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Unified (p,q) −analog of Apostol type polynomials of order α

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Abstract

Motivated by Kurt's work [Filomat 30 (4) 921-927, 2016], we first consider a class of a new generating function for (p,q)-analog of Apostol type polynomials of order α including Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α. By making use of their generating function, we derive some useful identities. We also introduce (p,q)-analog of Stirling numbers of second kind of order v by which we construct a relation including aforementioned polynomials.

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1. Introduction

During the last three decades, applications of *quantum calculus* based on q-numbers have been studied and investigated succesfully, densely and considerably (see [4, 8, 10, 15, 16, 19-21, 27, 30, 31]). In conjunction with the motivation and inspiration of these applications, with the introduction of the (p,q)-number, many mathematicians and physicists have extensively developed the theory of *post quantum calculus* based on (p,q)-numbers along the traditional lines of classical and quantum calculus. Certainly, these (p,q)-numbers cannot be derived only switching q by q/p in q-numbers. Conversely, (p,q)-numbers are native generalizations of q-numbers, since q-numbers may be obtained when $p=1$ in the definition of (p,q)-numbers (see [10]). In recent years, Corcino [5] studied on the (p,q)-extension of the binomial coefficients and also derived some properties parallel to those of the ordinary and q-binomial coefficients, comprised horizontal generating function, the triangular, vertical, and the horizontal recurrence relations, and the inverse and the orthogonality relationships. Duran *et al.*[6] considered (p,q)-analogs of Bernoulli polynomials, Euler polynomials and Genocchi polynomials and acquired the (p,q)-analogues of known earlier formulae. Duran and Acikgoz [7] gave (p,q)-analogue of the Apostol-Bernoulli, Euler and Genocchi polynomials and derived their some properties. Gupta [10] proposed the (p,q)-variant of the Baskakov-Kantorovich operators by means of (p,q)-integrals and also analyzed some approximation properties of them. Milovanović *et al.* [22] introduced a generalization of Beta functions under the (p,q)-calculus and committed the integral modification of the generalized Bernstien polynomials. Sadjang [26] satisfied some properties of the (p,q)-derivatives and the (p,q)-integrations. As an application, he presented two (p,q)-Taylor formulas for polynomials and derived the fundamental theorem of (p,q)-calculus.

The (p,q)-number is defined as

$$[n]_{p,q}=\frac{p^n-q^n}{p-q} \qquad (p\neq q)\,.$$

Notice that $[n]_{1,q} := [n]_q$ called as q-number in q-calculus.

The (p,q)-derivative operator given as

$$D_{p,q;x}f(x) := D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x} \quad (x \neq 0) \text{ with } (D_{p,q}f)(0) = f'(0) \tag{1.1}$$

is a lineer operator and satisfies the following properties

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) \tag{1.2}$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \tag{1.3}$$

The (p,q)-power basis is defined by

$$(x+a)_{p,q}^n := \begin{cases} (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), & \textit{if } n \geq 1, \\ 1, & \textit{if } n = 0, \end{cases}$$

or equivalently by

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}.$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ and $[n]_{p,q}!$ are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!} \; (n \geq k) \text{ and } [n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q} \; (n \in \mathbb{N})$$

with the initial condition $[0]_{p,q}! = 1$.

Let

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \text{ and } E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}$$

denote two types of exponential functions satisfying relations

$$e_{p,q}(x)E_{p,q}(-x) = 1 \text{ and } e_{p^{-1},q^{-1}}(x) = E_{p,q}(x) \tag{1.4}$$

and having the following (p,q)-derivative representations

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \text{ and } D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.5}$$

The definite (p,q)-integral for a function f is defined by

$$\int_0^a f(x)\,d_{p,q}x = (p-q)\,a\sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right) \tag{1.6}$$

with

$$\int_a^b f(x)\,d_{p,q}x = \int_0^b f(x)\,d_{p,q}x - \int_0^a f(x)\,d_{p,q}x.$$

For detail studies of (p,q)-calculus, one can look at [2, 5-8, 10, 11, 22, 26] and cited references therein.

Throughout the paper, let $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the set of all nonnegative integers, the set of all natural numbers, the set of all integers, the set of all real numbers and the set of all complex numbers.

