ON GENERALIZED k-FRACTIONAL DERIVATIVE OPERATOR

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ABSTRACT. The main objective of this paper is to introduce k-fractional derivative operator by using the definition of k-beta function. We establish some results related to the newly defined fractional operator such as Mellin transform and relations to k-hypergeometric and k-Appell's functions. Also, we investigate the k-fractional derivative of k-Mittag-Leffler and Wright hypergeometric functions.

1. INTRODUCTION

The classical beta function

$$\beta(\delta_1,\delta_2)=\int_0^\infty t^{\delta_1-1}(1-t)^{\delta_2-1}dt, (\Re(\delta_1)>0, \Re(\delta_2)>0) \tag{1.1}$$

and its relation with well known gamma function is given by

$$\beta(\delta_1,\delta_2)=\frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1+\delta_2)}, \Re(\delta_1)>0, \Re(\delta_2)>0.$$

The Gauss hypergeometric, confluent hypergeometric and Appell's functions which are respectively defined by(see [14])

$${}_2F_1(\delta_1,\delta_2;\delta_3;z)=\sum_{n=0}^{\infty}\frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n}\frac{z^n}{n!}, (|z|<1), \tag{1.2}$$

$$\Big(\delta_1,\delta_2,\delta_3\in\mathbb{C} \text{ and } \delta_3\neq 0,-1,-2,-3,\cdots\Big),$$

and

$${}_1\Phi_1(\delta_2;\delta_3;z)=\sum_{n=0}^{\infty}\frac{(\delta_2)_n}{(\delta_3)_n}\frac{z^n}{n!}, (|z|<1), \tag{1.3}$$

$$\Big(\delta_2,\delta_3\in\mathbb{C} \text{ and } \delta_3\neq 0,-1,-2,-3,\cdots\Big), \text{ respectively.}$$

The Appell's series or bivariate hypergeometric series is defined by

$$F_1(\delta_1,\delta_2,\delta_3;\delta_4;x,y)=\sum_{m,n=0}^{\infty}\frac{(\delta_1)_{m+n}(\delta_2)_m(\delta_3)_n x^m y^n}{(\delta_4)_{m+n}m!n!}; \tag{1.4}$$

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for all $\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}, \delta_4 \neq 0, -1, -2, -3, \cdots, \quad |x|, |y| < 1 < 1.$
The integral representation of hypergeometric, confluent hypergeometric and Appell's functions are respectively defined by

$$ {}_2F_1(\delta_1, \delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1}(1 - zt)^{-\delta_1} dt, \tag{1.5} $$

$$ \Big(\Re(\delta_3) > \Re(\delta_2) > 0, |\arg(1-z)| < \pi\Big), $$

and

$$ {}_1\Phi_1(\delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1} e^{zt} dt, \tag{1.6} $$

$$ \Big(\Re(\delta_3) > \Re(\delta_2) > 0\Big). $$

$$ F_1\Big(\delta_1, \delta_2, \delta_3; \delta_4; x, y\Big) = \frac{\Gamma(\delta_4)}{\Gamma(\delta_1)\Gamma(\delta_4 - \delta_1)} \int_0^1 t^{\delta_1 - 1}(1-t)^{\delta_4 - \delta_1 - 1}(1 - xt)^{-\delta_2}(1 - yt)^{-\delta_3} dt. \tag{1.7} $$

The $\mathtt{k}$-gamma function, $\mathtt{k}$-beta function and the $\mathtt{k}$-Pochhammer symbol introduced and studied by Diaz and Pariguan [1]. The integral representation of $\mathtt{k}$-gamma function and $\mathtt{k}$-beta function respectively given by

$$ \Gamma_{\mathtt{k}}(z) = \mathtt{k}^{\frac{z}{\mathtt{k}} - 1}\Gamma\left(\frac{z}{\mathtt{k}}\right) = \int_0^\infty t^{z-1} e^{-\frac{z^{\mathtt{k}}}{\mathtt{k}}} dt, \quad \Re(z) > 0, \mathtt{k} > 0 \tag{1.8} $$

$$ B_{\mathtt{k}}(x, y) = \frac{1}{\mathtt{k}} \int_0^1 t^{\frac{x}{\mathtt{k}} - 1}(1-t)^{\frac{y}{\mathtt{k}} - 1} dt, \quad \Re(x) > 0, \Re(y) > 0. \tag{1.9} $$

