Research on Some New Results Arising from Multiple q-Calculus

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Abstract

In this paper, we develop the theory of the multiple q-analogue of the Heine's binomial formula, chain rule and Leibnitz's rule. We also derive many useful definitions and results involving multiple q-antiderivative and multiple q-Jackson's integral. Finally, we list here multiple q-analogue of some elementary functions including trigonometric functions and hyperbolic functions. This may be a good consideration in developing the multiple q-calculus in combinatorics, number theory and other fields of mathematics.

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1. Introduction

In the year 1910, Jackson [11] first considered the q-difference calculus (or the so-called quantum calculus), which is an old subject. From Jackson's time to the present, this theory was widely-investigated in the theory of special functions, differential equations (also fractional differential equations), and other related theories: that is, quantum calculus (also known as q-calculus) was one of the most active area of research in the physics and mathematics. While one takes care of q-calculus with one base q, Nalci [5] concerned with multiple q-calculus for the functions including independent several variables. Thereby, the necessity of multiple q-calculus has been emerged in several physical and mathematical problems.

We now review briefly some concepts of the multiple q-calculus taken in [5].

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Throughout the paper, the indexes i and j will be considered as

$$i=1,2,\cdots,N \text{ and } j=1,2,\cdots,N.$$

Let $\overrightarrow{q}:=(q_1,q_2,\cdots,q_N)$. Then the multiple q-number (a generalization of q-number) is defined by

$$[n]_{q_i,q_j}:=\frac{q_i^n-q_j^n}{q_i-q_j}.$$

It is clear that $[n]_{q_i,q_j}=[n]_{q_j,q_i}$. These numbers are represented as

$$\left([n]_{q_i,q_j}\right)=\begin{pmatrix} [n]_{q_1,q_1} & [n]_{q_1,q_2} & \cdots & [n]_{q_1,q_N} \\ [n]_{q_2,q_1} & [n]_{q_2,q_2} & \cdots & [n]_{q_2,q_n} \\ \cdots & \cdots & \cdots & \cdots \\ [n]_{q_N,q_1} & [n]_{q_N,q_2} & \cdots & [n]_{q_N,q_N} \end{pmatrix} \tag{1.1}$$

where i denotes the number of rows and j denotes the number of columns.

One can see that the diagonal terms of the matrix can be considered as the limit $q_i\to q_j$: that is,

$$\lim_{q_i\to q_j}[n]_{q_i,q_j}=\lim_{q_i\to q_j}\frac{q_i^n-q_j^n}{q_i-q_j}=nq_i^{n-1}, \tag{1.2}$$

from which, by the Eqs. (1.1) and (1.2), we have

$$\left([n]_{q_i,q_j}\right)=\begin{pmatrix} nq_1^{n-1} & [n]_{q_1,q_2} & \cdots & [n]_{q_1,q_N} \\ [n]_{q_2,q_1} & nq_2^{n-1} & \cdots & [n]_{q_2,q_n} \\ \cdots & \cdots & \cdots & \cdots \\ [n]_{q_N,q_1} & [n]_{q_N,q_2} & \cdots & nq_N^{n-1} \end{pmatrix}.$$

In view of multiple q-calculus, multiple q-derivative is defined by the following linear operator:

$$D_{q_i,q_j}f(x)=\frac{f(q_ix)-f(q_jx)}{q_i-q_j}, \tag{1.3}$$

representing $N\times N$ matrix of multiple q-derivative operators $D:=\left(D_{q_i,q_j}\right)$ which is symmetric, $D_{q_i,q_j}=D_{q_j,q_i}$:

$$\left(D_{q_i,q_j}\right)=\begin{pmatrix} D_{q_1,q_1} & D_{q_1,q_2} & \cdots & D_{q_1,q_N} \\ D_{q_2,q_1} & D_{q_2,q_2} & \cdots & D_{q_2,q_N} \\ \cdots & \cdots & \cdots & \cdots \\ D_{q_N,q_1} & D_{q_N,q_2} & \cdots & D_{q_N,q_N} \end{pmatrix}.$$

The multiple q-analogue of $(x-a)^n$ (also can be called multiple q-binomial formula) is given by

$$(x-a)_{q_i,q_j}^n:=\begin{cases} (x-q_i^{n-1}a)(x-q_i^{n-2}q_ja)\cdots(x-q_iq_j^{n-2}a)(x-q_j^{n-1}a), & \text{if } n\geq 1 \\ 1, & \text{if } n=0 \end{cases} \tag{1.4}$$

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so that

$$(x-a)^n_{q_i,q_j} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j} (-1)^k (q_i q_j)^{\frac{k(k-1)}{2}} x^{n-k} a^k \qquad (xa = ax)$$

where the notations $\begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j}$ (called multiple q-Gauss Binomial coefficients) and $[n]_{q_i,q_j}!$ (called multiple q-factorial) are defined by

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j} &= \frac{[n]_{q_i,q_j}!}{[n-k]_{q_i,q_j}! \, [k]_{q_i,q_j}!} \qquad (n \geq k) \\ [n]_{q_i,q_j}! &= [n]_{q_i,q_j} [n-1]_{q_i,q_j} \cdots [2]_{q_i,q_j} [1]_{q_i,q_j} \qquad (n \in \mathbb{N}). \end{aligned}$$

The multiple q-exponential functions are introduced by

$$e_{q_i q_j}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q_i,q_j}!} x^n \tag{1.5}$$

$$E_{q_i q_j}(x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q_i,q_j}!} (q_i q_j)^{\frac{n(n-1)}{2}} x^n \tag{1.6}$$

whose multiple q-derivatives, respectively, are as follows:

$$D_{q_i,q_j} e_{q_i q_j}(x) = \sum_{n=0}^{\infty} \frac{D_{q_i,q_j} x^n}{[n]_{q_i,q_j}!} = \sum_{n=1}^{\infty} \frac{1}{[n-1]_{q_i,q_j}!} x^{n-1} = e_{q_i q_j}(x)$$

and

$$\begin{aligned} D_{q_i,q_j} E_{q_i q_j}(x) &= \sum_{n=0}^{\infty} (q_i q_j)^{\frac{n(n-1)}{2}} \frac{D_{q_i,q_j} x^n}{[n]_{q_i,q_j}!} \\ &= \sum_{n=1}^{\infty} (q_i q_j)^{\frac{(n-1)(n-2)}{2}} \frac{(q_i q_j)^{n-1} x^{n-1}}{[n-1]_{q_i,q_j}!} \\ &= E_{q_i q_j}(q_i q_j x). \end{aligned}$$

