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Applications of q-Umbral Calculus to Modified Apostol Type q-Bernoulli Polynomials

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Abstract- This article aims to identify the generating function of modified Apostol type q-Bernoulli polynomials. With the aid of this generating function, some properties of modified Apostol type q-Bernoulli polynomials are given. It is shown that aforementioned polynomials are q-Appell. Hence, we make use of these polynomials to have applications on q-Umbral calculus. From those applications, we derive some theorems in order to get Apostol type modified q-Bernoulli polynomials as a linear combination of some known polynomials which we stated in the paper.

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1. **Introduction**

Throughout this paper, we make use of the following standart notations: $\mathbb{N} := \{1, 2, 3, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

We now begin with the fundamental properties of q-calculus. Let q be chosen as a fixed real number between 0 and 1. The q-analogue of any number n is given by

$$[n]_q = \frac{1-q^n}{1-q}.$$

The expression

$$[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$$

means the q-factorial of n, and also let $n, k \in \mathbb{N}_0$, for $k \leq n$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$$

is called q-binomial coefficient. Note that $[0]_q! := 1$. The q-derivative of $f(x)$ is defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x} \qquad (0 < q < 1)\,. \tag{1.1}$$

If $q \to 1^-$, it becomes

$$\lim_{q \to 1^-} D_q f(x) = \frac{df(x)}{dx}$$

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representing familiar derivative of a function f, with respect to x. The Jackson definite q-integral of a function f is also defined by

$$\int_0^a f(x)d_qx = a(1-q)\sum_{j=0}^{\infty} f(q^j a)q^j.$$

The q-exponential functions are given by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \text{ and } E_q(t) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}t^n}{[n]_q!} \quad (t \in \mathbb{C} \text{ with } |t| < 1)$$

with the following equality

$$e_{q^{-1}}(t) = E_q(t).$$

These fundamental properties of q-calculus listed above are taken from the book [3].

By using an exponential function $e_q(x)$, Kupershmidt [10] defined the following q-Bernoulli polynomials

$$\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{t}{e_q(t)-1}e_q(xt).$$

In the case $x = 0$, $B_{n,q}(0) = B_{n,q}$ means the n-th q-Bernoulli number.

Very recently, Kurt [8] defined Apostol type q-Bernoulli polynomials of order α by making use of the following generating function:

$$\sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x,y,\lambda)\frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^{\alpha} e_q(xt)E_q(yt) \tag{1.2}$$

where $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{N}_0$. In this paper, we will study on the following polynomial $B_{n,q}^{(1)}(x,\lambda) := B_{n,q}(x,\lambda)$ which is given by special cases $\alpha = 1$ and $y = 0$ in (1.2):

$$\sum_{n=0}^{\infty} B_{n,q}(x,\lambda)\frac{t^n}{[n]_q!} = \frac{t}{\lambda e_q(t)-1}e_q(xt). \tag{1.3}$$

When $q \to 1$ in (1.3), it reduces to Apostol-Bernoulli polynomials, see [2,11].

We now review briefly the concept of q-umbral calculus. For the properties of q-umbral calculus, we refer the reader to see the references [1-4, 7, 13, 14].

Let $\mathbb{C}$ be a field of characteristic zero, and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f \mid f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}, \quad (a_k \in \mathbb{C}) \right\}.$$

Let $\mathbb{P}$ be the algebra of polynomials in the single variable x over the field complex numbers and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. In the q-Umbral calculus, $\langle L|p(x)\rangle$ means the action of a linear functional L on the polynomial $p(x)$. This operator has a linear property on $\mathbb{P}^*$ given by

$$\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$$

and

$$\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$$

for any constant c in $\mathbb{C}$.

The formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!} \tag{1.4}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad (n \geq 0). \tag{1.5}$$

Taking $f(t)=t^k$ in Eq. (1.4) and Eq. (1.5) gives

$$\langle t^k|x^n\rangle=[n]_q!\delta_{n,k},\quad (n,k\geq 0) \tag{1.6}$$

where

$$\delta_{n,k}=\begin{cases}1, & \text{if } n=k\\ 0, & \text{if } n\neq k\end{cases}.$$

Actually, any linear functional L in $\mathbb{P}^*$ has the form (1.4). That is, since

$$f_L(t)=\sum_{k=0}^{\infty}\langle L|x^k\rangle\frac{t^k}{[k]_q!},$$

we have

$$\langle f_L(t)|x^n\rangle=\langle L|x^n\rangle,$$

and so as linear functionals $L=f_L(t)$. Moreover, the map $L\to f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. From (1.5), we have

$$\langle e_q(yt)|x^n\rangle=y^n$$

and so

$$\langle e_q(yt)|p(x)\rangle=p(y)\quad (p(x)\in\mathbb{P}).$$

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t))=0$, then $f(t)$ is called an invertible series. A series $f(t)$ for which $o(f(t))=1$ will be called a delta series (*c.f.* [1-4, 7, 13, 14]).

If $f_1(t),...,f_m(t)$ are in $\mathcal{F}$, then

$$\langle f_1(t)...f_m(t)|x^n\rangle=\sum_{i_1+i_2+...+i_m=n}\binom{n}{i_1,...,i_m}_q\langle f_1(t)|x^{i_1}\rangle...\langle f_m(t)|x^{i_m}\rangle,$$

where

$$\binom{n}{i_1,\cdots,i_r}_q=\frac{[n]_q!}{[i_1]_q!\cdots[i_r]_q!}.$$

We use the notation t^k for the k-th q-derivative operator on $\mathbb{P}$ as follows:

$$t^kx^n=\begin{cases}\frac{[n]_q!}{[n-k]_q!}x^{n-k}, & k\leq n\\ 0, & k>n\end{cases}.$$

If $f(t)$ and $g(t)$ are in $\mathcal{F}$, then

$$\langle f(t)g(t)|p(x)\rangle=\langle f(t)|g(t)p(x)\rangle=\langle g(t)|f(t)p(x)\rangle$$

for all polynomials $p(x)$. Notice that for all $f(t)$ in $\mathcal{F}$, and for all polynomials $p(x)$

$$f(t)=\sum_{k=0}^{\infty}\langle f(t)|x^k\rangle\frac{t^k}{[k]_q!}\quad\text{and}\quad p(x)=\sum_{k=0}^{\infty}\langle t^k|p(x)\rangle\frac{x^k}{[k]_q!}. \tag{1.7}$$

Using (1.7), we obtain

$$p^{(k)}(x)=D_q^kp(x)=\sum_{l=k}^{\infty}\frac{\langle t^l|p(x)\rangle}{[l]_q!}x^{l-k}\prod_{s=1}^{k}[l-s+1]_q$$

providing

$$p^{(k)}(0)=\langle t^k|p(x)\rangle\quad\text{and}\quad\langle 1|p^{(k)}(x)\rangle=p^{(k)}(0). \tag{1.8}$$

Thus, from (1.8), we note that

$$t^kp(x)=p^{(k)}(x)=D_q^kp(x).$$

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Let $f(t) \in \mathcal{F}$ be a delta series and let $g(t) \in \mathcal{F}$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the following property

$$\langle g(t) f(t)^k | s_n(x) \rangle = [n]_q! \delta_{n,k} \qquad (n, k \geq 0) \tag{1.9}$$

which is called an orthogonality condition for any q-sheffer sequence, *cf.* [1-4, 7, 13, 14].

The sequence $s_n(x)$ is called the q-Sheffer sequence for the pair of $(g(t), f(t))$, or this $s_n(x)$ is q-Sheffer for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$.

Let $s_n(x)$ be q-Sheffer for $(g(t), f(t))$. Then for any $h(t)$ in $\mathcal{F}$, and for any polynomial $p(x)$, we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | s_k(x) \rangle}{[k]_q!} g(t) f(t)^k, \;\; p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k | p(x) \rangle}{[k]_q!} s_k(x) \tag{1.10}$$

and the sequence $s_n(x)$ is q-Sheffer for $(g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e_q(x \overline{f}(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{[n]_q!} \tag{1.11}$$

for all x in $\mathbb{C}$, where $\overline{f}\left(f(t)\right) = f\left(\overline{f}\left(t\right)\right) = t$.