Here, we recall the following relations (see [1]).

$$ B_{\mathtt{k}}(x, y) = \frac{\Gamma_{\mathtt{k}}(x)\Gamma_{\mathtt{k}}(y)}{\Gamma_{\mathtt{k}}(x + y)}, \tag{1.10} $$

$$ (z)_{n,\mathtt{k}} = \frac{\Gamma_{\mathtt{k}}(z + n\mathtt{k})}{\Gamma_{\mathtt{k}}(z)}, \tag{1.11} $$

where $(z)_{n,\mathtt{k}} = (z)(z + \mathtt{k})(z + 2\mathtt{k}) \cdots (z + (n-1)\mathtt{k}); \quad (z)_{0,\mathtt{k}} = 1$ and $\mathtt{k} > 0$
and

$$ \sum_{n=0}^{\infty} (\alpha)_{n,\mathtt{k}} \frac{z^n}{n!} = (1 - \mathtt{k}z)^{\frac{-\alpha}{\mathtt{k}}}. \tag{1.12} $$

These studies were followed by Mansour [6], Kokologiannaki [3], Krasniqi [4] and Merovci [7]. In 2012, Mubeen and Habibullah [8] defined the $\mathtt{k}$-hypergeometric function as

$$ {}_2F_{1,\mathtt{k}}\Big(\delta_1, \delta_2; \delta_3; z\Big) = \sum_{n=0}^{\infty} \frac{(\delta_1)_{n,\mathtt{k}}(\delta_2)_{n,\mathtt{k}}}{(\delta_3)_{n,\mathtt{k}}} \frac{z^n}{n!}, \tag{1.13} $$

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where $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$ and $\delta_3 \neq 0, -1, -2, \cdots$ and its integral representation is given by

$$ {}_2F_{1,\mathtt{k}}\Big(\delta_1, \delta_2; \delta_3; z\Big) = \frac{1}{\mathtt{k}\beta_{\mathtt{k}}(\delta_2, \delta_3 - \delta_2)} \int_0^1 t^{\frac{\delta_2}{\mathtt{k}}-1}(1-t)^{\frac{\delta_3-\delta_2}{\mathtt{k}}-1}(1-\mathtt{k}tz)^{-\frac{\delta_1}{\mathtt{k}}}dt. \quad (1.14)$$

The $\mathtt{k}$-Riemann-Liouville (R-L) fractional integral using $\mathtt{k}$-gamma function introduced in [9]:

$$\Big(I_{\mathtt{k}}^{\alpha} f(t)\Big)(x) = \frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(\alpha)} \int_0^x f(t)(x-t)^{\frac{\alpha}{\mathtt{k}}-1}dt, \mathtt{k}, \alpha \in \mathbb{R}^+. \quad (1.15)$$

The solution of some integral equations involving confluent $\mathtt{k}$-hypergeometric functions and $\mathtt{k}$-analogue of Kummer's first formula are given in [12, 13]. While the $\mathtt{k}$-hypergeometric and confluent $\mathtt{k}$-hypergeometric differential equations are introduced in [10].

In 2015, Mubeen et al. [11] introduced $\mathtt{k}$-Appell hypergeometric function as

$$F_{1,\mathtt{k}}(\delta_1, \delta_2, \delta_3; \delta_4; z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{(\delta_1)_{m+n,\mathtt{k}}(\delta_2)_{m,\mathtt{k}}(\delta_3)_{m,\mathtt{k}}}{(\delta_4)_{m+n,\mathtt{k}}} \frac{z_1^m z_2^n}{m!n!} \quad (1.16)$$

for all $\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}, \delta_4 \neq 0, -1, -2, -3, \cdots,\quad \max\{|z_1|, |z_2|\} < \frac{1}{\mathtt{k}}$ and $\mathtt{k} > 0$.