Apostol type polynomials and numbers firstly introduced by Apostol [1] and also Srivastava [29]. Motivated by their works, many mathematicians have studied and investigated Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and numbers, *cf.* [1, 7, 9, 12, 14, 15, 17, 18, 20, 23-25]. Also q-analogs of Apostol type polynomials and numbers were introduced and discussed by several authors, see [4, 16, 20]. Moreover, (p,q)-analog of Apostol type polynomials and numbers were defined and investigated by Duran and Acikgoz in [7]. Unification of classical polynomials and Apostol type polynomials were considered in [13, 14, 23-25]. Further, unification of q-Apostol type polynomials and numbers were studied in [15].

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The Apostol-Bernoulli polynomials $B_n(x;\lambda)$, the Apostol-Euler polynomials $E_n(x;\lambda)$ and the Apostol-Genocchi polynomials $G_n(x;\lambda)$ are defined by means of the following Taylor series expansions about $z=0$:

$$\sum_{n=0}^{\infty} B_n(x;\lambda)\frac{z^n}{n!}=\frac{z}{\lambda e^z-1}e^{xz},$$
$$(|z|<2\pi \ \textit{when}\ \lambda=1; |z|<|\log\lambda| \ \textit{when}\ \lambda\neq 1)$$
$$\sum_{n=0}^{\infty} E_n(x;\lambda)\frac{z^n}{n!}=\frac{2}{\lambda e^z+1}e^{xz}$$
$$(|z|<\pi \ \textit{when}\ \lambda=1; |z|<|\log(-\lambda)| \ \textit{when}\ \lambda\neq 1)$$

and

$$\sum_{n=0}^{\infty} G_n(x;\lambda)\frac{z^n}{n!}=\frac{2z}{\lambda e^z+1}e^{xz},$$
$$(|z|<\pi \ \textit{when}\ \lambda=1; |z|<|\log(-\lambda)| \ \textit{when}\ \lambda\neq 1)$$

see [1, 12, 17, 18] for further details. In the case when $\lambda=1$, aforementioned Apostol polynomials reduce to the classical forms *cf.* [3, 28, 29, 32].

(p,q)-analogs of Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order α, the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order α and the Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ of order α are, respectively, defined by Duran and Acikgoz [7], as follows:

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}=\left(\frac{z}{\lambda e_{p,q}(z)-1}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz),$$
$$(|z|<2\pi \ \textit{when}\ \lambda=1; |z|<|\log\lambda| \ \textit{when}\ \lambda\neq 1)$$
$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}=\left(\frac{2}{\lambda e_{p,q}(z)+1}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz)$$
$$(|z|<\pi \ \textit{when}\ \lambda=1; |z|<|\log(-\lambda)| \ \textit{when}\ \lambda\neq 1)$$

and

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x,y;\lambda:p,q)\frac{z^n}{[n]_{p,q}!}=\left(\frac{2z}{\lambda e_{p,q}(z)+1}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz)$$
$$(|z|<\pi \ \textit{when}\ \lambda=1; |z|<|\log(-\lambda)| \ \textit{when}\ \lambda\neq 1)$$

where λ and α are suitable (real or complex) parameters and $p,q\in\mathbb{C}$ with the condition $0<|q|<|p|\leq 1$.

In the next section, we perform to define the family of unified (p,q)-analog of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α and to investigate some properties of them. Moreover, we consider (p,q) analog of a new generalization of Stirling numbers of the second kind of order v by which we derive a relation including unified (p,q)-analog of Apostol type polynomials of order α.

2. Unified (p,q)-analog of Apostol type polynomials of order α

Inspired by the generating function [23]

$$f_{a,b}^{(\alpha)}(x;t;k,\beta):=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!}\ \ (k\in\mathbb{N}_0; a,b\in\mathbb{R}\backslash\{0\}; \alpha,\beta\in\mathbb{C})$$

under conditions of the convergence

(i) If $a^b>0$ and $k\in\mathbb{N}$, then $\left|t+b\log\left(\frac{\beta}{a}\right)\right|<2\pi; 1^{\alpha}:=1,\ x\in\mathbb{R},\beta\in\mathbb{C}$

(ii) If $a^b>0$ and $k=0$, then $0<\operatorname{Im}\left(t+b\log\left(\frac{\beta}{a}\right)\right)<2\pi;\ 1^{\alpha}:=1,\ x\in\mathbb{R},\beta\in\mathbb{C}$

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(iii) If $a^b<0$, then $\left|t+b\log\left(\frac{\beta}{a}\right)\right|<\pi$; $1^\alpha:=1$, $x\in\mathbb{R}, k\in\mathbb{N}_0$, $\beta\in\mathbb{C}$
in this paper, we consider the following Definition 1 based on (p,q)-numbers.

