ON GENERALIZED $k$-FRACTIONAL DERIVATIVE OPERATOR

GAUHAR RAHMAN, KOTTAKKARAN SOOPPY NISAR*, SHAHID MUBEEN

Abstract. The main objective of this paper is to introduce $k$-fractional derivative operator by using the definition of $k$-beta function. We establish some results related to the newly defined fractional operator such as Mellin transform and relations to $k$-hypergeometric and $k$-Appell’s functions. Also, we investigate the $k$-fractional derivative of $k$-Mittag-Leffler and Wright hypergeometric functions.

1. INTRODUCTION

The classical beta function

$$\beta(\delta_1, \delta_2) = \int_0^\infty t^{\delta_1-1}(1-t)^{\delta_2-1}dt, (\Re(\delta_1) > 0, \Re(\delta_2) > 0)$$

(1.1)

and its relation with well known gamma function is given by

$$\beta(\delta_1, \delta_2) = \frac{\Gamma(\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_1+\delta_2)}, \Re(\delta_1) > 0, \Re(\delta_2) > 0.$$

The Gauss hypergeometric, confluent hypergeometric and Appell’s functions which are respectively defined by (see [14])

$$2F_1(\delta_1, \delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n(\delta_2)_n}{(\delta_3)_n} \frac{z^n}{n!}, (|z| < 1),$$

(1.2)

$$\left( \delta_1, \delta_2, \delta_3 \in \mathbb{C} \text{ and } \delta_3 \neq 0, -1, -2, -3, \cdots \right),$$

and

$$1\Phi_1(\delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_2)_n}{(\delta_3)_n} \frac{z^n}{n!}, (|z| < 1),$$

(1.3)

$$\left( \delta_2, \delta_3 \in \mathbb{C} \text{ and } \delta_3 \neq 0, -1, -2, -3, \cdots \right),$$

respectively.

The Appell’s series or bivariate hypergeometric series is defined by

$$F_1(\delta_1, \delta_2, \delta_3; \delta_4; x, y) = \sum_{m,n=0}^{\infty} \frac{(\delta_1)_m(\delta_2)_m(\delta_3)_n}{(\delta_4)_{m+n}n!m!} x^m y^n;$$

(1.4)

2010 Mathematics Subject Classification. 33B15, 33C15, 33C05, 33C20, 33C65, 33E12, 26A33.

Key words and phrases. Beta function, $k$-beta function, Hypergeometric function, $k$-hypergeometric function, Mellin transform, fractional derivative, Appell’s function, $k$-Mittag-Leffler function.

*Corresponding author.
for all \(\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}, \delta_4 \neq 0, -1, -2, -3, \ldots\), \(|x|, |y| < 1 < 1\).

The integral representation of hypergeometric, confluent hypergeometric and Appell’s functions are respectively defined by

\[
2F_1(\delta_1, \delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1 - t)^{\delta_1 - \delta_2 - 1}(1 - zt)^{-\delta_1} dt, \quad (1.5)
\]

and

\[
1F_1(\delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1 - t)^{\delta_3 - \delta_2 - 1}e^{zt} dt, \quad (1.6)
\]

Here, we recall the following relations (see [1]).

\[
2F_1(\delta_1, \delta_2, \delta_3; \delta_4; x, y) = \frac{\Gamma(\delta_4)}{\Gamma(\delta_1)\Gamma(\delta_4 - \delta_1)} \int_0^1 t^{\delta_1 - 1}(1 - t)^{\delta_4 - \delta_1 - 1}(1 - xt)^{-\delta_2}(1 - yt)^{-\delta_3} dt. \quad (1.7)
\]

The k-gamma function, k-beta function and the k-Pochhammer symbol introduced and studied by Diaz and Pariguan [1]. The integral representation of k-gamma function and k-beta function respectively given by

\[
\Gamma_k(z) = k^{\frac{z}{k}} \Gamma\left(\frac{z}{k}\right) = \int_0^\infty t^{z - 1} e^{-\frac{t}{k}} dt, \quad \Re(z) > 0, k > 0 \quad (1.8)
\]

\[
B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}(1 - t)^{\frac{y}{k}-1} dt, \quad \Re(x) > 0, \Re(y) > 0. \quad (1.9)
\]

