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Article

NN-Harmonic Mean Aggregation Operators Based MCGDM Strategy in Neutrosophic Number Environment

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Abstract: The concept of neutrosophic number is a significant mathematical tool to deal with real scientific problems because it can tackle indeterminate and incomplete information which exists generally in real problems. In this article, we use neutrosophic numbers ($a + bI$), where a and bI denote determinate component and indeterminate component respectively. We explore the situations in which the input information is needed to express in terms of neutrosophic numbers. We define score functions and accuracy functions for ranking neutrosophic numbers. We then define a cosine function to determine unknown criteria weights. We define neutrosophic number harmonic mean operators and proved their basic properties. Then, we develop two novel MCGDM strategies using the proposed aggregation operators. We solve a numerical example to demonstrate the feasibility and effectiveness of the proposed two strategies. Sensitivity analysis with variation of "*I*" on neutrosophic numbers is performed to demonstrate how the preference ranking order of alternatives is sensitive to the change of "*I*". The efficiency of the developed strategies is ascertained by comparing the obtained results from the proposed strategies with the existing strategies in the literature.

Keywords: neutrosophic number; neutrosophic number harmonic mean operator (NNHMO); neutrosophic number weighted harmonic mean operator (NNWHMO); cosine function, score function; multi criteria group decision making

1. Introduction

Multi-criteria group decision making (MCGDM) is a significant branch of decision theory which has been commonly applied in many scientific fields such as medical diagnosis [1, 2], decision making [3, 4], supplier selection [5], etc. Because of the indeterminate information and the complexity of decision problems, it is difficult to express criteria in terms of crisp numbers. To tackle the difficulty, neutrosophic number (NN) [6, 7] is proposed in the literature. The NN consists of determinate component and an indeterminate component. So the NNs are more practical to deal with indeterminate and incomplete information in real world problems. The NN is expressed as the

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function $N = p + qI$ in which p is the determinate component and qI is the indeterminate component. If $N = qI$ i.e. the indeterminate part reaches the maximum label, the worst situation occurs. If $N = p$ i.e. the indeterminate part does not appear, the best situation occurs. Thus, application of NNs is more appropriate to deal with the indeterminate and incomplete information in practical decision making situations.

Information aggregation is an essential practice of accumulating relevant information from various sources. Harmonic mean is the reciprocal property of arithmetic mean. It is used to present for aggregation between the min and max operators. Harmonic mean is usually used as a mathematical tool to accumulate central tendency of information.

The harmonic mean (HM) is widely used in statistics to calculate central tendency of a set of data. Park et al. [8] proposed multi-attribute group decision making (MAGDM) strategy based on HM operators under uncertain linguistic environment. Wei [9] proposed MAGDM strategy based on fuzzy induced ordered weighted HM. In fuzzy environment, Xu [10] studied fuzzy weighted HM operator, fuzzy ordered weighted HM operator, and fuzzy hybrid HM operator and employed them for MADM problems. Ye [11] proposed multi-attribute decision making (MADM) strategy based on harmonic averaging projection for simplified neutrosophic sets (SNS) environment.

In NN environment, Liu and Liu [12] proposed NN generalized weighted power averaging operator for MAGDM. Zheng et al. [13] proposed MAGDM strategy based on NN generalized hybrid weighted averaging operator. Pramanik et al. [14] studied teacher selection strategy based on projection and bidirectional projection measures in NN environment.

Literature review reflects that MCGDM strategy using NNs has made little progress in real scientific and engineering fields. Therefore, it is necessary to explore new strategies to handle MCGDM problems in NN environment.

In this paper, we develop two MCGDM strategies based on neutrosophic number harmonic mean operator (NNHMO) and neutrosophic number weighted harmonic mean operator (NNWHMO) to solve MCGDM problems. The proposed strategies can handle the indeterminacy of information.

The paper is sequenced as follows. Section 2 presents some preliminaries of NNs and score and accuracy functions of NNs. Section 3 devotes NN harmonic mean operator (NNHMO) and NN weighted harmonic mean operator (NNWHMO). Section 4 defines cosine function to determine unknown criteria weights. Section 5 presents two novel decision making strategies based on NNHMO and NNWHMO. In section 6, a numerical example is presented to illustrate the proposed MCGDM strategies and the results show the feasibility of the proposed MCGDM strategies. Section 7 compares the obtained results derived from the proposed strategies and the existing strategies in NN environment. Finally, Section 8 concludes the paper with some remarks and future scope of research.

2. Preliminaries

In this section, the concepts of NNs, operations on NNs, score and accuracy functions of NNs are outlined.