An important property for the q-Sheffer sequence $s_n(x)$ having $(g(t), t)$ is the q-Appell sequence. It is also called q-Appell for $g(t)$ with the following consequence

$$s_n(x) = \frac{1}{g(t)} x^n \Leftrightarrow t s_n(x) = [n]_q s_{n-1}(x). \tag{1.12}$$

Further important property for q-Sheffer sequence $s_n(x)$ is as follows

$$s_n(x) \text{ is } q\text{-Appell for } g\left(t\right) \Leftrightarrow \frac{1}{g(t)} e_q(xt) = \sum_{k=0}^{\infty} s_n(x) \frac{t^n}{[n]_q!} \quad \left(x \in \mathbb{C}\right).$$

For having information about the properties of q-umbral theory, see [1-4, 7, 13, 14] and cited references therein.

Recently several authors have studied q-Bernoulli polynomials, q-Euler polynomials and various generalizations of these polynomials [1-15]. In the next section, we investigate modified Apostol type q-Bernoulli numbers and polynomials, and we apply these numbers and polynomials to q-umbral theory which is the systematic study of q-umbral algebra. Actually, we are motivated to write this paper from Kim's systematic works on q-umbral theory [4-7].

2. Modified Apostol type q-Bernoulli numbers and polynomials

Recall from (1.3) that

$$\sum_{n=0}^{\infty} B_{n,q}(x, \lambda) \frac{t^n}{[n]_q!} = \frac{t}{\lambda e_q(t) - 1} e_q(xt) \qquad (\lambda \neq 1). \tag{2.1}$$

Taking $t \rightarrow 0$ on the above gives $B_{0,q}(x, \lambda) = 0$. This shows that the generating function of these polynomials is not invertible. Therefore, we need to modify slightly Eq. (2.1) as follows

$$F_q^*\left(x, t\right) = \sum_{n=0}^{\infty} B_{n,q}^*(x, \lambda) \frac{t^n}{[n]_q!} = \frac{1}{\lambda e_q(t) - 1} e_q(xt)$$

representing

$$\frac{B_{n+1,q}(x, \lambda)}{[n+1]_q} = B_{n,q}^*(x, \lambda). \tag{2.2}$$

Here we called $B_{n,q}^*(x, \lambda)$ modified Apostol type q-Bernoulli polynomials. Now

$$\lim_{t \rightarrow 0} F_q^*\left(x, t\right) = B_{0,q}^*(x, \lambda) = \frac{1}{\lambda - 1} \neq 0 \qquad (\lambda \neq 1).$$

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This modification yields to being invertible for generating function of modified Apostol type q-Bernoulli polynomials. As a traditional for some special polynomials to be a number, in the case when $x = 0$, $B^*_{n,q}(0,\lambda) = B^*_{n,q}(\lambda)$ is called the modified Apostol type n-th q-Bernoulli number. Now we list some properties of modified Apostol type q-Bernoulli polynomials as follows.

From (2.2), we obtain

$$B^*_{n,q}(x,\lambda) = \sum_{k=0}^{n} \binom{n}{k}_q B^*_{k,q}(\lambda)\, x^{n-k} = \sum_{k=0}^{n} \binom{n}{k}_q B^*_{n-k,q}(\lambda)\, x^{k}. \tag{2.3}$$

By (2.2), the modified Apostol type q-Bernoulli numbers can be found by means of the following recurrence relation:

$$B^*_{0,q}(x,\lambda) = \frac{1}{\lambda-1} \quad \text{and} \quad \lambda B^*_{n,q}(1,\lambda) - B^*_{n,q}(\lambda) = \delta_{0,n}. \tag{2.4}$$

A few numbers are listed below:

$$B^*_{0,q}(\lambda) = \frac{1}{\lambda-1},\ B^*_{1,q}(\lambda) = \frac{-\lambda}{(\lambda-1)^2},\ B^*_{2,q}(\lambda) = \frac{\lambda(1+\lambda q)}{(\lambda-1)^3},$$
$$B^*_{3,q}(\lambda) = \frac{-\lambda(1+2\lambda q+2\lambda q^2+\lambda^2 q^3)}{(\lambda-1)^4}.$$

From (1.11) and (1.12), we have

$$B^*_{n,q}(x,\lambda) \sim (\lambda e_q(t)-1,t) \tag{2.5}$$

and

$$tB^*_{n,q}(x,\lambda) = [n]_q B^*_{n-1,q}(x,\lambda) = B_{n,q}(x,\lambda). \tag{2.6}$$

It follows from (2.6) that $B^*_{n,q}(x,\lambda)$ is q-Appell for $\lambda e_q(t)-1$.