Here, we recall the following relations (see [1]).

\[
B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)}, \quad (1.10)
\]

\[
(z)_{n,k} = \frac{\Gamma_k(z + nk)}{\Gamma_k(z)} \quad (1.11)
\]

where \((z)_{n,k} = (z)(z + k)(z + 2k) \cdots (z + (n - 1)k); \quad (z)_{0,k} = 1\) and \(k > 0\) and

\[
\sum_{n=0}^{\infty} (\alpha)_{n,k} \frac{z^n}{n!} = (1 - kz)^{-\alpha} \quad (1.12)
\]

These studies were followed by Mansour [6], Kokologiannaki [3], Krasniqi [4] and Merovci [7]. In 2012, Mubeen and Habibullah [8] defined the k-hypergeometric function as

\[
2F_{1,k}(\delta_1, \delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_1)_{n,k}(\delta_2)_{n,k} z^n}{(\delta_3)_{n,k} n!}, \quad (1.13)
\]
where $\delta_1, \delta_2, \delta_3 \in \mathbb{C}$ and $\delta_3 \neq 0, -1, -2, \cdots$ and its integral representation is given by

$$2F_1, k \left( \delta_1, \delta_2; \delta_3; z \right) = \frac{1}{k \beta_k(\delta_2, \delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1 - t)^{\delta_3 - \delta_2 - 1}(1 - kzt)^{\frac{\delta_2}{\delta_2}} dt.$$  

(1.14)

The $k$-Riemann-Liouville (R-L) fractional integral using $k$-gamma function introduced in [9]:

$$\left( I_k^\alpha f(t) \right)(x) = \frac{1}{k \Gamma_k(\alpha)} \int_0^x f(t)(x - t)^{\frac{\alpha}{\delta_2} - 1} dt, k, \alpha \in \mathbb{R}^+.$$  

(1.15)

The solution of some integral equations involving confluent $k$-hypergeometric functions and $k$-analogue of Kummer’s first formula are given in [12, 13]. While the $k$-hypergeometric and confluent $k$-hypergeometric differential equations are introduced in [10].

In 2015, Mubeen et al. [11] introduced $k$-Appell hypergeometric function as

$$F_{1,k}(\delta_1, \delta_2, \delta_3; \delta_4; z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{(\delta_1)_{m+n,k}(\delta_2)_{m,k}(\delta_3)_{m,k} z_1^m z_2^n}{(\delta_4)_{m+n,k} m!n!}$$  

(2.1)

for all $\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{C}, \delta_4 \neq 0, -1, -2, -3, \cdots$, max\{||z_1||, ||z_2||\} < $\frac{1}{k}$ and $k > 0$.

Also, they define its integral representation as

$$F_{1,k}(\delta_1, \delta_2, \delta_3; \delta_4; z_1, z_2) = \frac{1}{k \beta_k(\delta_1, \delta_4 - \delta_1)}$$

$$\times \int_0^1 t^{\delta_1 - 1}(1 - t)^{\delta_4 - \delta_1 - 1}(1 - kzt)^{-\frac{\delta_4}{\delta_4}} (1 - kzt)^{-\frac{\delta_3}{\delta_3}} dt.$$  

(2.2)

2. Extension of fractional derivative operator

In this section, we recall the definition of following fractional derivatives and give a new extension called Riemann-Liouville $k$-fractional derivative.