Under circumstance commutative x and y $(xy = yx)$, we have addition formula

$$e_{q_i q_j}(x+y)_{q_i q_j} = e_{q_i q_j}(x) E_{q_i q_j}(x). \tag{1.7}$$

The multiple q-integral (a generalization of Jackson's integral) is given by

$$\int f\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = (q_i - q_j) \sum_{k=0}^{\infty} \frac{q_j^k x}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} x\right). \tag{1.8}$$

Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a formal power series. Then it has multiple q-integral representation as follows:

$$\int f(x) \, d_{\frac{q_j}{q_i}} x = \sum_{k=0}^{\infty} q_i^{k+1} a_k \frac{x^{k+1}}{[k+1]_{q_i,q_j}} + C$$

where C is a constant.

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Taking here $q_i = q$ and $q_j = 1$ for indexes i and j in the case $N = 1$, then all notations given in this part reduce to the notations of the usual q-calculus (see, for details, [1], [2], [3], [4], [6], [8], [9], [10]).

Recently, Nalci [5] has represented multiple q-calculus and investigated many important notions and results in the course of developing multiple q-calculus along the traditional lines of q-calculus. In [7], Acikgoz *et al.* also considered some new identities involving a new class of some special polynomials in the light of multiple q-calculus. They also derived a further investigation of some new identities related to multiple q-Jackson integral.

In this paper, we develop the theory of the multiple q-analogue of the Heine's binomial formula, chain rule and Leibnitz's rule. We also derive many useful definitions and results involving multiple q-antiderivative and multiple q-Jackson's integral. Finally, we list here multiple q-analogue of some elementary functions including trigonometric functions and hyperbolic functions. This may be a good consideration in developing the multiple q-calculus in combinatorics, number theory and other fields of mathematics.

2. Generalizations of some elementary functions belonging to q-calculus

As it has been q-calculus, there doesn't exist a general chain rule for multiple q-derivatives. That is, if we consider the function $f(u(x))$, where $u = u(x) = \lambda x^{\mu}$ with λ, μ being constants, we have a chain rule as special cases:

$$\begin{aligned} D_{q_i,q_j}\left[f(u(x))\right] &= D_{q_i,q_j}\left[f\left(\lambda x^{\mu}\right)\right] = \frac{f(\lambda x^{\mu} q_i^{\mu}) - f(\lambda x^{\mu} q_j^{\mu})}{x(q_i - q_j)} \\ &= \frac{f(\lambda x^{\mu} q_i^{\mu}) - f(\lambda x^{\mu} q_j^{\mu})}{\lambda x^{\mu} q_i^{\mu} - \lambda x^{\mu} q_j^{\mu}} . \frac{\lambda x^{\mu} q_i^{\mu} - \lambda x^{\mu} q_j^{\mu}}{x(q_i - q_j)} \\ &= \frac{f(u q_i^{\mu}) - f(u q_j^{\mu})}{u q_i^{\mu} - u q_j^{\mu}} . \frac{u(q_i x) - u(q_j x)}{x(q_i - q_j)} \end{aligned}$$

which gives

$$D_{q_i,q_j}\ f(u(x)) = \left(D_{q_i^{\mu},q_j^{\mu}} f\right)(u(x)).D_{q_i,q_j} u(x). \tag{2.1}$$

Conversely, if we consider the function $u(x) = x^3 + x^2$ or $u(x) = \cos x$, the quantity $u(q_i x)$ and $u(q_j x)$ can not be derived in terms of u in a basic way, and thereby it is impossible to write a general chain rule.

Now let us investigate the derivative of the function $\frac{1}{(x-a)^n_{q_i,q_j}}$. For any integer n, we have

$$\begin{aligned} D_{q_i,q_j}\left(\frac{1}{(x-a)^n_{q_i,q_j}}\right) &= D_{q_i,q_j}\left(\frac{1}{\left(x - q_i^{-n}\left(q_i^n a\right)\right)^n_{q_i,q_j}}\right) \\ &= D_{q_i,q_j}(x - q_j^n q_i^n a)^{-n}_{q_i,q_j} \\ &= \left[-n\right]_{q_i,q_j}\left(x - \left(q_j q_i\right)^n a\right)^{-n-1}_{q_i,q_j} \\ &= -\left(q_j q_i\right)^{-n}\left[n\right]_{q_i,q_j}\left(x - \left(q_j q_i\right)^n a\right)^{-n-1}_{q_i,q_j}, \end{aligned}$$

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where

$$(x-q_j^n a)_{q_i,q_j}^{-n} = \frac{1}{(x-q_i^{-n}a)_{q_i,q_j}^n}.$$

By the similar way, we have for $n \geq 0$:

$$\begin{aligned} D_{q_i,q_j}(a-x)_{q_i,q_j}^n &= D_{q_i,q_j}\left((-1)^n (q_iq_j)^{\frac{n.(n-1)}{2}}(x-(q_iq_j)^{1-n}a)_{q_i,q_j}^n\right) \\ &= (-1)^n (q_iq_j)^{\frac{n.(n-1)}{2}} [n]_{q_i,q_j} (x-(q_iq_j)^{1-n}a)_{q_i,q_j}^{n-1} \\ &= -[n]_{q_i,q_j} (q_iq_j)^{n-1}(q_i^{-1}q_j^{-1}a-x)_{q_i,q_j}^{n-1} \\ &= -[n]_{q_i,q_j} (a-q_iq_jx)_{q_i,q_j}^{n-1} \end{aligned}$$

and

$$D_{q_i,q_j}\left(\frac{1}{(a-x)_{q_i,q_j}^n}\right) = \frac{-D_{q_i,q_j}(a-x)_{q_i,q_j}^n}{(a-q_ix)_{q_i,q_j}^n (a-q_jx)_{q_i,q_j}^n} \tag{2.2}$$