Definition 1. *Unified (p,q)-analog of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials of order α is defined as follows:*

$$\Upsilon_{a,b}^{(\alpha)}(x,y;z;k,\beta:p,q)=\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\frac{z^n}{[n]_{p,q}!}=\left(\frac{2^{1-k}z^k}{\beta^b e_{p,q}(z)-a^b}\right)^{\alpha}e_{p,q}(xz)\,E_{p,q}(yz)$$
$$\left(k\in\mathbb{N}_0;\ a,b\in\mathbb{R}/\{0\};\ \alpha,\beta\in\mathbb{C};\ 1^\alpha=1\right).$$

Here we investigate some special cases of $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ as follows:

$\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:1,q)=\mathcal{P}_{n,\beta,q}^{(\alpha)}(x,y,k,a,b)$ (*cf.*[15])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,\beta}^{(\alpha)}(x,0,k,a,b:1,q)=\mathcal{P}_{n,\beta}^{(\alpha)}(x,k,a,b)$ (*cf.*[23])
$\mathcal{P}_{n,\lambda}^{(\alpha)}(x,y,1,1,1:p,q)=\mathcal{B}_n^{(\alpha)}(x,y;\lambda:p,q)$ (*cf.*[7])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,\lambda}^{(\alpha)}(x,y,1,1,1:1,q)=B_n^{(\alpha)}(x+y;\lambda)$ (*cf.*[18])
$\mathcal{P}_{n,\lambda}^{(\alpha)}(x,y,0,-1,1:p,q)=\mathcal{E}_n^{(\alpha)}(x,y;\lambda:p,q)$ (*cf.*[7])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,\lambda}^{(\alpha)}(x,y,0,-1,1:1,q)=E_n^{(\alpha)}(x+y;\lambda)$ (*cf.*[18])
$\mathcal{P}_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x,y,1,-\frac{1}{2},1:p,q\right)=\mathcal{G}_n^{(\alpha)}(x,y;\lambda:p,q)$ (*cf.*[7])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x,y,1,-\frac{1}{2},1:1,q\right)=G_n^{(\alpha)}(x+y;\lambda)$ (*cf.*[23])
$\mathcal{P}_{n,1}^{(1)}(x,y,1,1,1:p,q)=\mathcal{B}_n(x,y:p,q)$ (*cf.*[6])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,1}^{(\alpha)}(x,0,1,1,1:1,q)=B_n^{(\alpha)}(x)$ (*cf.*[28])
$\mathcal{P}_{n,1}^{(1)}(x,y,0,-1,1:p,q)=\frac{2}{[2]_{p,q}}\mathcal{E}_n(x,y:p,q)$ (*cf.*[6])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,1}^{(\alpha)}(x,0,0,-1,1:1,q)=E_n^{(\alpha)}(x)$ (*cf.*[28])
$\mathcal{P}_{n,\frac{1}{2}}^{(1)}\left(x,y,1,-\frac{1}{2},1:p,q\right)=\frac{2}{[2]_{p,q}}\mathcal{G}_n(x,y:p,q)$ (*cf.*[6])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,\frac{1}{2}}^{(\alpha)}\left(x,0,1,-\frac{1}{2},1:1,q\right)=G_n^{(\alpha)}(x)$ (*cf.*[32])
$\lim\limits_{q\to1^-}\mathcal{P}_{n,1}^{(1)}(x,0,1,1,1:1,q)=B_n(x)$ (*cf.*[3])	$\lim\limits_{q\to1^-}\mathcal{P}_{n,1}^{(1)}(x,0,0,-1,1:1,q)=E_n(x)$ (*cf.*[3])

We note that
$$\mathcal{P}_{n,\beta}^{(1)}(x,y,k,a,b:p,q):=\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$$
which are called unified (p,q)-analog of Apostol type polynomials.

We give the following four Lemmas 1-4 without proofs, since they can be proved by using Definition 1.

