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A NOTE ON THE (p, q)-HERMITE POLYNOMIALS

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ABSTRACT. In this paper, we introduce a new generalization of the Hermite polynomials via (p, q)-exponential generating function and investigate several properties and relations for mentioned polynomials including derivative property, explicit formula, recurrence relation, integral representation. We also define a (p, q)-analogue of the Bernstein polynomials and acquire their some formulas. We then provide some (p, q)-hyperbolic representations of the (p, q)-Bernstein polynomials. In addition, we obtain a correlation between (p, q)-Hermite polynomials and (p, q)-Bernstein polynomials.

1. INTRODUCTION

During the last three decades, applications of *quantum calculus* based on q-numbers have been studied and investigated succesfully, densely and considerably (see [7, 8]). In conjunction with the motivation and inspiration of these applications and introduction of the (p, q)-numbers, many mathematicians and physicists have extensively developed the theory of *post quantum calculus* based on (p, q)-numbers along the traditional lines of classical and quantum calculus. Agyuz *et al.* [2] presented some novel results of multiplications of (p, q)-Bernstein polynomials and derived several new relations with related to (p, q)-Gamma and (p, q)-Beta functions. Duran *et al.* [3] introduced a new class of Bernoulli, Euler and Genocchi polynomials based on the (p, q)-calculus and investigated their many properties involving addition theorems, difference equations, derivative properties, recurrence relationships, and so on. Furthermore, they derived (p, q)-extension of Cheon's main result and (p, q)-analogue of the Srivastava and Pintér's addition theorem. Sadjang [9] investigated some properties of the (p, q)-derivative and the (p, q)-integration and presented two appropriate polynomials basis for the (p, q)-derivative, and then he derived various properties of these bases. As an application, he provided two (p, q)-Taylor formulas for polynomials. Furthermore, he stated the fundamental theorem of (p, q)-calculus and proved the formula of (p, q)-integration by part.

The (p, q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \qquad (0 < q < p \leqq 1).$$

Note that $[n]_{p,q} = p^{n-1} [n]_{q/p}$, where $[n]_{q/p}$ stands for q-number known as $[n]_{q/p} = \frac{(q/p)^n - 1}{(q/p) - 1}$. One can see that (p, q)-number is closely related to q-number with this relation $[n]_{p,q} = p^{n-1} [n]_{\frac{q}{p}}$. By appropriately using this obvious relation between the q-notation and its variant, the (p, q)-notation, most (if not all) of the (p, q)-results can be derived from the corresponding known q-results by merely changing the parameters and variables involved (see [4], [5]).

The (p, q)-derivative operator $D_{p,q;x} f(x)$ of a function f with respect to x given as

$$D_{p,q;x} f(x) := D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x} \quad (D_{p,q} f(x) \text{ when } x \neq 0; f'(0) \text{ when } x = 0) \tag{1.1}$$

is a lineer operator and satisfies the following property

$$D_{p,q}(f(x) g(x)) = f(px) D_{p,q} g(x) + g(qx) D_{p,q} f(x). \tag{1.2}$$

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The (p,q)-power basis is defined by

$$(x\oplus a)_{p,q}^{n}:=\begin{cases}(x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}) & \textit{if } n\geq 1,\\ 1 & \textit{if } n=0,\end{cases}$$

or equivalently by

$$=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{k}{2}}q^{\binom{n-k}{2}}x^{k}a^{n-k}.$$

where $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_{p,q}$ is given by

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}\ (n\geq k;n\in\mathbb{N}_0)$$

with $[n]_{p,q}!=[n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}$ and $[0]_{p,q}!=1$.

Let

$$e_{p,q}(x)=\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{x^n}{[n]_{p,q}!}$$

denote (p,q)-exponential function having the following (p,q)-derivative representation

$$D_{p,q}e_{p,q}(x)=e_{p,q}(px). \tag{1.3}$$

The definite (p,q)-integral for a function f is defined by

$$\int_0^a f(x)\,d_{p,q}x=(p-q)\,a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right) \tag{1.4}$$

with

$$\int_a^b f(x)\,d_{p,q}x=\int_0^b f(x)\,d_{p,q}x-\int_0^a f(x)\,d_{p,q}x.$$

To see further detailed studies and investigations for (p,q)-calculus, one can look at [2-5, 9] and cited references therein.