We now have the following theorem.

Theorem 1. *Let $p(x) \in \mathbb{P}$. We have*

$$\left\langle \frac{\lambda e_q(t)-1}{t} \mid p(x) \right\rangle = \lambda \int_0^1 p(u)d_q u.$$

Proof. From Eq. (2.5) and Eq. (2.6), we write

$$B^*_{n,q}(x,\lambda) = \frac{1}{\lambda e_q(t)-1} x^n \qquad (n \geq 0).$$

By (1.1) and (1.6), we obtain the following calculations

$$\begin{aligned} \left\langle \frac{\lambda e_q(t)-1}{t} \mid x^n \right\rangle &= \frac{1}{[n+1]_q} \left\langle \frac{\lambda e_q(t)-1}{t} \mid tx^{n+1} \right\rangle \\ &= \frac{1}{[n+1]_q} \langle \lambda e_q(t)-1 | x^{n+1} \rangle \\ &= \frac{\lambda}{[n+1]_q} = \lambda \int_0^1 x^n d_q x. \end{aligned} \tag{2.7}$$

Thus, from (2.7), we arrive at

$$\left\langle \frac{\lambda e_q(t)-1}{t} \mid p(x) \right\rangle = \lambda \int_0^1 p(u)d_q u \qquad (p(x) \in \mathbb{P})$$

which is desired result. □

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Example 1. *If we take $p(x) = B^*_{n,q}(x,\lambda)$ in Theorem 1, on the one hand, we derive*

$$\begin{aligned}\lambda\int_0^1 B^*_{n,q}(x,\lambda)d_qx &= \left\langle \frac{\lambda e_q(t)-1}{t} \mid B^*_{n,q}(x,\lambda)\right\rangle \\ &= \left\langle 1 \mid \frac{\lambda e_q(t)-1}{t}\frac{tB^*_{n+1,q}(x,\lambda)}{[n+1]_q}\right\rangle \\ &= \frac{1}{[n+1]_q}\langle t^0|x^{n+1}\rangle = [n]_q!\delta_{n+1,0}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda[n+1]_q\int_0^1 B^*_{n,q}(x,\lambda)d_qx &= \lambda[n+1]_q\int_0^1\sum_{k=0}^n\binom{n}{k}_q B^*_{n-k,q}(\lambda)x^kd_qx \\ &= \lambda[n+1]_q\sum_{k=0}^n\binom{n}{k}_q B^*_{n-k,q}(\lambda)\int_0^1 x^kd_qx \\ &= \lambda\sum_{k=0}^n\binom{n+1}{k+1}_q B^*_{n-k,q}(\lambda).\end{aligned}$$

Thus we have the following interesting property for modified Apostol type q-Bernoulli numbers derived from Theorem 1 for $n \geq 0$:

$$\sum_{k=0}^n\binom{n+1}{k+1}_q B^*_{n-k,q}(\lambda) = 0$$

which can be also generated by Eq.(2.3) and Eq.(2.4).

The following is an immediate result emerging from (1.10) and (2.5) that

$$\begin{aligned}p(x) &= \sum_{k=0}^\infty\frac{[n+1]_q}{[k]_q!}\left\langle\frac{\lambda e_q(t)-1}{t}t^k \mid p(x)\right\rangle B^*_{k,q}(x,\lambda) \\ &= \sum_{k=0}^\infty\frac{[n+1]_q}{[k]_q!}\left\langle\frac{\lambda e_q(t)-1}{t} \mid t^kp(x)\right\rangle B^*_{k,q}(x,\lambda) \\ &= \lambda\sum_{k=0}^\infty\frac{[n+1]_q}{[k]_q!}B^*_{k,q}(x,\lambda)\int_0^1 t^kp(x)d_qx.\end{aligned}$$

By choosing suitable polynomials $p(x)$, one can derive some interesting results. So we omit to give examples, and so we now take care of a fundamental property in q-umbral theory which is stated below by Theorem 2.