Lemma 1. *We have*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)&=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\,x^{n-j}p^{\binom{n-j}{2}},\\&=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\,y^{n-j}q^{\binom{n-j}{2}},\\&=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(0,0,k,a,b:p,q)\,(x+y)_{p,q}^{n-j}.\end{aligned}\tag{2.1}$$

Lemma 2. *(Addition property) For $\alpha,\mu\in\mathbb{N}$, $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ satisfies the following relation:*

$$\mathcal{P}_{n,\beta}^{(\alpha+\mu)}(x,y,k,a,b:p,q)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{p,q}\mathcal{P}_{j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\,\mathcal{P}_{n-j,\beta}^{(\mu)}(0,y,k,a,b:p,q).$$

It follows from Lemma 2 that

$$\mathcal{P}_{n,\beta}^{(0)}(x,y,k,a,b:p,q)=(x+y)_{p,q}^{n}.\tag{2.2}$$

Lemma 3. *(Derivative properties) We have*

$$\begin{aligned}D_{p,q;x}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)&=[n]_{p,q}\,\mathcal{P}_{n-1,\beta}^{(\alpha)}(px,y,k,a,b:p,q),\\D_{p,q;y}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)&=[n]_{p,q}\,\mathcal{P}_{n-1,\beta}^{(\alpha)}(x,qy,k,a,b:p,q).\end{aligned}$$

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Lemma 4. *(Difference equation) We have*

$$\mathcal{P}_{n,\beta}^{(\alpha-1)}(0,y,k,a,b:p,q)=\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\left(\beta^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(1,y,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\right), \quad (2.3)$$

$$\mathcal{P}_{n,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q)=\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\left(\beta^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(x,0,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(x,-1,k,a,b:p,q)\right).$$

From Lemma 1 and Lemma 3, we obtain the following Theorem 1.

Theorem 1. *We have*

$$\frac{[n+k]_{p,q}!}{2^{k-1}[n]_{p,q}!}\mathcal{P}_{n,\beta}^{(\alpha-1)}(0,y,k,a,b:p,q)=\beta^b\sum_{j=0}^{n+k}\begin{bmatrix}n+k\\ j\end{bmatrix}_{p,q}p^{\binom{n+k-j}{2}}\mathcal{P}_{j,\beta}^{(\alpha)}(0,y,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}^{(\alpha)}(0,y,k,a,b:p,q). \quad (2.4)$$

Corollary 1. *Upon setting $\alpha=1$ in Eq. (2.4) gives the following relation*

$$y^n=\frac{2^{k-1}[n]_{p,q}!}{q^{\binom{n}{2}}[n+k]_{p,q}!}\left(\beta^b\sum_{j=0}^{n+k}\begin{bmatrix}n+k\\ j\end{bmatrix}_{p,q}p^{\binom{n+k-j}{2}}\mathcal{P}_{j,\beta}(0,y,k,a,b:p,q)-a^b\mathcal{P}_{n+k,\beta}(0,y,k,a,b:p,q)\right).$$

Here is a recurrence relation of unified (p,q)-analog of Apostol type polynomials by the following theorem.

Theorem 2. *The following relationship holds true for $\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$:*

$$a^b\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)=\beta^b\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_{p,q}q^{\binom{n-j}{2}}\mathcal{P}_{j,\beta}(x,y,k,a,b:p,q)-\frac{[n]_{p,q}!}{[n-k]_{p,q}!}2^{1-k}(x+y)_{p,q}^{n-k}.$$

Proof. Since

$$\frac{a^b}{\left(\beta^b e_{p,q}(z)-a^b\right)e_{p,q}(z)}=\frac{\beta^b}{\beta^b e_{p,q}(z)-a^b}-\frac{1}{e_{p,q}(z)},$$

we have

$$\begin{aligned}\frac{2^{1-k}z^k a^b e_{p,q}(xz)E_{p,q}(yz)}{\left(\beta^b e_{p,q}(z)-a^b\right)e_{p,q}(z)} &= \frac{2^{1-k}z^k\beta^b e_{p,q}(xz)E_{p,q}(yz)}{\beta^b e_{p,q}(z)-a^b}-\frac{2^{1-k}z^k e_{p,q}(xz)E_{p,q}(yz)}{e_{p,q}(z)},\\ a^b\frac{2^{1-k}z^k}{\beta^b e_{p,q}(z)-a^b}e_{p,q}(xz)E_{p,q}(yz) &= \beta^b\frac{2^{1-k}z^k e_{p,q}(xz)E_{p,q}(yz)}{\beta^b e_{p,q}(z)-a^b}e_{p,q}(z)-2^{1-k}z^k e_{p,q}(xz)E_{p,q}(yz).\end{aligned}$$

From here we derive that

$$\begin{aligned}&a^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\frac{z^n}{[n]_{p,q}!}\\ =\;&\beta^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}-2^{1-k}\sum_{n=0}^{\infty}(x+y)_{p,q}^n\frac{z^{n+k}}{[n]_{p,q}!}.\end{aligned}$$

Using Cauchy product and then equating the coefficients of $\frac{z^n}{[n]_{p,q}}$ completes the proof. □

We provide now the following explicit formula for unified (p,q)-analog of Apostol type polynomials of order α.