Also, they define its integral representation as

$$F_{1,\mathtt{k}}\Big(\delta_1, \delta_2, \delta_3; \delta_4; z_1, z_2\Big) = \frac{1}{\mathtt{k}\beta(\delta_1, \delta_4 - \delta_1)} \times \int_0^1 t^{\frac{\delta_1}{\mathtt{k}}-1}(1-t)^{\frac{\delta_4-\delta_1}{\mathtt{k}}-1}(1-\mathtt{k}z_1t)^{-\frac{\delta_2}{\mathtt{k}}}(1-\mathtt{k}z_2t)^{-\frac{\delta_3}{\mathtt{k}}}dt \quad (1.17)$$

2. Extension of fractional derivative operator

In this section, we recall the definition of following fractional derivatives and give a new extension called Riemann-Liouville $\mathtt{k}$-fractional derivative.

Definition 2.1. *The well-known R-L fractional derivative of order μ is defined by*

$$\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1}dt, \Re(\mu) > 0. \quad (2.1)$$

For the case $m-1 < \Re(\mu) < m$ where $m = 1, 2, \cdots,$ it follows

$$\begin{aligned}\mathfrak{D}_x^{\mu}\{f(x)\} &= \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\Big\{f(x)\Big\} \\ &= \frac{d^m}{dx^m}\Big\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}dt\Big\}, \Re(\mu) > 0. \quad (2.2)\end{aligned}$$

In the following, we define Riemann-Liouville $\mathtt{k}$-fractional derivative of order μ as

Definition 2.2.

$${}_{\mathtt{k}}\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)} \int_0^x f(t)(x-t)^{-\frac{\mu}{\mathtt{k}}-1}dt, \Re(\mu) > 0, \mathtt{k} \in \mathbb{R}^+. \quad (2.3)$$

For the case $m-1 < \Re(\mu) < m$ where $m = 1, 2, \cdots,$ it follows

$${}_{\mathtt{k}}\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{d^m}{dx^m}\,{}_{\mathtt{k}}\mathfrak{D}_x^{\mu-m\mathtt{k}}\Big\{f(x)\Big\}$$

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$$=\frac{d^m}{dx^m}\Big\{\frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu+m\mathtt{k})}\int_0^x f(t)(x-t)^{-\frac{\mu}{\mathtt{k}}+m-1}dt\Big\}, \Re(\mu)>0. \tag{2.4}$$

Note that for $\mathtt{k}=1$, definition 2.2 reduces to the classical R-L fractional derivative operator given in definition 2.1.
Now, we are ready to prove some theorems by using the new definition 2.2.

Theorem 2.1. *The following formula holds true,*

$${}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{z^{\frac{\eta}{\mathtt{k}}}\}=\frac{z^{\frac{\eta-\mu}{\mathtt{k}}}}{\Gamma_{\mathtt{k}}(-\mu)}\beta_{\mathtt{k}}(\eta+\mathtt{k},-\mu), \Re(\mu)>0. \tag{2.5}$$

Proof. From (2.3), we have

$${}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{z^{\frac{\eta}{\mathtt{k}}}\}=\frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^z t^{\frac{\eta}{\mathtt{k}}}(z-t)^{-\frac{\mu}{\mathtt{k}}-1}\,dt. \tag{2.6}$$

Substituting $t=uz$ in (2.6), we get

$$\begin{aligned}{}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{z^{\frac{\eta}{\mathtt{k}}}\} &= \frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^1 (uz)^{\frac{\eta}{\mathtt{k}}}(z-uz)^{-\frac{\mu}{\mathtt{k}}-1}\,zdu\\ &= \frac{z^{\frac{\eta-\mu}{\mathtt{k}}}}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^1 u^{\frac{\eta}{\mathtt{k}}}(1-u)^{-\frac{\mu}{\mathtt{k}}-1}\,du.\end{aligned}$$

