the number of place holders representing that number in short-term memory. Three other analogical ways of quantifying sets of discrete objects are briefly described below.

4.1.1 One-to-One Correspondence

Before using any numerical symbols, children can judge the equality or inequality of two collections of objects by setting up a one-to-one correspondence between the items in one collection and those in the other. Here, the analogy resides in the spatial correspondence of the collections being compared. This is one of the procedures that Piaget and Szeminska (1941) used to test for the conservation of number by getting the child to agree on the numerical equality of two rows of tokens before changing one of them. Piaget did not, however, grant this procedure an important role in the construction of number, precisely because, before the age of 6 or 7, the equality it permits is not conserved when the spatial spread of the two collections is changed. Yet when children become capable, at the age of about two or three years, of mapping

[3] In the literature, these processes are in fact usually called non-symbolic or non-verbal, which emphasizes what they are *not* and advantageously avoids having to make a statement about what they *are*. In my mind, they are analogical, but this is clearly a point of discussion.

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the elements of two collections, they are abstracting an identity relation that paves the way to the notion of exact equality.[4] This was demonstrated in Mix, Moore, and Holcomb's (2011) experiment, where 3-year-old children's ability to judge numerical equivalency (assessed by showing them two cards and asking them to choose which one had the same number of objects as the target card) improved considerably when the children were first given toys designed to stimulate a one-to-one mapping activity (e.g., objects made up of two parts that fit together).

4.1.2 Early Finger Counting

Another way to use one-to-one correspondence to determine how many elements there are in a set is to put up as many fingers as there are elements in the set; this is a peculiar procedure, however, in that it can be regarded as an instance of embodied cognition (Bender & Beller, 2011) because the fingers are part of the body. The finger-based representation of the number of objects also rests on an analogy, i.e., it uses as many fingers as there are objects. It implicitly involves several properties of numbers, which is an aid to understanding them later on. The order relation is intrinsic to the sequence of fingers held up (Brissiaud, 1992; Roesch, Moeller, 2015), and so is the successor function:[5] the same unit — a finger — separates each element from its successor.

There are good reasons, then, to contend that the representation of numbers with fingers contributes to the development of numerical cognition. This was shown in an experiment (Gunderson et al., 2015a) in which 4-year-old children who had not yet learned the concept of cardinality had to state the number of objects displayed on cards ("What's on this card?" task). In one of the experimental conditions, the children had to reply with a number word; in the other, they had to put up the corresponding number of fingers. The results indicated that response accuracy (measured by how close the response was to the correct answer) was greater in the gestural modality, whether the number was small (1 to 4) or large (5 to 10). Moreover, in cases where both response modalities were used at the same time by the child, if the two responses didn't match, the finger response was the more accurate one. As the authors stated, "These results show that children convey numerical information in gesture that they cannot

[4] This is not yet a relation of numerical equality because if, say, the identity is modified by replacing an item in one of the two collections by an item that is not identical to it, the relation of numerical equality is disrupted in the child's eyes (Izard, Streri, & Spelke, 2014).

[5] The successor function is a rule establishing the existence of a minimal quantity — one — that corresponds to the minimal distance between two consecutive numbers.

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convey in speech, and raise the possibility that number gestures play a functional role in children's development of number concepts" (p. 14).

4.1.3 The Sequence of Number Words

Between the ages of two and three, children learn from people in their surroundings to recite the list of the first few number words. At this stage, the list is merely an unbreakable string of sounds, recited by heart (Fuson, 1988). The words that compose the sound string start becoming separate entities when the child learns to imitate the procedure consisting of saying each word while pointing to a different object. At this point, even if the number words are individualized, they do not yet have the properties of numerical symbols. However, the order in which they are uttered, which is intrinsic to the numerical sequence recited in the past, is analogous to the order relation that structures numbers, and as such can promote its acquisition and comprehension.

4.2 Symbolic Processing

Symbolic processing relies on arbitrary symbols (e.g., number words, Arabic numerals) to represent exact numbers of objects. It enables us to directly perform mental quantification operations on these symbols. Unlike the analogical type of processing considered above, it is not based on any kind of resemblance between the symbol and the quantity it represents. The advantage of this route is that the relationships between the symbols are only those defined by the formal rules governing the numerical system. The disadvantage is that it is difficult to mentally represent the exact quantity that corresponds, by convention, to each symbol and makes it meaningful.

4.2.1 Subitizing

Subitizing is the rapid apprehension, without counting, of small numbers of objects, from 1 to 3 and sometimes 4. It is very similar to the parallel individuation process presented above and is probably an extension of it (Piazza, Fumarola, Chinello, Melcher, 2011). The essential difference is that this kind of rapid apprehension of number is accompanied by verbalization of the corresponding number word (the cardinal of the collection), which requires minimal access to language. This explains why subitizing is rarely observed in children under two. It supplies no information about the ordinal properties of numbers but constitutes an initial form of mapping between a number word and the cardinal of the perceived set.