Theorem 2. *Let n be nonnegative integer. Then we have*

$$\left\langle\frac{e_q(t)-1}{t} \mid B^*_{n,q}(x,\lambda)\right\rangle = \int_0^1 B^*_{n,q}(u,\lambda)d_qu.$$

Proof. From (2.3), we first obtain

$$\begin{aligned}\int_x^{x+y}B^*_{n,q}(u,\lambda)d_qu &= \sum_{k=0}^n\binom{n}{k}_q B^*_{n-k,q}(\lambda)\frac{1}{[k+1]_q}\left\{(x+y)^{k+1}-x^{k+1}\right\} \qquad (2.8)\\ &= \frac{1}{[n+1]_q}\sum_{k=0}^n\binom{n+1}{k+1}_q B^*_{n-k,q}(\lambda)\left\{(x+y)^{k+1}-x^{k+1}\right\} \\ &= \frac{1}{[n+1]_q}\left(B^*_{n+1,q}(x+y,\lambda)-B^*_{n+1,q}(x,\lambda)\right).\end{aligned}$$

Thus, by applying (2.8), we get

$$\begin{aligned}\left\langle \frac{e_q(t)-1}{t} \mid B^*_{n,q}(x,\lambda)\right\rangle &= \frac{1}{[n+1]_q}\left\langle \frac{e_q(t)-1}{t} \mid tB^*_{n+1,q}(x,\lambda)\right\rangle \\ &= \frac{1}{[n+1]_q}\left\{B^*_{n+1,q}(1,\lambda) - B^*_{n+1,q}(\lambda)\right\} \\ &= \int_0^1 B^*_{n,q}(u,\lambda)\,d_q u.\end{aligned} \tag{2.9}$$

Comparing Eq. (2.8) with Eq. (2.9), we complete the proof of this theorem. □

The following theorem is useful to derive any polynomial as a linear combination of modified Apostol type q-Bernoulli polynomials.

Theorem 3. *For $q(x) \in P_n$, let*

$$q(x) = \sum_{k=0}^{n} b_{k,q} B^*_{k,q}(x,\lambda).$$

Then

$$b_{k,q} = \frac{1}{[k]_q!}\left\{\lambda q^{(k)}(1) - q^{(k)}(0)\right\}.$$

Proof. It follows from (1.9) that

$$\left\langle (\lambda e_q(t)-1)\,t^k | B^*_{n,q}(x,\lambda)\right\rangle = [n]_q!\delta_{n,k} \qquad (n,k \geq 0). \tag{2.10}$$

We now consider the following sets of polynomials of degree less than or equal to n:

$$\mathbb{P}_n = \{q(x) \in \mathbb{C}[x] |\deg q(x) \leq n\}.$$

For $q(x) \in \mathbb{P}_n$, we further consider that

$$q(x) = \sum_{k=0}^{n} b_{k,q} B^*_{k,q}(x,\lambda). \tag{2.11}$$

Combining (2.10) with (2.11), it becomes

$$\begin{aligned}\left\langle (\lambda e_q(t)-1)\,t^k | q(x)\right\rangle &= \sum_{l=0}^{n} b_{l,q}\left\langle (\lambda e_q(t)-1)\,t^k \mid B^*_{l,q}(x,\lambda)\right\rangle \\ &= \sum_{l=0}^{n} b_{l,q}\,[l]_q!\delta_{l,k} = [k]_q! b_{k,q}.\end{aligned} \tag{2.12}$$

Thus, from (2.12), we have

$$b_{k,q} = \frac{1}{[k]_q!}\left\langle (\lambda e_q(t)-1)\,t^k | q(x)\right\rangle = \frac{1}{[k]_q!}\left\{\lambda q^{(k)}(1) - q^{(k)}(0)\right\},$$

where $q^{(k)}(x) = D_q^k q(x)$. Thus the proof is completed. □

When we choose $q(x) = E_{n,q}(x)$, we have the following corollary which is given by its proof.

