SERIES REPRESENTATION OF POWER FUNCTION

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Abstract. In this paper we discuss a problem of generalization of binomial distributed triangle, that is sequence A287326 in OEIS. The main property of A287326 that it returns a perfect cube \( n \) as sum of \( n \)-th row terms over \( k \), \( 0 \leq k \leq n-1 \) or \( 1 \leq k \leq n \), by means of its symmetry. In this paper we have derived a similar triangles in order to receive powers \( m = 5 \), \( 7 \) as row items sum and generalized obtained results in order to receive every odd-powered monomial \( n^{2m+1} \), \( m \geq 0 \) as sum of row terms of corresponding triangle.

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1. Structure of the manuscript

The problem of finding expansions of monomials, binomials, trinomials, etc. is classical and a lot of theorems have been found, the most prominent examples are Binomial Theorem [2], Multinomial theorem, Wozpitsky Identity [30], Stirling numbers of second kind identity, etc. In this paper we try to solve the classical problem of finding expansions of monomials. We start from binomial distributed triangle A287326 [11] in OEIS. The main property of A287326 that it returns a perfect cube \( n \) as \( n \)-th row sum, starting from \( 0, \ldots, n-1 \) or from \( 1, \ldots, n \) by means of its symmetry. Therefore, the following question stated:

- Can we find similar to A287326 triangles in order to receive monomial \( n^t \), \( t > 3 \) as sum of row terms? In other words, can A287326 be generalized in order to receive monomial \( n^t \), \( t > 3 \) as sum of row terms?

Date: May 25, 2018.
Key words and phrases. Power function, Monomial, Binomial coefficient, Binomial Theorem, Finite difference, Perfect cube, Pascal’s triangle, Series representation, Faulhaber’s formula.
Finding an analogs for $t = 5, 7$ in section 3, we answer to above questions positively. Could this process be continued for each $t = 1, 3, 5, 7...$ similarly? Positive answer to this question is given by theorem (3.29).

2. Introduction

Let describe the derivation of the sequence A287326 in OEIS. Sequence A287326 returns the perfect cube $n$ as row sum over $k$, $0 \leq k \leq n - 1$, as well as sum over $1 \leq k \leq n$, by means of its symmetry. First, consider a difference table of perfect cubes ([4], eq. 7)

\[
\begin{array}{cccccc}
 n & \Delta^0(n^3) & \Delta^1(n^3) & \Delta^2(n^3) & \Delta^3(n^3) \\
 0 & 0 & 1 & 6 & 6 \\
 1 & 1 & 7 & 12 & 6 \\
 2 & 8 & 19 & 18 & 6 \\
 3 & 27 & 37 & 24 & 6 \\
 4 & 64 & 61 & 30 & 6 \\
 5 & 125 & 91 & 36 & 6 \\
 6 & 216 & 127 & 42 & 6 \\
 7 & 343 & 169 & 48 & 6 \\
 8 & 512 & 217 & 54 & 6 \\
 9 & 729 & 271 & & \\
 10 & 1000 & & & \\
\end{array}
\]

Table 1: Difference table of perfect cubes $n$, $0 \leq n \leq 10$ up to $3^{\text{rd}}$ order.

Reviewing above table, we have noticed that

\[
\begin{align*}
\Delta(0^3) &= 1 + 6 \cdot 0 = 6 \binom{1}{2} + \binom{1}{0} \\
\Delta(1^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 = 6 \binom{2}{2} + \binom{2}{0} \\
\Delta(2^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 = 6 \binom{3}{2} + \binom{3}{0} \\
\Delta(3^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + 6 \cdot 3 = 6 \binom{4}{2} + \binom{4}{0} \\
\vdots \\
\Delta(n^3) &= 1 + 6 \cdot 0 + 6 \cdot 1 + 6 \cdot 2 + \cdots + 6 \cdot n = 6(n+1) + \binom{n+1}{0}
\end{align*}
\]

Above difference identity is closely related to Faulhaber’s sum of cubes, where $n^3 = 6\binom{n+1}{3} + \binom{n+1}{1}$, see ([21], p. 9). Note that $\Delta^2(n^3)$ could be found similarly using above identity $\Delta^2(n^3) = 6\binom{n+2}{5} + \binom{n+1}{1}$.

Property 2.3. (Generalized finite difference of power using Faulhaber’s formula). Consider the identities, ([21], p. 9).

\[
\begin{align*}
n^1 &= \binom{n}{1} \\
n^3 &= 6\binom{n+1}{3} + \binom{n}{1} \\
n^5 &= 120\binom{n+2}{5} + 30\binom{n+1}{3} + \binom{n}{1}
\end{align*}
\]
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We can find the first order finite difference of odd power as decreasing the variable of corresponding binomial coefficients by 1, for example

\[
\begin{align*}
\Delta n^1 &= \binom{n}{0} \\
\Delta n^3 &= 6 \binom{n+1}{2} + \binom{n}{0} \\
\Delta n^5 &= 120 \binom{n+2}{4} + 30 \binom{n+1}{2} + \binom{n}{0}
\end{align*}
\]