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Theorem 3. *The unified polynomial* $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ *holds the following relation:*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \sum_{j=0}^{n}\begin{bmatrix} n \\ j\end{bmatrix}_{p,q}\frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\mathcal{P}_{n-j,\beta}^{(\alpha)}(0,0,k,a,b:p,q)\\ &\cdot\left(\beta^b\sum_{s=0}^{j+k}\begin{bmatrix} j+k \\ s\end{bmatrix}_{p,q}p^{\binom{j+k-s}{2}}\mathcal{P}_{s,\beta}(x,y,k,a,b:p,q)-a^b\mathcal{P}_{j+k,\beta}(x,y,k,a,b:p,q)\right).\end{aligned}$$

Proof. The proof of this theorem is derived from the Eq. (2.1) and Theorem 2. So we omit the proof. □

The (p,q)-integral representations of $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ are given in the following theorem.

Theorem 4. *(Integral representations) We have*

$$\begin{aligned}\int_u^v \mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\,d_{p,q}x &= \frac{\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(\frac{v}{p},y,k,a,b:p,q\right)-\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(\frac{u}{p},y,k,a,b:p,q\right)}{[n+1]_{p,q}},\\ \int_u^v \mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)\,d_{p,q}y &= \frac{\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(x,\frac{v}{q},k,a,b:p,q\right)-\mathcal{P}_{n+1,\beta}^{(\alpha)}\left(x,\frac{u}{q},k,a,b:p,q\right)}{[n+1]_{p,q}}.\end{aligned}$$

Proof. By using Lemma 3 and Eq. (1.6), the proof can be easily proved. So we omit it. □

The following theorem involves in the recurrence relationship for unified (p,q)-analog of Apostol type polynomials of order α.

Theorem 5. *(Recurrence relationship) The following equality is true for* $n,k\in\mathbb{N}_0$*:*

$$\begin{aligned}&\beta^b\sum_{j=0}^{n}\begin{bmatrix} n \\ j\end{bmatrix}_{p,q}p^{\binom{n-j}{2}}m^j\mathcal{P}_{j,\beta}^{(\alpha)}(x,0,k,a,b:p,q)-a^b\sum_{j=0}^{n}\begin{bmatrix} n \\ j\end{bmatrix}_{p,q}p^{\binom{n-j}{2}}m^j\mathcal{P}_{j,\beta}^{(\alpha)}(x,-1,k,a,b:p,q) \quad (2.5)\\ &=\frac{2^{1-k}[n]_{p,q}!}{[n-k]_{p,q}!}\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j\end{bmatrix}_{p,q}p^{\binom{n-k-j}{2}}m^{j+k}\mathcal{P}_{j,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q).\end{aligned}$$

Proof. Based on the proof technique of Mahmudov in [19], we consider

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q)\,m^n\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}\\ &=\left(\frac{2^{1-k}(mz)^k}{\beta^b e_{p,q}(mz)-a^b}\right)^{\alpha}\frac{\beta^b e_{p,q}(mz)-a^b}{2^{1-k}(mz)^k}e_{p,q}(mxz)\,E_{p,q}(-mz)\,e_{p,q}(z)\\ &=\frac{2^{k-1}}{(mz)^k}\left[\beta^b\left(\frac{2^{1-k}(mz)^k}{\beta^b e_{p,q}(mz)-a^b}\right)^{\alpha}e_{p,q}(mxz)\,e_{p,q}(z)\right.\\ &\left.-a^b\left(\frac{2^{1-k}(mz)^k}{\beta^b e_{p,q}(mz)-a^b}\right)^{\alpha}e_{p,q}(xmz)\,E_{p,q}(-mz)\,e_{p,q}(z)\right]\\ &=\frac{2^{k-1}}{(mz)^k}\left[\beta^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\,m^n\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}m^{-n}\frac{z^n}{[n]_{p,q}!}\right.\\ &\left.-a^b\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha)}(x,-1,k,a,b:p,q)\,m^n\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{z^n}{[n]_{p,q}!}\right].\end{aligned}$$