Corollary 1. *Let $n \geq 2$. Then*

$$\begin{aligned}E_{n,q}(x) &= (\lambda q - 1)B^*_{n,q}(x,\lambda) + [n]_q\left(\frac{\lambda+1}{2}\right)B^*_{n-1,q}(x,\lambda) \\ &\quad -(\lambda+1)\sum_{k=0}^{n-2}\binom{n}{k}_q E_{n-k,q}B^*_{k,q}(x,\lambda).\end{aligned}$$

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Proof. Recall that the q-Euler polynomials $E_{n,q}(x)$ are defined by

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{[n]_q!} = \frac{[2]_q}{e_q(t)+1}e_q(xt) \quad (c.f.\ [12,15])$$

which in turn yields to

$$E_{n,q}(x) \sim \left(\frac{e_q(t)+1}{[2]_q}, t\right) \qquad (n \geq 0)$$

and

$$tE_{n,q}(x) = [n]_q E_{n-1,q}(x).$$

Set

$$q(x) = E_{n,q}(x) \in \mathbb{P}_n.$$

Then it becomes

$$E_{n,q}(x) = \sum_{k=0}^{n} b_{k,q} B^*_{k,q}(x,\lambda). \tag{2.13}$$

Let us now evaluate the coefficients $b_{k,q}$ as follows

$$\begin{aligned} b_{k,q} &= \frac{1}{[k]_q!}\left\langle (\lambda e_q(t)-1)\, t^k \mid E_{n,q}(x)\right\rangle \\ &= \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{[k]_q!}\langle \lambda e_q(t) - 1 \mid E_{n-k,q}(x)\rangle \\ &= \binom{n}{k}_q \langle \lambda e_q(t) - 1 \mid E_{n-k,q}(x)\rangle \\ &= \binom{n}{k}_q (\lambda E_{n-k,q}(1) - E_{n-k,q}), \end{aligned}$$

where $E_{n,q} := E_{n,q}(0)$ are called q-Euler numbers satisfying the following property

$$E_{n,q}(1) + E_{n,q} = [2]_q\, \delta_{0,n} \tag{2.14}$$

with the conditions $E_{0,q} = 1$ and $E_{1,q} = -\frac{1}{2}$. By (2.13) and (2.14), we have

$$\begin{aligned} E_{n,q}(x) &= b_{n,q}B^*_{n,q}(x,\lambda) + b_{n-1,q}B^*_{n-1,q}(x,\lambda) + \sum_{k=0}^{n-2} b_{k,q}B^*_{k,q}(x,\lambda) \\ &= (\lambda q - 1)B^*_{n,q}(x,\lambda) + [n]_q\left(\frac{\lambda+1}{2}\right)B^*_{n-1,q}(x,\lambda) \\ &\quad - (\lambda+1)\sum_{k=0}^{n-2}\binom{n}{k}_q E_{n-k,q}B^*_{k,q}(x,\lambda). \end{aligned}$$

□

Recall from (1.2) that Apostol type q-Bernoulli polynomials of order r are given by the following generating function, for $y = 0$ (see [8]):

$$\sum_{n=0}^{\infty} B^{(r)}_{n,q}(x,\lambda)\frac{t^n}{[n]_q!} = \left(\frac{t}{\lambda e_q(t)-1}\right)^r e_q(xt),$$

where $t \in \mathbb{C}$ and $r \in \mathbb{N}_0$. If t approaches to 0 on the above, it yields to $B^{(\alpha)}_{0,q}(x,\lambda) = 0$, which means that the generating function of $B^{(\alpha)}_{n,q}(x,\lambda)$ is not invertible. So, we need to modify slightly Eq. (2.1), as follows

$$\widetilde{F}^{(r)}_q(x,t) = \sum_{n=0}^{\infty}\widetilde{B}^{(r)}_{n,q}(x,\lambda)\frac{t^n}{[n]_q!} = \left(\frac{1}{\lambda e_q(t)-1}\right)^r e_q(xt). \tag{2.15}$$

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The polynomials $\widetilde{B}_{n,q}^{(r)}(x,\lambda)$ may be called as modified Apostol type q-Bernoulli polynomials of higher order.