2.1. NNs [5, 6]

NN consists of a determinate component x and an indeterminate component yI, and mathematically is expressed as $z = x + yI$ for $x, y \in R$, where I is indeterminacy interval and R is the set

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of real numbers. A NN z can be specified as a possible interval number, denoted by $z = [x + yI^L, x + yI^U]$ for $z \in Z$ (Z is set of all NNs) and $I \in [I^L, I^U]$. The interval $I \in [I^L, I^U]$ is considered as an indeterminate interval.

- If $yI = 0$, then z is degenerated to the determinate component $z = x$
- If $x = 0$, then z is degenerated to the indeterminate component $z = yI$
- If $I^L = I^U$, then z is degenerated to a real number.

Let two NNs be $z_1 = x_1 + y_1I$ and $z_2 = x_2 + y_2I$ for $z_1, z_2 \in Z$, and $I \in [I^L, I^U]$. Some basic operational rules for z_1 and z_2 are presented as follows:

(1) $I^2 = I$

(2) $I.0 = 0$

(3) I/I = Undefined

(4) $z_1 + z_2 = x_1 + x_2 + (y_1 + y_2)I = [x_1 + x_2 + (y_1 + y_2)I^L, x_1 + x_2 + (y_1 + y_2)I^U]$

(5) $z_1 - z_2 = x_1 - x_2 + (y_1 - y_2)I = [x_1 - x_2 + (y_1 - y_2)I^L, x_1 - x_2 + (y_1 - y_2)I^U]$

(6) $z_1 \times z_2 = x_1x_2 + (x_1y_2 + x_2y_1)I + y_1y_2I^2 = x_1x_2 + (x_1y_2 + x_2y_1 + y_1y_2)I$

(7) $\dfrac{z_1}{z_2} = \dfrac{x_1 + y_1I}{x_2 + y_2I} = \dfrac{x_1}{x_2} + \dfrac{x_2y_1 - x_1y_2}{x_2(x_2 + y_2)}I;\ x_2 \neq 0, x_2 \neq -y_2$

(8) $\dfrac{1}{z_1} = \dfrac{1 + 0.I}{x_1 + y_1I} = \dfrac{1}{x_1} + \dfrac{-y_1}{x_1(x_1 + y_1)}I;\ x_1 \neq 0, x_1 \neq -y_1$

(9) $z_1^2 = x_1^2 + (2x_1y_1 + y_1^2)I$

(10) $\lambda z_1 = \lambda x_1 + \lambda y_1 I$

Definition 1. For any NN $z = x + yI = [x + yI^L, x + yI^U]$, ($x$ and y not both zeroes), its score and accuracy functions are defined, respectively, as follows:

$$S(z) = \left| \frac{x + y(I^U - I^L)}{2\sqrt{x^2 + y^2}} \right| \tag{1}$$

$$A(z) = 1 - \exp\left\langle -\left| x + y(I^U - I^L) \right| \right\rangle \tag{2}$$

Theorem 1. Both score function $S(z)$ and accuracy function $A(z)$ are bounded.

Proof.

$x, y \in R$ and $I \in [0, 1]$, $0 \le \dfrac{x}{\sqrt{x^2 + y^2}} \le 1$, $0 \le \dfrac{y(I^U - I^L)}{\sqrt{x^2 + y^2}} \le 1$

$$\Rightarrow 0 \le \left| \frac{x + y(I^U - I^L)}{\sqrt{x^2 + y^2}} \right| \le 2 \Rightarrow 0 \le \left| \frac{x + y(I^U - I^L)}{2\sqrt{x^2 + y^2}} \right| \le 1 \Rightarrow 0 \le S(z) \le 1.$$

Since $0 \le S(z) \le 1$, score function is bounded.

Again,

$$0 \le \exp\left\langle -\left| x + y(I^U - I^L) \right| \right\rangle \le 1 \Rightarrow -1 \le -\exp\left\langle -\left| x + y(I^U - I^L) \right| \right\rangle \le 0 \Rightarrow 0 \le 1 - \exp\left\langle -\left| x + y(I^U - I^L) \right| \right\rangle \le 1$$

Since $-1 \le A(z) \le 1$, accuracy function is bounded.

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Definition 2. Let two NNs be $z_1 = x_1+y_1I = [x_1+y_1I^L, x_1+y_1I^U]$, and $z_2 = x_2+y_2I = [x_2+y_2I^L, x_2+y_2I^U]$, then the following comparative relations hold:

- If $S(z_1) > S(z_2)$, then $z_1 > z_2$
- If $S(z_1) = S(z_2)$ and $A(z_1) < A(z_2)$, then $z_1 < z_2$
- If $S(z_1) = S(z_2)$ and $A(z_1) = A(z_2)$, then $z_1 = z_2$.

