

## OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR THE GENERALIZED $k$ - $g$ -FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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ABSTRACT. Let  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$  on  $(a, b)$ . For the Lebesgue integrable function  $f : (a, b) \rightarrow \mathbb{C}$ , we define the  $k$ - $g$ -left-sided fractional integral of  $f$  by

$$S_{k,g,a+}f(x) = \int_a^x k(g(x) - g(t)) g'(t) f(t) dt, \quad x \in (a, b)$$

and the  $k$ - $g$ -right-sided fractional integral of  $f$  by

$$S_{k,g,b-}f(x) = \int_x^b k(g(t) - g(x)) g'(t) f(t) dt, \quad x \in [a, b),$$

where the kernel  $k$  is defined either on  $(0, \infty)$  or on  $[0, \infty)$  with complex values and integrable on any finite subinterval.

In this paper we establish some Ostrowski and trapezoid type inequalities for the  $k$ - $g$ -fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

### 1. INTRODUCTION

Assume that the kernel  $k$  is defined either on  $(0, \infty)$  or on  $[0, \infty)$  with complex values and integrable on any finite subinterval. We define the function  $K : [0, \infty) \rightarrow \mathbb{C}$  by

$$K(t) := \begin{cases} \int_0^t k(s) ds & \text{if } 0 < t, \\ 0 & \text{if } t = 0. \end{cases}$$

As a simple example, if  $k(t) = t^{\alpha-1}$  then for  $\alpha \in (0, 1)$  the function  $k$  is defined on  $(0, \infty)$  and  $K(t) := \frac{1}{\alpha} t^\alpha$  for  $t \in [0, \infty)$ . If  $\alpha \geq 1$ , then  $k$  is defined on  $[0, \infty)$  and  $K(t) := \frac{1}{\alpha} t^\alpha$  for  $t \in [0, \infty)$ .

Let  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$  on  $(a, b)$ . For the Lebesgue integrable function  $f : (a, b) \rightarrow \mathbb{C}$ , we define the  $k$ - $g$ -left-sided fractional integral of  $f$  by

$$(1.1) \quad S_{k,g,a+}f(x) = \int_a^x k(g(x) - g(t)) g'(t) f(t) dt, \quad x \in (a, b)$$

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1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

*Key words and phrases.* Generalized Riemann-Liouville fractional integrals, Hadamard fractional integrals, Functions of bounded variation, Ostrowski type inequalities, Trapezoid inequalities.

and the  $k$ - $g$ -right-sided fractional integral of  $f$  by

$$(1.2) \quad S_{k,g,b-}f(x) = \int_x^b k(g(t) - g(x))g'(t)f(t)dt, \quad x \in [a, b].$$

If we take  $k(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ , where  $\Gamma$  is the *Gamma function*, then

$$(1.3) \quad \begin{aligned} S_{k,g,a+}f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t) dt \\ &=: I_{a+,g}^\alpha f(x), \quad a < x \leq b \end{aligned}$$

and

$$(1.4) \quad \begin{aligned} S_{k,g,b-}f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t) dt \\ &=: I_{b-,g}^\alpha f(x), \quad a \leq x < b, \end{aligned}$$

which are the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function  $f$  with respect to another function  $g$  on  $[a, b]$  as defined in [21, p. 100]

For  $g(t) = t$  in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function  $g(t) = \ln t$  we have the *Hadamard fractional integrals* [21, p. 111]

$$(1.5) \quad H_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[ \ln \left( \frac{x}{t} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x \leq b$$

and

$$(1.6) \quad H_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[ \ln \left( \frac{t}{x} \right) \right]^{\alpha-1} \frac{f(t) dt}{t}, \quad 0 \leq a < x < b.$$

One can consider the function  $g(t) = -t^{-1}$  and define the "*Harmonic fractional integrals*" by

$$(1.7) \quad R_{a+}^\alpha f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha} t^{\alpha+1}}, \quad 0 \leq a < x \leq b$$

and

$$(1.8) \quad R_{b-}^\alpha f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t) dt}{(t-x)^{1-\alpha} t^{\alpha+1}}, \quad 0 \leq a < x < b.$$

Also, for  $g(t) = \exp(\beta t)$ ,  $\beta > 0$ , we can consider the " *$\beta$ -Exponential fractional integrals*"

$$(1.9) \quad E_{a+,\beta}^\alpha f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t) dt,$$

for  $a < x \leq b$  and

$$(1.10) \quad E_{b-,\beta}^\alpha f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t) dt,$$

for  $a \leq x < b$ .

If we take  $g(t) = t$  in (1.1) and (1.2), then we can consider the following  $k$ -*fractional integrals*

$$(1.11) \quad S_{k,a+}f(x) = \int_a^x k(x-t)f(t)dt, \quad x \in (a, b]$$

and

$$(1.12) \quad S_{k,b-}f(x) = \int_x^b k(t-x)f(t)dt, \quad x \in [a, b].$$

In [24], Raina studied a class of functions defined formally by

$$(1.13) \quad \mathcal{F}_{\rho,\lambda}^\sigma(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \quad |x| < R, \quad \text{with } R > 0$$

for  $\rho, \lambda > 0$  where the coefficients  $\sigma(k)$  generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$(1.14) \quad \mathcal{J}_{\rho,\lambda,a+;w}^\sigma f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(w(x-t)^\rho) f(t) dt, \quad x > a$$

where  $\rho, \lambda > 0$ ,  $w \in \mathbb{R}$  and  $f$  is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$(1.15) \quad \mathcal{J}_{\rho,\lambda,b-;w}^\sigma f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(w(t-x)^\rho) f(t) dt, \quad x < b$$

where  $\rho, \lambda > 0$ ,  $w \in \mathbb{R}$  and  $f$  is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for  $k(t) = t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma(wt^\rho)$  we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [22], Kirane and Torebek introduced the following *exponential fractional integrals*

$$(1.16) \quad \mathcal{T}_{a+}^\alpha f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t) dt, \quad x > a$$

and

$$(1.17) \quad \mathcal{T}_{b-}^\alpha f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t) dt, \quad x < b$$

where  $\alpha \in (0, 1)$ .

We observe that for  $k(t) = \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha}t\right)$ ,  $t \in \mathbb{R}$  we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$  on  $(a, b)$ . We can define the more general exponential fractional integrals

$$(1.18) \quad \mathcal{T}_{g,a+}^\alpha f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t) dt, \quad x > a$$

and

$$(1.19) \quad \mathcal{T}_{g,b-}^\alpha f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t) dt, \quad x < b$$

where  $\alpha \in (0, 1)$ .

Let  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$  on  $(a, b)$ . Assume that  $\alpha > 0$ . We can also define the *logarithmic fractional integrals*

$$(1.20) \quad \mathcal{L}_{g,a+}^\alpha f(x) := \int_a^x (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t)) g'(t) f(t) dt,$$

for  $0 < a < x \leq b$  and

$$(1.21) \quad \mathcal{L}_{g,b-}^{\alpha} f(x) := \int_x^b (g(t) - g(x))^{\alpha-1} \ln(g(t) - g(x)) g'(t) f(t) dt,$$

for  $0 < a \leq x < b$ , where  $\alpha > 0$ . These are obtained from (1.11) and (1.12) for the kernel  $k(t) = t^{\alpha-1} \ln t$ ,  $t > 0$ .