Applying definition (1.9) to the above equation, we get the desired result. □

Theorem 2.2. *Let $\Re(\mu)>0$ and suppose that the function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z)=\sum_{n=0}^{\infty}a_nz^n$ where $|z|<\rho$ for some $\rho\in\mathbb{R}^+$. Then*

$${}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{f(z)\}=\sum_{n=0}^{\infty}a_n\,{}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{z^n\}. \tag{2.7}$$

Proof. Using the series expansion of the function $f(z)$ in (2.3) gives

$${}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{f(z)\}=\frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^z\sum_{n=0}^{\infty}a_nt^n(z-t)^{-\frac{\mu}{\mathtt{k}}-1}\,dt.$$

As the series is uniformly convergent on any closed disk centered at the origin with its radius smaller then ρ, therefore the series so does on the line segment from 0 to a fixed z for $|z|<\rho$. Thus it guarantee terms by terms integration as follows

$$\begin{aligned}{}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{f(z)\} &= \sum_{n=0}^{\infty}a_n\Big\{\frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^z t^n(z-t)^{-\frac{\mu}{\mathtt{k}}-1}\,dt\\ &= \sum_{n=0}^{\infty}a_n\,{}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\{z^n\},\end{aligned}$$

which is the required proof. □

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Theorem 2.3. *The following result holds true:*

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}z)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{\Gamma_{\mathrm{k}}(\eta)}{\Gamma_{\mathrm{k}}(\mu)} z^{\frac{\mu}{\mathrm{k}}-1}\, {}_2F_{1,\mathrm{k}}\Big(\beta,\eta;\mu;z\Big), \tag{2.8}$$

where $\Re(\mu) > \Re(\eta) > 0$ *and* $|z| < 1$.

Proof. By direct calculation, we have

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}z)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\int_0^z t^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}t)^{-\frac{\beta}{\mathrm{k}}}(z-t)^{\frac{\mu-\eta}{\mathrm{k}}-1}dt$$
$$ = \frac{z^{\frac{\mu-\eta}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\int_0^z t^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}t)^{-\frac{\beta}{\mathrm{k}}}(1-\frac{t}{z})^{\frac{\mu-\eta}{\mathrm{k}}-1}dt.$$

Substituting $t = zu$ in the above equation, we get

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}z)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{z^{\frac{\mu}{\mathrm{k}}-1}}{\mathrm{k}\Gamma_{\mathrm{k}}(\mu-\eta)}\int_0^1 u^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}uz)^{-\frac{\beta}{\mathrm{k}}}(1-u)^{\frac{\mu-\eta}{\mathrm{k}}-1}zdu.$$

Applying (1.14) and after simplification we get the required proof. □

Theorem 2.4. *The following result holds true:*

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{\Gamma_{\mathrm{k}}(\eta)}{\Gamma_{\mathrm{k}}(\mu)} z^{\frac{\mu}{\mathrm{k}}-1} F_{1,\mathrm{k}}\Big(\eta,\alpha,\beta;\mu;az,bz\Big), \tag{2.9}$$

where $\Re(\mu) > \Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\max\{|az|,|bz|\} < \frac{1}{\mathrm{k}}$.