Example 1. Let three NNs be $z_1 = 10 + 2I$, $z_2 = 12$ and $z_3 = 12 + 5I$ and $I \in [0, 0.2]$. Then,

$S(z_1) = 0.5099$, $S(z_2) = 0.5$, $S(z_3) = 0.5577$, $A(z_1) = 0.999969$, $A(z_2) = 0.999994$, $A(z_3) = 0.999997$.

We see that, $S(z_1) \succ S(z_2) = S(z_3)$, and $A(z_3) \succ S(z_2)$.

Using definition 2, we conclude that, $z_1 \succ z_3 \succ z_2$.

3. NN- harmonic mean operator (NNHMO)

Definition 3. Let $z_i = x_i + y_iI$ (i = 1, 2, …, n) be a collection of NNs. Then the NNHMO is defined as follows:

$$\text{NNHMO}(z_1, z_2, \cdots, z_n) = n.\left(\sum_{i=1}^{n}(z_i)^{-1}\right)^{-1} \tag{3}$$

Theorem 2. Let $z_i = x_i + y_iI$ (i = 1, 2, …, n) be a collection of NNs. The aggregated value of the $\text{NNHMO}(z_1, z_2, \cdots, z_n)$ operator is also a NN.

Proof. $\text{NNHMO}(z_1, z_2, \cdots, z_n) \;=\; n.\left(\sum_{i=1}^{n}(z_i)^{-1}\right)^{-1}$

$$= n.\left(\sum_{i=1}^{n}\frac{1}{x_i} + \frac{-y_i}{x_i(x_i+y_i)}I\right)^{-1};\; x_i \neq 0, x_i \neq -y_i$$

$$= n.\left(\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}I\right)^{-1};\; x_i \neq 0, x_i \neq -y_i$$

$$= n.\left(\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}I\right)^{-1};\; x_i \neq 0, x_i \neq -y_i$$

$$= \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} + \frac{-n.\sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}}{\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{i=1}^{n}\frac{1}{x_i} + \sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}\right)}I;\; \sum_{i=1}^{n}\frac{1}{x_i} \neq 0, \sum_{i=1}^{n}\frac{1}{x_i} \neq -\sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}$$

It shows that NNHMO is also a NN.

Definition 4. Let $z_i = x_i + y_iI$ (i = 1, 2, …, n) be a collection of NNs. Then the NN- weighted harmonic mean (NNWHMO) is defined as follows:

$$\text{NNWHMO}(z_1, z_2, \cdots, z_n) = \left(\sum_{i=1}^{n} w_i.(z_i)^{-1}\right)^{-1} \tag{4}$$

Theorem 3. Let $z_i = x_i + y_iI$ (i = 1, 2, …, n) be a collection of NNs. The aggregated value of the $\text{NNWHMO}(z_1, z_2, \cdots, z_n)$ operator is also a NN.

Proof. $\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) = \left(\sum_{i=1}^{n} w_i.(z_i)^{-1}\right)^{-1}$

$$= \left(\sum_{i=1}^{n} w_i\left(\frac{1}{x_i} + \frac{-y_i}{x_i(x_i+y_i)}I\right)\right)^{-1};\ x_i \neq 0, x_i \neq -y_i$$

$$= \left(w_i.\sum_{i=1}^{n}\frac{1}{x_i} + w_i.\sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}I\right)^{-1};\ x_i \neq 0, x_i \neq -y_i$$

$$= \frac{1}{w_i.\sum_{i=1}^{n}\frac{1}{x_i}} + \frac{-w_i.\sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}}{\left(w_i.\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(w_i.\sum_{i=1}^{n}\frac{1}{x_i} + w_i.\sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}\right)}I;\ w_i.\sum_{i=1}^{n}\frac{1}{x_i} \neq 0, w_i.\sum_{i=1}^{n}\frac{1}{x_i} \neq -w_i.\sum_{i=1}^{n}\frac{-y_i}{x_i(x_i+y_i)}; \sum_{i=1}^{n} w_i = 1.$$

It shows that NNWHMO is also a NN.