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4.2.2 Counting

Counting is a complex process whose role was underestimated by Piaget but reevaluated by Greco (1962). We are indebted to Gelman and Gallistel (1978) for having brought back into the foreground the role of counting in the genesis of the notion of number. These authors identified five principles that must be obeyed to count correctly. The most important ones are one-to-one correspondence, stable order, and cardinality. The principle of one-to-one correspondence states that the items to be counted must be mapped, via a one-to-one correspondence, to the set of number tags used to count (e.g., the set of number words). The stable order principle states that the number tags have a fixed order. The cardinality principle states that the last number used in a count represents the cardinality of the items counted. In the spirit of the neo-nativist era of the seventies, Gelman and Gallistel assumed that these principles were spawned by an innate cognitive structure specific to number. Later research (Fuson, 1988; Wynn, 1990), however, did not confirm their innateness. Rather, they are acquired gradually in the course of early childhood and hardly ever show up in counting behavior before the age of four years.

The principle of cardinality is assessed using the "Give me N" task (Wynn, 1990). The experimenter places the child in front of a set of objects and asks the child to give him/her N objects. Cardinality is considered to be acquired when the child starts to count and, upon arriving at the number N (and only at that moment), gives the experimenter the set of objects just counted. Understanding that the cardinal of a collection also and necessarily includes all the cardinals of smaller collections is a much more advanced stage that no doubt requires mastery of logical operations such as class inclusion, to which Piaget granted a unique role. Children who master the principle of cardinality have begun to understand that each of the number words in the sequence they know refers to an exact quantity that is specific to it.

The various processes reviewed above, both analogical and symbolic, have a shared function, that of quantification, but they perform it by processing different information. Plurality of processing opens up the possibility of interactions, a potential source of development (Lautrey, 2014).

5. Interaction

How are these different processes related to each other in the course of development? Are they independent or interdependent? If they are interdependent, are the relationships between them symmetrical or asymmetrical? If they interact, what types of interactions are involved, conflicting ones or mutually supportive ones?

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These questions will be addressed below for the two processing systems reviewed here, the analogical system of approximate number representation or ANS, and the symbolic system of exact representation. Both are subject to developmental change over time: ANS acuity increases with age and so do numerical skills. Both give rise to large individual differences, that is, ANS-acuity differences and performance differences in initial numerical skills (e.g., number list, counting, cardinality, small-number operations). Within the past few years, many experiments have looked at these individual differences, generally using correlation methods, to determine how the two systems are related. I will begin with an overview of the results of studies demonstrating the impact of the ANS on the symbolic system. Then I will present the results of studies showing the opposite impact.

5.1 Does ANS Have an Effect on the Development of the Symbolic System?

It is not possible here to describe all of the studies on how the ANS affects numerical-skill development, but the reader will find reviews of this question in Mussolin et al. (2016) and in two meta-analyses (Fazio et al., 2014; Chen & Li, 2014). The conclusions of these studies are convergent, so to summarize the results, I will rely solely on the Chen and Li (2014) meta-analysis (which is the most comprehensive).

The cross-sectional studies meta-analyzed dealt with 31 studies involving 36 independent samples. Many factors differed across experiments, including the age of the participants, the covariables controlled, the tasks used to assess ANS acuity, the indexes calculated to establish its signature, and the tasks employed to assess mathematical abilities. A positive correlation was found in 35 of the 36 samples and was significant in 20 of them. The mean correlation, .24, was not very high but significant. None of the factors just mentioned had a significant impact on the magnitude of the correlation.

The existence in the cross-sectional studies of a correlation between ANS and numerical skills does not, however, tell us anything about the direction of the relationship between the two variables. Longitudinal studies conducted to find out whether individual differences in ANS acuity at time t predict performance differences in numerical skills at time $t+1$ are better suited to determining the direction of the relationship. Eight longitudinal studies involving 11 samples were included in the Chen and Li (2014) meta-analysis. Once the covariables measuring more general cognitive abilities were controlled, the mean correlation between ANS acuity and numerical-skill performance was .25, which is similar in magnitude to that found in the cross-sectional research.

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The structural analysis of the relationships between the variables provides further information. Chu et al. (2015) showed that among the children they examined at the age of four and then again at the end of the school year, ANS acuity at 4 years predicted numerical skills at the end of the year. However, this relationship was fully mediated by the relationship between performance differences on the cardinality task and differences in numerical skill level. Insofar as the ANS precedes cardinality, this result suggests that differences in ANS acuity are the source of differences in the acquisition of cardinality which, in turn, contribute to differences in numerical skills.

Other convincing data on the direction of the relationship between the ANS and symbolic arithmetic can be found in training experiments. Hyde et al. (2014) gave first graders two training sessions on non-symbolic numerical approximation (approximate addition of sets of points or approximate comparison of numbers). The results showed that in both cases, the children's performance on a test of exact symbolic arithmetic was significantly better than that of the children in the control groups. Similar results were found by Obersteiner, Reiss, and Ufer (2013). Park and Brannon (2013), who studied adult participants, also showed that training in approximate addition and subtraction of sets of points raised both ANS acuity and performance on a test of exact addition and subtraction.