Proof. To prove(2.9), we use the power series expansion

$$ (1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,\mathrm{k}}(\beta)_{n,\mathrm{k}}\frac{(az)^m}{m!}\frac{(bz)^n}{n!}.$$

Now, applying Theorem 2.1, we obtain

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}}\}$$
$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,\mathrm{k}}(\beta)_{n,\mathrm{k}}\frac{(a)^m}{m!}\frac{(b)^n}{n!}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}+m+n-1}\}$$
$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,\mathrm{k}}(\beta)_{n,\mathrm{k}}\frac{(a)^m}{m!}\frac{(b)^n}{n!}\frac{\beta_{\mathrm{k}}(\eta+m\mathrm{k}+n\mathrm{k},\mu-\eta)}{\Gamma_{\mathrm{k}}(\mu-\eta)}z^{\frac{\mu}{\mathrm{k}}+m+n-1}$$
$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_{m,k}(\beta)_{n,k}\frac{(a)^m}{m!}\frac{(b)^n}{n!}\frac{\Gamma_k(\eta+mk+nk)}{\Gamma_k(\mu+mk+nk)}z^{\frac{\mu}{k}+m+n-1}.$$

In view of (1.16), we get

$$ {}_{\mathrm{k}}\mathfrak{D}_z^{\eta-\mu}\{z^{\frac{\eta}{\mathrm{k}}-1}(1-\mathrm{k}az)^{-\frac{\alpha}{\mathrm{k}}}(1-\mathrm{k}bz)^{-\frac{\beta}{\mathrm{k}}}\} = \frac{\Gamma_{\mathrm{k}}(\eta)}{\Gamma_{\mathrm{k}}(\mu)} z^{\frac{\mu}{\mathrm{k}}-1} F_{1,\mathrm{k}}\Big(\eta,\alpha,\beta;\mu;az,bz\Big).$$

□

Theorem 2.5. *The following Mellin transform formula holds true:*

$$ M\Big\{e^{-x}\,{}_{\mathrm{k}}\mathfrak{D}_z^{\mu}(z^{\frac{\eta}{\mathrm{k}}});s\Big\} = \frac{\Gamma(s)}{\Gamma_{\mathrm{k}}(-\mu)}\beta_{\mathrm{k}}(\eta+\mathrm{k},-\mu)z^{\frac{\eta-\mu}{\mathrm{k}}}, \tag{2.10}$$

where $\Re(\eta) > -1$, $\Re(\mu) > 0$, $\Re(s) > 0$.

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Proof. Applying the Mellin transform on definition (2.3), we have

$$M\Big\{e^{-x}\,{}_{\mathtt{k}}\mathfrak{D}^{\mu}_{z}(z^{\frac{\eta}{\mathtt{k}}});s\Big\}=\int_0^\infty x^{s-1}e^{-x}\,{}_{\mathtt{k}}\mathfrak{D}^{\mu}_{z}(z^{\eta});s\Big\}dx$$
$$=\frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^\infty x^{s-1}e^{-x}\Big\{\int_0^z t^{\frac{\eta}{\mathtt{k}}}(z-t)^{-\frac{\mu}{\mathtt{k}}-1}\,dt\Big\}dx$$
$$=\frac{z^{-\frac{\mu}{\mathtt{k}}-1}}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^\infty x^{s-1}e^{-x}\Big\{\int_0^z t^{\frac{\eta}{\mathtt{k}}}(1-\frac{t}{z})^{-\frac{\mu}{\mathtt{k}}-1}\,dt\Big\}dx$$
$$=\frac{z^{\frac{\eta-\mu}{\mathtt{k}}}}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^\infty x^{s-1}e^{-x}\Big\{\int_0^1 u^{\frac{\eta}{\mathtt{k}}}(1-u)^{-\frac{\mu}{\mathtt{k}}-1}\,du\Big\}dx$$

Interchanging the order of integrations in above equation, we get

$$M\Big\{e^{-x}\,{}_{\mathtt{k}}\mathfrak{D}^{\mu}_{z}(z^{\frac{\eta}{\mathtt{k}}});s\Big\}=\frac{z^{\frac{\eta-\mu}{\mathtt{k}}}}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\int_0^1 u^{\frac{\eta}{\mathtt{k}}}(1-u)^{-\frac{\mu}{\mathtt{k}}-1}\Big(\int_0^\infty x^{s-1}e^{-x}\,dx\Big)du.$$
$$=\frac{z^{\frac{\eta-\mu}{\mathtt{k}}}}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\Gamma(s)\int_0^1 u^{\frac{\eta}{\mathtt{k}}}(1-u)^{-\frac{\mu}{\mathtt{k}}-1}du$$
$$=\frac{\Gamma(s)}{\Gamma_{\mathtt{k}}(-\mu)}\beta_{\mathtt{k}}(\eta+\mathtt{k},-\mu)z^{\frac{\eta-\mu}{\mathtt{k}}},$$