For  $\alpha = 1$  we get

$$(1.22) \quad \mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) dt, \quad 0 < a < x \leq b$$

and

$$(1.23) \quad \mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) dt, \quad 0 < a \leq x < b.$$

For  $g(t) = t$ , we have the simple forms

$$(1.24) \quad \mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) dt, \quad 0 < a < x \leq b,$$

$$(1.25) \quad \mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) dt, \quad 0 < a \leq x < b,$$

$$(1.26) \quad \mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) dt, \quad 0 < a < x \leq b$$

and

$$(1.27) \quad \mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) dt, \quad 0 < a \leq x < b.$$

In the recent paper [18] we obtained the following Ostrowski and trapezoid type inequalities for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function  $f$  with respect to another function  $g$  on  $[a, b]$ .

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{C}$  be a function of bounded variation on  $[a, b]$ . Also let  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$*

on  $(a, b)$ . Then we have

$$\begin{aligned}
 (1.28) \quad & \left| I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b) \right. \\
 & \quad \left. - \frac{1}{\Gamma(\alpha+1)} [(g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha] f(x) \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) \mathcal{V}_t^x(f) dt + \int_x^b (g(b) - g(t))^{\alpha-1} g'(t) \mathcal{V}_x^t(f) dt \right] \\
 & \leq \frac{1}{\Gamma(\alpha+1)} \left[ (g(x) - g(a))^\alpha \mathcal{V}_a^x(f) + (g(b) - g(x))^\alpha \mathcal{V}_x^b(f) \right] \\
 & \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \begin{array}{l} \left[ \frac{1}{2}(g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^\alpha \mathcal{V}_a^b(f); \\ ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} \left( (\mathcal{V}_a^x(f))^q + (\mathcal{V}_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[ \frac{1}{2} \mathcal{V}_a^b(f) + \frac{1}{2} \left| \mathcal{V}_a^x(f) - \mathcal{V}_x^b(f) \right| \right] ((g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha) \end{array} \right.
 \end{aligned}$$

and

$$\begin{aligned}
 (1.29) \quad & \left| I_{a+,g}^\alpha f(x) + I_{b-,g}^\alpha f(x) \right. \\
 & \quad \left. - \frac{1}{\Gamma(\alpha+1)} [(g(x) - g(a))^\alpha f(a) + (g(b) - g(x))^\alpha f(b)] \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[ \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \mathcal{V}_a^t(f) dt + \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \mathcal{V}_t^b(f) dt \right] \\
 & \leq \frac{1}{\Gamma(\alpha+1)} \left[ (g(x) - g(a))^\alpha \mathcal{V}_a^x(f) + (g(b) - g(x))^\alpha \mathcal{V}_x^b(f) \right] \\
 & \leq \frac{1}{\Gamma(\alpha+1)} \left\{ \begin{array}{l} \left[ \frac{1}{2}(g(b) - g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^\alpha \mathcal{V}_a^b(f); \\ ((g(x) - g(a))^{\alpha p} + (g(b) - g(x))^{\alpha p})^{1/p} \left( (\mathcal{V}_a^x(f))^q + (\mathcal{V}_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[ \frac{1}{2} \mathcal{V}_a^b(f) + \frac{1}{2} \left| \mathcal{V}_a^x(f) - \mathcal{V}_x^b(f) \right| \right] ((g(x) - g(a))^\alpha + (g(b) - g(x))^\alpha) \end{array} \right.
 \end{aligned}$$

for any  $x \in (a, b)$ .

For applications to the classical Riemann-Liouville fractional integrals, Hadamard fractional integrals and Harmonic fractional integrals see [18].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [19]-[32] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski and trapezoid type inequalities for the  $k$ - $g$ -fractional integrals of functions of bounded

variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

## 2. SOME IDENTITIES FOR THE OPERATOR $S_{k,g,a+,b-}$

For  $k$  and  $g$  as at the beginning of Introduction, we consider the mixed operator

$$(2.1) \quad \begin{aligned} S_{k,g,a+,b-} f(x) &:= \frac{1}{2} [S_{k,g,a+} f(x) + S_{k,g,b-} f(x)] \\ &= \frac{1}{2} \left[ \int_a^x k(g(x) - g(t)) g'(t) f(t) dt + \int_x^b k(g(t) - g(x)) g'(t) f(t) dt \right] \end{aligned}$$

for the Lebesgue integrable function  $f : (a, b) \rightarrow \mathbb{C}$  and  $x \in (a, b)$ .

The following two parameters representation for the operator  $S_{k,g,a+,b-}$  holds:

**Lemma 1.** *With the above assumptions for  $k$ ,  $g$  and  $f$  we have*

$$(2.2) \quad \begin{aligned} S_{k,g,a+,b-} f(x) &= \frac{1}{2} [\gamma K(g(b) - g(x)) + \lambda K(g(x) - g(a))] \\ &\quad + \frac{1}{2} \int_a^x k(g(x) - g(t)) g'(t) [f(t) - \lambda] dt \\ &\quad + \frac{1}{2} \int_x^b k(g(t) - g(x)) g'(t) [f(t) - \gamma] dt \end{aligned}$$

for any  $\lambda, \gamma \in \mathbb{C}$ .

*Proof.* We have, by taking the derivative over  $t$  and using the chain rule, that

$$[K(g(x) - g(t))] = K'(g(x) - g(t)) (g(x) - g(t))' = -k(g(x) - g(t)) g'(t)$$

for  $t \in (a, x)$  and

$$[K(g(t) - g(x))] = K'(g(t) - g(x)) (g(t) - g(x))' = k(g(t) - g(x)) g'(t)$$

for  $t \in (x, b)$ .

Therefore, for any  $\lambda, \gamma \in \mathbb{C}$  we have

$$(2.3) \quad \begin{aligned} &\int_a^x k(g(x) - g(t)) g'(t) [f(t) - \lambda] dt \\ &= \int_a^x k(g(x) - g(t)) g'(t) f(t) dt - \lambda \int_a^x k(g(x) - g(t)) g'(t) dt \\ &= S_{k,g,a+} f(x) + \lambda \int_a^x [K(g(x) - g(t))] dt \\ &= S_{k,g,a+} f(x) + \lambda [K(g(x) - g(t))]_a^x = S_{k,g,a+} f(x) - \lambda K(g(x) - g(a)) \end{aligned}$$

and

$$\begin{aligned}
 (2.4) \quad & \int_x^b k(g(t) - g(x)) g'(t) [f(t) - \gamma] dt \\
 &= \int_x^b k(g(t) - g(x)) g'(t) f(t) dt - \gamma \int_x^b k(g(t) - g(x)) g'(t) dt \\
 &= S_{k,g,b-} f(x) - \gamma \int_x^b [K(g(t) - g(x))]_x' dt \\
 &= S_{k,g,b-} f(x) - \gamma [K(g(t) - g(x))]_x^b = S_{k,g,b-} f(x) - \gamma K(g(b) - g(x))
 \end{aligned}$$

for  $x \in (a, b)$ .

If we add the equalities (2.3) and (2.4) and divide by 2 then we get the desired result (2.2).  $\square$

**Corollary 1.** *With the above assumptions for  $k$ ,  $g$  and  $f$  we have the Ostrowski type identity*

$$\begin{aligned}
 (2.5) \quad S_{k,g,a+,b-} f(x) &= \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \\
 &\quad + \frac{1}{2} \int_a^x k(g(x) - g(t)) g'(t) [f(t) - f(x)] dt \\
 &\quad + \frac{1}{2} \int_x^b k(g(t) - g(x)) g'(t) [f(t) - f(x)] dt
 \end{aligned}$$

and the trapezoid type identity

$$\begin{aligned}
 (2.6) \quad S_{k,g,a+,b-} f(x) &= \frac{1}{2} [K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)] \\
 &\quad + \frac{1}{2} \int_a^x k(g(x) - g(t)) g'(t) [f(t) - f(a)] dt \\
 &\quad + \frac{1}{2} \int_x^b k(g(t) - g(x)) g'(t) [f(t) - f(b)] dt
 \end{aligned}$$

for any  $x \in (a, b)$ .