**Definition 2.1.** The well-known R-L fractional derivative of order $\mu$ is defined by

$$D_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x - t)^{-\mu - 1} dt, \Re(\mu) > 0.$$  

(2.1)

For the case $m - 1 < \Re(\mu) < m$ where $m = 1, 2, \cdots$, it follows

$$D_x^{\mu}\{f(x)\} = \frac{d^m}{dx^m} D_x^{\mu-m}\{f(x)\}$$

$$= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu + m)} \int_0^x f(t)(x - t)^{-\mu + m - 1} dt \right\}, \Re(\mu) > 0.$$  

(2.2)

In the following, we define Riemann-Liouville $k$-fractional derivative of order $\mu$ as

**Definition 2.2.**

$$kD_x^{\mu}\{f(x)\} = \frac{1}{k \Gamma_k(-\mu)} \int_0^x f(t)(x - t)^{-\mu - 1} dt, \Re(\mu) > 0, k \in \mathbb{R}^+.$$  

(2.3)

For the case $m - 1 < \Re(\mu) < m$ where $m = 1, 2, \cdots$, it follows

$$kD_x^{\mu}\{f(x)\} = \frac{d^m}{dx^m} kD_x^{\mu-m}\{f(x)\}$$
Note that for $k = 1$, definition 2.2 reduces to the classical R-L fractional derivative operator given in definition 2.1.

Now, we are ready to prove some theorems by using the new definition 2.2.

**Theorem 2.1.** The following formula holds true,

$$kD_z^\mu\{z_\frac{n-k}{k}\} = \frac{z^\frac{n-k}{k}}{\Gamma_k(-\mu)} \beta_k(n+k,-\mu), \Re(\mu) > 0. \quad (2.5)$$

**Proof.** From (2.3), we have

$$kD_z^\mu\{z_\frac{n-k}{k}\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^\frac{n-k}{k} (z-t)^{-\frac{\mu}{k}-1} dt. \quad (2.6)$$

Substituting $t = uz$ in (2.6), we get

$$kD_z^\mu\{z_\frac{n-k}{k}\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^1 u^\frac{n-k}{k} (z-uz)^{-\frac{\mu}{k}-1} du.$$

Applying definition (1.9) to the above equation, we get the desired result.

**Theorem 2.2.** Let $\Re(\mu) > 0$ and suppose that the function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^\infty a_n z^n$ where $|z| < \rho$ for some $\rho \in \mathbb{R}^+$. Then

$$kD_z^\mu\{f(z)\} = \sum_{n=0}^\infty a_n kD_z^\mu\{z^n\}. \quad (2.7)$$

**Proof.** Using the series expansion of the function $f(z)$ in (2.3) gives

$$kD_z^\mu\{f(z)\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^z \sum_{n=0}^\infty a_n t^n (z-t)^{-\frac{\mu}{k}-1} dt.$$

As the series is uniformly convergent on any closed disk centered at the origin with its radius smaller than $\rho$, therefore the series so does on the line segment from 0 to a fixed $z$ for $|z| < \rho$. Thus it guarantee terms by terms integration as follows

$$kD_z^\mu\{f(z)\} = \sum_{n=0}^\infty a_n \left\{ \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^n (z-t)^{-\frac{\mu}{k}-1} dt \right\}$$

$$= \sum_{n=0}^\infty a_n kD_z^\mu\{z^n\},$$

which is the required proof.
ON GENERALIZE $k$-FRACTIONAL DERIVATIVE OPERATOR

**Theorem 2.3.** The following result holds true:

$$
\kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} (1 - kz)^{-\frac{\mu}{k}} \} = \frac{\Gamma_k(\eta)}{\Gamma_k(\mu)} z^{\frac{\eta}{k} - 1} \sum_{\eta=0}^{\mu-n} F_{1,k} \left( \beta, \eta; \mu; z \right),
$$  \hspace{1cm} (2.8)

where $\Re(\mu) > \Re(\eta) > 0$ and $|z| < 1$.