Matching coefficients $\frac{z^n}{[n]_{p,q}!}$ of both sides gives desired result. □

In the case $\alpha = 1$ in Theorem 5, we get the following result

$$\beta^b \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} m^j \mathcal{P}_{j,\beta}(x,0,k,a,b:p,q) - a^b \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} p^{\binom{n-j}{2}} m^j \mathcal{P}_{j,\beta}(x,-1,k,a,b:p,q) \tag{2.6}$$
$$= \frac{2^{1-k}[n]_{p,q}!}{[n-k]_{p,q}!} \sum_{j=0}^{n-k} \begin{bmatrix} n-k \\ j \end{bmatrix}_{p,q} p^{\binom{n-k-j}{2}} m^{j+k} (x-1)_{p,q}^{j}.$$

Now we are in a position to state some recurrence relationships for the unified (p,q)-analog of Apostol type polynomials as follows.

Theorem 6. *The following recurrence relation holds true for $n,k \in \mathbb{N}_0$ and $x,y \in \mathbb{R}$:*

$$\mathcal{P}_{n+1,\beta}(x,y,k,a,b:p,q) = yq^k p^{n-k} \mathcal{P}_{n,\beta}\left(\frac{q}{p}x, \frac{q}{p}y, k, a, b : p, q\right) \tag{2.7}$$
$$+p^{n+1-k} \frac{[k]_{p,q}}{[n+1]_{p,q}} \mathcal{P}_{n+1,\beta}(x,y,k,a,b:p,q) + xq^k p^{n-k} \mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$$
$$-2^{k-1}\beta^b \frac{[n]_{p,q}!}{[n+k]_{p,q}!} \sum_{j=0}^{n+k} \begin{bmatrix} n+k \\ j \end{bmatrix}_{p,q} \mathcal{P}_{j,\beta}(x,y,k,a,b:p,q)\, q^j p^{n-j} \mathcal{P}_{n+k-j,\beta}(1,0,k,a,b:p,q),$$

Proof. By using the same method of Kurt's work [15], for $\alpha = 1$ in Definition 1, applying (p,q)-derivative operator to $\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)$, with respect to z, and using (1.2) and (1.3) yields to

$$\begin{aligned}
& D_{p,q;z} \Upsilon_{a,b}^{(1)}(x;y;z;k,\beta:p,q) \\
= \; & D_{p,q;z}\left\{ \frac{2^{1-k} z^k}{\beta^b e_{p,q}(z) - a^b} e_{p,q}(xz) E_{p,q}(yz) \right\} \\
= \; & 2^{1-k} \left\{ \frac{\left(\beta^b e_{p,q}(qz) - a^b\right) D_{p,q;z}\left(z^k e_{p,q}(xz) E_{p,q}(yz)\right) - (qz)^k e_{p,q}(qxz) E_{p,q}(qyz) D_{p,q;z}\left(\beta^b e_{p,q}(z) - a^b\right)}{\left(\beta^b e_{p,q}(pz) - a^b\right)\left(\beta^b e_{p,q}(qz) - a^b\right)} \right\} \\
= \; & 2^{1-k} \frac{q^k z^k \left[x e_{p,q}(xpz) E_{p,q}(ypz) + y e_{p,q}(xqz) E_{p,q}(yqz)\right] + e_{p,q}(xpz) E_{p,q}(ypz) [k]_{p,q} z^{k-1}}{\left(\beta^b e_{p,q}(pz) - a^b\right)} \\
& -2^{1-k} \frac{(qz)^k e_{p,q}(qxz) E_{p,q}(qyz)}{\left(\beta^b e_{p,q}(qz) - a^b\right)} \cdot \frac{\beta^b e_{p,q}(pz)}{\left(\beta^b e_{p,q}(pz) - a^b\right)}.
\end{aligned}$$