It thus seems reasonable to conclude that the system of approximate number representation affects the development of the symbolic system of exact representation.

5.2 Does Learning the Symbolic System Have an Effect on ANS Acuity?

The correlation found in the cross-sectional studies analyzed above could also be due to an effect of numerical-skill acquisition on ANS acuity, but it does not demonstrate this effect.

Here again, longitudinal studies offer less ambiguous information. In Chen and Li's (2014) meta-analysis, four longitudinal studies (with five samples) tested the relationship in this direction. The mean correlation was .23, which is comparable to the value found in the longitudinal studies where the relationship was tested in the other direction. Here again also, an analysis of the structure of the relationships at play can supply more precise information. Mussolin et al. (2014) assessed numerical skills and ANS acuity in children at the age of 4 years and then again seven months later. Using the cross-correlation method, they showed that numerical skills at 4 years predicted ANS acuity at the end of the school year, but not the opposite.

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Other convincing data on the relationship in this direction can be found in experiments analyzing the effects of numerical-system learning on ANS acuity. Opfer and Siegler (2012) studied the developmental time course of the function linking the representation of the magnitude of numbers to their real magnitude. Children of different grades in school were asked to position numbers from different numerical intervals (0 to 10, 10 to 100, 100 to 1000, etc.) on a continuous line bounded on both ends. Their results showed that the logarithmic function became a linear function as school grade increased, but only interval by interval. For example, for the interval 0-100, the function was logarithmic for the preschoolers and linear for the second graders, but for these same second graders, the function was still logarithmic for the interval 0-1000, and so on. Hence, learning the rule of succession — whereby each number leads to the next by iteration of one unit and all intervals between two consecutive numbers are equal — is not transferred in an immediate way to the approximate representation of all numbers. It is more likely that the experience acquired through manipulation of the numbers in the interval being learned in each school grade transforms the ANS-based representation of approximate magnitudes.

Cross-cultural studies shed another type of light on this issue. Piazza et al. (2013) studied the approximate estimation of numerosities in a native Amazon population, the Mundurucù. The language of these people has a very limited lexicon of number words and they have no symbolic way whatsoever to process discrete quantities. They can, however, compare the numerosities of two sets, or estimate their approximate sum. The authors compared adults who had gone to school to those who had not, on a task involving the approximate estimation of numerosities. The results showed that for all participants, schooled or not, the function that linked the discrimination capacity to the size of the compared numerosities looked very much like the logarithmic function already found in industrialized nations. However, Weber's ratio was significantly lower — indicating greater acuity — among the Mundurucù who had been taught arithmetic, and acuity grew as the number of years of schooling rose.

It thus seems reasonable, here also, to conclude that learning the symbolic system affects the developmental time course of the ANS, in particular by improving its acuity and changing the shape (from logarithmic to linear) of the function that links approximate representations of magnitude to their real magnitudes.

If each process that performs the quantification function has an impact on the unfolding of each of the others, then the relationship between them is one of mutual support, and together they form a dynamic system capable of self-organization (van de Maas et al., 2006). Furthermore,

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whenever several processes fulfill the same function, they can also be related by substitution, i.e., substitution of one for another, depending on the situation and the individual.

6. Substitution

Acquiring the principle of cardinality is decisive in the acquisition of the numerical system. It is at this time, between the ages of three and a half and four and a half, that children understand that each of the numerical symbols they know (number words, digits) represents an exact quantity, one that is specific to it. To grasp this, children must map these symbols to the analogical representations of quantity they have at their disposal. By the time they are two and a half, on average, children know that number words refer to quantities, but while they know and can learn to recite these words, they do not know the correspondence between the words and the quantities. First, they learn the meaning of "1", but it will take them several months to learn the meaning of "2", and then several additional months to learn the meaning of "3". The discovery of the meaning of numbers larger than 3 or 4 marks the transition to another stage and is based on a different process. This capacity is a testimony to the fact that the child has discovered the principles of counting, in particular that of cardinality, and has become capable of generalizing them to larger numbers. This occurs between the ages of three and four, on average, and the criterion is the child's ability, on the "Give me N" task, to give the exact quantity corresponding to the number requested by the experimenter, for a number greater than four.

There is an ongoing debate about the pathway taken by children to arrive at the principle of cardinality. For some authors (e.g., Le Corre, Carey, 2007; Carey et al., 2017), the transition rests on the "parallel individuation system" (PIS), the only one capable of enabling the child to map the number words from "one" to "three" or "four" to the exact quantities to which they correspond. For others (Dehaene, 2011; Feigenson, et al. 2004; Piazza, 2010; Wynn, 1992), the mapping can be achieved via the approximate number system (ANS).

6.1 Hypothesis of the ANS-to-Word Pathway

As soon as children begin to learn a list of numerical symbols, they can be asked to compare two members of the list (e.g., two digits) and to indicate the larger (or smaller) of the two. Their responses resemble those obtained — well before they have knowledge of numerical symbols — when children are asked to indicate which is the largest of two sets of points or objects. The function that links representations of the magnitude of numerical symbols to their real

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magnitudes is the logarithmic function considered to be the signature of the approximate representation of quantities, furnished by the ANS.