**Proof.** By direct calculation, we have

$$\kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} (1 - kz)^{-\frac{\mu}{k}} \} = \frac{1}{\kappa \Gamma_k(\mu - \eta)} \int_0^z t^{\frac{\eta}{k} - 1} (1 - kt)^{-\frac{\mu}{k}} dt
$$

$$= \frac{z^{\frac{\eta}{k} - 1}}{\kappa \Gamma_k(\mu - \eta)} \int_0^z t^{\frac{\eta}{k} - 1} (1 - kt)^{-\frac{\mu}{k}} (1 - \frac{t}{z})^{\frac{\mu-n}{k}} dt.
$$

Substituting $t = zu$ in the above equation, we get

$$\kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} (1 - kz)^{-\frac{\mu}{k}} \} = \frac{z^{\frac{\eta}{k} - 1}}{\kappa \Gamma_k(\mu - \eta)} \int_0^1 u^{\frac{\eta}{k} - 1} (1 - kuz)^{-\frac{\mu}{k}} (1 - u)^{\frac{\mu-n}{k}} du.
$$

Applying (1.14) and after simplification we get the required proof. \hfill $\square$

**Theorem 2.4.** The following result holds true:

$$
\kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} (1 - kaz)^{-\frac{\mu}{k}} (1 - kbz)^{-\frac{\mu}{k}} \} = \frac{\Gamma_k(\eta)}{\Gamma_k(\mu)} z^{\frac{\eta}{k} - 1} \sum_{\eta=0}^{\mu-n} \Gamma(\eta, \alpha, \beta; \mu; az, bz),
$$  \hspace{1cm} (2.9)

where $\Re(\mu) > \Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\max\{|az|, |bz|\} < \frac{1}{k}$.

**Proof.** To prove (2.9), we use the power series expansion

$$(1 - kuz)^{-\frac{\mu}{k}} (1 - u)^{\frac{\mu-n}{k}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{m,k} (\beta)_{n,k} \frac{(az)^m (bz)^n}{m! n!}.
$$

Now, applying Theorem 2.1, we obtain

$$\kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} (1 - kaz)^{-\frac{\mu}{k}} (1 - kbz)^{-\frac{\mu}{k}} \}
$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{m,k} (\beta)_{n,k} \frac{(a)^m (b)^n}{m! n!} \kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k} + m+n-1} \}
$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{m,k} (\beta)_{n,k} \frac{(a)^m (b)^n \beta_k(\eta + mk + nk, \mu - \eta)}{m! n! \Gamma_k(\mu - \eta)} z^{\frac{\eta}{k} + m+n-1}
$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\alpha)_{m,k} (\beta)_{n,k} \frac{(a)^m (b)^n \Gamma_k(\eta + mk + nk)}{m! n! \Gamma_k(\mu + mk + nk)} z^{\frac{\eta}{k} + m+n-1}.
$$

In view of (1.16), we get

$$\kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} (1 - kaz)^{-\frac{\mu}{k}} (1 - kbz)^{-\frac{\mu}{k}} \} = \frac{\Gamma_k(\eta)}{\Gamma_k(\mu)} z^{\frac{\eta}{k} - 1} \sum_{\eta=0}^{\mu-n} F_{1,k} \left( \eta, \alpha, \beta; \mu; az, bz \right).
$$

\hfill $\square$

**Theorem 2.5.** The following Mellin transform formula holds true:

$$
M\left\{ e^{-x} \kappa D_z^{\eta-\mu} \{ z^{\frac{\eta}{k}} \} ; s \right\} = \frac{\Gamma(s)}{\Gamma_k(-\mu)} \beta_k(\eta + k, -\mu) z^{\frac{\eta}{k}},
$$  \hspace{1cm} (2.10)

where $\Re(\eta) > -1$, $\Re(\mu) > 0$, $\Re(s) > 0$.  