Throughout the paper, let $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote, respectively, the set of all nonnegative integers, the set of all natural numbers, the set of all integers, the set of all real numbers and the set of all complex numbers.

The classical Hermite polynomials are defined by the following exponential generating function to be

$$\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}=e^{2xt-t^2}\quad(\text{see [6], [7], [8]}). \tag{1.5}$$

In the following parts, we introduce a new generalization of the Hermite polynomials based on the (p,q)-numbers via an exponential generating function and investigate several properties and relations for mentioned polynomials including derivative property, explicit formula, recurrence relation, integral representation. We also define a (p,q)-analogue of the Bernstein polynomials and acquire their some formulas. We then provide some (p,q)-hyperbolic representations of the (p,q)-Bernstein polynomials. In addition, we obtain a correlation between (p,q)-Hermite polynomials and (p,q)-Bernstein polynomials.

2. MAIN RESULTS

For a long time, the Hermite polynomials and its various generalizations have been extensively studied and investigated by many mathematicians and physicists (see [6-8] and cited references therein).

We introduce (p,q)-extension of the Hermite polynomials via the following (p,q)-exponential generating function

$$\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}=e_{p,q}\left([2]_{p,q}\,xt\right)e_{p,q}\left(-t^2\right). \tag{2.1}$$

Remark 1. *When $p=1$, the (p,q)-Hermite polynomials reduce to the q-Hermite polynomials defined by* [8]

$$\sum_{n=0}^{\infty} H_{n,q}(x)\frac{t^n}{[n]_q!}=e_q\left([2]_{p,q}xt\right)e_q\left(-t^2\right).$$

Remark 2. *When $q\to p=1$, the (p,q)-Hermite polynomials reduce to the classical Hermite polynomials defined by (1.5).*

Now we give a fundamental property (known also as explicit formula) of the (p,q)-Hermite polynomials $H_{n,p,q}(x)$ by the following theorem.

Theorem 1. *The explicit formula based on (p,q)-numbers for $H_{n,p,q}(x)$ is given below:*

$$H_{n,p,q}(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\left([2]_{p,q}x\right)^{n-2k}(-1)^k[n]_{p,q}!}{[n-2k]_{p,q}![k]_{p,q}!} \tag{2.2}$$

where $\lfloor\cdot\rfloor$ means the greatest integer function.

Proof. Using

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor \frac{n}{2}\rfloor}A(m,n-2m)\qquad\text{(see [6])}$$

and (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} &= e_{p,q}\left([2]_{p,q}xt\right)e_{p,q}\left(-t^2\right)\\ &= \left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{\left([2]_{p,q}xt\right)^n}{[n]_{p,q}!}\right)\left(\sum_{n=0}^{\infty}p^{\binom{n}{2}}\frac{\left(-t^2\right)^n}{[n]_{p,q}!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\frac{\left([2]_{p,q}x\right)^{n-2k}[n]_{p,q}!(-1)^k}{[n-2k]_{p,q}![k]_{p,q}!}\right)\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which gives the asserted formula (2.2) by comparing the coefficients $t^n/[n]_{p,q}!$ of both sides above. □