Continue similarly, we can express each difference of order \( t \geq 1 \). The coefficients \{1, 6, 120, 30, 1\} in above identities are generated by

\[V_{n,k} = \frac{1}{r} \sum_{j=0}^{r} (-1)^j \binom{2r}{j} (r-j)^2n,\]

where \( r = n-k+1 \), this formula was provided by Peter Luschny in [27]. Therefore, for every odd \( t > 0 \) and \( m \geq 0 \), we have

\[\Delta^t n^{2m+1} = \sum_{0 \leq k \leq m \atop l \text{ is even}} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 0 \text{ and odd}\]

Let be \( m \geq 0, t > 1 \) and even, then

\[\Delta^t n^{2m+1} = \sum_{0 \leq k \leq m \atop l \geq 2(m-k)+1-t} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 1 \text{ and even}\]

Let show finite differences, set \( m \geq 1, t > 1 \), then we have finite difference identity

\[\Delta^t n^{2m} = \sum_{0 \leq k \leq m \atop l \leq 2(m-k)+1-t} \frac{1}{r} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 0 \text{ and odd}\]

And

\[\Delta^t n^{2m} = \sum_{0 \leq k \leq m \atop l \leq 2(m-k)+1-t} \frac{1}{r} V_{m,k} \binom{n+m-k}{l}, \text{ if } t > 1 \text{ and even}\]

By the identity \( \sum_{k=0}^{n-1} \Delta n^m = n^m \), we have right to represent perfect cube \( n \) as

\[n^3 = 6 \binom{1}{2} + \binom{1}{0} + 6 \binom{2}{2} + \binom{2}{0} + 6 \binom{3}{2} + \binom{3}{0} + \cdots + 6^{(n+1)} + \binom{n+1}{0}\]

Let rewrite it again and display every binomial coefficient as summation \( \binom{n+1}{2} = 1 + 2 + \cdots + n \), then

\[n^3 = (1 + 6 \cdot 0) + (1 + 6 \cdot 0 + 6 \cdot 1) + \cdots + (1 + 6 \cdot 0 + \cdots + 6 \cdot (n-1))\]

Particularizing above expression, we get

\[n^3 = n + (n-0) \cdot 6 \cdot 0 + (n-1) \cdot 6 \cdot 1 + \cdots + (n - (n-1)) \cdot 6 \cdot (n-1)\]

Provided that \( n \) is natural. Now we apply a compact sigma notation on (2.6), thus

\[n^3 = n + \sum_{1 \leq k \leq n} 6k(n-k)\]
As sum $\sum_{1 \leq k \leq n} 6k(n - k)$ consists of $n$ terms, we have right to move $n$ in (2.7) under sigma notation, we get

\[ n^3 = \sum_{1 \leq k \leq n} 6k(n - k) + 1 \]

(2.8)

**Property 2.9.** (Proof of symmetry). Let be a sets $A(n) := \{1, 2, \ldots, n\}$, $B(n) := \{0, 1, \ldots, n\}$, $C(n) := \{0, 1, \ldots, n - 1\}$, let be expression (2.8) defined as

\[ M(n, C(n)) \overset{\text{def}}{=} \sum_{k \in C(n)} 6k(n - k) + 1 \]

where $x$ is natural-valued variable and $C(n)$ is iteration set of (2.8), then we have equality

\[ (2.10) \quad M(n, A(n)) = M(n, C(n)) \]

Let review and define expression (2.6) as

\[ U(n, C(n)) \overset{\text{def}}{=} n + 6 \cdot \sum_{k \in C(n)} k(n - k) \]

then

\[ (2.11) \quad U(n, A(n)) = U(n, B(n)) = U(n, C(n)) \]

Other words, changing of iteration sets of (2.6) and (2.8) by $A(n)$, $B(n)$, $C(n)$ and $A(n)$, $C(n)$, respectively, doesn't change resulting value for each natural $x$.

**Proof.** Let be a plot $y(n, k) = 6k(n - k) + 1$, $k \in \mathbb{R}$, $0 \leq k \leq 10$, given $n = 10$

![Figure 2. Plot of $6k(n - k) + 1$, $k \in \mathbb{R}$, $0 \leq k \leq n$, where $n = 10$.](image)

Obviously, being a parabolic function, it’s symmetrical over $\frac{n}{2}$, hence equivalent $M(n, A(n)) = M(n, C(n))$ follows. Reviewing (2.6) and denote $u(n, k) = kn - k^2$, we can conclude, that $u(n, 0) = u(n, n) = 0$, then equality of $U(n, A(n)) = U(n, B(n)) = U(n, C(n))$ immediately follows. This completes the proof. $\square$

Review above property (2.9). Let be an example of triangle built using
**Definition 2.12.** For every \( n \geq 0 \)

\[
L_1(n, k) \overset{\text{def}}{=} 6k(n-k) + 1, \quad 0 \leq k \leq n
\]

over \( n \) from 0 to \( n = 4 \), where \( n \) denotes corresponding row and \( k \) shows the item of row \( n \).

\[
\begin{align*}
\text{Row 0:} & \quad 1 \\
\text{Row 1:} & \quad 1 \quad 1 \\
\text{Row 2:} & \quad 1 \quad 7 \quad 1 \\
\text{Row 3:} & \quad 1 \quad 13 \quad 13 \quad 1 \\
\text{Row 4:} & \quad 1 \quad 19 \quad 25 \quad 19 \quad 1
\end{align*}
\]

Figure 3. Triangle generated by \( L_1(n, k) \) from 0 to \( n = 4 \), sequence A287326 in OEIS [11].