which completes the proof. □

Theorem 2.6. *The following Mellin transform formula holds true:*

$$M\Big\{e^{-x}\,{}_{\mathtt{k}}\mathfrak{D}^{\mu}_{z}((1-\mathtt{k}z)^{-\frac{\alpha}{\mathtt{k}}});s\Big\}=\frac{z^{-\frac{\mu}{\mathtt{k}}}\Gamma(s)}{\Gamma_{\mathtt{k}}(-\mu)}\beta_{\mathtt{k}}(\mathtt{k},-\mu)\,{}_2F_{1,\mathtt{k}}\Big(\alpha,\mathtt{k};-\mu+\mathtt{k};z\Big),\tag{2.11}$$

where $\Re(\alpha)>0,\ \Re(\mu)<0,\ \Re(s)>0,$ *and* $|z|<1.$

Proof. Using the power series for $(1-\mathtt{k}z)^{-\frac{\alpha}{\mathtt{k}}}$ and applying Theorem 2.5 with $\eta=n\mathtt{k}$, we can write

$$M\Big\{e^{-x}\,{}_{\mathtt{k}}\mathfrak{D}^{\mu}_{z}((1-\mathtt{k}z)^{-\frac{\alpha}{\mathtt{k}}});s\Big\}=\sum_{n=0}^\infty\frac{(\alpha)_{n,\mathtt{k}}}{n!}M\Big\{e^{-x}\,{}_{\mathtt{k}}\mathfrak{D}^{\mu}_{z}(z^{n});s\Big\}$$
$$=\frac{\Gamma(s)}{\mathtt{k}\Gamma_{\mathtt{k}}(-\mu)}\sum_{n=0}^\infty\frac{(\alpha)_{n,\mathtt{k}}}{n!}\beta_{\mathtt{k}}(n\mathtt{k}+\mathtt{k},-\mu)z^{n-\frac{\mu}{\mathtt{k}}}$$
$$=\frac{\Gamma(s)z^{-\frac{\mu}{\mathtt{k}}}}{\Gamma_{\mathtt{k}}(-\mu)}\sum_{n=0}^\infty\beta_{\mathtt{k}}(n\mathtt{k}+\mathtt{k},-\mu)\frac{(\alpha)_{n,\mathtt{k}}z^n}{n!}$$
$$=\Gamma(s)z^{-\frac{\mu}{k}}\sum_{n=0}^\infty\frac{\Gamma_k(k+nk)}{\Gamma_k(-\mu+k+nk)}\frac{(\alpha)_{n,k}z^n}{n!}$$
$$=\frac{\Gamma(s)}{\Gamma_k(-\mu+k)}z^{-\frac{\mu}{k}}\sum_{n=0}^\infty\frac{(k)_{n,k}}{(-\mu+k)_{n,k}}\frac{(\alpha)_{n,k}z^n}{n!}$$
$$=\frac{\Gamma(s)z^{-\frac{\mu}{\mathtt{k}}}}{\Gamma_{\mathtt{k}}(-\mu)}\beta_{\mathtt{k}}(\mathtt{k},-\mu)\,{}_2F_{1,\mathtt{k}}\Big(\alpha,\mathtt{k};-\mu+\mathtt{k};z\Big),$$

which is the required proof. □

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Theorem 2.7. *The following result holds true:*