The existence of a mapping between numerical symbols and the approximate representations of their magnitude is inferred from this type of result. This has also been demonstrated in a more direct way in experiments where children are asked to choose the number that best corresponds to the numerosity of a given set of points, or to choose the set of points that best corresponds to a given number (Mundy & Gilmore, 2009). These experiments showed that starting at the age of six, children are able to map symbolic and analogical representations in both directions, and that this ability improves with age. It would seem as if the number words are represented by their approximate position on a mental number line that has a left-to-right orientation (in cultures that read from left to right) and is compressed on the right.

The spatial properties of this mental number line have been demonstrated by experiments where the spatial positions of two to-be-compared numbers are varied. Reaction time is shorter and the percentage of correct responses is greater when the larger number is located to the right of the smaller one than in the opposite case. Dehaene, Bossini, and Giraux (1993) called this effect the "spatial numerical association of response codes" (SNARC). Because of the fact that later on the list means a larger set, the order relation between the symbols is intrinsic to the mental number line.

However, this way of representing numbers remains approximate, while the notion of cardinality rests on an exact correspondence between each number and the quantity it represents. Authors who advocate the role of the ANS in the acquisition of cardinality hypothesize that children begin mapping the first few number words because they are the most frequent in the language. Moreover, the quantities corresponding to these numbers are related to each other in ways that make them discriminable by 3-year-olds. At this age, children are capable of discriminating ratios of 3:4, so they must also be able to discriminate 1 from 2, and 2 from 3, since the ratios of these comparisons (1:2 and 2:3) are easier than 3:4. Moving up from these small numbers to the following ones can be achieved by realizing that going from one number to the next involves adding one object.

6.2 Hypothesis of the PIS-to-Word Pathway

Carey and her colleagues (Le Corre, Carey, 2007; Carey et al., 2017) of course agree that numerical symbols are mapped to approximate representations of their magnitude, but they do not agree that this is what leads to cardinality. Their hypothesis is that this type of mapping can

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only take place once the principle of cardinality has been acquired. One of the reasons for their reluctance is that cardinality is based on an exact correspondence between the quantity and the symbol that represents it, yet mapping via the ANS only supplies approximate representations. On the other hand, as we have seen above, PIS gives the exact representation outright for the numbers from 1 to 3 or 4. The parallel individuation of the elements of a small set nevertheless only furnishes fleeting representations in short-term memory, whereas representations of the magnitudes of number words must be permanently stored in long-term memory. To account for the fact that the mapping done in short-term memory is transferred to long-term memory, Carey and colleagues hypothesize that the representation of a set is enriched by knowledge of other sets of the same size (e.g., "me" for 1, "Mommy and Daddy" for 2, "Mommy, Daddy, and me" for 3). This process is called the "enriched parallel individuation system". It is assumed to enable mapping of each of the first three or four numbers to an exact representation of the quantities associated with them. When going from the exact representation of the number 1 to that of the number 2, and likewise for 2 to 3, the child has the opportunity, here also, to grasp that going from a given number word to the next involves adding 1. Cardinality, discovered on small numbers, thanks to PIS, would then be generalized to become the cardinality principle that counting obeys.

6.3 Experiments Aimed at Choosing Between These Two Hypotheses

Experiments devised to decide which is the better hypothesis about the route to cardinality have given rise to results that are both contradictory and difficult to interpret. This section begins by summarizing the findings, and then looks at how the model of substitution might help resolve the contradictions. The first two experiments presented validate the hypothesis of a PIS-based pathway to number words; the last three validate the hypothesis of an ANS-based pathway.

6.3.1 Le Corre and Carey (2007)

Le Corre and Carey's (2007) experiment was designed around three main ideas. The first is that there should be a gap in the developmental progression of the notion of cardinality between small numbers (from 1 to 3 or 4) and the numbers that follow them. The second is that if this pathway rests on ANS cardinality, we should find its signature. The third is that approximate ANS-based mapping most certainly exists for large numbers, but it should only be found *after* the principle of cardinality has been acquired.

In this study, Le Corre and Carey tested 116 children (mean age 3;11, range 3;0-5;7) on two tasks, one ("Give me N") designed to detect their knowledge of the cardinality of numbers, the

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other ("Fast Cards") designed to see whether the signature of the ANS would be found in the children's matching behavior. In "Give me N", those who responded correctly only for N = 1 were labelled 1-knowers, and so on up to 4-knowers. Those who responded correctly only for the small numbers (1 to 4) were put in a group called "subset-knowers". Those who responded correctly for larger numbers, at least up to 5 or 6, were put in a group called "cardinality principle knowers" (CP-knowers). In "Fast Cards", the children had to say the number word corresponding to the number of elements (circles) on the card shown by the experimenter. The number of circles varied between 1 and 10, but only the responses given for the larger numbers (5 to 10) were analyzed. To prevent the children from counting the circles, each card was shown for only one second.