$$= \frac{[n]_{q_i,q_j}(a-q_iq_jx)_{q_i,q_j}^{n-1}}{(a-q_ix)_{q_i,q_j}^n (a-q_jx)_{q_i,q_j}^n}$$

$$= \frac{[n]_{q_i,q_j}}{(a-q_i^nx)(a-q_jx)_{q_i,q_j}^n} \tag{2.3}$$

$$= \frac{[n]_{q_i,q_j}}{(a-q_jx)_{q_i,q_j}^{n+1}}. \tag{2.4}$$

Taking the value $a=1$ in the Eq. (2.2), we derive multiple q-derivative of k-th order as follows:

$$D_{q_i,q_j}^k\left(\frac{1}{(1-x)_{q_i,q_j}^n}\right) = \frac{[n]_{q_i,q_j}[n+1]_{q_i,q_j}\cdots[n+k-1]_{q_i,q_j}}{(1-q_j^kx)_{q_i,q_j}^{n+k}}. \tag{2.5}$$

In the case $x=0$ in the Eq. (2.5), it gives

$$[n]_{q_i,q_j}[n+1]_{q_i,q_j}\cdots[n+k-1]_{q_i,q_j}. \tag{2.6}$$

By the Eq. (2.6), we have, i.e., a Taylor expansion for $\frac{1}{(1-x)_{q_i,q_j}^n}$ about $x=0$:

$$\begin{aligned} \frac{1}{(1-x)_{q_i,q_j}^n} &= \sum_{k=0}^{\infty} \frac{[n]_{q_i,q_j}[n+1]_{q_i,q_j}\cdots[n+k-1]_{q_i,q_j}}{[k]_{q_i,q_j}!} x^k \\ &= \sum_{k=0}^{\infty} \frac{(1-Q^n)_Q^k}{(1-Q)_Q^k} q_i^{(n-k)k} x^k \qquad \left(Q=\frac{q_j}{q_i}\right) \end{aligned}$$

which is called Heine's multiple q-Binomial formula.

We now give the multiple q-analogue of Leibnitz rule as follows.

Theorem 1. *Let $f(x)$ and $g(x)$ be n-times multiple q-differentiable functions. Then $(fg)(x)$ is also n-times multiple q-differentiable and*

$$D^n_{q_i,q_j}(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j} D^k_{q_i,q_j}(f)\left(xq_i^{n-k}\right) D^{n-k}_{q_i,q_j}(g)\left(xq_j^k\right).$$

Proof. The theorem is proved by mathematical induction method. Firstly, for $n = 1$, one can without difficulty see that Theorem 1 is true. Assume that Theorem 1 is true for $n = m$. Then it holds also true for $n = m + 1$ using the case $n = m$ and the Eq. (2.1), because

$$\begin{aligned}
D^{m+1}_{q_i,q_j}(fg)(x) &= D_{q_i,q_j}\left(D^m_{q_i,q_j}(fg)(x)\right) \\
&= D_{q_i,q_j}\left(\sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i,q_j} D^k_{q_i,q_j}(f)\left(xq_i^{m-k}\right) D^{m-k}_{q_i,q_j}(g)\left(xq_j^k\right)\right) \\
&= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i,q_j} \Big[q_i^{m-k} D^{k+1}_{q_i,q_j}(f)\left(xq_i^{m-k}\right) D^{m-k}_{q_i,q_j}(g)\left(xq_j^{k+1}\right) \\
&\quad + D^k_{q_i,q_j}(f)\left(xq_i^{m+1-k}\right) D^{m+1-k}_{q_i,q_j}(g)\left(xq_j^k\right) q_j^k\Big] \\
&= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q_i,q_j} D^k_{q_i,q_j}(f)\left(xq_i^{m+1-k}\right)_{q_i,q_j} D^{m+1-k}_{q_i,q_j}(g)\left(xq_j^k\right) q_j^k \\
&\quad + \sum_{k=1}^{m+1} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q_i,q_j} q_i^{m+1-k} D^k_{q_i,q_j}(f)\left(xq_i^{m+1-k}\right) D^{m+1-k}_{q_i,q_j}(g)\left(xq_j^k\right) \\
&= f\left(xq_i^{m+1}\right) D^{m+1}_{q_i,q_j}(g)(x) \\
&\quad + \sum_{k=1}^{m} \left\{ q_j^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i,q_j} + q_i^{m+1-k} \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q_i,q_j} \right\} D^k_{q_i,q_j}(f)\left(xq_i^{m+1-k}\right) D^{m+1-k}_{q_i,q_j}(g)\left(xq_j^k\right) \\
&\quad + D^{m+1}_{q_i,q_j}(f)(x)\, g\left(xq_j^{m+1}\right) \\
&= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q_i,q_j} D^k_{q_i,q_j}(f)\left(xq_i^{m+1-k}\right) D^{m+1-k}_{q_i,q_j}(g)\left(xq_j^k\right).
\end{aligned}$$

□

Corollary 1. *Each multiple q-binomial coefficient is a polynomial including the parameters q_i and q_j of degree $k(n-k)$ whose leading coefficient is 1.*

Proof. To prove this, we firstly consider for any nonnegative integer n, as follows:

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_{q_i,q_j} = \begin{bmatrix} n \\ n \end{bmatrix}_{q_i,q_j} = 1,$$

which is obviously a polynomial. By making use of multiple q-Pascal rules and induction on n, for any $1 \leq k \leq n-1$, $\begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j}$ is the sum of two polynomials, hence is itself a polynomial. The multiple q-binomial coefficient has the following explicit expression:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j} = \frac{[n]_{q_i,q_j}[n-1]_{q_i,q_j}\cdots[n-k+1]_{q_i,q_j}}{[k]_{q_i,q_j}[k-1]_{q_i,q_j}\cdots[1]_{q_i,q_j}} = \frac{\left(q_i^n-q_j^n\right)\left(q_i^{n-1}-q_j^{n-1}\right)\cdots\left(q_i^{n-k+1}-q_j^{n-k+1}\right)}{\left(q_i^k-q_j^k\right)\left(q_i^{k-1}-q_j^{k-1}\right)\cdots(q_i-q_j)}. \tag{2.7}$$