$$ {}_{\mathtt{k}}\mathfrak{D}_z^{\eta-\mu}\left[z^{\frac{\eta}{\mathtt{k}}-1}E^{\mu}_{\mathtt{k},\gamma,\delta}(z)\right] = \frac{z^{\frac{\mu}{\mathtt{k}}-1}}{\mathtt{k}\Gamma_{\mathtt{k}}(\mu-\eta)}\sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\gamma n+\delta)}\beta_{\mathtt{k}}(\eta+n\mathtt{k},\mu-\eta)\frac{z^n}{n!}, \tag{2.12} $$

where $\gamma,\delta,\mu\in\mathbb{C}$, $\Re(p)>0$, $\Re(q)>0$, $\Re(\mu)>\Re(\eta)>0$, $\Re(\lambda)>0$, $\Re(\rho)>0$ *and* $E^{\mu}_{\mathtt{k},\gamma,\delta}(z)$ *is* k*-Mittag-Leffler function (see [2]) defined as:*

$$ E^{\mu}_{\mathtt{k},\gamma,\delta}(z) = \sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\gamma n+\delta)}\frac{z^n}{n!}. \tag{2.13} $$

Proof. Using (2.13), the left-hand side of (2.12) can be written as

$$ {}_{\mathtt{k}}\mathfrak{D}_z^{\eta-\mu}\left[z^{\frac{\eta}{\mathtt{k}}-1}E^{\mu}_{\mathtt{k},\gamma,\delta}(z)\right] = {}_{\mathtt{k}}\mathfrak{D}_z^{\eta-\mu}\left[z^{\frac{\eta}{\mathtt{k}}-1}\left\{\sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\gamma n+\delta)}\frac{z^n}{n!}\right\}\right]. $$

By Theorem 2.2, we have

$$ {}_{\mathtt{k}}\mathfrak{D}_z^{\eta-\mu}\left[z^{\frac{\eta}{\mathtt{k}}-1}E^{\mu}_{\mathtt{k},\gamma,\delta}(z)\right] = \sum_{n=0}^{\infty}\frac{(\mu)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\gamma n+\delta)}\left\{{}_{\mathtt{k}}\mathfrak{D}_z^{\mu}\left[z^{\frac{\eta}{\mathtt{k}}+n-1}\right]\right\}. $$

In view of Theorem 2.1, we get the required proof. □

Theorem 2.8. *The following result holds true:*

$$ {}_{\mathtt{k}}\mathfrak{D}_z^{\eta-\mu}\left\{z^{\frac{\eta}{\mathtt{k}}-1}{}_m\Psi_n\left[\begin{matrix}(\alpha_i,A_i)_{1,m};\\ \\ (\beta_j,B_j)_{1,n};\end{matrix}\;\middle|z\right]\right\} = \frac{z^{\frac{\mu}{\mathtt{k}}-1}}{\mathtt{k}\Gamma_{\mathtt{k}}(\mu-\eta)} \times\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_i n)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_j n}\beta_{\mathtt{k}}(\eta+n\mathtt{k},\mu-\eta)\frac{z^n}{n!}, \tag{2.14} $$

where $\Re(p)>0$, $\Re(q)>0$, $\Re(\mu)>\Re(\eta)>0$, $\Re(\lambda)>0$, $\Re(\rho)>0$ *and* ${}_m\Psi_n(z)$ *is the Fox-Wright function defined by (see [5], pages 56-58)*

$$ {}_m\Psi_n(z) = {}_m\Psi_n\left[\begin{matrix}(\alpha_i,A_i)_{1,m};\\ \\ (\beta_j,B_j)_{1,n};\end{matrix}\;\middle|z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{m}\Gamma(\alpha_i+A_i n)}{\prod_{j=1}^{n}\Gamma(\beta_j+B_j n}\frac{z^n}{n!}. \tag{2.15} $$

Proof. Applying Theorem 2.1 and followed the same procedure used in Theorem 2.7, we get the desired result. □

3. Concluding remarks

In this paper, we established k-fractional derivative operator. If letting $\mathtt{k}\to 1$ then all the results established in this paper will reduce to the results related to the classical Reimann-Liouville fractional derivative operator.

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