As hypothesized, the ANS signature in the "Fast Cards" task was found only among the CP-knowers. However, their response distribution as a function of the number of circles (5 to 10) was bimodal. Only half of these children could be described by the increasing function characteristic of number word mapping to the ANS. They were called "matchers" and the others, "non-matchers". Comparison of these two subgroups showed that, on average, the matchers were six months older than the non-matchers.

According to the authors, the fact that the signature of the ANS was not found among the subset-knowers but only among the CP-knowers shows that the pathway toward cardinality does not rely on the ANS. It was only *after* the principle of cardinality had been acquired that mapping of large numbers to an approximate representation of their magnitude was observed here. Furthermore, the six-month age difference between the non-matching CP-knowers and the matching CP-knowers was interpreted as confirming that there is indeed a gap between the PIS-based and ANS-based ways of representing quantities.

6.3.2 Carey, Shusterman, et al. (2017)

In a more recent study, Carey, Shusterman, and their colleagues used a learning phase to further validate their hypotheses. They selected children who were 3-knowers, hence ones situated at the boundary between the acquisition of cardinality for small numbers and the generalization of this principle to larger numbers. They taught the children to associate a number word to the corresponding approximate quantity (the number of circles on a card). The training was done with the number word "four" in one experiment and with the number word "ten" in another. On the test phase, the children had to say the number word learned when they thought it corresponded to the number of circles shown on the card. Their scores were a little above the

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chance level for 4, but not for 10. The authors concluded, once again, that generalization of cardinality from small to large numbers is not achieved by mapping number words to the ANS.

6.3.3 Wagner and Johnson (2011)

Wagner and Johnson had 35 children (mean age 4;1, range 3;0 to 5;4) perform the "Give me N" task. Unlike Le Corre and Carey (2007), they did not stop task execution after the last correct response but went up to the number 10 for all children. Another difference is that they did not use a separate task to look for a potential ANS effect on the responses — as Le Corre and Carey did with "Fast Cards" — but used the responses on the "Give me N" task. The independent variable was the difference between the number requested by the experimenter and the number corresponding to the child's knowledge level. For example, if the requested number was 5 and the child's last correct response was at level K=3, the difference was 5-3 = 2. The results showed that the mean number of objects given by the child increased with the magnitude of the difference, as defined here. The standard deviation also increased and the coefficient of variation (standard deviation divided by the mean) was constant, which is the signature of the ANS. The authors concluded that the ANS played a role in the representation of the magnitude of number words whose cardinality the children did not know. This is the opposite of what Le Corre and Carey found. Note that in Wagner and Johnson's analysis of the difference scores — and this is critical for understanding what follows — the children who were subset-knowers (those who had not yet acquired cardinality) were not distinguished from the CP-knowers (those who were assumed to have acquired cardinality).

Wagner and Johnson's findings contradict Le Corre and Carey's, but the tasks used to study the mapping were not the same. "Fast Cards" consists of presenting a set of items and asking the child to say the number word that corresponds to it, whereas "Give me N" consists of saying a number word and asking the child to give the corresponding number of items. The direction of the required mapping is thus ANS to number word in the former case and number word to ANS in the latter.

6.3.4 Odic, Le Corre, and Halberda (2015)

In an attempt to shed light on the contradictory results of the above studies, Odic, Le Corre, and Halberda had 62 children (mean age 3;6, range 2;7-4;6) perform both tasks: the mapping task in the ANS-to-word direction ("Fast Cards") and the mapping task in the word-to-ANS direction (an adapted version of the "Give me N" task). Instead of handing the experimenter the requested number of objects, the children had to indicate that number by patting a stuffed tiger's

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head the corresponding number of times. The game was called "Pat the Tiger". The numbers 1 to 4 were used to familiarize the children with the task, so only those trials pertaining to numbers greater than 5 were analyzed.

The results of the "Fast Cards" experiment replicated Le Corre and Carey's findings: in the ANS-to-word direction, the function linking the magnitude of the set of points to the magnitude of the number word produced had a positive slope only for some of the CP-knowers. The results of the "Pat the Tiger" task, in which the mapping was in the word-to-ANS direction, replicated Wagner and Johnson's results: the slope of the function linking the number of pats by the children to the number word requested by the experimenter was positive and significantly different from zero; this was true not only for the CP-knowers but also for the 2-knowers and the 3-knowers (analyzed separately here, which Wagner and Johnson did not do). This allowed the authors to conclude that "before children have become CP-knowers, they are able to map from a discrete number word representation, e.g. ten, to a region on the continuous ANS mental number line" (p. 118). Importantly — and once again, this is critical for understanding what follows — this experiment studied mapping with large numbers only (5 to 10).