For  $x = \frac{a+b}{2}$  we can consider

$$\begin{aligned}
 (2.7) \quad M_{k,g,a+,b-} f &:= S_{k,g,a+,b-} f \left( \frac{a+b}{2} \right) \\
 &= \frac{1}{2} \int_a^{\frac{a+b}{2}} k \left( g \left( \frac{a+b}{2} \right) - g(t) \right) g'(t) f(t) dt \\
 &\quad + \frac{1}{2} \int_{\frac{a+b}{2}}^b k \left( g(t) - g \left( \frac{a+b}{2} \right) \right) g'(t) f(t) dt.
 \end{aligned}$$

By (2.5) we have the representation

$$\begin{aligned}
 (2.8) \quad M_{k,g,a+,b-}f &= \frac{1}{2} \left[ K \left( g(b) - g \left( \frac{a+b}{2} \right) \right) + K \left( g \left( \frac{a+b}{2} \right) - g(a) \right) \right] f \left( \frac{a+b}{2} \right) \\
 &+ \frac{1}{2} \int_a^{\frac{a+b}{2}} k \left( g \left( \frac{a+b}{2} \right) - g(t) \right) g'(t) \left[ f(t) - f \left( \frac{a+b}{2} \right) \right] dt \\
 &+ \frac{1}{2} \int_{\frac{a+b}{2}}^b k \left( g(t) - g \left( \frac{a+b}{2} \right) \right) g'(t) \left[ f(t) - f \left( \frac{a+b}{2} \right) \right] dt
 \end{aligned}$$

and (2.6) we have

$$\begin{aligned}
 (2.9) \quad M_{k,g,a+,b-}f &= \frac{1}{2} \left[ K \left( g(b) - g \left( \frac{a+b}{2} \right) \right) f(b) + K \left( g \left( \frac{a+b}{2} \right) - g(a) \right) f(a) \right] \\
 &+ \frac{1}{2} \int_a^{\frac{a+b}{2}} k \left( g \left( \frac{a+b}{2} \right) - g(t) \right) g'(t) [f(t) - f(a)] dt \\
 &+ \frac{1}{2} \int_{\frac{a+b}{2}}^b k \left( g(t) - g \left( \frac{a+b}{2} \right) \right) g'(t) [f(t) - f(b)] dt.
 \end{aligned}$$

If  $g$  is a function which maps an interval  $I$  of the real line to the real numbers, and is both continuous and injective then we can define the  $g$ -mean of two numbers  $a, b \in I$  as

$$M_g(a, b) := g^{-1} \left( \frac{g(a) + g(b)}{2} \right).$$

If  $I = \mathbb{R}$  and  $g(t) = t$  is the *identity function*, then  $M_g(a, b) = A(a, b) := \frac{a+b}{2}$ , the *arithmetic mean*. If  $I = (0, \infty)$  and  $g(t) = \ln t$ , then  $M_g(a, b) = G(a, b) := \sqrt{ab}$ , the *geometric mean*. If  $I = (0, \infty)$  and  $g(t) = \frac{1}{t}$ , then  $M_g(a, b) = H(a, b) := \frac{2ab}{a+b}$ , the *harmonic mean*. If  $I = (0, \infty)$  and  $g(t) = t^p$ ,  $p \neq 0$ , then  $M_g(a, b) = M_p(a, b) := \left( \frac{a^p + b^p}{2} \right)^{1/p}$ , the *power mean with exponent  $p$* . Finally, if  $I = \mathbb{R}$  and  $g(t) = \exp t$ , then

$$M_g(a, b) = LME(a, b) := \ln \left( \frac{\exp a + \exp b}{2} \right),$$

the *LogMeanExp function*.

Using the  $g$ -mean of two numbers we can introduce

$$\begin{aligned}
 (2.10) \quad P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f(M_g(a, b)) \\
 &= \frac{1}{2} \int_a^{M_g(a,b)} k \left( \frac{g(a) + g(b)}{2} - g(t) \right) g'(t) f(t) dt \\
 &+ \frac{1}{2} \int_{M_g(a,b)}^b k \left( g(t) - \frac{g(a) + g(b)}{2} \right) g'(t) f(t) dt.
 \end{aligned}$$



Using (2.5) and (2.6) we have the representations

$$(2.11) \quad \begin{aligned} P_{k,g,a+,b-}f &= K \left( \frac{g(b) - g(a)}{2} \right) f(M_g(a, b)) \\ &+ \frac{1}{2} \int_a^{M_g(a,b)} k \left( \frac{g(a) + g(b)}{2} - g(t) \right) g'(t) [f(t) - f(M_g(a, b))] dt \\ &+ \frac{1}{2} \int_{M_g(a,b)}^b k \left( g(t) - \frac{g(a) + g(b)}{2} \right) g'(t) [f(t) - f(M_g(a, b))] dt \end{aligned}$$

and

$$(2.12) \quad \begin{aligned} P_{k,g,a+,b-}f &= K \left( \frac{g(b) - g(a)}{2} \right) \frac{f(b) + f(a)}{2} \\ &+ \frac{1}{2} \int_a^{M_g(a,b)} k \left( \frac{g(a) + g(b)}{2} - g(t) \right) g'(t) [f(t) - f(a)] dt \\ &+ \frac{1}{2} \int_{M_g(a,b)}^b k \left( g(t) - \frac{g(a) + g(b)}{2} \right) g'(t) [f(t) - f(b)] dt. \end{aligned}$$

### 3. SOME IDENTITIES FOR THE DUAL OPERATOR $\check{S}_{k,g,a+,b-}$

Observe that

$$(3.1) \quad S_{k,g,x+}f(b) = \int_x^b k(g(b) - g(t)) g'(t) f(t) dt, \quad x \in [a, b]$$

and

$$(3.2) \quad S_{k,g,x-}f(a) = \int_a^x k(g(t) - g(a)) g'(t) f(t) dt, \quad x \in (a, b].$$

Define also the mixed operator

$$(3.3) \quad \begin{aligned} \check{S}_{k,g,a+,b-}f(x) &:= \frac{1}{2} [S_{k,g,x+}f(b) + S_{k,g,x-}f(a)] \\ &= \frac{1}{2} \left[ \int_x^b k(g(b) - g(t)) g'(t) f(t) dt + \int_a^x k(g(t) - g(a)) g'(t) f(t) dt \right] \end{aligned}$$

for any  $x \in (a, b)$ .

**Lemma 2.** *With the above assumptions for  $k$ ,  $g$  and  $f$  we have*

$$(3.4) \quad \begin{aligned} \check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2} [\lambda K(g(b) - g(x)) + \gamma K(g(x) - g(a))] \\ &+ \frac{1}{2} \int_a^x k(g(t) - g(a)) g'(t) [f(t) - \gamma] dt \\ &+ \frac{1}{2} \int_x^b k(g(b) - g(t)) g'(t) [f(t) - \lambda] dt \end{aligned}$$

for any  $\lambda, \gamma \in \mathbb{C}$ .