Proof. Applying the Mellin transform on definition (2.3), we have
\[ M\left\{e^{-x} \mathcal{D}_z^\mu(z^\mu); s\right\} = \int_0^\infty x^{s-1} e^{-x} \mathcal{D}_z^\mu(z^\mu); s\right\} dx \]
\[ = -\frac{1}{k \Gamma_k(-\mu)} \int_0^\infty x^{s-1} e^{-x} \left\{ \int_0^z t^{\mu}(z-t)^{-\mu-1} dt \right\} dx \]
\[ = \frac{z^{-\frac{\mu}{\mu}-1}}{k \Gamma_k(-\mu)} \int_0^\infty x^{s-1} e^{-x} \left\{ \int_0^z \frac{t^{\mu}}{(1 - \frac{t}{z})^{-\mu-1}} dt \right\} dx \]
\[ = \frac{z^{-\frac{\mu}{\mu}-1}}{k \Gamma_k(-\mu)} \int_0^\infty x^{s-1} e^{-x} \left\{ \int_0^1 u^\mu (1 - u)^{-\mu-1} du \right\} dx \]
Interchanging the order of integrations in above equation, we get
\[ M\left\{e^{-x} \mathcal{D}_z^\mu(z^\mu); s\right\} = \frac{z^{-\frac{\mu}{\mu}-1}}{k \Gamma_k(-\mu)} \Gamma(s) \int_0^1 u^\mu (1 - u)^{-\mu-1} du \]
\[ = \frac{\Gamma(s)}{\Gamma_k(-\mu)} \beta_k(\eta + k, -\mu) z^{-\frac{\mu}{\mu}-1}, \]
which completes the proof. \(\square\)

Theorem 2.6. The following Mellin transform formula holds true:
\[ M\left\{e^{-x} \mathcal{D}_z^\mu((1 - k z)^{-\frac{\mu}{\mu}}); s\right\} = \frac{z^{-\frac{\mu}{\mu}} \Gamma(s)}{\Gamma_k(-\mu)} \beta_k(k, -\mu) \ {}_2F_1(k, \mu, -\mu + k; z), \] (2.11)
where \(\Re(\alpha) > 0, \Re(\mu) < 0, \Re(s) > 0, \text{ and } |z| < 1.\)

Proof. Using the power series for \((1 - k z)^{-\frac{\mu}{\mu}}\) and applying Theorem 2.5 with \(\eta = nk\), we can write
\[ M\left\{e^{-x} \mathcal{D}_z^\mu((1 - k z)^{-\frac{\mu}{\mu}}); s\right\} = \sum_{n=0}^\infty \frac{(\alpha)_{n,k}}{n!} M\left\{e^{-x} \mathcal{D}_z^\mu(z^n); s\right\} \]
\[ = \frac{\Gamma(s)}{\Gamma_k(-\mu)} \sum_{n=0}^\infty \frac{(\alpha)_{n,k}}{n!} \beta_k(nk + k, -\mu) z^{n-\frac{\mu}{\mu}} \]
\[ = \frac{\Gamma(s)}{\Gamma_k(-\mu)} \sum_{n=0}^\infty \beta_k(nk + k, -\mu) \frac{(\alpha)_{n,k}}{n!} \frac{z^n}{n!} \]
\[ = \frac{\Gamma(s)}{\Gamma_k(-\mu)} \frac{z^{-\frac{\mu}{\mu}}}{z^{-\frac{\mu}{\mu}}} \sum_{n=0}^\infty \frac{(k)_{n,k}}{(-\mu + k)_{n,k}} \frac{(\alpha)_{n,k} z^n}{n!} \]
\[ = \frac{\Gamma(s)}{\Gamma_k(-\mu)} \beta_k(k, -\mu) \ {}_2F_1(k, \mu, -\mu + k; z), \]
which is the required proof. \(\square\)
Theorem 2.7. The following result holds true:

\[ kD^\eta_\mu \left[ z^{\frac{\mu}{\gamma} - 1} E^\mu_{\gamma, \delta}(z) \right] = \frac{z^{\mu - 1}}{k \Gamma_k(\mu - \eta)} \sum_{n=0}^{\infty} \frac{\Gamma(\eta + nk, \mu - \eta)}{\Gamma_k(\gamma n + \delta)} z^n, \quad (2.12) \]

where \( \gamma, \delta, \mu, k \in \mathbb{C}, \Re(p) > 0, \Re(q) > 0, \Re(\mu) > \Re(\eta) > 0, \Re(\lambda) > 0, \Re(\rho) > 0 \) and \( E^\mu_{\gamma, \delta}(z) \) is \( k \)-Mittag-Leffler function (see [2]) defined as:

\[ E^\mu_{\gamma, \delta}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{nk}}{\Gamma_k(\gamma n + \delta)} \frac{z^n}{n!}, \quad (2.13) \]