Note that \( n \)-th row sum of Triangle (2.14) over \( 0 \leq k \leq n - 1 \) returns perfect cube \( n \). We can see that each row with respect to variable \( n = 0, 1, 2, 3, 4, \ldots \), has Binomial distribution of row terms. One could compare Triangle (2.14) with Pascal’s triangle [1], [12]

\[
\begin{align*}
\text{Row 0:} & \quad 1 \\
\text{Row 1:} & \quad 1 \quad 1 \\
\text{Row 2:} & \quad 1 \quad 2 \quad 1 \\
\text{Row 3:} & \quad 1 \quad 3 \quad 3 \quad 1 \\
\text{Row 4:} & \quad 1 \quad 4 \quad 6 \quad 4 \quad 1
\end{align*}
\]

Figure 4. Pascal’s triangle read by rows, sequence A007318 in OEIS [1].

Let us approach to show a few properties of triangle (2.14) and \( L_1(n, k) \).

**Properties 2.15. Properties of triangle (2.14).**

1. Summation of items \( L_1(n, k) \) of \( n \)-th row of triangle (2.14) over \( k \) from 0 to \( n - 1 \) returns perfect cube \( n \) as follows

\[
\sum_{1 \leq k \leq n} L_1(n, k) = n^3
\]

2. Relation between \( \alpha_{0,n} \) and \( \alpha_{1,n} \)

\[
\alpha_{0,n+1} = \alpha_{1,n}, \quad n \geq 1
\]

3. First item of each row’s number corresponding to central polygonal numbers sequence \( a(n) = \frac{n^2 + n + 2}{2} \) (sequence A000124 in OEIS [13]) returns finite difference of consequent perfect cubes. For example, let be a \( k \)-th row of triangle (2.14), such that \( k = \frac{n^2 + n + 2}{2} \), \( n = 0, 1, 2, \ldots \), then item

\[
L_1 \left( \frac{n^2 + n + 2}{2}, 1 \right) = (n+1)^3 - n^3
\]

4. Items of (2.14) have Binomial distribution over rows.
Linear recurrence, for every \( k \) and \( n > 0 \)

\[
2L_1(n, k) = L_1(n+1, k) + L_1(n-1, k)
\]

This linear recurrence is direct result of second order binomial transform of \( L_1(n, k) \) over \( n \).

Linear recurrence, for each \( n > k \)

\[
2L_1(n, k) = L_1(2n-k, k) + L_1(2n-k, 0)
\]

From (1.24) for every \( n \geq 0 \) follows

\[
\sum_{1 \leq k \leq n} L_1(n, k) = \sum_{1 \leq k \leq n} L_1 \left( \frac{n^2 + n + 2}{2}, 1 \right) = n^3
\]

Triangle (2.14) is symmetric, i.e

\[
L_1(n, k) = L_1(n, n-k)
\]

Property 2.22. (Generalized binomial series by means of identity (2.16). Let review identity (2.16) in sense of

\[
\sum_{1 \leq k \leq t} L_1(n, k) = \alpha_{0,t} n - \beta_{0,t}
\]

By property (2.9) we rewrite above expression as

\[
\sum_{0 \leq k \leq t} L_1(n, k) = \alpha_{1,t} n - \beta_{1,t}
\]

where subscripts 0, t and 1, t denote the ranges of summation, respectively. Running over \( t > 0 \) above identities produce sets of coefficients \( \{\alpha_{0,t}\}, \{\beta_{0,t}\}, \{\alpha_{1,t}\} \) and \( \{\beta_{1,t}\} \). Below table shows initial terms of these sequences

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \alpha_{0,t} )</th>
<th>( \beta_{0,t} )</th>
<th>( \alpha_{1,t} )</th>
<th>( \beta_{1,t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>4</td>
<td>18</td>
<td>28</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>27</td>
<td>36</td>
<td>81</td>
</tr>
<tr>
<td>4</td>
<td>36</td>
<td>80</td>
<td>60</td>
<td>176</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>175</td>
<td>90</td>
<td>325</td>
</tr>
<tr>
<td>6</td>
<td>90</td>
<td>324</td>
<td>126</td>
<td>540</td>
</tr>
<tr>
<td>7</td>
<td>126</td>
<td>539</td>
<td>168</td>
<td>833</td>
</tr>
<tr>
<td>8</td>
<td>168</td>
<td>832</td>
<td>216</td>
<td>1216</td>
</tr>
<tr>
<td>9</td>
<td>216</td>
<td>1215</td>
<td>270</td>
<td>1701</td>
</tr>
<tr>
<td>10</td>
<td>270</td>
<td>1700</td>
<td>330</td>
<td>2300</td>
</tr>
</tbody>
</table>

Table 5. Array of coefficients \( \alpha_{0,t,n} \), \( \beta_{0,t,n} \) given \( n = 1, \ldots, 10 \).