In conjunction with some basic mathematical computations, we have

$$\sum_{n=0}^{\infty} \mathcal{P}_{n+1,\beta}(x,y,k,a,b:p,q) \frac{z^n}{[n]_{p,q}!} = x\frac{q^k}{p^k} \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\, p^n \frac{z^n}{[n]_{p,q}!}$$
$$+y\frac{q^k}{p^k} \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}\left(\frac{q}{p}x, \frac{q}{p}y, k, a, b : p, q\right) p^n \frac{z^n}{[n]_{p,q}!} + \frac{[k]_{p,q}}{p^k} \sum_{n=0}^{\infty} \mathcal{P}_{n,\beta}(x,y,k,a,b:p,q)\, p^n \frac{z^{n-1}}{[n]_{p,q}!}$$
$$-2^{k-1}\frac{\beta^b}{p^k} \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{p,q} \mathcal{P}_{j,\beta}(x,y,k,a,b:p,q)\, q^j p^{n-k} \mathcal{P}_{n-j,\beta}(x,y,k,a,b:p,q) \right\} \frac{z^n}{[n]_{p,q}!}.$$

Checking against the coefficients of $\frac{z^n}{[n]_{p,q}!}$, we obtain the desired result. ☐

We now give the following Theorem 7.

Theorem 7. *For $n \in \mathbb{N}_0$ and $x, y \in \mathbb{R}$, the following formulas are valid:*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\begin{bmatrix} n+k \\ s \end{bmatrix}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n} \\ &\cdot\left\{\beta^b\sum_{j=0}^{s}\begin{bmatrix} s \\ j \end{bmatrix}_{p,q}p^{\binom{j}{2}}m^{-j}\mathcal{P}_{s-j,\beta}^{(\alpha)}(x,0,k,a,b:p,q) - a^b\mathcal{P}_{s,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\right\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\begin{bmatrix} n+k \\ s \end{bmatrix}_{p,q}\mathcal{P}_{n+k-s,\beta}(mx,0,k,a,b:p,q)\,m^{s-n} \\ &\cdot\left\{\beta^b\sum_{j=0}^{s}\begin{bmatrix} s \\ j \end{bmatrix}_{p,q}\mathcal{P}_{s-j,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\,p^{\binom{j}{2}}m^{-j} - a^b\mathcal{P}_{s,\beta}^{(\alpha)}(0,y,k,a,b:p,q)\right\}.\end{aligned}$$

Proof. This proof can be made by using the same method of Mahmudov [19]. So we omit it. □

Combining Theorem 5 with Theorem 7 gives the following theorem.

Theorem 8. *We have*

$$\begin{aligned}\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\begin{bmatrix} n+k \\ s \end{bmatrix}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n} \\ &\cdot\left\{\frac{2^{1-k}[s]_{p,q}!}{m^s[s-k]_{p,q}!}\sum_{j=0}^{s-k}\begin{bmatrix} s-k \\ j \end{bmatrix}_{p,q}p^{\binom{s-k-j}{2}}m^{j+k}\mathcal{P}_{j,\beta}^{(\alpha-1)}(x,-1,k,a,b:p,q)\right. \\ &\left.+a^b\sum_{j=0}^{s}\begin{bmatrix} s \\ j \end{bmatrix}_{p,q}p^{\binom{s-j}{2}}m^j\mathcal{P}_{j,\beta}^{(\alpha)}(x,-1,k,a,b:p,q) - a^b\mathcal{P}_{s,\beta}^{(\alpha)}(x,0,k,a,b:p,q)\right\}.\end{aligned}$$

In the case when $\alpha = 1$ in Theorem 8, we have the following corollary.

Corollary 2. *We have*

$$\begin{aligned}\mathcal{P}_{n,\beta}(x,y,k,a,b:p,q) &= \frac{2^{k-1}[n]_{p,q}!}{[n+k]_{p,q}!}\sum_{s=0}^{n+k}\begin{bmatrix} n+k \\ s \end{bmatrix}_{p,q}\mathcal{P}_{n+k-s,\beta}(0,my,k,a,b:p,q)\,m^{s-n} \\ &\cdot\left\{\frac{2^{1-k}[s]_{p,q}!}{m^s[s-k]_{p,q}!}\sum_{j=0}^{s-k}\begin{bmatrix} s-k \\ j \end{bmatrix}_{p,q}p^{\binom{s-k-j}{2}}m^{j+k}(x-1)_{p,q}^j\right. \\ &\left.+a^b\sum_{j=0}^{s}\begin{bmatrix} s \\ j \end{bmatrix}_{p,q}p^{\binom{s-j}{2}}m^j\mathcal{P}_{j,\beta}(x,-1,k,a,b:p,q) - a^b\mathcal{P}_{s,\beta}(x,0,k,a,b:p,q)\right\}.\end{aligned}$$

Let us define (p,q)-analog of Stirling numbers of the second kind of order v as follows.