Proof. Using (2.13), the left-hand side of (2.12) can be written as

\[ kD^\eta_\mu \left[ z^{\frac{\mu}{\gamma} - 1} E^\mu_{\gamma, \delta}(z) \right] = kD^\eta_\mu \left[ z^{\frac{\mu}{\gamma} - 1} \left\{ \sum_{n=0}^{\infty} \frac{(\mu)_{nk}}{\Gamma_k(\gamma n + \delta)} z^n \right\} \right]. \]

By Theorem 2.2, we have

\[ kD^\eta_\mu \left[ z^{\frac{\mu}{\gamma} - 1} E^\mu_{\gamma, \delta}(z) \right] = \sum_{n=0}^{\infty} \frac{(\mu)_{nk}}{\Gamma_k(\gamma n + \delta)} \left\{ kD^\eta_\mu \left[ z^{\frac{\mu}{\gamma} + n - 1} \right] \right\}. \]

In view of Theorem 2.1, we get the required proof. \( \square \)

Theorem 2.8. The following result holds true:

\[ kD^\eta_\mu \left\{ z^{\frac{\mu}{\gamma} - 1} m \Psi_n \left[ \begin{array}{c} (\alpha_i, A_i)_{1, m} \\ (\beta_j, B_j)_{1, n} \end{array} \right] \right\} = \frac{z^{\mu - 1}}{k \Gamma_k(\mu - \eta)} \]

\[ \times \sum_{n=0}^{\infty} \prod_{i=1}^{m} \Gamma(\alpha_i + A_in) \prod_{j=1}^{n} \Gamma(\beta_j + Bjn) \beta_k(\eta + nk, \mu - \eta) z^n, \quad (2.14) \]

where \( \Re(p) > 0, \Re(q) > 0, \Re(\mu) > \Re(\eta) > 0, \Re(\lambda) > 0, \Re(\rho) > 0 \) and \( m \Psi_n(z) \) is the Fox-Wright function defined by (see [5], pages 56-58)

\[ m \Psi_n(z) = m \Psi_n \left[ \begin{array}{c} (\alpha_i, A_i)_{1, m} \\ (\beta_j, B_j)_{1, n} \end{array} \right] = \sum_{n=0}^{\infty} \prod_{i=1}^{m} \Gamma(\alpha_i + A_in) \frac{z^n}{n!}. \quad (2.15) \]

Proof. Applying Theorem 2.1 and followed the same procedure used in Theorem 2.7, we get the desired result. \( \square \)

3. Concluding remarks

In this paper, we established \( k \)-fractional derivative operator. If letting \( k \to 1 \) then all the results established in this paper will reduce to the results related to the classical Reimann-Liouville fractional derivative operator.
G. RAHMAN, K.S. NISAR, S. MUBEEN

REFERENCES


Gauhar Rahman: Department of Mathematics, International Islamic University, Islamabad, Pakistan
E-mail address: gauhar55uom@gmail.com

Kottakkaran Sooppy Nisar: Department of Mathematics, College of Arts and Science-Wadi Aldawaser, 11991, Prince Sattam Bin Abdulaziz University, Alkharj, Kingdom of Saudi Arabia
E-mail address: n.sooppy@psau.edu.sa; ksnisar1@gmail.com
ON GENERALIZE $k$-FRACTIONAL DERIVATIVE OPERATOR

Shahid Mubeen: Department of Mathematics, University of Sargodha, Sargodha, Pakistan

E-mail address: smjhand@gmail.com