Therefore, perfect cube \( n \) could be rewritten as binomials of the form

\[
n^3 = \begin{cases} 
\alpha_{0,n-1} n - \beta_{0,n-1}, & \text{if } t = n - 1; \\
\alpha_{1,n} n - \beta_{1,n}, & \text{if } t = n 
\end{cases}
\]

By the main power property, for every \( m \in \mathbb{N} \)

\[
n^m = \begin{cases} 
\alpha_{0,n-1} n^{m-2} - \beta_{0,n-1} n^{m-3}, & \text{if } t = n - 1; \\
\alpha_{1,n} n^{m-2} - \beta_{1,n} n^{m-3}, & \text{if } t = n 
\end{cases}
\]
We denote above equation as

\[ n^m = \alpha_{0,1,n-1,n}n^{m-2} - \beta_{0,1,n-1,n}n^{m-3} \]

Let rewrite the right part of above expression regarding to itself as recursion

\[ n^m = \alpha_{0,1,n-1,n}(\alpha_{0,1,n-1,n}n^{m-4} - \beta_{0,1,n-1,n}n^{m-5}) \]
\[ - \beta_{0,1,n-1,n}(\alpha_{0,1,n-1,n}n^{m-5} - \beta_{0,1,n-1,n}n^{m-6}) \]
\[ = \alpha_{0,1,n-1,n}^2n^{m-4} - 2\alpha_{0,1,n-1,n}\beta_{0,1,n-1,n}n^{m-5} + \beta_{0,1,n-1,n}^2n^{m-6} \]

We can observe corresponding binomial coefficient present before each \( \alpha_{0,1,n-1,n} \) times \( \beta_{0,1,n-1,n} \). Continuous \( j \)-times recursion gives

\[ n^m = \sum_{k \geq 0} (-1)^k \binom{j}{k} \alpha_{0,1,n-1,n}^{j-k} \beta_{0,1,n-1,n}^{m-2j-k}, \quad j \geq 0 \]

Sequences \( \alpha_{1,t}, \alpha_{0,t>1} \) are generated by \( 3n^2 + 3n \), sequence [A028896] in OEIS, [23]. Sequence \( \beta_{1,t} \) is generated by \( 2n^3 + 3n^2 \), sequence [A275709] in OEIS, [20].

In this section we have reached binomial distributed triangle (2.14), such that perfect cube \( n \) could be found as sum of \( n \)-th row terms of (2.14). Therefore, the follow question is stated

**Question 2.23.** Can we find similar to [A287326] triangles in order to receive monomial \( n^t \), \( t > 3 \) as sum of row terms? Is it exist \( L_v(n,k) \), \( v \neq 1 \), such that

\[ n^t \equiv \sum_{1 \leq k \leq n} L_v(n,k), \quad v \neq t \]

3. **Generalization of sequence A287326**

In order to get analogs of Triangle (2.14) one should solve a system of equations, where unknowns are coefficients of polynomial and variable of polynomial is \( k(n-k) \). Let show a triangle generated by \( L_2(n,k) \), such that sum of \( n \)-th row terms returns \( n^5 \).

**Example 3.1.** We suspect that \( n \)-th row of triangle is generated by

\[ L_2(n,k) = A_{2,2}(n-k)^2k^2 + A_{2,1}(n-k)k + A_{2,0} \]

where \( A_{2,2}, A_{2,1}, A_{2,0} \) are unknown coefficients and \( n \geq 0, \quad 0 \leq k \leq n \). Assume that for every \( n \geq 0, \quad m \geq 0 \) holds

\[ \sum_{1 \leq k \leq n} L_2(n,k) \equiv n^5 \]
In more explicit view

\[(3.4) \quad A_{2.2} \sum_{1 \leq k \leq n} k^2(n - k)^2 + A_{2.1} \sum_{1 \leq k \leq n} k(n - k) + A_{2.0}n\]

\[= A_{2.2} \sum_{1 \leq k \leq n} k^2(n^2 - 2nk + k^2) + A_{2.1} \sum_{1 \leq k \leq n} kn - k^2 + A_{2.0}n\]

\[= A_{2.2} \sum_{1 \leq k \leq n} k^2n^2 - 2nk^3 + k^4 + A_{2.1} \sum_{1 \leq k \leq n} kn - k^2 + A_{2.0}n\]

\[= A_{2.2}n^2 \sum_{1 \leq k \leq n} k^2 - 2A_{2.2}n \sum_{1 \leq k \leq n} k^3 + A_{2.2} \sum_{1 \leq k \leq n} k^4 + A_{2.1}n \sum_{1 \leq k \leq n} k\]

\[- A_{2.1} \sum_{1 \leq k \leq n} k^2 + A_{2.0}n\]

Thus, we have received expression containing sums of powers of successive natural numbers, where powers are \(\{1, 2, 3, 4\}\). By the Faulhaber’s formula [7], the following identities hold

\[(3.5) \quad \sum_{1 \leq k \leq n} k = \frac{n^2 + n}{2},\]

\[(3.6) \quad \sum_{1 \leq k \leq n} k^2 = \frac{2n^3 + 3n^2 + n}{6},\]

\[(3.7) \quad \sum_{1 \leq k \leq n} k^3 = \frac{n^4 + 2n^3 + n^2}{4},\]

\[(3.8) \quad \sum_{1 \leq k \leq n} k^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}.