The contradiction observed in the preceding studies was thus replicated here, where both tasks were performed by the same children. The authors set forth two hypotheses to account for the difference in the difficulty of the two tasks. The first was that the difference could be due to the lag commonly observed in language development between production and comprehension. In the word-to-ANS task, one must *comprehend* the number word in order to map a quantity to it. In the ANS-to-word direction, one must *produce* the number word corresponding to a quantity. In language acquisition, comprehension precedes production, so the same may hold true here. The second hypothesis was that in the ANS-to-word direction, the approximate representation of the magnitude of a quantity is a region on the mental number line to which several number words can be mapped; hence the difficulty of choosing a single one. In the word-to-ANS direction, only one number word is given, so there is less uncertainty about how many items to give, especially since the child can add or take away some of those already given, according to the sensation of numerosity provoked by the set being produced. This hypothesis only holds, however, if we agree that the representation of the magnitude of a number word is less approximate than the representation of the magnitude of a set of points, which is doubtful.

6.3.5 Gunderson, Spaepen, and Levine (2015b)

A final study, by Gunderson, Spaepen, and Levine, added to the complexity of the findings. These authors tested children whose knower level (assessed on the "Give me N" task) was

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between 0 and 4 (subset-knowers). The children performed the "Give me N" task and the "Fast Cards" task. In both tasks, they had to map the numbers between 1 and 10, but their responses were analyzed separately for the small numbers (1 to 4) and the large numbers (5 to 10).

Their first experiment was with 47 children (mean age 3.44, range 2.99 - 4.18). For both tasks and both directions (word-to-ANS and ANS-to-word), the data analysis yielded an increasing function for the numbers 1 to 4 but not for the numbers 5 to 10. This is a new finding. It was not available in the Wagner and Johnson study because the data analysis did not distinguish between the subset-knowers and the CP-knowers, nor in the other two studies, which, while separating the subset-knowers from the others, only looked at the numbers from 5 to 10. So this new finding does not contradict any of the previous studies, but adds new information by showing that the subset-knower children, who had not yet acquired the principle of cardinality, nevertheless had an approximate representation of the magnitude of the number just above their knower level, and albeit less frequently, of the number just above that. On the other hand, the fact that an increasing function was not found in either direction for the numbers 5 to 10, farther away from their knower level, is contradictory to the Odic et al. (2015) results. In that study, the authors found an increasing function for the numbers 5 to 10 in the word-to-ANS direction, even among the subset-knowers. Thus, no difference in difficulty was noted between the two mapping directions.

To be able to observe the behavior of subset-knowers whose ANS development was more advanced, Gunderson et al. conducted a second experiment with older subset-knowers. To find children who were older but still had not acquired the cardinality principle, they recruited participants from nursery schools in a neighborhood with a lower sociocultural level than in the first experiment. The sample consisted of 79 children whose mean age was 4;2 (range 3;11-5;5). The children performed the same tasks, and in the same conditions, as in the first experiment. The results replicated those of the first experiment for the numbers 1 to 4, i.e., there was an increasing function in both tasks. The new finding this time was that for the numbers 5 to 10, there was an increasing function on the "Fast Dots" task, where the mapping direction was ANS-to-word, but not on the "Give me N" task, where the direction was word-to-ANS. This means that there was a difference in difficulty between the two mapping directions, but it was the opposite of the one found by Odic et al. and therefore cannot be readily explained on the basis of the same hypotheses.

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6.4 Substitution and Different Developmental Pathways

The experiments reported in the preceding section were aimed at choosing between two hypotheses about the developmental pathway taken by children to acquire the principle of cardinality, one that stresses the role of the PIS and the other, the role of the ANS. The results are complex, surprising, and sometimes contradictory. The discrepancies can be explained, however, if we agree that the pathway to cardinality does not rest on only one of these two systems (PIS or ANS), to the exclusion of the other. The general picture painted by the above results suggests that the relationship between these two systems, which perform the same function, is one of substitution.

The first reason is that the signature of the ANS was found in the phase that precedes the acquisition of the cardinality principle, at a point where the advocates of the PIS think that only this system can play a role. In the sole experiment where small-number mapping (1 to 4) was tested, the ANS signature was found in both mapping directions. This result was obtained in Experiment 1 by Gunderson et al. and replicated in Experiment 2. The subset-knowers had an approximate representation of the magnitude of number words located beyond their knowledge level. Thus, in the phase that precedes the acquisition of cardinality, the two types of processes coexisted in their repertoire and both could be used to map number words and set sizes.

The second reason is that — once again, among the subset-knowers — the mapping direction had an impact on success. Mapping between number words and quantities from 5 to 10 is more difficult, and the children did not succeed in all cases. In two of the experiments (Odic et al.; Wagner & Johnson), they only succeeded in the word-to-ANS direction. In another study (Gunderson et al.), they only succeeded in the ANS-to-word direction.

The third reason is that individual differences in mapping difficulty were also found. In the Gunderson et al. study, large number mapping (5 to 10) in the ANS-to-word direction was not mastered in Experiment 1 whereas it was mastered in Experiment 2. All of the children examined in those two experiments were subset-knowers, but there was an age difference (between the two experiments) that was linked to the differing sociocultural levels of the families.

The fourth reason is that the situation interacted with individual differences: mapping difficulty among these children was found in the opposite direction to that observed in the Odic et al. experiment. For these children, ANS-to-word mapping was the more difficult of the two, whereas in Odic et al., it was word-to-ANS mapping.