*Proof.* We have, by taking the derivative over  $t$  and using the chain rule, that

$$[K(g(b) - g(t))]' = K'(g(b) - g(t))(g(b) - g(t))' = -k(g(b) - g(t))g'(t)$$

for  $t \in (x, b)$  and

$$[K(g(t) - g(a))]' = K'(g(t) - g(a))(g(t) - g(a))' = k(g(t) - g(a))g'(t)$$

for  $t \in (a, x)$ .

For any  $\lambda, \gamma \in \mathbb{C}$  we have

$$\begin{aligned} (3.5) \quad & \int_x^b k(g(b) - g(t))g'(t)[f(t) - \lambda] dt \\ &= \int_x^b k(g(b) - g(t))g'(t)f(t) dt - \lambda \int_x^b k(g(b) - g(t))g'(t) dt \\ &= S_{k,g,x+f}(b) + \lambda \int_x^b [K(g(b) - g(t))]' dt \\ &= S_{k,g,x+f}(b) - \lambda K(g(b) - g(x)) \end{aligned}$$

and

$$\begin{aligned} (3.6) \quad & \int_a^x k(g(t) - g(a))g'(t)[f(t) - \gamma] dt \\ &= \int_a^x k(g(t) - g(a))g'(t)f(t) dt - \gamma \int_a^x k(g(t) - g(a))g'(t) dt \\ &= \int_a^x k(g(t) - g(a))g'(t)f(t) dt - \gamma \int_a^x [K(g(t) - g(a))]' dt \\ &= \int_a^x k(g(t) - g(a))g'(t)f(t) dt - \gamma K(g(x) - g(a)) \end{aligned}$$

for  $x \in (a, b)$ .

If we add the equalities (3.5) and (3.6) and divide by 2 then we get the desired result (3.4).  $\square$

**Corollary 2.** *With the assumptions of Lemma 2 we have the Ostrowski type identity*

$$\begin{aligned} (3.7) \quad \check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \\ &\quad + \frac{1}{2} \int_a^x k(g(t) - g(a))g'(t)[f(t) - f(x)] dt \\ &\quad + \frac{1}{2} \int_x^b k(g(b) - g(t))g'(t)[f(t) - f(x)] dt \end{aligned}$$

and the trapezoid identity

$$\begin{aligned} (3.8) \quad \check{S}_{k,g,a+,b-}f(x) &= \frac{1}{2} [K(g(b) - g(x))f(b) + K(g(x) - g(a))f(a)] \\ &\quad + \frac{1}{2} \int_a^x k(g(t) - g(a))g'(t)[f(t) - f(a)] dt \\ &\quad + \frac{1}{2} \int_x^b k(g(b) - g(t))g'(t)[f(t) - f(b)] dt \end{aligned}$$

for  $x \in (a, b)$ .

For  $x = \frac{a+b}{2}$  we can consider

$$(3.9) \quad \begin{aligned} \check{M}_{k,g,a+,b-}f &:= \check{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \\ &= \frac{1}{2} \int_{\frac{a+b}{2}}^b k(g(b) - g(t)) g'(t) f(t) dt \\ &\quad + \frac{1}{2} \int_a^{\frac{a+b}{2}} k(g(t) - g(a)) g'(t) f(t) dt. \end{aligned}$$

Using the equalities (3.7) and (3.8), we have

$$(3.10) \quad \begin{aligned} \check{M}_{k,g,a+,b-}f &= \frac{1}{2} \left[ K \left( g(b) - g\left(\frac{a+b}{2}\right) \right) + K \left( g\left(\frac{a+b}{2}\right) - g(a) \right) \right] f\left(\frac{a+b}{2}\right) \\ &\quad + \frac{1}{2} \int_a^{\frac{a+b}{2}} k(g(t) - g(a)) g'(t) \left[ f(t) - f\left(\frac{a+b}{2}\right) \right] dt \\ &\quad + \frac{1}{2} \int_{\frac{a+b}{2}}^b k(g(b) - g(t)) g'(t) \left[ f(t) - f\left(\frac{a+b}{2}\right) \right] dt \end{aligned}$$

and

$$(3.11) \quad \begin{aligned} \check{M}_{k,g,a+,b-}f &= \frac{1}{2} \left[ K \left( g(b) - g\left(\frac{a+b}{2}\right) \right) f(b) + K \left( g\left(\frac{a+b}{2}\right) - g(a) \right) f(a) \right] \\ &\quad + \frac{1}{2} \int_a^{\frac{a+b}{2}} k(g(t) - g(a)) g'(t) [f(t) - f(a)] dt \\ &\quad + \frac{1}{2} \int_{\frac{a+b}{2}}^b k(g(b) - g(t)) g'(t) [f(t) - f(b)] dt. \end{aligned}$$

Using the  $g$ -mean of two numbers we can introduce

$$(3.12) \quad \begin{aligned} \check{P}_{k,g,a+,b-}f &:= \check{S}_{k,g,a+,b-}f(M_g(a,b)) \\ &= \frac{1}{2} \int_{M_g(a,b)}^b k(g(b) - g(t)) g'(t) f(t) dt \\ &\quad + \frac{1}{2} \int_a^{M_g(a,b)} k(g(t) - g(a)) g'(t) f(t) dt. \end{aligned}$$

Using the equalities (3.7) and (3.8), we have

$$(3.13) \quad \begin{aligned} \check{P}_{k,g,a+,b-}f &= K \left( \frac{g(b) - g(a)}{2} \right) f(M_g(a,b)) \\ &\quad + \frac{1}{2} \int_a^{M_g(a,b)} k(g(t) - g(a)) g'(t) [f(t) - f(M_g(a,b))] dt \\ &\quad + \frac{1}{2} \int_{M_g(a,b)}^b k(g(b) - g(t)) g'(t) [f(t) - f(M_g(a,b))] dt \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} \check{P}_{k,g,a+,b-}f &= K \left( \frac{g(b) - g(a)}{2} \right) \frac{f(b) + f(a)}{2} \\ &+ \frac{1}{2} \int_a^{M_g(a,b)} k(g(t) - g(a)) g'(t) [f(t) - f(a)] dt \\ &+ \frac{1}{2} \int_{M_g(a,b)}^b k(g(b) - g(t)) g'(t) [f(t) - f(b)] dt. \end{aligned}$$

#### 4. TRAPEZOID FUNCTIONAL $T_{k,g,a+,b-}$

We can also introduce the functional

$$(4.1) \quad \begin{aligned} T_{k,g,a+,b-}f &:= \frac{1}{2} [S_{k,g,a+}f(b) + S_{k,g,b-}f(a)] \\ &= \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) f(t) dt. \end{aligned}$$

We have:

**Lemma 3.** *With the assumption of Lemma 1, we have*

$$(4.2) \quad \begin{aligned} T_{k,g,a+,b-}f &= K(g(b) - g(a)) \delta \\ &+ \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) [f(t) - \delta] dt \end{aligned}$$

for any  $\delta \in \mathbb{C}$ .