Now we substitute above identities to (3.4), respectively, we get

\[A_{2.2}n^2 \frac{2n^3 + 3n^2 + n}{6} - 2A_{2.2}n \frac{n^4 + 2n^3 + n^2}{4} + A_{2.2} \frac{6n^5 + 15n^4 + 10n^3 - n}{30}\]

\[+ A_{2.1}n \frac{n^2 + n}{2} - A_{2.1} \frac{2n^3 + 3n^2 + n}{6} + A_{2.0}n\]

Particularizing the elements of above expression and moving them under the common divisor, we get

\[(3.9) \quad \frac{A_{2.2}n^5 - A_{2.2}n + 30A_{2.0}}{30} + A_{2.1} \left(\frac{n^3 - n}{6}\right)\]

We have to remember that expression (3.9) is the left side of the input equation (2.2). Therefore,

\[(3.10) \quad \frac{A_{2.2}n^5 - A_{2.2}n + 30A_{2.0}}{30} + A_{2.1} \left(\frac{n^3 - n}{6}\right) = n^5, \quad n \geq 0\]

In order to satisfy (3.10) for each natural \(n\), coefficients \(A_{2.0}, A_{2.1}, A_{2.2}\) should be a solutions of following system of equations

\[
\begin{align*}
&\frac{1}{30}A_{2.2} = 1 \\
&A_{2.1} = 1 \\
&30A_{2.0} - A_{2.2} = 0
\end{align*}
\]
The only solution of above system is $A_{2,2} = 30$, $A_{2,1} = 0$, $A_{2,0} = 1$. Hereby, $L_2(n,k)$ takes the form

$$L_2(n,k) = 30k^2(n - k)^2 + 1$$

And for each natural $n$ holds

$$\sum_{1 \leq k \leq n} 30k^2(n - k)^2 + 1 = n^5$$

Let show initial rows of triangle built by $L_2(n,k)$

\[
\begin{array}{cccc}
1 \\
& 1 \\
1 & 31 & 1 \\
1 & 271 & 481 & 271 & 1 \\
1 & 480 & 1081 & 1081 & 481 & 1 \\
& & & & & \\
\end{array}
\]

Figure 6. Triangle generated by $L_2(n,k)$, $0 \leq k \leq n$, sequence [A300656] in OEIS.

Similarly, finding the coefficients $A_{3,0}, A_{3,1}, A_{3,2}, A_{3,3}$ in

$$L_3(n,k) = A_{3,3}k^3(n - k)^3 + A_{3,2}k^2(n - k)^2 + A_{3,1}k(n - k) + A_{3,0}$$

we get $A_{3,3} = 140$, $A_{3,2} = -14$, $A_{3,1} = 0$, $A_{3,0} = 1$, therefore, for each $n \geq 0$ holds

$$\sum_{1 \leq k \leq n} 140k^3(n - k)^3 - 14k^2(n - k)^2 + 1 = n^7$$

Below we show a few initial rows of triangle built by $L_3(n,k)$

\[
\begin{array}{cccc}
1 \\
& 1 \\
1 & 127 & 1 \\
1 & 3793 & 8905 & 3793 & 1 \\
1 & 8905 & 30157 & 30157 & 8905 & 1 \\
& & & & & \\
\end{array}
\]

Figure 7. Triangle generated by $L_3(n,k)$, $0 \leq k \leq n$, sequence [A300785] in OEIS.

We assume now that generalization of A287326 holds for odd powers only. To generalize our sequences A287326, A300656, A300785 for every odd power $2m+1$, $m = 0, 1, 2, \ldots$ we have to review the generating functions of corresponding sequences, that is

$$\sum_{1 \leq k \leq n} \sum_{0 \leq j \leq m} A_{m,j}k^j(n - k)^j = n^{2m+1}, \ m = 1, 2, 3$$
Where $A_{m,j}$ are unknown coefficients of polynomials (2.1) and (2.13).

**Definition 3.18.** Let define the part of (2.1) as
\[
\sum_{0 \leq j \leq m} A_{m,j}k^j(n-k)^j \overset{\text{def}}{=} L_m(n,k) \overset{\text{def}}{=} \sum_{0 \leq j \leq m} A_{m,j}T^j(n,k)
\]
where
\[
T(n,k) \overset{\text{def}}{=} k(n-k).
\]
Note that $L_m(n,k)$ is generalization of definitions (2.12) for $m = 1$ and (3.11) for $m = 2$, respectively.