Definition 2. *(p,q)-analog of Stirling numbers $S_{p,q}(n,v;a,b,\beta)$ of the second kind of order v is defined by means of the following generating function:*

$$\sum_{n=0}^{\infty}S_{p,q}(n,v;a,b,\beta)\frac{z^n}{[n]_{p,q}!} = \frac{\left(\beta^b e_{p,q}(z) - a^b\right)^v}{[v]_{p,q}!}.$$

A correlation between the family of unified polynomials $\mathcal{P}_{n,\beta}^{(\alpha)}(x,y,k,a,b:p,q)$ and the generalized (p,q)-Stirling numbers $S_{p,q}(n,v;a,b,\beta)$ of the second kind of order v is presented in following Theorem 9.

Theorem 9. *The following relationship*

$$\mathcal{P}_{n-vk,\beta}^{(\alpha)}\left(x,y,k,a,b:p,q\right)=2^{(k-1)v}\frac{[v]_{p,q}!}{[vk]_{p,q}!}\sum_{j=0}^{n}\frac{\begin{bmatrix} n \\ j \end{bmatrix}_{p,q}}{\begin{bmatrix} n \\ kv \end{bmatrix}_{p,q}}\mathcal{P}_{j,\beta}^{(\alpha+v)}\left(x,y,k,a,b:p,q\right)S_{p,q}\left(n-j,v;a,b,\beta\right)$$

is true.

Proof. It follows from Definition 2 that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha)}\left(x,y,k,a,b:p,q\right)\frac{z^n}{[n]_{p,q}!}\\
=&\left(\frac{2^{1-k}z^k}{\beta^b e_{p,q}\left(z\right)-a^b}\right)^{\alpha}e_{p,q}\left(xz\right)E_{p,q}\left(yz\right)\frac{\left(\beta^b e_{p,q}\left(z\right)-a^b\right)^v}{\left(2^{1-k}z^k\right)^v[v]_{p,q}!}\frac{\left(2^{1-k}z^k\right)^v[v]_{p,q}!}{\left(\beta^b e_{p,q}\left(z\right)-a^b\right)^v}\\
=&\frac{[v]_{p,q}!}{\left(2^{1-k}z^k\right)^v}\sum_{n=0}^{\infty}\mathcal{P}_{n,\beta}^{(\alpha+v)}\left(x,y,k,a,b:p,q\right)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}S_{p,q}\left(n,v;a,b,\beta\right)\frac{z^n}{[n]_{p,q}!}\\
=&2^{(k-1)v}[v]_{p,q}!\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{p,q}\mathcal{P}_{j,\beta}^{(v+\alpha)}\left(x,y,k,a,b:p,q\right)S_{p,q}\left(n-j,v;a,b,\beta\right)\frac{z^{n-vk}}{[n]_{p,q}!}.
\end{aligned}$$

Equating the coefficients z^{n-vk} of both sides gives the desired result. □

In the case when $\alpha=0$ in Theorem 9, we have the following corollary.

Corollary 3. *The following correlation holds true:*

$$2^{(1-k)v}\frac{[vk]_{p,q}!}{[v]_{p,q}!}\left(x+y\right)_{p,q}^{n-vk}=\sum_{j=0}^{n}\frac{\begin{bmatrix} n \\ j \end{bmatrix}_{p,q}}{\begin{bmatrix} n \\ vk \end{bmatrix}_{p,q}}\mathcal{P}_{j,\beta}^{(v)}\left(x,y,k,a,b:p,q\right)S_{p,q}\left(n-j,v;a,b,\beta\right).$$

3. Conclusion

In this paper, we have introduced unified (p,q)-analog of Apostol type polynomials of order α. We have also analyzed some properties of them including addition property, derivative properties, recurrence relationships, integral representations and so on. By defining the generalized (p,q)-Stirling numbers of the second kind of order v, a correlation between these numbers and unified (p,q)-analog of Apostol type polynomials of order α is obtained. We note that the results obtained here reduce to known results of unified q-polynomials when $p=1$. Also, when $q\to p=1$, our results in this paper turn into the unified Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials.

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