*Proof.* Observe that

$$\begin{aligned} &\int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) dt \\ &= \int_a^b k(g(b) - g(t)) g'(t) dt + \int_a^b k(g(t) - g(a)) g'(t) dt \\ &= - \int_a^b [K(g(b) - g(t))] dt + \int_a^b [K(g(t) - g(a))] dt \\ &= -K(g(b) - g(t))|_a^b + K(g(t) - g(a))|_a^b \\ &= K(g(b) - g(a)) + K(g(b) - g(a)) = 2K(g(b) - g(a)). \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) [f(t) - \delta] dt \\ &= \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) f(t) dt \\ &\quad - \frac{1}{2} \delta \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) dt \\ &= T_{k,g,a+,b-}f - \delta K(g(b) - g(a)), \end{aligned}$$

which proves the desired equality (4.2).  $\square$

**Corollary 3.** *With the assumptions of Lemma 3 we have the Ostrowski type identity*

$$(4.3) \quad \begin{aligned} T_{k,g,a+,b-}f &= K(g(b) - g(a))f(x) \\ &+ \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))]g'(t)[f(t) - f(x)]dt \end{aligned}$$

for any  $x \in [a, b]$  and the trapezoid identity

$$(4.4) \quad \begin{aligned} T_{k,g,a+,b-}f &= K(g(b) - g(a)) \frac{f(a) + f(b)}{2} \\ &+ \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))]g'(t) \left[ f(t) - \frac{f(a) + f(b)}{2} \right] dt. \end{aligned}$$

We observe that for  $x = \frac{a+b}{2}$  we obtain from (4.3) that

$$(4.5) \quad \begin{aligned} T_{k,g,a+,b-}f &= K(g(b) - g(a))f\left(\frac{a+b}{2}\right) \\ &+ \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))]g'(t) \left[ f(t) - f\left(\frac{a+b}{2}\right) \right] dt. \end{aligned}$$

## 5. INEQUALITIES FOR FUNCTIONS OF BOUNDED VARIATION

We considered the cumulative function  $K : [0, \infty) \rightarrow \mathbb{C}$  by

$$K(t) := \begin{cases} \int_0^t k(s) ds & \text{if } 0 < t, \\ 0 & \text{if } t = 0. \end{cases}$$

We also define the function  $\mathbf{K} : [0, \infty) \rightarrow [0, \infty)$  by

$$\mathbf{K}(t) := \begin{cases} \int_0^t |k(s)| ds & \text{if } 0 < t, \\ 0 & \text{if } t = 0. \end{cases}$$

We observe that if  $k$  takes nonnegative values on  $(0, \infty)$ , as it does in some of the examples in Introduction, then  $\mathbf{K}(t) = K(t)$  for  $t \in [0, \infty)$ .

**Theorem 2.** *Assume that the kernel  $k$  is defined either on  $(0, \infty)$  or on  $[0, \infty)$  with complex values and integrable on any finite subinterval. Let  $f : [a, b] \rightarrow \mathbb{C}$  be a function of bounded variation on  $[a, b]$  and  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$  on  $(a, b)$ . Then we have the Ostrowski type*

inequality

$$\begin{aligned}
 (5.1) \quad & \left| S_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \right| \\
 & \leq \frac{1}{2} \left[ \int_x^b |k(g(t) - g(x))| \bigvee_x^t(f) g'(t) dt + \int_a^x |k(g(x) - g(t))| \bigvee_t^x(f) g'(t) dt \right] \\
 & \leq \frac{1}{2} \left[ \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) \right] \\
 & \leq \frac{1}{2} \begin{cases} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b(f); \\ [\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))]^{1/p} \left( (\bigvee_a^x(f))^q + (\bigvee_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))] \left[ \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{cases}
 \end{aligned}$$

and the trapezoid type inequality

$$\begin{aligned}
 (5.2) \quad & \left| S_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)] \right| \\
 & \leq \frac{1}{2} \left[ \int_a^x |k(g(x) - g(t))| \bigvee_a^t(f) g'(t) dt + \int_x^b |k(g(t) - g(x))| \bigvee_t^b(f) g'(t) dt \right] \\
 & \leq \frac{1}{2} \left[ \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) \right] \\
 & \leq \frac{1}{2} \begin{cases} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b(f); \\ [\mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a))]^{1/p} \\ \times \left( (\bigvee_a^x(f))^q + (\bigvee_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ [\mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a))] \\ \times \left[ \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{cases}
 \end{aligned}$$

for any  $x \in (a, b)$ .

*Proof.* Using the equality (2.5) we have

$$\begin{aligned}
 (5.3) \quad & \left| S_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \right| \\
 & \leq \frac{1}{2} \left| \int_a^x k(g(x) - g(t)) g'(t) [f(t) - f(x)] dt \right| \\
 & + \frac{1}{2} \left| \int_x^b k(g(t) - g(x)) g'(t) [f(t) - f(x)] dt \right| \\
 & \leq \frac{1}{2} \int_a^x |k(g(x) - g(t)) g'(t) [f(t) - f(x)]| dt \\
 & + \frac{1}{2} \int_x^b |k(g(t) - g(x)) g'(t) [f(t) - f(x)]| dt \\
 & = \frac{1}{2} \int_a^x |k(g(x) - g(t))| |f(x) - f(t)| g'(t) dt \\
 & + \frac{1}{2} \int_x^b |k(g(t) - g(x))| |f(t) - f(x)| g'(t) dt \\
 & =: B(x)
 \end{aligned}$$

for  $x \in (a, b)$ .

Since  $f$  is of bounded variation, then

$$|f(x) - f(t)| \leq \bigvee_t^x(f) \leq \bigvee_a^x(f) \text{ for } a < t \leq x \leq b$$

and

$$|f(t) - f(x)| \leq \bigvee_x^t(f) \leq \bigvee_x^b(f) \text{ for } a \leq x \leq t < b.$$

Therefore

$$\begin{aligned}
 B(x) & \leq \frac{1}{2} \int_a^x |k(g(x) - g(t))| \bigvee_t^x(f) g'(t) dt \\
 & + \frac{1}{2} \int_x^b |k(g(t) - g(x))| \bigvee_x^t(f) g'(t) dt \\
 & \leq \frac{1}{2} \bigvee_a^x(f) \int_a^x |k(g(x) - g(t))| g'(t) dt \\
 & + \frac{1}{2} \bigvee_x^b(f) \int_x^b |k(g(t) - g(x))| g'(t) dt \\
 & =: C(x)
 \end{aligned}$$

for  $x \in (a, b)$ .

We have, by taking the derivative over  $t$  and using the chain rule, that

$$[\mathbf{K}(g(x) - g(t))] = \mathbf{K}'(g(x) - g(t)) (g(x) - g(t))' = -|k(g(x) - g(t))| g'(t)$$

for  $t \in (a, x)$  and

$$[\mathbf{K}(g(t) - g(x))] = \mathbf{K}'(g(t) - g(x)) (g(t) - g(x))' = |k(g(t) - g(x))| g'(t)$$

for  $t \in (x, b)$ .

Then

$$\int_a^x |k(g(x) - g(t))| g'(t) dt = - \int_a^x [\mathbf{K}(g(x) - g(t))]'' dt = \mathbf{K}(g(x) - g(a))$$

and

$$\int_x^b |k(g(t) - g(x))| g'(t) dt = \int_x^b [\mathbf{K}(g(t) - g(x))]'' dt = \mathbf{K}(g(b) - g(x))$$

giving that

$$C(x) = \frac{1}{2} \left[ \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) \right],$$

for  $x \in (a, b)$ , which proves the first and the second inequality in (5.1).