Since both the numerator and denominator of (2.7) are polynomials based on the parameters q_i and q_j with leading coefficient 1, so is their quotient. Finally, the degree of $\begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j}$ in q_i and q_j is the difference of the degrees of numerator and denominator: that is,

$$[n+(n-1)+\cdots(n-k+1)]-[k+(k-1)+\cdots 1]=(n-k)\,k.$$

Therefore, we conclude the proof of the corollary. □

In the other hand, we consider Corollary 1 in the following form:

$$\begin{aligned} &\alpha_0 q_j^{(n-k)k}+\alpha_1 q_j^{(n-k)k-1}q_i+\cdots+\alpha_{k(n-k)-1}q_j q_i^{(n-k)k-1}+\alpha_{k(n-k)}q_i^{(n-k)k} \\ &= \frac{\left(q_i^n-q_j^n\right)\left(q_i^{n-1}-q_j^{n-1}\right)\cdots\left(q_i^{n-k+1}-q_j^{n-k+1}\right)}{\left(q_i^k-q_j^k\right)\left(q_i^{k-1}-q_j^{k-1}\right)\cdots(q_i-q_j)}. \end{aligned} \tag{2.8}$$

By changing q_i by $\frac{1}{q_i}$ and q_j by $\frac{1}{q_j}$, also multiply both sides by $(q_iq_j)^{(n-k)k}$, it is easy to observe that the right-hand side will be unchanged, while the left-hand side,

$$\alpha_0 q_i^{(n-k)k}+\alpha_1 q_i^{(n-k)k-1}q_j+\cdots+\alpha_{k(n-k)-1}q_i q_j^{(n-k)k-1}+\alpha_{k(n-k)}q_j^{(n-k)k}, \tag{2.9}$$

has the sequence of coefficient α_m reversed in order. If we compare the coefficients of the Eqs. (2.8) and (2.9), we acquire the coefficients in the polynomial expression of $\begin{bmatrix} n \\ k \end{bmatrix}_{q_i,q_j}$ that are symmetric:

$$\begin{aligned} \alpha_0 &= \alpha_{(n-k)k} \\ \alpha_1 &= \alpha_{(n-k)k-1} \\ \alpha_2 &= \alpha_{(n-k)k-2} \\ &\vdots \end{aligned}$$

i.e, $\alpha_m=\alpha_{(n-k)k-m}$.

Note that the multiple q-binomial coefficients also have combinatorial interpretations like q-binomial coefficients.

3. Multiple q-Antiderivative

Some information and useful methods in this section will be utilized from the book [6].

Definition 1. *The function $F(x)$ is a q-antiderivative of $f(x)$ if $D_{q_i,q_j}F(x)=f(x)$. It is shown by*

$$\int f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x.$$

Proposition 1. *Let $0<\frac{q_j}{q_i}<1$. Then, any function $f(x)$ has at most one multiple q-antiderivative which is continuous at $x=0$, up to adding a constant.*

Proof. Let us consider F_1 and F_2 as two multiple q-antiderivatives of f, which are both continuous at 0. Let $\varpi=F_1-F_2$, which also must be continuous at 0. Moreover

$$D_{q_i,q_j}\varpi(x)=D_{q_i,q_j}(F_1(x)-F_2(x))=f(x)-f(x)=0$$

implies that $\varpi(q_ix)=\varpi(q_jx)$ for any x. For some $U>0$, let

$$\begin{aligned} s &= \inf\left\{\varpi(x)\mid \frac{q_j}{q_i}U\le x\le U\right\}, \\ S &= \sup\left\{\varpi(x)\mid \frac{q_j}{q_i}U\le x\le U\right\}, \end{aligned}$$

which may be infinity if ϖ is unbounded above and/or below. It should be clear that because of $s\neq S$, $\varpi(0)$ can not be both s and S. It is not problem that we select s or S, so we can suppose $\varpi(0)\neq s$. By the definition of continuous at $x=0$, for every $\epsilon>0$ there exists a $\delta>0$ such that

$$s+\varepsilon\notin\varpi(0,\delta).$$

However there exists for some sufficiently N such that $\left(\frac{q_j}{q_i}\right)^N U<\delta$, which implies that

$$s+\varepsilon\in(s,S)\subset\varpi\left[\frac{q_j}{q_i}U,U\right]=\varpi\left[\left(\frac{q_j}{q_i}\right)^{N+1}U,\left(\frac{q_j}{q_i}\right)^{N}U\right]\subset\varpi(0,\delta),$$

bringing about a contradiction. So, we have $s=S$, ϖ is a constant in that $\varpi\left[\frac{q_j}{q_i}U,U\right]$, which shows that F_1-F_2 is also constant everywhere. □

4. Multiple q-Jackson Integral

By the expression of the Eq. (1.8), we develop a more general formula:

$$\begin{aligned} \int f\left(\frac{x}{q_i}\right)D_{q_i,q_j}g\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x &= (q_i-q_j)\sum_{k=0}^{\infty}\frac{q_j^kx}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}x\right)D_{q_i,q_j}g\left(\frac{q_j^k}{q_i^{k+1}}x\right) \\ &= (q_i-q_j)\sum_{k=0}^{\infty}\frac{q_j^kx}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}x\right)\frac{g\left(\frac{q_j^k}{q_i^k}x\right)-g\left(\frac{q_j^{k+1}}{q_i^{k+1}}x\right)}{(q_i-q_j)\frac{q_j^k}{q_i^{k+1}}x}, \end{aligned}$$

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and also

$$\int f\left(\frac{x}{q_i}\right) D_{q_i,q_j} g\left(\frac{x}{q_i}\right) d_{\frac{q_j}{q_i}} x = \sum_{k=0}^{\infty} f\left(\frac{q_j^k}{q_i^{k+1}} x\right)\left(g\left(\frac{q_j^k}{q_i^k} x\right) - g\left(\frac{q_j^{k+1}}{q_i^{k+1}} x\right)\right).$$