For example, generating functions of sequences A287326, A300656, A300785 are
\[
\begin{align*}
L_1(n,k) &= 1 + 6k(n-k), \quad \text{for A287326} \\
L_2(n,k) &= 1 - 6k(n-k) + 30k^2(n-k)^2, \quad \text{for A300656} \\
L_3(n,k) &= 1 - 14k(n-k) - 6k^2(n-k)^2 + 140k^3(n-k)^3, \quad \text{for A300785}
\end{align*}
\]
Where coefficients $A_{m,j}$, for $m = 1, 2, 3$ are $\{A_{1,j}\}_{j=0}^1 = \{1, 6\}$, $\{A_{2,j}\}_{j=0}^2 = \{1, 0, 30\}$, $\{A_{3,j}\}_{j=0}^3 = \{1, -14, 0, 140\}$ in definitions of generating functions of A287326, A300656, A300785, respectively. To generalize above result in order to receive monomial $n^{2m+1}$ as $\sum_{i=0}^{\infty} L_m(n,k) = n^{2m+1}$, $m = 0, 1, 2, \ldots$ one has to solve the system of equations. Complete set of coefficients $\{A_{m,0}, \ldots, A_{m,m}\}$ such that $\sum_{1 \leq k \leq n} L_m(n,k) = n^{2m+1}$, $m \geq 0$ holds can be found solving follow system of equations
\[
\begin{align*}
L_m(1,0) &= 1^{2m+1} \\
L_m(2,0) + L_m(2,1) &= 2^{2m+1} \\
L_m(3,0) + L_m(3,1) + L_m(3,2) &= 3^{2m+1} \\
& \vdots \\
L_m(r,0) + L_m(r,1) + \cdots + L_m(r,r-1) &= r^{2m+1}, \quad r \geq m
\end{align*}
\]
List of solutions of system (2.4) is split and assigned to OEIS under the numbers A302971 (numerators of $A_{m,j}$) and A304042 (denominators of $A_{m,j}$). To reach recurrent formula of $A_{m,j}$, first let fix the unused values $A_{m,j} = 0$, for $j < 0$ or $j > m$, so we don’t need to care about the summation range for $j$, then by expanding $(n-k)^j$ and using Faulhaber’s formula [7], we get
\[
\begin{align*}
\sum_{k=0}^{n-1} (n-k)^j k^j &= \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} \binom{j}{i} n^{i-j} (-1)^j k^{i+j} \\
&= \sum_{i,t} \binom{j}{i} n^{i-j} \frac{(-1)^i}{i+j+1} \left[ \sum_{t} \binom{i+j+1}{t} B_t n^{i+j+1-t} - B_{i+j+1} \right] \\
&= \sum_{i,t} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} B_t n^{2j+1-t} - \sum_{i,t} \binom{j}{i} \frac{(-1)^i}{i+j+1} B_{i+j+1} n^{j-i}
\end{align*}
\]
\[\text{(*)} \quad \text{(*)}\]

\[\text{1One can produce a list of solutions of system (2.4) up to } t = 11 \text{ using Mathematica code solutions_system_2_4.txt [24].}\]
where \( B_t \) are Bernoulli numbers [13]. Now, we notice that

\[
\sum_{i,t}^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} = \begin{cases} 
\frac{1}{(2j+1)(2j)}, & \text{if } t = 0; \\
\frac{(-1)^j}{t} \binom{j}{t+j}, & \text{if } t > 0
\end{cases}
\]

In particular, the last sum is zero for \( 0 < t \leq j \). Now we substitute the terms from right part of (3.25) into (\(*\)), thus

\[
\sum_{i,t}^{\infty} \binom{j}{i} \frac{(-1)^i}{i+j+1} \binom{i+j+1}{t} B_t n^{2j+1-t} = \frac{1}{(2j+1)(2j)} + \sum_{t>0} \frac{(-1)^j}{t} \binom{j}{2j-t+1} B_t n^{2j+1-t}
\]

Therefore, (3.24) takes the form

\[
(\ast) \sum_{k=0}^{n-1} (n-k)^j k^j = \frac{1}{(2j+1)(2j)} n^{2j+1} + \sum_{t>0} \frac{(-1)^j}{t} \binom{j}{2j-t+1} B_t n^{2j+1-t} - \sum_{t} \binom{j}{i} \frac{(-1)^i}{i+j+1} B_{i+j+1} n^{j-i}
\]

Now, we keep our attention to (\ast\) and we have to remember that if the sum over some variable \( i \) contains \( \binom{j}{i} \), then instead of limiting its summation range to \( i = 0, \ldots, j \), we can let \( i = -\infty, \ldots, +\infty \) since \( \binom{j}{i} = 0 \) for \( i \) outside the range \( i = 0, \ldots, j \) (i.e., when \( i < 0 \) or \( i > j \)). It’s much easier to review such sum as summing from \( -\infty \) to \( +\infty \) (unless specified otherwise), where only a finite number of terms are nonzero, this fact is discussed in [28] as well. To combine or cancel identical terms across the two sums in (\ast\) more easily, we introduce \( \ell = 2j+1-t \) to (\ast\) and \( \ell = j-i \) to (\circ\), we get

\[
(\ast) \sum_{k=0}^{n-1} (n-k)^j k^j = \frac{1}{(2j+1)(2j)} n^{2j+1} + \sum_{\ell=-\infty}^{\infty} \frac{(-1)^j}{2j+1-\ell} \binom{j}{\ell} B_{2j+1-\ell} n^{2j+1-\ell}
\]