The first few (p,q)-Hermite polynomials are listed via (2.2) below:

$$\begin{aligned}H_{0,p,q}(x) &= 1,\\ H_{1,p,q}(x) &= [2]_{p,q}x,\\ H_{2,p,q}(x) &= [2]_{p,q}^2x^2-[2]_{p,q},\\ H_{3,p,q}(x) &= [2]_{p,q}^3x^3-[3]_{p,q}[2]_{p,q}^2x,\\ H_{4,p,q}(x) &= [2]_{p,q}^4x^4-[4]_{p,q}[3]_{p,q}[2]_{p,q}^2x^2+[4]_{p,q}[3]_{p,q},\\ H_{5,p,q}(x) &= [2]_{p,q}^5x^5-[5]_{p,q}[4]_{p,q}[2]_{p,q}^3x^3+[5]_{p,q}!x,\\ H_{6,p,q}(x) &= [2]_{p,q}^6x^6-[6]_{p,q}[5]_{p,q}[2]_{p,q}^4x^4+[6]_{p,q}!x^2-[6]_{p,q}[5]_{p,q}[4]_{p,q}.\end{aligned}$$

Notice that upon setting $q\to p=1$ on the above gives the classical Hermite polynomials.

From Eq. (2.2), we get the following corollary.

Corollary 1. *For $n\in\mathbb{N}_0$, we have*

$$H_{2n,p,q}(0)=\frac{[2n]_{p,q}!(-1)^n}{[n]_{p,q}!}\ \text{and}\ H_{2n+1,p,q}(0)=0.$$

The following proposition is a symmetric property for $H_{n,p,q}(x)$.

Proposition 1. *For $n \in \mathbb{N}_0$, we have*

$$H_{n,p,q}(-x) = (-1)^n H_{n,p,q}(x). \tag{2.3}$$

Proof. We readily obtain that

$$\begin{aligned}\sum_{n=0}^{\infty} H_{n,p,q}(-x)\frac{t^n}{[n]_{p,q}!} &= e_{p,q}\left([2]_{p,q}(-x)t\right)e_{p,q}\left(-t^2\right)\\ &= e_{p,q}\left([2]_{p,q}x(-t)\right)e_{p,q}\left(-(-t)^2\right)\\ &= \sum_{n=0}^{\infty}(-1)^n H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!},\end{aligned}$$

which yields to the claimed result (2.3). □

Now we research some behaviours of $H_{n,p,q}(x)$ by applying (p,q)-derivative operator with respect to x and t respectively.

Theorem 2. *We have*

$$D_{p,q;x}H_{n,p,q}(x) =: H'_{n,p,q}(x) = [2]_{p,q}[n]_{p,q}H_{n-1,p,q}(px). \tag{2.4}$$

Proof. Applying the (p,q)-derivative operator $D_{p,q}$ (1.1) to the both sides of (2.1) with respect to x and using (1.2), we acquire

$$\begin{aligned}\sum_{n=0}^{\infty} D_{p,q;x}H_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} &= D_{p,q;x}e_{p,q}\left([2]_{p,q}xt\right)e_{p,q}\left(-t^2\right)\\ &= [2]_{p,q}te_{p,q}\left([2]_{p,q}pxt\right)e_{p,q}\left(-t^2\right).\end{aligned}$$

By comparing the coefficients $\frac{t^n}{[n]_{p,q}!}$ of both sides above, we get the alleged result (2.4). □

The immediate results of the Eq. (2.4) are stated below:

$$D_{p,q;x}H_{2n,p,q}(0) = 0 \text{ and } D_{p,q;x}H_{2n+1,p,q}(0) = [2]_{p,q}\frac{[2n+1]_{p,q}!(-1)^n}{[n]_{p,q}!}.$$

Another result of the Eq. (2.4) is given for $m < n$ as follows:

$$D^m_{p,q;x}H_{n,p,q}(x) =: H^{(m)}_{n,p,q}(x) = p^{\binom{m}{2}}\frac{[2]^m_{p,q}[n]_{p,q}!}{[n-m]_{p,q}!}H_{n-m,p,q}(p^m x),$$

where $D^m_{p,q;x}$ shows (p,q)-derivative operator of order m as $D^m_{p,q;x} = D^{m-1}_{p,q;x}D_{p,q;x}$.

In order to state Theorem 3, we need the following lemma.