The last part of (4.2) is obvious by making use of the elementary Hölder type inequalities for positive real numbers  $c, d, m, n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m, n\}(c + d); \\ (m^p + n^p)^{1/p} (c^q + d^q)^{1/q} \text{ with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

Further, by the identity (2.6) we have, as above,

$$\begin{aligned} & \left| S_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)] \right| \\ & \leq \frac{1}{2} \int_a^x |k(g(x) - g(t))| |f(t) - f(a)| g'(t) dt \\ & \quad + \frac{1}{2} \int_x^b |k(g(t) - g(x))| |f(t) - f(b)| g'(t) dt \\ & \leq \frac{1}{2} \int_a^x |k(g(x) - g(t))| \bigvee_a^t(f) g'(t) dt \\ & \quad + \frac{1}{2} \int_x^b |k(g(t) - g(x))| \bigvee_t^b(f) g'(t) dt \\ & \leq \frac{1}{2} \bigvee_a^x(f) \int_a^x |k(g(x) - g(t))| g'(t) dt \\ & \quad + \frac{1}{2} \bigvee_x^b(f) \int_x^b |k(g(t) - g(x))| g'(t) dt \\ & = \frac{1}{2} \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) + \frac{1}{2} \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f), \end{aligned}$$

which proves (5.2). □



The following particular case for the functional

$$\begin{aligned} P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f(M_g(a,b)) \\ &= \frac{1}{2} \int_a^{M_g(a,b)} k \left( \frac{g(b)+g(a)}{2} - g(t) \right) g'(t) f(t) dt \\ &\quad + \frac{1}{2} \int_{M_g(a,b)}^b k \left( g(t) - \frac{g(b)+g(a)}{2} \right) g'(t) f(t) dt. \end{aligned}$$

is of interest:

**Corollary 4.** *With the assumptions of Theorem 2 we have*

$$\begin{aligned} (5.4) \quad & \left| P_{k,g,a+,b-}f - K \left( \frac{g(b)-g(a)}{2} \right) f(M_g(a,b)) \right| \\ & \leq \frac{1}{2} \int_{M_g(a,b)}^b \left| k \left( g(t) - \frac{g(b)+g(a)}{2} \right) \right| \bigvee_{M_g(a,b)}^t (f) g'(t) dt \\ & \quad + \frac{1}{2} \int_a^{M_g(a,b)} \left| k \left( \frac{g(b)+g(a)}{2} - g(t) \right) \right| \bigvee_t^{M_g(a,b)} (f) g'(t) dt \\ & \leq \frac{1}{2} \mathbf{K} \left( \frac{g(b)-g(a)}{2} \right) \bigvee_b^b (f) \end{aligned}$$

and

$$\begin{aligned} (5.5) \quad & \left| P_{k,g,a+,b-}f - K \left( \frac{g(b)-g(a)}{2} \right) \frac{f(b)+f(a)}{2} \right| \\ & \leq \frac{1}{2} \int_a^{M_g(a,b)} \left| k \left( \frac{g(b)+g(a)}{2} - g(t) \right) \right| \bigvee_a^t (f) g'(t) dt \\ & \quad + \frac{1}{2} \int_{M_g(a,b)}^b \left| k \left( g(t) - \frac{g(b)+g(a)}{2} \right) \right| \bigvee_t^b (f) g'(t) dt \\ & \leq \frac{1}{2} \mathbf{K} \left( \frac{g(b)-g(a)}{2} \right) \bigvee_b^b (f). \end{aligned}$$

We have:

**Theorem 3.** *With the assumptions of Theorem 2 we have the Ostrowski type inequality*

$$\begin{aligned}
 (5.6) \quad & \left| \check{S}_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \right| \\
 & \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| \bigvee_t^x (f) g'(t) dt + \frac{1}{2} \int_x^b |k(g(b) - g(t))| \bigvee_x^t (f) g'(t) dt \\
 & \leq \frac{1}{2} \left[ \mathbf{K}(g(b) - g(x)) \bigvee_x^b (f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x (f) \right] \\
 & \leq \frac{1}{2} \begin{cases} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b (f); \\ \left[ \mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a)) \right]^{1/p} \left( (\bigvee_a^x (f))^q + (\bigvee_x^b (f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[ \mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a)) \right] \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \end{cases}
 \end{aligned}$$

and the trapezoid inequality

$$\begin{aligned}
 (5.7) \quad & \left| \check{S}_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)] \right| \\
 & \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| \bigvee_a^t (f) g'(t) dt + \frac{1}{2} \int_x^b |k(g(b) - g(t))| \bigvee_t^b (f) g'(t) dt \\
 & \leq \frac{1}{2} \left[ \mathbf{K}(g(b) - g(x)) \bigvee_x^b (f) + \mathbf{K}(g(x) - g(a)) \bigvee_a^x (f) \right] \\
 & \leq \frac{1}{2} \begin{cases} \max \{ \mathbf{K}(g(b) - g(x)), \mathbf{K}(g(x) - g(a)) \} \bigvee_a^b (f); \\ \left[ \mathbf{K}^p(g(b) - g(x)) + \mathbf{K}^p(g(x) - g(a)) \right]^{1/p} \left( (\bigvee_a^x (f))^q + (\bigvee_x^b (f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[ \mathbf{K}(g(b) - g(x)) + \mathbf{K}(g(x) - g(a)) \right] \left[ \frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \end{cases}
 \end{aligned}$$

for any  $x \in (a, b)$ .

*Proof.* Using the identity (3.7) we have

$$\begin{aligned}
& \left| \check{S}_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) + K(g(x) - g(a))] f(x) \right| \\
& \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| |f(t) - f(x)| g'(t) dt \\
& \quad + \frac{1}{2} \int_x^b |k(g(b) - g(t))| |f(t) - f(x)| g'(t) dt \\
& \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| \bigvee_t^x(f) g'(t) dt \\
& \quad + \frac{1}{2} \int_x^b |k(g(b) - g(t))| \bigvee_x^t(f) g'(t) dt \\
& \leq \frac{1}{2} \bigvee_a^x(f) \int_a^x |k(g(t) - g(a))| g'(t) dt \\
& \quad + \frac{1}{2} \bigvee_x^b(f) \int_x^b |k(g(b) - g(t))| g'(t) dt \\
& = \frac{1}{2} \left[ \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) + \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) \right],
\end{aligned}$$

for any  $x \in (a, b)$ , which proves (5.6).

By the identity (3.8) we have

$$\begin{aligned}
& \left| \check{S}_{k,g,a+,b-} f(x) - \frac{1}{2} [K(g(b) - g(x)) f(b) + K(g(x) - g(a)) f(a)] \right| \\
& \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| |f(t) - f(a)| g'(t) dt \\
& \quad + \frac{1}{2} \int_x^b |k(g(b) - g(t))| |f(b) - f(t)| g'(t) dt \\
& \leq \frac{1}{2} \int_a^x |k(g(t) - g(a))| \bigvee_a^t(f) g'(t) dt \\
& \quad + \frac{1}{2} \int_x^b |k(g(b) - g(t))| \bigvee_t^b(f) g'(t) dt \\
& \leq \frac{1}{2} \bigvee_a^x(f) \int_a^x |k(g(t) - g(a))| g'(t) dt \\
& \quad + \frac{1}{2} \bigvee_x^b(f) \int_x^b |k(g(b) - g(t))| g'(t) dt \\
& = \frac{1}{2} \left[ \mathbf{K}(g(x) - g(a)) \bigvee_a^x(f) + \mathbf{K}(g(b) - g(x)) \bigvee_x^b(f) \right]
\end{aligned}$$

for any  $x \in (a, b)$ , which proves (5.7). □

Also, we have the particular inequalities for

$$\begin{aligned}\check{P}_{k,g,a+,b-f} &:= \check{S}_{k,g,a+,b-f}(M_g(a,b)) \\ &= \frac{1}{2} \int_{M_g(a,b)}^b k(g(b) - g(t)) g'(t) f(t) dt \\ &\quad + \frac{1}{2} \int_a^{M_g(a,b)} k(g(t) - g(a)) g'(t) f(t) dt.\end{aligned}$$