Theorem 2. *Let $q_i, q_j \in (0,1)$ with $0 < \frac{q_j}{q_i} < 1$ and let $|f(x)x^{\tau}|$ be bounded on the interval $(0, A]$ for some $0 \le \tau < 1$. Then the Jackson integral defined by (1.8) converges to a function $F(x)$ on $(0, A]$, which is a multiple q-antiderivative of $f(x)$. Moreover, $F(x)$ is continuous at $x = 0$ with $F(0) = 0$.*

Proof. Suppose $|f(x)x^{\tau}| < M$ on $(0, A]$ and fix $0 < x \le A$. Then for $k \ge 0$,

$$\begin{aligned}
\left| f\left(\frac{q_j^k}{q_i^{k+1}} x\right)\left(\frac{q_j^k}{q_i^{k+1}} x\right)^{\tau} \right| &< M \\
\left| f\left(\frac{q_j^k}{q_i^{k+1}} x\right)\right| \left|\left(\frac{q_j^k}{q_i^{k+1}} x\right)^{\tau} \right| &< M \\
\left| f\left(\frac{q_j^k}{q_i^{k+1}} x\right)\right| &< M\left(\frac{q_j^k}{q_i^{k+1}} x\right)^{-\tau}.
\end{aligned}$$

Hence, for any $0 < x \le A$, we get

$$\left|\left(\frac{q_j^k}{q_i^{k+1}}\right) f\left(\frac{q_j^k}{q_i^{k+1}} x\right)\right| < M\left(\frac{q_j^k}{q_i^{k+1}}\right)\left(\frac{q_j^k}{q_i^{k+1}} x\right)^{-\tau} = Mx^{-\tau}\frac{1}{\left(q_i^{1-\tau}\right)}\left(\frac{q_j^{1-\tau}}{q_i^{1-\tau}}\right)^k. \tag{4.1}$$

If we write in the following sum including Jackson integral that is majorized by a convergent geometric series. Then, (1.8) converges pointwise to some functions. Namely, one can see without difficulty that $F(0) = 0$. It is the fact that $F(x)$ is continuous at $x = 0$, i.e., $F(x)$ approaches zero as $x \longrightarrow 0$ using (4.1), for $0 < x \le A$ as

$$\begin{aligned}
\left|(q_i - q_j)\sum_{k=0}^{\infty}\frac{q_j^k x}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} x\right)\right| &< |q_i - q_j|\,|x|\sum_{k=0}^{\infty}\frac{q_j^k}{q_i^{k+1}} f\left(\frac{q_j^k}{q_i^{k+1}} x\right) \\
&< |q_i - q_j|\,|x|\sum_{k=0}^{\infty} Mx^{-\tau}\frac{1}{\left(q_i^{1-\tau}\right)}\left(\frac{q_j^{1-\tau}}{q_i^{1-\tau}}\right)^k \\
&< |q_i - q_j|\frac{1}{\left(q_i^{1-\tau}\right)}\frac{Mx^{1-\tau}}{1 - \left(\frac{q_j}{q_i}\right)^{1-\tau}}.
\end{aligned}$$

□

We now give the following theorem in order to verify $F(x)$ being a multiple q-antiderivative of $f(x)$.

Theorem 3. *The definition of q-multiple Jackson integral given in (1.8) presents a q-antiderivatives of $f(x)$.*

Proof. It is sufficient to check that

$$\begin{aligned} D_{q_i,q_j}F(x) &= \frac{1}{(q_i-q_j)x}\left((q_i-q_j)\sum_{\tau=0}^{\infty}\frac{q_j^{\tau}x}{q_i^{\tau}}f\left(\frac{q_j^{\tau}}{q_i^{\tau}}x\right)-(q_i-q_j)\sum_{\tau=0}^{\infty}\frac{q_j^{\tau+1}x}{q_i^{\tau+1}}f\left(\frac{q_j^{\tau+1}}{q_i^{\tau+1}}x\right)\right) \\ &= \sum_{\tau=0}^{\infty}\frac{q_j^{\tau}}{q_i^{\tau}}f\left(\frac{q_j^{\tau}}{q_i^{\tau}}x\right)-\sum_{\tau=0}^{\infty}\frac{q_j^{\tau+1}}{q_i^{\tau+1}}f\left(\frac{q_j^{\tau+1}}{q_i^{\tau+1}}x\right) \\ &= f(x). \end{aligned}$$

This completes the proof of the Theorem. □

Notice that the multiple q-differentiation is valid provided that $x\in(0,A]$ and $0<\frac{q_j}{q_i}<1$, then $x\frac{q_j}{q_i}\in(0,A]$.

By Proposition 1, if the hypothesis of Theorem 2 is satisfied, the q-multiple Jackson integral gives the unique multiple q-antiderivative being continuous at $x=0$, up to adding a constant. On the other hand, if we know that $F(x)$ is a multiple q-antiderivative of $f(x)$ and $F(x)$ is continuous at $x=0$, $F(x)$ must be given, up to adding a constant. By q-multiple Jackson's formula (1.8), since a partial sum of the q-multiple Jackson integral is

$$\begin{aligned} (q_i-q_j)\sum_{\tau=0}^{N}\frac{q_j^{\tau}x}{q_i^{\tau+1}}f\left(\frac{q_j^{\tau}}{q_i^{\tau+1}}x\right) &= (q_i-q_j)\sum_{\tau=0}^{N}\frac{q_j^{\tau}x}{q_i^{\tau+1}}D_{q_i,q_j}F(x)\mid_{\frac{q_j^{\tau}}{q_i^{\tau+1}}x} \\ &= \sum_{\tau=0}^{N}\left[F\left(\frac{q_j^{\tau}}{q_i^{\tau}}x\right)-F\left(\frac{q_j^{\tau+1}}{q_i^{\tau+1}}x\right)\right] \\ &= F(x)-F\left(\frac{q_j^{N+1}}{q_i^{N+1}}x\right), \end{aligned}$$

approaching to $F(x)-F(0)$ as $N\longrightarrow\infty$, by the continuity of $F(x)$ at the case $x=0$.

We now give an example to see in which the q-multiple Jackson formula fails.