Notice that

$$\lim_{t\to 0}\widetilde{F}_q^{(r)}(x,t)=\widetilde{B}_{n,q}^{(r)}(x,\lambda)=\left(\frac{1}{\lambda-1}\right)^r\neq 0\qquad(\lambda\neq 1),$$

which implies an invertible for generating function of modified Apostol type q-Bernoulli polynomials of higher order. In the case $x=0$, $\widetilde{B}_{n,q}^{(r)}(0,\lambda):=\widetilde{B}_{n,q}^{(r)}(\lambda)$ may be called the modified Apostol type q-Bernoulli numbers.

Let

$$g^r(t,\lambda)=(\lambda e_q(t)-1)^r.$$

It is clear that $g^r(t,\lambda)$ is an invertible series. It follows from (2.15) that $\widetilde{B}_{n,q}^{(r)}(x,\lambda)$ is q-Appell for $(\lambda e_q(t)-1)^r$. So, by (1.12), we have

$$\widetilde{B}_{n,q}^{(r)}(x,\lambda)=\frac{1}{g^r(t,\lambda)}x^n,$$

and

$$t\widetilde{B}_{n,q}^{(r)}(x,\lambda)=[n]_q\widetilde{B}_{n-1,q}^{(r)}(x,\lambda).$$

Thus, we have

$$\widetilde{B}_{n,q}^{(r)}(x,\lambda)\sim((\lambda e_q(t)-1)^r,t).$$

By (1.5) and (2.15), we get

$$\left\langle\frac{1^r}{(\lambda e_q(t)-1)^r}e_q(yt)|x^n\right\rangle=\widetilde{B}_{n,q}^{(r)}(y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}_q\widetilde{B}_{n-l,q}^{(r)}(\lambda)\,y^l. \tag{2.16}$$

Here we find that

$$\begin{aligned}\left\langle\left(\frac{1}{\lambda e_q(t)-1}\right)^r\mid x^n\right\rangle &= \left\langle\frac{1}{\lambda e_q(t)-1}\times\cdots\times\frac{1}{\lambda e_q(t)-1}\mid x^n\right\rangle \qquad (2.17)\\ &= \sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\cdots,i_r}_q B_{i_1,q}^*(\lambda)\cdots B_{i_r,q}^*(\lambda).\end{aligned}$$

By using (2.16), we have

$$\left\langle\left(\frac{1}{\lambda e_q(t)-1}\right)^r|x^n\right\rangle=\widetilde{B}_{n,q}^{(r)}(\lambda). \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 4. *Let n be nonnegative integer. Then we have*

$$\widetilde{B}_{n,q}^{(r)}(\lambda)=\sum_{i_1+\cdots+i_r=n}\binom{n}{i_1,\cdots,i_r}_q\prod_{j=1}^{r}B_{i_j,q}^*(\lambda).$$

Set

$$q(x)=\widetilde{B}_{n,q}^{(r)}(x,\lambda)\in\mathbb{P}_n.$$

Then, by Theorem 3, we write

$$\widetilde{B}_{n,q}^{(r)}(x,\lambda)=\sum_{k=0}^{n}b_{k,q}B_{k,q}^*(x,\lambda), \tag{2.19}$$

where the coefficient $b_{k,q}$ is given by

$$\begin{aligned}b_{k,q} &= \frac{1}{[k]_q!}\left\langle(\lambda e_q(t)-1)\,t^k\mid q(x)\right\rangle \qquad (2.20)\\ &= \binom{n}{k}_q\left\langle(\lambda e_q(t)-1)\mid\widetilde{B}_{n-k,q}^{(r)}(x,\lambda)\right\rangle\\ &= \binom{n}{k}_q\left(\lambda\widetilde{B}_{n-k,q}^{(r)}(1,\lambda)-\widetilde{B}_{n-k,q}^{(r)}(\lambda)\right).\end{aligned}$$