Example 2. Let two NNs be $z_1 = 3 + 2I$ and $z_2 = 2 + I$ and $I \in [0, 0.2]$. Then,

$$\mathrm{NNHMO}(z_1, z_2) = 2\left(\frac{1}{z_1} + \frac{1}{z_2}\right)^{-1} = 2\left(\frac{1}{3+2I} + \frac{1}{2+I}\right)^{-1} = 2.4 + 0.635I.$$

Example 3. Let two NNs be $z_1 = 3 + 2I$ and $z_2 = 2 + I$, $I \in [0, 0.2]$ and $w_1 = 0.4$, $w_2 = 0.6$, then,

$$\mathrm{NNWHMO}(z_1, z_2) = \left(w_1\frac{1}{z_1} + w_2\frac{1}{z_2}\right)^{-1} = \left(0.4\frac{1}{3+2I} + 0.6\frac{1}{2+I}\right)^{-1} = 2.308 + 1.370I.$$

The NNHMO operator and the NNWHMO operator satisfy the following properties.

P1. ***Idempotent law***: If $z_i = z$ for i = 1, 2, …, n then, $\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) = z$ and $\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) = z$

Proof. For, $z_i = z$,

$$\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) = n.\left(\sum_{i=1}^{n}(z_i)^{-1}\right)^{-1} = n.\left(\sum_{i=1}^{n}(z)^{-1}\right)^{-1} = \frac{n}{n.z^{-1}} = z.$$

$$\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) = \left(\sum_{i=1}^{n} w_i.(z_i)^{-1}\right)^{-1} = \left(\sum_{i=1}^{n} w_i.(z)^{-1}\right)^{-1} = \left(\sum_{i=1}^{n} w_i.\right)^{-1}.\left(z^{-1}\right)^{-1} = z.$$

P2. ***Boundedness***: Both the operators are bounded.

Proof. If $z_{\min} = \min(z_1, z_2, \cdots, z_n)$, $z_{\max} = \max(z_1, z_2, \cdots, z_n)$ for i = 1, 2, …, n then, $z_{\min} \leq \mathrm{NNHMO}(z_1, z_2, \cdots, z_n) \leq z_{\max}$ and $z_{\min} \leq \mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) \leq z_{\max}$.

P3. ***Monotonicity***: If $z_i \leq z_i^*$ for i = 1, 2, …, n then, $\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) \leq \mathrm{NNHMO}(z_1^*, z_2^*, \cdots, z_n^*)$ and

$\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) \leq \mathrm{NNWHMO}(z_1^*, z_2^*, \cdots, z_n^*)$.

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Proof. $\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) - \mathrm{NNHMO}(z_1^*, z_2^*, \cdots, z_n^*)$

$= n.\left(\sum_{i=1}^{n}(z_i)^{-1}\right)^{-1} - n.\left(\sum_{i=1}^{n}(z_i^*)^{-1}\right)^{-1} \leq 0$, for $z_i \leq z_i^*$, for i = 1, 2, …, n.

Again,

$\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) - \mathrm{NNWHMO}(z_1^*, z_2^*, \cdots, z_n^*)$

$= \left(\sum_{i=1}^{n} w_i.(z_i)^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i.(z_i^*)^{-1}\right)^{-1} \leq 0$, for $z_i \leq z_i^*$ (i = 1, 2, …, n).

This proves the monotonicity of the functions $\mathrm{NNHMO}(z_1, z_2, \cdots, z_n)$ and $\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n)$.

P4. ***Commutativity***: If $(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$ be any permutation of $(z_1, z_2, \cdots, z_n)$ then,

$\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) = \mathrm{NNHMO}(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$ and $\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) = \mathrm{NNWHMO}(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$

Proof. $\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) - \mathrm{NNHMO}(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$

$= n.\left(\sum_{i=1}^{n}(z_i)^{-1}\right)^{-1} - n.\left(\sum_{i=1}^{n}(z_i^*)^{-1}\right)^{-1} = 0$, because, $(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$ is any permutation of $(z_1, z_2, \cdots, z_n)$.

Hence, we have $\mathrm{NNHMO}(z_1, z_2, \cdots, z_n) = \mathrm{NNHMO}(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$.

Again,

$\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) - \mathrm{NNWHMO}(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$

$= \left(\sum_{i=1}^{n} w_i.(z_i)^{-1}\right)^{-1} - \left(\sum_{i=1}^{n} w_i.(z_i^*)^{-1}\right)^{-1} = 0$, because, $(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$ is any permutation of $(z_1, z_2, \cdots, z_n)$.

Hence, we have $\mathrm{NNWHMO}(z_1, z_2, \cdots, z_n) = \mathrm{NNWHMO}(z_1^\circ, z_2^\circ, \cdots, z_n^\circ)$.

4. Cosine function for determining unknown criteria weights

When criteria weights are completely unknown to decision makers, the entropy measure [15] can be used to calculate criteria weights. Biswas et al. [16] employed entropy measure for MADM problems to determine completely unknown attribute weights of single valued neutrosophic sets (SVNSs). Literature review reflects that, strategy to determine unknown weights in NN environment is yet to appear. In this paper, we propose a cosine function to determine unknown criteria weights.