Let $f(x)=1/x$. We have

$$\int\frac{1}{x}d_{\frac{q_j}{q_i}}x=\frac{(q_i-q_j)}{\log(\frac{q_j}{q_i})}\log(x)$$

since

$$D_{q_i,q_j}\log x=\frac{\log(q_ix)-\log(q_jx)}{(q_i-q_j)x}=\frac{\log(\frac{q_j}{q_i})}{(q_i-q_j)}\frac{1}{x}.$$

However, the q-multiple Jackson formula gives

$$\int\frac{1}{x}d_{\frac{q_j}{q_i}}x=\frac{(q_i-q_j)}{q_i}\sum_{k=0}^{\infty}1=\infty.$$

Finally, the formula fails because $f(x)x^{\tau}$ is not bounded for any $0\leq\tau<1$. Note that $\log x$ is not continuous at the case $x=0$.

5. Multiple q-Trigonometric Functions

The multiple q-analogues of the sine, cosine, tangent and cotangent functions can be defined in the same manner with their well known Euler expressions of the exponential functions.

Definition 2. *Let* $\mathbf{i} = \sqrt{-1}$. *Then two pairs of multiple q-trigonometric functions are defined by*

$$\begin{array}{|l|l|}\hline \sin_{q_i,q_j} x := \dfrac{e_{q_iq_j}(\mathbf{i}x) - e_{q_iq_j}(-\mathbf{i}x)}{2\mathbf{i}} & \mathrm{SIN}_{q_i,q_j} x := \dfrac{E_{q_iq_j}(\mathbf{i}x) - E_{q_iq_j}(-\mathbf{i}x)}{2\mathbf{i}} \\ \hline \cos_{q_i,q_j} x := \dfrac{e_{q_iq_j}(\mathbf{i}x) + e_{q_iq_j}(-\mathbf{i}x)}{2} & \mathrm{COS}_{q_i,q_j} x := \dfrac{E_{q_iq_j}(\mathbf{i}x) + E_{q_iq_j}(-\mathbf{i}x)}{2} \\ \hline \tan_{q_i,q_j} x := \dfrac{\sin_{q_i,q_j} x}{\cos_{q_i,q_j} x} & \mathrm{TAN}_{q_i,q_j} x := \dfrac{\mathrm{SIN}_{q_i,q_j} x}{\mathrm{COS}_{q_i,q_j} x} \\ \hline \cot_{q_i,q_j} x := \dfrac{\cos_{q_i,q_j} x}{\sin_{q_i,q_j} x} & \mathrm{COT}_{q_i,q_j} x := \dfrac{\mathrm{COS}_{q_i,q_j} x}{\mathrm{SIN}_{q_i,q_j} x}. \\ \hline \end{array} \tag{5.1}$$

We now give a representation of $N \times N$ matrix of $\sin_{q_i,q_j}$ including sin functions elements as in the following form.

$$\left(\sin_{q_i,q_j} x\right) = \begin{pmatrix} \sin_{q_1,q_1} x & \sin_{q_1,q_2} x & \cdots & \sin_{q_1,q_N} x \\ \sin_{q_2,q_1} x & \sin_{q_2,q_2} x & \cdots & \sin_{q_2,q_N} x \\ \cdots & \cdots & \cdots & \cdots \\ \sin_{q_N,q_1} x & \sin_{q_N,q_2} x & \cdots & \sin_{q_N,q_N} x \end{pmatrix}.$$

Note that one can represent $N \times N$ matrix of other multiple q-trigonometric functions as in the above.

Definition 3. *Two pairs of multiple q-hyperbolic functions are defined by*

$$\begin{array}{|l|l|}\hline \sinh_{q_i,q_j} x = \dfrac{e_{q_iq_j}(x) - e_{q_iq_j}(-x)}{2} & \mathrm{SINH}_{q_i,q_j} x = \dfrac{E_{q_iq_j}(x) - E_{q_iq_j}(-x)}{2} \\ \hline \cosh_{q_i,q_j} x = \dfrac{e_{q_iq_j}(x) + e_{q_iq_j}(-x)}{2} & \mathrm{COSH}_{q_i,q_j} x = \dfrac{E_{q_iq_j}(x) + E_{q_iq_j}(-x)}{2} \\ \hline \tanh_{q_i,q_j} x = \dfrac{\sinh_{q_i,q_j} x}{\cosh_{q_i,q_j} x} & \mathrm{TANH}_{q_i,q_j} x = \dfrac{\mathrm{SINH}_{q_i,q_j} x}{\mathrm{COSH}_{q_i,q_j} x} \\ \hline \coth_{q_i,q_j} x = \dfrac{\cosh_{q_i,q_j} x}{\sinh_{q_i,q_j} x} & \mathrm{COTH}_{q_i,q_j} x = \dfrac{\mathrm{COSH}_{q_i,q_j} x}{\mathrm{SINH}_{q_i,q_j} x}. \\ \hline \end{array} \tag{5.2}$$

By Definition 3, we readily see that

$$\begin{array}{|l|l|}\hline e_{q_iq_j}(x) = \cosh_{q_i,q_j} x + \sinh_{q_i,q_j} x & E_{q_iq_j}(x) = \mathrm{COSH}_{q_i,q_j} x + \mathrm{SINH}_{q_i,q_j} x \\ \hline \end{array}$$

We now give here a representation of $N \times N$ matrix of $\sinh_{q_i,q_j}$ including sinh functions elements as in the following form

$$\left(\sinh_{q_i,q_j} x\right) = \begin{pmatrix} \sinh_{q_1,q_1} x & \sinh_{q_1,q_2} x & \cdots & \sinh_{q_1,q_N} x \\ \sinh_{q_2,q_1} x & \sinh_{q_2,q_2} x & \cdots & \sinh_{q_2,q_N} x \\ \cdots & \cdots & \cdots & \cdots \\ \sinh_{q_N,q_1} x & \sinh_{q_N,q_2} x & \cdots & \sinh_{q_N,q_N} x \end{pmatrix}.$$

Note that one can represent $N \times N$ matrix of other multiple q-hyperbolic functions as in the above.