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Insofar as these experiments were not designed to test for substitution relationships, their interpretation can only be hypothetical. One hypothesis could be that, among subset-knower children, the activation of one or the other of the two systems on a mapping task depends in part on the mapping direction. The affordance of the ANS would be better in the ANS-to-word direction, where an approximate quantity that has to be mapped to a word, acts as a prime. The affordance of the PIS would be better in the word-to-ANS direction where the prime is a number word that has to be mapped to an approximate quantity. The same would apply to individual differences, depending on which system is more efficient. Children for whom the ANS has better acuity would perform better in the ANS-to-word direction, whereas those who are more at ease with the PIS and the language would perform better in the word-to-ANS direction.

In fact, in Gunderson et al.'s Experiment 2, the children were selected on the basis of their age so as to include participants who would perform better with the ANS. This could be the very reason why the more favorable situation for them was the one where the mapping was primed by the ANS; and *vice versa* for the younger children. Another hypothesis can be forwarded. In this same group, a difference in sociocultural level was linked to the difference in age. Children with a low sociocultural level are known to perform less well in the verbal domain than in the spatial domain. Given that the mental number line used by the ANS has spatial properties, ANS priming of the mapping may be more effective in this group than verbal priming. This hypothesis could be tested by having children carry out tasks that assess ANS acuity, verbal aptitude, and spatial aptitude independently.

The results for the children categorized as CP-knowers are irrelevant to assessing the relationships between the two systems, since numbers above 4 are not processed by the PIS, but they are interesting nonetheless. Large-number mapping (5 to 10) was studied in the ANS-to-word direction by Le Corre and Carey and by Odic et al. It was studied in the word-to-ANS direction also by Odic et al. The results were the same in all three cases: there was no effect of mapping direction and the differences between the individuals were developmental in nature (among the CP-knowers, the matchers were six months older, on average, than the non-matchers).

7. Compensation and Atypical Development

Whenever a deficiency, whether genetic or accidental, is such that one of the co-functioning processes is absent from the child's repertoire or is inefficient, the existence of substitution relationships enables another process to take its place, either partially or totally. The word *compensation* is more suitable in this case than the word substitution, because a substitution

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relationship is reciprocal. We find compensation relationships in certain kinds of atypical development.

Children with Williams Syndrome, for example, have a unique developmental profile in which verbal abilities are less impaired, relatively speaking, than spatial abilities. Their acquisition of arithmetic is delayed. Ansari et al. (2003) wondered whether these children could acquire numerical skills via the same route as typically developing children. They focused in particular on the acquisition of the principle of cardinality.

These authors studied two groups of children matched on spatial abilities: a group of typically developing children, mean age 3;5, and a group of Williams Syndrome children, mean age 7;6. Both groups had taken verbal aptitude tests. The cardinality principle was in the process of being acquired in each group, with comparable mean performance levels and dispersions. The most interesting finding is that in the typically developing group, individual differences in the scores on the cardinality task "Give me N" were linked to individual differences in spatial aptitude, whereas in the Williams group, the cardinality scores were linked to differences in verbal aptitude. This suggests that the two groups follow different pathways to arrive at the principle of cardinality.

Van Herwegen et al.'s (2008) later study sheds an interesting light on this issue. These authors had a group of nine Williams children (mean age 2;11) perform a task involving the approximate estimation of large numerosities, and a task involving the exact discrimination of small numerosities. The children were capable of exactly discriminating the small numerosities but were unable to discriminate the large numerosities. This result suggests that the ANS, known to be operational shortly after birth in typically developing children, has not yet been acquired by Williams children who are nearly three years old. Insofar as the mental number line of the ANS has spatial properties, it could very well be that the ANS deficit of these children is linked to their spatial impairment. The PIS, where words are mapped with exact quantities, may not be affected as much because these children's language abilities are less altered than their spatial ones. The pathway taken by Ansari et al.'s children with Williams Syndrome, who, in spite of their impairment, were able to construct the notion of cardinality, is thus likely to be a route that relies on their relatively well-spared abilities, namely, the PIS and language.

Why, then, are Williams children so far behind in learning the principle of cardinality, knowing that the PIS is available in their repertoire? Undoubtedly, it is because this system can only partially compensate for ANS deficits. If this is indeed the case, then the above finding points

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in the same direction as those analyzed in the preceding section. It shows — *a contrario* — that typical development relies on both systems.

Deaf individuals who lack a conventional language (spoken or signed) exhibit the opposite deficit. These individuals, called homesigners, devise their own gestures to communicate. Their gestures are not used as a tally system, and homesigners do not have the equivalent of a counting list or a counting routine. When they communicate about the magnitude of sets of objects, they are accurate for sets from 1 to 3. For sets from 4 to 20, they are approximately, but not exactly, correct. They understand that each set has an exact numerical value, but they do not have an errorless way of arriving at a gestural representation of that value. Their responses are centered on the target value, with a small dispersion around it (Spaepen et al., 2011). These facts confirm the specific role of the PIS in the exact representation of the numbers from 1 to 3. They also show that acquisition of the cardinality principle is not a necessary condition for the development of the ANS. The system of approximate representation can partly compensate for an impairment in a conventional (spoken or signed) language, but it is also clear that both systems are necessary for optimal efficiency of the quantification function.