Now, using the definition of \( A_{m,j} \), we obtain the following identity for polynomials in \( n \)

\[
\sum_{j} A_{m,j} \frac{1}{(2j+1)(2j)} n^{2j+1} + 2 \sum_{j, \text{odd } \ell} A_{m,j} \frac{(-1)^j}{2j+1-\ell} B_{2j+1-\ell} n^{2j+1-\ell} \\
= n^{2m+1}.
\]

Taking the coefficient of \( n^{2m+1} \) in above expression, we get \( A_{m,m} = (2m+1)(2m) \), and taking the coefficient of \( x^{2d+1} \) for an integer \( d \) in the range \( m/2 \leq d < m \) we
get \(A_{m,d} = 0\). Taking the coefficient of \(n^{2d+1}\) in (2.8) for \(m/4 \leq d < m/2\), we get
\[
A_{m,d} \frac{1}{(2d+1)!} \frac{\binom{2d}{d}}{d} + 2(2m+1)\binom{2m}{m} \frac{(-1)^m}{2m-2d} B_{2m-2d} = 0,
\]
i.e.
\[
A_{m,d} = (-1)^{m-1} \frac{(2m+1)!}{d!d!m!} \frac{1}{(m-2d-1)!m-d} B_{2m-2d}.
\]
Continue similarly, we can express \(A_{m,j}\) for each integer \(j\) in range \(m/2^{s+1} \leq j < m/2^s\) (iterating consecutively \(s = 1, 2, \ldots\)) via previously determined values of \(A_{m,d}, d < j\) as follows
\[
A_{m,j} = (2j + 1) \left(\frac{2j}{j}\right) \sum_{d=2j+1}^{m} A_{m,d} \left(\frac{d}{2j+1}\right) \frac{(-1)^{d-1}}{d-j} B_{2d-2j}.
\]
The same formula holds also for \(m = 0\). Note that in above sum \(m\) have to be \(m \geq 2j + 1\) to return nonzero term \(A_{m,j}\).

**Definition 3.27.** We define here a generalized sequence of coefficients \(A_{m,j}\), such that
\[
\sum_{k=0}^{n-1} \sum_{j=0}^{m} A_{m,j}(n-k)^j k^j = n^{2m+1}, \quad n \geq 0, \quad m = 0, 1, 2, \ldots
\]
\[
A_{m,j} := \begin{cases} 
0, & \text{if } j < 0 \text{ or } j > m \\
(2j + 1)\left(\frac{2j}{j}\right) \sum_{d=2j+1}^{m} A_{m,d} \left(\frac{d}{2j+1}\right) \frac{(-1)^{d-1}}{d-j} B_{2d-2j}, & \text{if } 0 \leq j < m \\
(2j + 1)\left(\frac{2j}{j}\right), & \text{if } j = m
\end{cases}
\]

Five initial rows of triangle generated by \(A_{m,j}\) are
\[
\begin{align*}
1 & \quad 1 \\
1 & \quad 6 \\
1 & \quad 0 \quad 30 \\
1 & \quad -14 \quad 0 \quad 140 \\
1 & \quad -120 \quad 0 \quad 0 \quad 630 \\
1 & \quad -1386 \quad 660 \quad 0 \quad 0 \quad 2772
\end{align*}
\]

... Figure 8. Triangle generated by \(A_{m,j}\), \(0 \leq j \leq m\), sequences \([A302971]\) (numerators of \(A_{m,j}\)) and \([A304042]\) (denominators of \(A_{m,j}\)). Note that starting from row \(m \geq 11\) the terms of Triangle \([3.28]\) consist fractional numbers, for example, \(A_{11,1} = 800361655623, 6\). One can find complete list of the numerators and denominators of \(A_{m,j}\) in OEIS under the identifiers A302971 and A304042, respectively, see \([17][18]\). To verify the terms that definition \([3.27]\) produces one should refer to Mathematica code\(^{2}\). Hereby, let be theorem

**Theorem 3.29.** For every positive integers \(n\) and \(m\) holds
\[
\sum_{1 \leq k \leq n} \sum_{j} A_{m,j} k^j (n-k)^j = n^{2m+1}
\]
One can verify results concerning above theorem via Mathematica code\(^\text{[2]}\). Therefore, theorem (3.29) answers to the question question (2.23) positively, since for every \(m \geq 0\) exists a triangle, generated by \(\sum_j A_{m,j} k^j (n-k)^j = n^{2m+1}\), such that odd power \(n^{2m+1}\) can be reached as sum of \(n\)-th row of corresponding triangle over \(k\) and \[A287326\] is partial case for \(m = 1\).