Lemma 1. *We have*

$$D_{p,q;t}e_{p,q}\left(-t^2\right) = -pte_{p,q}\left(-(pt)^2\right) - qte_{p,q}\left(-\left(t\sqrt{pq}\right)^2\right). \tag{2.5}$$

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Proof. We observe that

$$\begin{aligned} D_{p,q;t}e_{p,q}\left(-t^2\right) &= \sum_{n=0}^{\infty}(-1)^n p^{\binom{n}{2}}\frac{D_{p,q;t}t^{2n}}{[n]_{p,q}!} = \sum_{n=1}^{\infty}(-1)^n p^{\binom{n}{2}}\frac{[2n]_{p,q}\,t^{2n-1}}{[n]_{p,q}!} \\ &= -\sum_{n=0}^{\infty}(-1)^n p^{\binom{n}{2}+n}\frac{t^{2n+1}}{[n]_{p,q}!}\frac{[2n+2]_{p,q}}{[n+1]_{p,q}} \\ &= -\sum_{n=0}^{\infty}(-1)^n p^{\binom{n}{2}+n}\frac{t^{2n+1}}{[n]_{p,q}!}\left(p^{n+1}+q^{n+1}\right) \\ &= -pt\sum_{n=0}^{\infty}(-1)^n p^{\binom{n}{2}}\frac{(pt)^{2n}}{[n]_{p,q}!} - qt\sum_{n=0}^{\infty}(-1)^n p^{\binom{n}{2}}\frac{\left(t\sqrt{pq}\right)^{2n}}{[n]_{p,q}!} \\ &= -te_{p,q}\left(-(tp)^2\right) - qte_{p,q}\left(-\left(t\sqrt{pq}\right)^2\right). \end{aligned}$$

□

Theorem 3. *We have*

$$H_{n+1,p,q}(x) = [2]_{p,q}\,xp^{\frac{n}{2}}H_{n,p,q}(x\sqrt{p}) - p^n[n]_{p,q}H_{n-1,p,q}\left(\frac{q}{p}x\right) - q^{\frac{n+1}{2}}p^{\frac{n-1}{2}}[n]_{p,q}H_{n-1,p,q}\left(\sqrt{\frac{q}{p}}x\right). \quad (2.6)$$

Proof. Applying (1.1) to the both sides of (2.1), we obtain

$$LHS = \sum_{n=0}^{\infty}H_{n,p,q}(x)\frac{D_{p,q;t}t^n}{[n]_{p,q}!} = \sum_{n=1}^{\infty}H_{n,p,q}(x)\frac{t^{n-1}}{[n-1]_{p,q}!} = \sum_{n=0}^{\infty}H_{n+1,p,q}(x)\frac{t^n}{[n]_{p,q}!}$$

and, by using (1.3) and (2.5),

$$\begin{aligned} RHS &= D_{p,q;t}\left(e_{p,q}\left([2]_{p,q}\,xt\right)e_{p,q}\left(-t^2\right)\right) \\ &= e_{p,q}\left(-\left(t\sqrt{p}\right)^2\right)[2]_{p,q}\,xe_{p,q}\left([2]_{p,q}\,x\sqrt{p}\left(t\sqrt{p}\right)\right) \\ &\quad + e_{p,q}\left([2]_{p,q}\,qxt\right)\left(-pte_{p,q}\left(-(tp)^2\right) - qte_{p,q}\left(-\left(t\sqrt{pq}\right)^2\right)\right) \\ &= [2]_{p,q}\,x\sum_{n=0}^{\infty}H_{n,p,q}(x\sqrt{p})\frac{\left(t\sqrt{p}\right)^n}{[n]_{p,q}!} - p\sum_{n=0}^{\infty}H_{n,p,q}\left(\frac{q}{p}x\right)p^n\frac{t^{n+1}}{[n]_{p,q}!} \\ &\quad - q\sum_{n=0}^{\infty}H_{n,p,q}\left(x\sqrt{\frac{q}{p}}\right)(pq)^{\frac{n}{2}}\frac{t^{n+1}}{[n]_{p,q}!}. \end{aligned}$$

Comparing LHS and RHS gives the asserted result (2.6). □

As a result of (2.4) and (2.6), we give the following identity for (p,q)-Hermite polynomials.