8. Conclusion

The concept of number is not constructed solely by the coordination of logical operations, as Piaget thought (Piaget & Szeminska, 1941). Nor does it result from the actualization of an innate cognitive structure that harbors the principles of counting, as Gelman and Gallistel thought (1978). These two theories, one constructivist, the other nativist, made major contributions to furthering our knowledge of the genesis of number in children, but later work pointed out their limitations. The cognitive processes underlying numerical development are in fact more numerous and more diverse. Some rely on an analogical representation of quantity, others on a symbolic representation. Some make use of language, others do not. Some are best suited to representing small quantities, others to representing large numerosities. Some supply an approximate representation of quantity, others an exact representation. Although these diverse cognitive processes treat different information, they perform a common function (hence the term "co-functional") — namely, the quantification of sets of discrete objects.

Two of them, the ANS and the PIS, have a status of their own due to their anteriority on the phylogenetic and ontogenetic levels. We know very little about the exact role played by each of these core systems in the orchestration of the processes that take effect later in the development of numerical cognition. We can nevertheless assume that in situations that call upon the quantification function, all processes that have affordances in these situations are

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activated. At the beginning of child development, this only concerns the core systems, but as soon as other processes become operational, a growing number of co-functional processes are activated in quantification situations. During this phase — and by virtue of their anteriority — the neural structures that have been supporting the function's primitive form would act as an attractor to which the other simultaneously activated neural structures would connect up. The system gradually formed in this manner is not built through the coordination of isolated, initially unrelated schemes, as Piaget thought. Rather, it is formed by the integration of new cognitive capacities that serve a preexisting function, around which they themselves are structured at the same time as they are transforming and reconstructing the function.

Two phases must be distinguished in this reconstruction process: invention and learning. During the invention phase, human beings gradually discovered new possibilities for quantification, ones that were more precise and more efficient, opened up by the evolution of the brain in their species. Some examples of inventions are when a human being got the idea to have a stone, a finger, a notch, or a stick correspond to each member of his herd, or imagined assigning a name or a gesture to an exact quantity, etc. This phase of the reconstruction process took thousands of years and continues today. It gave rise to the number-based culture in which the children whose development we are now studying are immersed. The second form of reconstruction is not void of inventions — which for children are rediscoveries — but relies much more heavily on learning, and for this reason, takes only a few years. In a cultural environment where numerical information is an integral part of daily life, situations confronting children with quantification problems activate the available primitive quantification systems, granted, but they also activate all cognitive processes that can help in understanding the language, gestures, signs, etc. used in those situations by people in their surroundings. The co-activation of all of the processes that are co-functional but treat different information about quantity, has implications not only from the developmental point of view but also from the point of view of variability.

From the developmental standpoint, any process can affect the unfolding of any of the other co-functional processes, since each one treats information that the others cannot access directly on their own. If this is the case, the interactions between them can take on the form of relations of mutual support, and form a dynamic system capable of self-organization. This is a source of development that differs from those rooted in the maturation of the nervous system and in learning. The above findings on the reciprocal impact of the approximate number system and the symbolic system are compatible with the mutual-support hypothesis. It would be interesting

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to find out whether this type of interaction can be generalized to all cognitive processes that perform the quantification function.

From the standpoint of variability, a system containing a plurality of co-functional processes also opens up the possibility of substitution relationships. Reuchlin's (1978) model of substitution can account for the contradictions between the results of experiments designed to show that only one of two co-functional processing systems, ANS or PIS, is at play in the child's pathway toward cardinality. Substitution relationships can give rise to two sorts of variability in behavior. Firstly, we have within-individual variability, which is situation-dependent and stems from the fact that a given co-functional process does not necessarily have the same affordance for all situations (note in passing that situation-dependent alternation between the various processes promotes their interaction). Secondly, we have between-individual variability, which stems from the fact that in a given situation, not all individuals necessarily select the same process among those available in their repertoire, to perform the function in question. These different forms of variability are manifestations of the plasticity engendered by the existence of multiple co-functional processes. Variability, here, is not seen as a peculiarity that must be neutralized in order to gain access to the general laws of cognitive functioning and development, but as a consequence of those laws. It follows that studying the different forms of variability opens up an original window on the study of the general laws of cognitive functioning and development.

To look through this window, one must adopt a slightly different methodological approach from that most frequently used in developmental research. Rather than examining each co-functional process separately, one needs to study the dynamics of the system they form. This requires a longitudinal study of within-individual variation at the developmental level attained by each child in each co-functional process, plus the study of the stability of each child's within-individual variation profile over time, plus the study of between-individual variation in these within-child variations. The pathway differences that this approach should uncover would be doubly interesting. Theoretically, these differences will tell us about the degrees of freedom that exist along the developmental pathway leading to the reconstruction of the cognitive function. Practically, better knowledge of the various possible pathways could help in devising learning methods suited to each route.

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