**Corollary 5.** *With the assumptions of Theorem 2 we have*

$$\begin{aligned}(5.8) \quad & \left| \check{P}_{k,g,a+,b-f} - K \left( \frac{g(b) - g(a)}{2} \right) \frac{f(b) + f(a)}{2} \right| \\ & \leq \frac{1}{2} \int_a^{M_g(a,b)} |k(g(t) - g(a))| \bigvee_t^{M_g(a,b)} (f) g'(t) dt \\ & \quad + \frac{1}{2} \int_{M_g(a,b)}^b |k(g(b) - g(t))| \bigvee_{M_g(a,b)}^t (f) g'(t) dt \\ & \leq \frac{1}{2} \mathbf{K} \left( \frac{g(b) - g(a)}{2} \right) \bigvee_b^b (f)\end{aligned}$$

and

$$\begin{aligned}(5.9) \quad & \left| \check{P}_{k,g,a+,b-f} - K \left( \frac{g(b) - g(a)}{2} \right) \frac{f(b) + f(a)}{2} \right| \\ & \leq \frac{1}{2} \int_a^{M_g(a,b)} |k(g(t) - g(a))| \bigvee_a^t (f) g'(t) dt \\ & \quad + \frac{1}{2} \int_{M_g(a,b)}^b |k(g(b) - g(t))| \bigvee_t^b (f) g'(t) dt \\ & \leq \frac{1}{2} \mathbf{K} \left( \frac{g(b) - g(a)}{2} \right) \bigvee_b^b (f).\end{aligned}$$

Finally, we have the following result for the trapezoid functional

$$\begin{aligned}T_{k,g,a+,b-f} &:= \frac{1}{2} [S_{k,g,a+} f(b) + S_{k,g,b-} f(a)] \\ &= \frac{1}{2} \int_a^b [k(g(b) - g(t)) + k(g(t) - g(a))] g'(t) f(t) dt.\end{aligned}$$

**Theorem 4.** *With the assumptions of Theorem 2 we have the trapezoid type inequality*

$$(5.10) \quad \left| T_{k,g,a+,b-f} - K(g(b) - g(a)) \frac{f(a) + f(b)}{2} \right| \leq \frac{1}{2} \mathbf{K}(g(b) - g(a)) \bigvee_a^b (f).$$

*Proof.* From the identity (4.4) we have

$$\begin{aligned}
 (5.11) \quad & \left| T_{k,g,a+,b-} f - K(g(b) - g(a)) \frac{f(a) + f(b)}{2} \right| \\
 & \leq \frac{1}{2} \int_a^b |k(g(b) - g(t)) + k(g(t) - g(a))| \left| f(t) - \frac{f(a) + f(b)}{2} \right| g'(t) dt \\
 & \leq \frac{1}{2} \int_a^b [|k(g(b) - g(t))| + |k(g(t) - g(a))|] \left| f(t) - \frac{f(a) + f(b)}{2} \right| g'(t) dt \\
 & =: D.
 \end{aligned}$$

Since  $f : [a, b] \rightarrow \mathbb{C}$  is of bounded variation, then for any  $t \in [a, b]$  we have

$$\begin{aligned}
 \left| f(t) - \frac{f(a) + f(b)}{2} \right| &= \left| \frac{f(t) - f(a) + f(t) - f(b)}{2} \right| \\
 &\leq \frac{1}{2} [|f(t) - f(a)| + |f(b) - f(t)|] \leq \frac{1}{2} \bigvee_a^b(f).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 D &\leq \frac{1}{4} \bigvee_a^b(f) \int_a^b [|k(g(b) - g(t))| + |k(g(t) - g(a))|] g'(t) dt \\
 &= \frac{1}{4} \bigvee_a^b(f) [\mathbf{K}(g(b) - g(a)) + \mathbf{K}(g(b) - g(a))] = \frac{1}{2} \mathbf{K}(g(b) - g(a)) \bigvee_a^b(f),
 \end{aligned}$$

which proves the desired result (5.10). □

### 6. EXAMPLE FOR AN EXPONENTIAL KERNEL

The above inequalities may be written for all the particular fractional integrals introduced in the introduction.

If we take, for instance  $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$ , where  $\Gamma$  is the *Gamma function*, then we recapture the results for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function  $f$  with respect to another function  $g$  on  $[a, b]$  as outlined in [18].

For  $\alpha, \beta \in \mathbb{R}$  we consider the kernel  $k(t) := \exp[(\alpha + \beta i)t]$ ,  $t \in \mathbb{R}$ . We have

$$K(t) = \frac{\exp[(\alpha + \beta i)t] - 1}{(\alpha + \beta i)}, \text{ if } t \in \mathbb{R}$$

for  $\alpha, \beta \neq 0$ .

Also, we have

$$|k(s)| := |\exp[(\alpha + \beta i)s]| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$\mathbf{K}(t) = \int_0^t \exp(\alpha s) ds = \frac{\exp(\alpha t) - 1}{\alpha} \text{ if } 0 < t,$$

for  $\alpha \neq 0$ .

Let  $f : [a, b] \rightarrow \mathbb{C}$  be a function of bounded variation on  $[a, b]$  and  $g$  be a strictly increasing function on  $(a, b)$ , having a continuous derivative  $g'$  on  $(a, b)$ . We have

$$(6.1) \quad \begin{aligned} \mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(x) &= \frac{1}{2} \int_a^x \exp[(\alpha + \beta i)(g(x) - g(t))] g'(t) f(t) dt \\ &+ \frac{1}{2} \int_x^b \exp[(\alpha + \beta i)(g(t) - g(x))] g'(t) f(t) dt \end{aligned}$$

for  $x \in (a, b)$ .

If  $g = \ln h$  where  $h : [a, b] \rightarrow (0, \infty)$  is a strictly increasing function on  $(a, b)$ , having a continuous derivative  $h'$  on  $(a, b)$ , then we can consider the following operator as well

$$(6.2) \quad \begin{aligned} \kappa_{h,a+,b-}^{\alpha+\beta i} f(x) &:= \mathcal{E}_{\ln h,a+,b-}^{\alpha+\beta i} f(x) \\ &= \frac{1}{2} \left[ \int_a^x \left( \frac{h(x)}{h(t)} \right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t) dt + \int_x^b \left( \frac{h(t)}{h(x)} \right)^{\alpha+\beta i} \frac{h'(t)}{h(t)} f(t) dt \right], \end{aligned}$$

for  $x \in (a, b)$ .