Corollary 2. *We have*

$$H_{n+1,p,q}(x) = [2]_{p,q}\,xp^{\frac{n}{2}}H_{n,p,q}(x\sqrt{p}) - \frac{p^{n+2}}{q^2+pq}H'_{n,p,q}\left(qp^{-2}x\right) - \frac{q^{\frac{n}{2}}p^{\frac{n}{2}+1}}{p+q}H'_{n,p,q}\left(q^{1/2}p^{-3/2}x\right).$$

The (p,q)-integral representation of the (p,q)-Hermite polynomials is given by the following theorem.

Theorem 4. *We have*

$$\int_a^b H_{n,p,q}(x)\,d_{p,q}x = p\frac{H_{n+1,p,q}\left(\frac{b}{p}\right) - H_{n+1,p,q}\left(\frac{a}{p}\right)}{[2]_{p,q}\,[n+1]_{p,q}}.$$

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Proof. Since

$$\int_a^b D_{p,q}f(x)\,d_{p,q}x = f(b) - f(a) \text{ (see [4])} \tag{2.7}$$

in view of Theorem 2 and using Eqs. (1.4) and (2.7), we obtain

$$\begin{aligned}\int_a^b H_{n,p,q}(x)\,d_{p,q}x &= \frac{p}{[2]_{p,q}[n+1]_{p,q}}\int_a^b H'_{n+1,p,q}\left(\frac{x}{p}\right)d_{p,q}x\\ &= p\frac{H_{n+1,p,q}\left(\frac{b}{p}\right) - H_{n+1,p,q}\left(\frac{a}{p}\right)}{[2]_{p,q}[n+1]_{p,q}}.\end{aligned}$$

Therefore, we complete the proof of this theorem. □

We now give the following theorem.

Theorem 5. *We have*

$$\sum_{n=0}^{\infty} H_{n,p,q}(y)\frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} p^{\binom{n}{2}-\binom{2n}{2}}\frac{(-1)^n}{[n]_{p,q}!}[2]_{p,q}^{-2n} D^{2n}_{p,q;x} e_{p,q}\left(p^{-2n}[2]_{p,q}\,yt\right), \tag{2.8}$$

where $y = -p^{2n}x$.

Proof. We consider that

$$[2]_{p,q}^{-n} D^n_{p,q;x} e_{p,q}\left(-[2]_{p,q}\,tx\right) = p^{\binom{n}{2}}(-t)^n e_{p,q}\left(-p^n[2]_{p,q}\,tx\right),$$

and hence,

$$\sum_{n=0}^{\infty} p^{\binom{n}{2}-\binom{2n}{2}}\frac{(-1)^n}{[n]_{p,q}!}[2]_{p,q}^{-2n} D^{2n}_{p,q;x} e_{p,q}\left(p^{-2n}[2]_{p,q}\,yt\right) = e_{p,q}\left(-t^2\right)e_{p,q}\left([2]_{p,q}\,yt\right),$$

where $y = -p^{2n}x$. Thus, we obtain the claimed result (2.8). □

3. FURTHER REMARKS

Let us introduce the (p,q)-Bernstein operator of order n for $f \in C[0,1]$ given by

$$B_{n,p,q}(f\,|x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}(x\,|p,q) \quad (n,k \in \mathbb{N}),$$

where the (p,q)-Bernstein polynomial of degree n is defined by

$$B_{k,n}(x\,|p,q) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}[x]^k_{p,q}(1-x)^{n-k} \quad (n,k \in \mathbb{N} \text{ with } 0 < k \leqq n). \tag{3.1}$$

By (3.1), the generating function of the (p,q)-Bernstein polynomials is given by

$$\sum_{n=k}^{\infty} B_{k,n}(x\,|p,q)\frac{t^n}{[n]_{p,q}!} = \frac{t^k[x]^k_{p,q}}{[k]_{p,q}!}e_{p,q}(t(1-x)).$$