We now list intriguing identities for trigonometric and hyperbolic functions under the theory of multiple q-theory as follows.

$\sin_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{q_i,q_j}!} x^{2n+1}$	$\sinh_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{[2n+1]_{q_i,q_j}!}$
$\mathrm{SIN}_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n+1]_{q_i,q_j}!} (q_i q_j)^{\frac{(2n+1)2n}{2}} x^{2n+1}$	$\mathrm{SINH}_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(q_i q_j)^{\frac{(2n+1)2n}{2}} x^{2n+1}}{[2n+1]_{q_i,q_j}!}$
$\cos_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{q_i,q_j}!} x^{2n}$	$\cosh_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{[2n]_{q_i,q_j}!}$
$\mathrm{COS}_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{[2n]_{q_i,q_j}!} (q_i q_j)^{\frac{2n(2n-1)}{2}} x^{2n}$	$\mathrm{COSH}_{q_i,q_j} x = \sum_{n=0}^{\infty} \frac{(q_i q_j)^{\frac{2n(2n-1)}{2}} x^{2n}}{[2n]_{q_i,q_j}!}$
$\sec_{q_i,q_j} x := \frac{1}{\cos_{q_i,q_j} x}$	$\csc_{q_i,q_j} x := \frac{1}{\sin_{q_i,q_j} x}$
$\mathrm{SEC}_{q_i,q_j} x := \frac{1}{\mathrm{COS}_{q_i,q_j} x}$	$\mathrm{CSC}_{q_i,q_j} x := \frac{1}{\mathrm{SIN}_{q_i,q_j} x}$
$\mathrm{sech}_{q_i,q_j} x := \frac{1}{\cosh_{q_i,q_j} x}$	$\mathrm{csch}_{q_i,q_j} x := \frac{1}{\sinh_{q_i,q_j} x}$
$\mathrm{SECH}_{q_i,q_j} x := \frac{1}{\mathrm{COSH}_{q_i,q_j} x}$	$\mathrm{CSCH}_{q_i,q_j} x := \frac{1}{\mathrm{SINH}_{q_i,q_j} x}$

$\sinh_{q_i,q_j} x = -\mathbf{i} \sin_{q_i,q_j}(\mathbf{i}x)$ $\mathrm{SINH}_{q_i,q_j} x = -\mathbf{i}\, \mathrm{SIN}_{q_i,q_j}(\mathbf{i}x)$	$q_i \to 1$ and $q_j \to q$ for $N = 1$	$\sinh x = -\mathbf{i} \sin(\mathbf{i}x)$
$\cosh_{q_i,q_j} x = \cos_{q_i,q_j}(\mathbf{i}x)$ $\mathrm{COSH}_{q_i,q_j} x = \mathrm{COS}_{q_i,q_j}(\mathbf{i}x)$	$q_i \to 1$ and $q_j \to q$ for $N = 1$	$\cosh x = \cos(\mathbf{i}x)$
$\tanh_{q_i,q_j} x = -\mathbf{i} \tan_{q_i,q_j}(\mathbf{i}x)$ $\mathrm{TANH}_{q_i,q_j} x = -\mathbf{i}\, \mathrm{TAN}_{q_i,q_j}(\mathbf{i}x)$	$q_i \to 1$ and $q_j \to q$ for $N = 1$	$\tanh x = -\mathbf{i} \tan(\mathbf{i}x)$
$\coth_{q_i,q_j} x = \mathbf{i} \cot_{q_i,q_j}(\mathbf{i}x)$ $\mathrm{COTH}_{q_i,q_j} x = \mathbf{i}\, \mathrm{COT}_{q_i,q_j}(\mathbf{i}x)$	$q_i \to 1$ and $q_j \to q$ for $N = 1$	$\coth x = \mathbf{i} \cot(\mathbf{i}x)$

$e_{q_iq_j}(x+y)_{q_i,q_j} = \cosh_{q_i,q_j}(x+y)_{q_i,q_j} + \sinh_{q_i,q_j}(x+y)_{q_i,q_j}$
$E_{q_iq_j}(x+y)_{q_i,q_j} = \mathrm{COSH}_{q_i,q_j}(x+y)_{q_i,q_j} + \mathrm{SINH}_{q_i,q_j}(x+y)_{q_i,q_j}$
$\sinh_{q_i,q_j}(x+y)_{q_i,q_j} = \sinh_{q_i,q_j} x\,\mathrm{COSH}_{q_i,q_j} y + \cosh_{q_i,q_j} x\,\mathrm{SINH}_{q_i,q_j} y$
$\cosh_{q_i,q_j}(x+y)_{q_i,q_j} = \cosh_{q_i,q_j} x\,\mathrm{COSH}_{q_i,q_j} y + \sinh_{q_i,q_j} x\,\mathrm{SINH}_{q_i,q_j} y$
$\mathrm{SINH}_{q_i,q_j}(x+y)_{q_i,q_j} = \sinh_{q_i,q_j} x\,\mathrm{COSH}_{q_i,q_j} y + \cosh_{q_i,q_j} x\,\mathrm{SINH}_{q_i,q_j} y$
$\mathrm{COSH}_{q_i,q_j}(x+y)_{q_i,q_j} = \cosh_{q_i,q_j} x\,\mathrm{COSH}_{q_i,q_j} y + \sinh_{q_i,q_j} x\,\mathrm{SINH}_{q_i,q_j} y$
$\sin_{q_i,q_j}(x+\mathbf{i}y)_{q_i,q_j} = \sin_{q_i,q_j} x\,\mathrm{COSH}_{q_i,q_j} y + \mathbf{i}\cos_{q_i,q_j} x\,\mathrm{SINH}_{q_i,q_j} y$
$\cos_{q_i,q_j}(x+\mathbf{i}y)_{q_i,q_j} = \cos_{q_i,q_j} x\,\mathrm{COSH}_{q_i,q_j} y + \mathbf{i}\sin_{q_i,q_j} x\,\mathrm{SINH}_{q_i,q_j} y$