By using the inequality (5.1) we have for  $x \in (a, b)$  that

$$(6.3) \quad \begin{aligned} & \left| \mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(x) \right. \\ & \left. - \frac{1}{2} \left[ \frac{\exp[(\alpha + \beta i)(g(b) - g(x))] - 1 + \exp[(\alpha + \beta i)(g(x) - g(a))] - 1}{(\alpha + \beta i)} \right] f(x) \right| \\ & \leq \frac{1}{2} \left[ \int_x^b \exp(\alpha(g(t) - g(x))) g'(t) \bigvee_x^t(f) dt + \int_a^x \exp(\alpha(g(x) - g(t))) g'(t) \bigvee_t^x(f) dt \right] \\ & \leq \frac{1}{2} \left[ \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \bigvee_x^b(f) + \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \bigvee_a^x(f) \right] \\ & \leq \frac{1}{2} \begin{cases} \max \left\{ \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha}, \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right\} \bigvee_a^b(f); \\ \left[ \left( \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \right)^p + \left( \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right)^p \right]^{1/p} \left( \bigvee_a^x(f) \right)^q + \left( \bigvee_x^b(f) \right)^q \right]^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[ \frac{\exp(\alpha(g(b) - g(x))) - 1 + \exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right] \left[ \frac{1}{2} \bigvee_a^b(f) + \frac{1}{2} \left| \bigvee_a^x(f) - \bigvee_x^b(f) \right| \right] \end{cases} \end{aligned}$$

for  $\alpha, \beta \in \mathbb{R}$  with  $\alpha \neq 0$ .

By using the inequality (5.2) we also have for  $x \in (a, b)$  that

$$\begin{aligned}
 (6.4) \quad & \left| \mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(x) \right. \\
 & \left. - \frac{1}{2} \left[ \frac{(\exp[(\alpha + \beta i)(g(b) - g(x))] - 1) f(b) + (\exp[(\alpha + \beta i)(g(x) - g(a))] - 1) f(a)}{(\alpha + \beta i)} \right] \right| \\
 & \leq \frac{1}{2} \left[ \int_a^x \exp(\alpha(g(t) - g(x))) g'(t) \mathcal{V}_a^t(f) dt + \int_x^b \exp(\alpha(g(x) - g(t))) g'(t) \mathcal{V}_t^b(f) dt \right] \\
 & \leq \frac{1}{2} \left[ \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \mathcal{V}_x^b(f) + \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \mathcal{V}_a^x(f) \right] \\
 & \leq \frac{1}{2} \begin{cases} \max \left\{ \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha}, \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right\} \mathcal{V}_a^b(f); \\ \left[ \left( \frac{\exp(\alpha(g(b) - g(x))) - 1}{\alpha} \right)^p + \left( \frac{\exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right)^p \right]^{1/p} \\ \times \left( (\mathcal{V}_a^x(f))^q + (\mathcal{V}_x^b(f))^q \right)^{1/q} \\ \text{with } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[ \frac{\exp(\alpha(g(b) - g(x))) - 1 + \exp(\alpha(g(x) - g(a))) - 1}{\alpha} \right] \\ \times \left[ \frac{1}{2} \mathcal{V}_a^b(f) + \frac{1}{2} \left| \mathcal{V}_a^x(f) - \mathcal{V}_x^b(f) \right| \right] \end{cases}
 \end{aligned}$$

for  $\alpha, \beta \in \mathbb{R}$  with  $\alpha \neq 0$ .

If we denote

$$\begin{aligned}
 \bar{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f & := \mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(M_g(a, b)) \\
 & = \frac{1}{2} \int_a^x \exp \left[ (\alpha + \beta i) \left( \frac{g(b) + g(a)}{2} - g(t) \right) \right] g'(t) f(t) dt \\
 & \quad + \frac{1}{2} \int_x^b \exp \left[ (\alpha + \beta i) \left( g(t) - \frac{g(b) + g(a)}{2} \right) \right] g'(t) f(t) dt
 \end{aligned}$$

then by (5.4) and (5.5) we have the simpler results

$$\begin{aligned}
 (6.5) \quad & \left| \bar{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f - \frac{\exp \left[ (\alpha + \beta i) \frac{g(b) - g(a)}{2} \right] - 1}{(\alpha + \beta i)} f(M_g(a, b)) \right| \\
 & \leq \frac{1}{2} \int_{M_g(a,b)}^b \exp \left( \alpha \left( g(t) - \frac{g(b) + g(a)}{2} \right) \right) g'(t) \mathcal{V}_{M_g(a,b)}^t(f) dt \\
 & \quad + \frac{1}{2} \int_a^{M_g(a,b)} \exp \left( \alpha \left( \frac{g(b) + g(a)}{2} - g(t) \right) \right) g'(t) \mathcal{V}_t^{M_g(a,b)}(f) dt \\
 & \leq \frac{1}{2} \frac{\exp \left( \alpha \left( \frac{g(b) - g(a)}{2} \right) \right) - 1}{\alpha} \mathcal{V}_b^b(f)
 \end{aligned}$$

and

$$\begin{aligned}
 (6.6) \quad & \left| \bar{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f - \frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right] - 1}{(\alpha+\beta i)} \frac{f(b) + f(a)}{2} \right| \\
 & \leq \frac{1}{2} \int_a^{M_g(a,b)} \exp\left(\alpha\left(g(t) - \frac{g(b)+g(a)}{2}\right)\right) g'(t) \bigvee_a^t(f) dt \\
 & \quad + \frac{1}{2} \int_{M_g(a,b)}^b \exp\left(\alpha\left(\frac{g(b)+g(a)}{2} - g(t)\right)\right) g'(t) \bigvee_t^b(f) dt \\
 & \leq \frac{1}{2} \frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right) - 1}{\alpha} \bigvee_b^b(f).
 \end{aligned}$$

In particular, if we take in (6.5) and (6.6)  $g = \ln t$ ,  $t \in [a, b] \subset (0, \infty)$ , then by using the notation  $G(\gamma, \delta) := \sqrt{\gamma\delta}$  for the *geometric mean* of the positive real numbers  $\gamma, \delta > 0$  we have

$$\begin{aligned}
 (6.7) \quad & \left| \bar{\kappa}_{a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{b}{a}\right)^{\alpha+\beta i} - 1}{(\alpha+\beta i)} f(G(a, b)) \right| \\
 & \leq \frac{1}{2} \int_{G(a,b)}^b \left(\frac{t}{G(a,b)}\right)^\alpha \frac{1}{t} \bigvee_{G(a,b)}^t(f) dt \\
 & \quad + \frac{1}{2} \int_a^{G(a,b)} \left(\frac{G(a,b)}{t}\right)^\alpha \frac{1}{t} \bigvee_t^{G(a,b)}(f) dt \\
 & \leq \frac{1}{2} \frac{\left(\frac{b}{a}\right)^\alpha - 1}{\alpha} \bigvee_b^b(f)
 \end{aligned}$$

and

$$\begin{aligned}
 (6.8) \quad & \left| \bar{\kappa}_{a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{b}{a}\right)^{\alpha+\beta i} - 1}{(\alpha+\beta i)} \frac{f(b) + f(a)}{2} \right| \\
 & \leq \frac{1}{2} \int_{G(a,b)}^b \left(\frac{G(a,b)}{t}\right)^\alpha \frac{1}{t} \bigvee_t^b(f) dt \\
 & \quad + \frac{1}{2} \int_a^{G(a,b)} \left(\frac{t}{G(a,b)}\right)^\alpha \frac{1}{t} \bigvee_a^t(f) dt \\
 & \leq \frac{1}{2} \frac{\left(\frac{b}{a}\right)^\alpha - 1}{\alpha} \bigvee_b^b(f),
 \end{aligned}$$

where

$$\bar{\kappa}_{a+,b-}^{\alpha+\beta i} f := \frac{1}{2} \int_{G(a,b)}^b \left(\frac{t}{G(a,b)}\right)^{\alpha+\beta i} \frac{1}{t} f(t) dt + \frac{1}{2} \int_a^{G(a,b)} \left(\frac{G(a,b)}{t}\right)^{\alpha+\beta i} \frac{1}{t} f(t) dt.$$



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