From here, it is obvious that

$$B_{k,n}(1-x\,|p,q) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}}[1-x]^k_{p,q}x^{n-k}. \tag{3.2}$$

Remark 3. *We have*

$$\begin{aligned}\lim_{\substack{q\to 1^- \\ p=1}}\left(\frac{t^k\left[x\right]_{p,q}^k}{\left[k\right]_{p,q}!}e_{p,q}\left(t\left(1-x\right)\right)\right) &= \frac{\left(tx\right)^k}{k!}e^{t(1-x)} \\ &= \sum_{n=k}^{\infty}B_{k,n}\left(x\right)\frac{t^n}{n!}\end{aligned}$$

which is already introduced by Acikgoz and Araci [1]*.*

A triangular recurrence relation for the (p,q)-Bernstein polynomials is given in the following theorem.

Theorem 6. *For $x\in[0,1]$ and $0\leqq k\leqq n$, the (p,q)-Bernstein polynomials satisfy the following formula*

$$B_{k,n}\left(x\left|p,q\right.\right)=\left(1-x\right)p^{n-1}B_{k,n-1}\left(x\left|p,q\right.\right)+q^{n-k}\left[x\right]_{p,q}B_{k-1,n-1}\left(x\left|p,q\right.\right). \tag{3.3}$$

Proof. Using the (p,q)-Pascal rule given by

$$\begin{bmatrix} n \\ k\end{bmatrix}_{p,q}=p^k\begin{bmatrix} n-1 \\ k\end{bmatrix}_{p,q}+q^{n-k}\begin{bmatrix} n-1 \\ k-1\end{bmatrix}_{p,q}\text{ (see [4], [9])}$$

and (3.1), we observe that

$$\begin{aligned}B_{k,n}\left(x\left|p,q\right.\right) &= \begin{bmatrix} n \\ k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\left[x\right]_{p,q}^k\left(1-x\right)^{n-k} \\ &= \left(p^k\begin{bmatrix} n-1 \\ k\end{bmatrix}_{p,q}+q^{n-k}\begin{bmatrix} n-1 \\ k-1\end{bmatrix}_{p,q}\right)p^{\binom{n-k}{2}}\left[x\right]_{p,q}^k\left(1-x\right)^{n-k} \\ &= p^k\begin{bmatrix} n-1 \\ k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\left[x\right]_{p,q}^k\left(1-x\right)^{n-k}+q^{n-k}\begin{bmatrix} n-1 \\ k-1\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\left[x\right]_{p,q}^k\left(1-x\right)^{n-k} \\ &= \left(1-x\right)p^{n-1}B_{k,n-1}\left(x\left|p,q\right.\right)+q^{n-k}\left[x\right]_{p,q}B_{k-1,n-1}\left(x\left|p,q\right.\right),\end{aligned}$$

which is the wanted formula (3.3). □

Theorem 7. *The following identity holds true for $x\in[0,1]$ and $k,n\in\mathbb{N}$ with $k\leqq n$:*

$$B_{n+k,2n+k}\left(x\left|p,q\right.\right)=\frac{\left[2n+k\right]_{p,q}!\left[k\right]_{p,q}!}{\left[n+k\right]_{p,q}!\left[n+k\right]_{p,q}!}\left[x\right]_{p,q}^n B_{k,n+k}\left(x\left|p,q\right.\right). \tag{3.4}$$