$\cos_{q_i,q_j}(0+\mathbf{i}y)_{q_i,q_j} = \mathrm{COSH}_{q_i,q_j} y$	$\mathrm{COS}_{q_i,q_j}(0+\mathbf{i}y)_{q_i,q_j} = \sinh_{q_i,q_j} y$
$\sinh_{q_i,q_j}(\mathbf{i}y+0)_{q_i,q_j} = \mathbf{i}\sin_{q_i,q_j} y$	$\cosh_{q_i,q_j}(\mathbf{i}y+0)_{q_i,q_j} = \cos_{q_i,q_j} y$
$\sin_{q_i,q_j}(x+\mathbf{i}0)_{q_i,q_j} = \sin_{q_i,q_j} x$	$\cos_{q_i,q_j}(x+\mathbf{i}0)_{q_i,q_j} = \cos_{q_i,q_j} x$
$\mathrm{SIN}_{q_i,q_j}(x+\mathbf{i}0)_{q_i,q_j} = \mathrm{SIN}_{q_i,q_j} x$	$\mathrm{COS}_{q_i,q_j}(x+\mathbf{i}0)_{q_i,q_j} = \mathrm{COS}_{q_i,q_j} x$
$\sin_{q_i,q_j}(\mathbf{i}y+0)_{q_i,q_j} = \mathbf{i}\sinh_{q_i,q_j} y$	$\cos_{q_i,q_j}(\mathbf{i}y+0)_{q_i,q_j} = \cosh_{q_i,q_j} y$
$\sin_{q_i,q_j}(0+\mathbf{i}y)_{q_i,q_j} = \mathbf{i}\,\mathrm{SINH}_{q_i,q_j} y$	$\mathrm{SIN}_{q_i,q_j}(0+\mathbf{i}y)_{q_i,q_j} = \mathbf{i}\sinh_{q_i,q_j} y$

$\sin_{q_i,q_j}(-x) = -\sin_{q_i,q_j} x$	$\mathrm{SIN}_{q_i,q_j}(-x) = -\mathrm{SIN}_{q_i,q_j} x$
$\cos_{q_i,q_j}(-x) = \cos_{q_i,q_j} x$	$\mathrm{COS}_{q_i,q_j}(-x) = \mathrm{COS}_{q_i,q_j} x$
$\tan_{q_i,q_j}(-x) = -\tan_{q_i,q_j} x$	$\mathrm{TAN}_{q_i,q_j}(-x) = -\mathrm{TAN}_{q_i,q_j} x$
$\cot_{q_i,q_j}(-x) = -\cot_{q_i,q_j} x$	$\mathrm{COT}_{q_i,q_j}(-x) = -\mathrm{COT}_{q_i,q_j} x$
$\sec_{q_i,q_j}(-x) = \sec_{q_i,q_j} x$	$\mathrm{SEC}_{q_i,q_j}(-x) = \mathrm{SEC}_{q_i,q_j} x$
$\csc_{q_i,q_j}(-x) = -\csc_{q_i,q_j} x$	$\mathrm{CSC}_{q_i,q_j}(-x) = -\mathrm{CSC}_{q_i,q_j} x$
$\sinh_{q_i,q_j}(-x) = -\sinh_{q_i,q_j} x$	$\mathrm{SINH}_{q_i,q_j}(-x) = -\mathrm{SINH}_{q_i,q_j} x$
$\cosh_{q_i,q_j}(-x) = \cosh_{q_i,q_j} x$	$\mathrm{COSH}_{q_i,q_j}(-x) = \mathrm{COSH}_{q_i,q_j} x$
$\tanh_{q_i,q_j}(-x) = -\tanh_{q_i,q_j} x$	$\mathrm{TANH}_{q_i,q_j}(-x) = -\mathrm{TANH}_{q_i,q_j} x$
$\coth_{q_i,q_j}(-x) = -\coth_{q_i,q_j} x$	$\mathrm{COTH}_{q_i,q_j}(-x) = -\mathrm{COTH}_{q_i,q_j} x$
$\mathrm{sech}_{q_i,q_j}(-x) = \mathrm{sech}_{q_i,q_j} x$	$\mathrm{SECH}_{q_i,q_j}(-x) = \mathrm{SECH}_{q_i,q_j} x$
$\mathrm{csch}_{q_i,q_j}(-x) = -\mathrm{csch}_{q_i,q_j} x$	$\mathrm{CSCH}_{q_i,q_j}(-x) = -\mathrm{CSCH}_{q_i,q_j} x$

$D_{q_i,q_j}\sin_{q_i,q_j}x=\cos_{q_i,q_j}x$	$\int\sin_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=-\cos_{q_i,q_j}x+C$
$D_{q_i,q_j}\mathrm{SIN}_{q_i,q_j}x=\mathrm{COS}_{q_i,q_j}(q_iq_jx)$	$\int\mathrm{SIN}_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=-q_iq_j\,\mathrm{COS}_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$
$D_{q_i,q_j}\cos_{q_i,q_j}x=-\sin_{q_i,q_j}x$	$\int\cos_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\sin_{q_i,q_j}x+C$
$D_{q_i,q_j}\mathrm{COS}_{q_i,q_j}x=-\mathrm{SIN}_{q_i,q_j}(q_iq_jx)$	$\int\mathrm{COS}_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=q_iq_j\,\mathrm{SIN}_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$
$D_{q_i,q_j}\sinh_{q_i,q_j}x=\cosh_{q_i,q_j}$	$\int\sinh_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\cosh_{q_i,q_j}+C$
$D_{q_i,q_j}\mathrm{SINH}_{q_i,q_j}x=\mathrm{COSH}_{q_i,q_j}(q_iq_jx)$	$\int\mathrm{SINH}_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=q_iq_j\,\mathrm{COSH}_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$
$D_{q_i,q_j}\cosh_{q_i,q_j}=\sinh_{q_i,q_j}x$	$\int\cosh_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\sinh_{q_i,q_j}x+C$
$D_{q_i,q_j}\mathrm{COSH}_{q_i,q_j}x=\mathrm{SINH}_{q_i,q_j}(q_iq_jx)$	$\int\mathrm{COSH}_{q_i,q_j}\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=q_iq_j\,\mathrm{SINH}_{q_i,q_j}\left(\frac{x}{q_iq_j}\right)+C$

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