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From the Eq. (2.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\lambda\widetilde{B}_{n,q}^{(r)}\left(1,\lambda\right)-\widetilde{B}_{n,q}^{(r)}\left(\lambda\right)\right)\frac{t^{n}}{[n]_{q}!} &= \left(\frac{1}{\lambda e_{q}\left(t\right)-1}\right)^{r}\left(\lambda e_{q}\left(t\right)-1\right)\\ &= \left(\frac{1}{\lambda e_{q}\left(t\right)-1}\right)^{r-1}\\ &= \sum_{n=0}^{\infty}\widetilde{B}_{n,q}^{(r-1)}\left(\lambda\right)\frac{t^{n}}{[n]_{q}!}.\end{aligned}$$

By comparing the coefficients $\frac{t^n}{[n]_q!}$ in the above equation, we get

$$\lambda\widetilde{B}_{n,q}^{(r)}(1,\lambda)-\widetilde{B}_{n,q}^{(r)}\left(\lambda\right)=\widetilde{B}_{n,q}^{(r-1)}\left(\lambda\right). \tag{2.21}$$

From the Eqs. (2.19), (2.20) and (2.21), we get the following theorem.

Theorem 5. *Let $n\in\mathbb{N}_0$ and $r\in\mathbb{N}_0$. Then*

$$\widetilde{B}_{n,q}^{(r)}(x,\lambda)=\sum_{k=0}^{n}\binom{n}{k}_{q}\widetilde{B}_{n-k,q}^{(r-1)}(\lambda)B_{k,q}^{*}(x,\lambda).$$

Let us assume that

$$q(x)=\sum_{k=0}^{n}b_{k,q}^{r}\widetilde{B}_{k,q}^{(r)}(x,\lambda)\in\mathbb{P}_{n}. \tag{2.22}$$

We use a similar method in order to find the coefficient $b_{k,q}^r$ as same as Theorem 3. So we omit the details and give the following equality:

$$b_{k,q}^{r}=\frac{1}{[k]_q!}\sum_{l=0}^{r}\binom{r}{l}_{q}\lambda^{l}(-1)^{r-l}\sum_{m\geq 0}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}_{q}\frac{1}{[m]_q!}q^{(m+k)}(0).$$

By (2.22) and coefficient $b_{k,q}^r$, we state the following theorem.

Theorem 6. *For $n\in\mathbb{N}_0$, let*

$$q(x)=\sum_{k=0}^{n}b_{k,q}^{r}\widetilde{B}_{k,q}^{(r)}(x,\lambda)\in\mathbb{P}_{n}.$$

Then

$$\begin{aligned}b_{k,q}^{r} &= \frac{1}{[k]_q!}\left\langle\left(\lambda e_{q}\left(t\right)-1\right)t^{k}\mid q\left(x\right)\right\rangle\\ &= \frac{1}{[k]_q!}\sum_{m\geq 0}\sum_{l=0}^{r}\binom{r}{l}_{q}\lambda^{l}(-1)^{r-l}\sum_{i_1+\cdots+i_l=m}\binom{m}{i_1,\cdots,i_l}_{q}\frac{1}{[m]_q!}q^{(m+k)}(0),\end{aligned}$$

where $q^{(k)}(x)=D_q^k q(x)$.

Let us consider $q(x)=B_{n,q}^{*}(x,\lambda)\in\mathbb{P}_n$. Then, by Theorem 6, we have

$$B_{n,q}^{*}(x,\lambda)=\sum_{k=0}^{n}b_{k,q}^{r}\widetilde{B}_{k,q}^{(r)}(x,\lambda). \tag{2.23}$$

From Theorem 6 and (2.23), we acquire the following theorem.

Theorem 7. *For $n, r \in \mathbb{N}_0$, the following equality holds true:*

$$B^*_{n,q}(x,\lambda) = \sum_{k=0}^{n} \left(\sum_{m=0}^{n-k} \sum_{l=0}^{r} \sum_{i_1+\cdots+i_l=m} (-1)^{r-l} \lambda^l \binom{r}{l}_q \binom{m}{i_1,\cdots,i_l}_q \right.$$
$$\left. \times \binom{m+k}{m}_q \binom{n}{m+k}_q B^*_{n-m-k,q}(\lambda) \right) \widetilde{B}^{(r)}_{k,q}(x,\lambda).$$

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