Proof. From (3.1), we calculate that

$$\begin{aligned}B_{n+k,2n+k}\left(x\left|p,q\right.\right) &= \begin{bmatrix} 2n+k \\ n+k\end{bmatrix}_{p,q}p^{\binom{n}{2}}\left[x\right]_{p,q}^{n+k}\left(1-x\right)^n \\ &= \frac{\left[2n+k\right]_{p,q}!\left[k\right]_{p,q}!\left[n\right]_{p,q}!}{\left[n\right]_{p,q}!\left[n+k\right]_{p,q}!\left[n+k\right]_{p,q}!}\left[x\right]_{p,q}^n\frac{\left[n+k\right]_{p,q}!}{\left[n\right]_{p,q}!\left[k\right]_{p,q}!}p^{\binom{n}{2}}\left[x\right]_{p,q}^k\left(1-x\right)^n,\end{aligned}$$

which gives the asserted result (3.4). □

The (p,q)-hyperbolic sine and cosine functions are defined by

$$\sinh_{p,q}x=\frac{e_{p,q}(x)-e_{p,q}(-x)}{2}\text{ and }\cosh_{p,q}x=\frac{e_{p,q}(x)+e_{p,q}(-x)}{2}\text{ (see [4]).} \tag{3.5}$$

Theorem 8. *We have for $x\in(0,1]$*

$$\begin{aligned}\sinh_{p,q}\left(t\left(1-x\right)\right) &= \frac{1}{2\left[x\right]_{p,q}^k}\left(\sum_{n=0}^{\infty}\left(1-\left(-1\right)^n\right)\frac{B_{k,n+k}\left(x\left|p,q\right.\right)}{\left[{n+k \atop k}\right]_{p,q}}\frac{t^n}{\left[n\right]_{p,q}!}\right), \\ \cosh_{p,q}\left(t\left(1-x\right)\right) &= \frac{1}{2\left[x\right]_{p,q}^k}\left(\sum_{n=0}^{\infty}\left(1+\left(-1\right)^n\right)\frac{B_{k,n+k}\left(x\left|p,q\right.\right)}{\left[{n+k \atop k}\right]_{p,q}}\frac{t^n}{\left[n\right]_{p,q}!}\right).\end{aligned} \tag{3.6}$$

Proof. Since

$$\begin{aligned}
\sinh_{p,q}(t(1-x)) &= \frac{[k]_{p,q}!}{2t^k [x]_{p,q}^k}\left(\frac{t^k [x]_{p,q}^k e_{p,q}(t(1-x)) - t^k [x]_{p,q}^k e_{p,q}(-t(1-x))}{[k]_{p,q}!}\right) \\
&= \frac{[k]_{p,q}!}{2t^k [x]_{p,q}^k}\left(\sum_{n=k}^{\infty} B_{k,n}(x|p,q)\frac{t^n}{[n]_{p,q}!} - \sum_{n=k}^{\infty} B_{k,n}(x|p,q)(-1)^{n-k}\frac{t^n}{[n]_{p,q}!}\right) \\
&= \frac{[k]_{p,q}!}{2t^k [x]_{p,q}^k}\left(\sum_{n=k}^{\infty}\left(1-(-1)^{n-k}\right) B_{k,n}(x|p,q)\frac{t^n}{[n]_{p,q}!}\right) \\
&= \frac{[k]_{p,q}!}{2 [x]_{p,q}^k}\left(\sum_{n=0}^{\infty}\left(1-(-1)^{n}\right) B_{k,n+k}(x|p,q)\frac{t^n}{[n+k]_{p,q}!}\right) \\
&= \frac{1}{2 [x]_{p,q}^k}\left(\sum_{n=0}^{\infty}\left(1-(-1)^{n}\right) \frac{B_{k,n+k}(x|p,q)}{\left[\begin{smallmatrix} n+k \\ k \end{smallmatrix}\right]_{p,q}}\frac{t^n}{[n]_{p,q}!}\right),
\end{aligned}$$

we get the desired result (3.6) The other can be shown similarly. □

Here we give a correlation between (p,q)-Hermite polynomials and (p,q)-Bernstein polynomials.

Theorem 9. *The following correlation is valid for $x \in [0,1)$*

$$H_{n,p,q}(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{[2]_{p,q}^{n-2k}(-1)^k p^{-\binom{n-2k}{2}} [2k]_{p,q}!}{[k]_{p,q}! [1-x]_{p,q}^{2k}} B_{2k,n}(1-x|p,q).$$

Proof. The proof of this theorem follows from (2.2) and (3.2). □

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