3.1. Properties of \(L_m(n,k)\) and \(A_{m,j}\). Here we show a few properties of definition \(L_m(n,k)\), some of them correlates with properties of partial case \(L_1(n,k)\) in 2.15

1. Sum of \(A_{m,j}\), \(m \geq 0\) gives
   \[
   \sum_{j \geq 0} A_{m,j} = 2^{2m+1} - 1
   \]

2. Similarly to particular property (1.28), items of \(\{L_m(n,k)\}_{k=0}^{n}\), \(m \geq 0\) is symmetric, i.e
   \[
   L_m(n,k) = L_m(n,n-k),\ n \geq 0,\ 0 \leq k \leq n
   \]

3. From (2) for every \(n \geq 0\), \(m \geq 0\) immediately follows
   \[
   \sum_{1 \leq j \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = \sum_{0 \leq j \leq n-1} \sum_{j \geq 0} A_{m,j} T^j(n,k)
   \]

4. \(A_{m,m}, m = 0, 1, 2, \ldots\) are terms of \[A002457\]

5. For every \(m \geq 0\)
   \[
   A_{m,0} = 1
   \]

6. For each \(m \geq 0\)
   \[
   \sum_{j \geq 0} A_{m,j} = \sum_{j \geq 0} \left(\begin{array}{c} 2m+1 \\ j \end{array}\right) - 1
   \]
   \[
   \sum_{1 \leq j \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = n + \sum_{2 \leq j \leq n} \sum_{j \geq 1} A_{m,j} T^j(n,k)
   \]

7. For each even power \(2m\), \(m \geq 0\) and \(n \in \mathbb{Z}\) we have
   \[
   \sum_{1 \leq k \leq n} \sum_{j \geq 0} \frac{1}{n} A_{m,j} T^j(n,k) = n^{2m}
   \]

8. Forward and inverse summation identity
   \[
   \sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = \sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,m-j} T^{m-j}(n,k)
   \]

\[^\text{expression_2.1.txt}\]
3.2. **Example of use.** Recall existing pattern

\[
\begin{array}{cccc}
1 & 6 & 0 & 30 \\
1 & 0 & 140 & 0 \\
1 & -120 & 0 & 630 \\
1 & -1386 & 660 & 0 & 2772 \\
\end{array}
\]

(3.30)

Figure 9. Triangle generated by \(A_{m,j}\), \(0 \leq j \leq m\).

By received formula \(\sum_{k=0}^{n-1} \sum_{j \geq 0} A_{m,j} T^j(n, k) = n^{2m+1}\) each line of above triangle being multiplied by \(T^j(n, k)\) and summed up to \(n\) or \(n - 1\) over \(k\) from 0 or 1, respectively, will result odd power of \(n\), depending on which row of \(A_{m,j}\), \(0 \leq j \leq m\) is applied. Consider the case \(n = 3\), \(m = 2\), we introduce triangle built using \(T(n, k)\), \(1 \leq k \leq n\),

\[
\begin{array}{cccc}
0 & 1 & 0 \\
2 & 2 & 0 \\
3 & 4 & 3 & 0 \\
\end{array}
\]

(3.31)

Figure 10. Triangle generated by \(T(n, k)\), \(1 \leq k \leq n\), sequence A094053, [29] in OEIS.

Then,

\[
3^{2\cdot2+1} = 1 + 0 \cdot 2^1 + 30 \cdot 2^2 \\
+ 1 + 0 \cdot 2^1 + 30 \cdot 2^2 \\
+ 1 + 0 \cdot 0^1 + 30 \cdot 0^2 \\
= 121 + 121 + 1 = 243
\]

We’ve highlighted the terms of \(A_{2,j}\) and \(T(3, k)\) with different colors to be more easily to see regularity. Result we received are terms of the third row of triangle [A300656]

4. **Acknowledgements**

We would like to thank to Dr. Max Alekseyev (Department of Mathematics and Computational Biology, George Washington University), Dr. Hansruedi Widmer, Dr. Ron Knott (Visiting Fellow, Dept. of Mathematics at University of Surrey) and Mr. Albert Tkaczyk for very useful comments, suggestions and help during writing of this article. Also, we’d like to thank to OEIS editors Michel Marcus, Peter Luschny, Jon E. Schoenfield and others for their patient, faithful and volunteer work and for useful comments and suggestions during edition of sequences, concerned with this manuscript.
5. Conclusion

In this paper particular pattern, that is binomial distributed triangle $A_{287326}$ in OEIS, which shows perfect cube $n$ as sum of row terms over $0 \leq k \leq n-1$ or $1 \leq k \leq n$ is generalized. Firstly, we discussed analogs of $A_{287326}$ for powers $2^m+1 = 5, 7$, sequences $A_{300656}, A_{300785}$, respectively, then we derived coefficients $A_{m,j}$, such that for every $n \geq 0$ and $m \geq 0$ holds

$$\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = n^{2m+1}$$

where $A_{m,j}$ is defined by definition (3.27). Therefore, question question (2.23) is answered positively. Section 3 is totally dedicated to complete and extended derivation of identity $\sum_{1 \leq k \leq n} \sum_{j \geq 0} A_{m,j} T^j(n,k) = n^{2m+1}$. Properties of triangle (2.14) and $L_m(n,k)$ are shown in properties 2.15 and subsection 3.1, respectively. Relation between Faulhaber’s sum $\sum n^m$ and finite differences of power are shown in 2.3.

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