Definition 5. The cosine function of a NN $P = x_{ij} + y_{ij}I = [x_{ij} + y_{ij}I^L, x_{ij} + y_{ij}I^U]$, (i = 1, 2, ..., m; j = 1, 2, ..., n) is defined as follows:

$$COS_j(P) = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{\pi}{2}\left(\left|\frac{y_{ij}}{\sqrt{x_{ij}^2 + y_{ij}^2}}\right|\right), \text{ (x_{ij} and y_{ij} are not both zeroes)} \quad (5)$$

The weight structure is defined as follows.

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$$w_j = \frac{COS_j(P)}{\sum_{j=1}^{n} COS_j(P)};\ j = 1, 2, ..., n\ \&\ \sum_{j=1}^{n} w_j = 1 \quad (6)$$

The cosine function $COS_j(P)$ satisfies the following properties:

P1. $COS_j(P) = 1$, if $y_{ij} = 0$.

P2. $COS_j(P) = 0$, if $x_{ij} = 0$

P3. $COS_j(P) \geq COS_j(Q)$, if x_{ij} of $P > x_{ij}$ of Q or y_{ij} of $P < y_{ij}$ of Q or both.

Proof.

P1. $y_{ij} = 0 \Rightarrow COS_j(P) = \frac{1}{n}\sum_{i=1}^{n}[\cos 0] = 1$

P2. $x_{ij} = 0 \Rightarrow COS_j(P) = \frac{1}{n}\sum_{i=1}^{n}\left[\cos\frac{\pi}{2}\right] = 0$

P3. For, x_{ij} of $P > x_{ij}$ of Q

$\Rightarrow$ Determinate part of $P >$ Determinate part of Q

$\Rightarrow COS_j(Q) < COS_j(P)$.

For, y_{ij} of $P < y_{ij}$ of Q

$\Rightarrow$ Indeterminacy part of $P <$ Indeterminacy part of Q

$\Rightarrow COS_j(Q) > COS_j(P)$.

For, x_{ij} of $P > x_{ij}$ of Q and y_{ij} of $P < y_{ij}$ of Q

$\Rightarrow$ (Real part of $P >$ Real part of Q) & (Indeterminacy part of $P <$ Indeterminacy part of Q)

$\Rightarrow COS_j(Q) > COS_j(P)$.

Example 4. Let two NNs be $z_1 = 3 + 2I$, and $z_2 = 3 + 5I$, then, $COS(z_1) = 0.9066$, $COS(z_2) = 0.7817$.
Example 5. Let two NNs be $z_1 = 3 + I$, and $z_2 = 7 + I$, then, $COS(z_1) = 0.9693$, $COS(z_2) = 0.9938$.
Example 6. Let two NNs be $z_1 = 10 + 2I$, and $z_2 = 2 + 10I$, then, $COS(z_1) = 0.9882$, $COS(z_2) = 0.7178$.

5. Multi-criteria group decision making strategies based on NNHMO and NNWHMO

Two MCGDM strategies using the NNHMO and NNWHMO respectively are developed in this section. Suppose that $L = \{L_1, L_2, . . . , L_m\}$ is a set of alternatives, $C = \{C_1, C_2, . . . , C_n\}$ is a set of criteria and $DM = \{DM_1, DM_2, . . . , DM_k\}$ is a set of decision makers. Decision makers' assessment for each alternative L_i will be based on each criterion C_j. All the assessment values are expressed by NNs. Steps of decision making strategies based on proposed NNHMO and NNWHMO to solve MCGDM problems are presented as follows.

5.1. MCGDM Strategy 1 (based on NNHMO)

The strategy 1 is presented (see Figure 1) using the following six steps.

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Step 1. Determine the relation between alternatives and criteria

Each decision maker forms a NN decision matrix. The relation between the alternative L_i (i = 1, 2, ..., m) and the criterion C_j (j = 1, 2, ..., n) is presented in Table 1.

Table 1. The relation between alternatives and criteria in terms of NNs

$$DM_k[L\,|\,C] = \begin{matrix} \\ L_1 \\ L_2 \\ \vdots \\ L_m \end{matrix} \begin{pmatrix} C_1 & C_2 & \cdots & C_n \\ \langle x_{11}+y_{11}I\rangle_k & \langle x_{12}+y_{12}I\rangle_k & \cdots & \langle x_{1n}+y_{1n}I\rangle_k \\ \langle x_{21}+y_{21}I\rangle_k & \langle x_{22}+y_{22}I\rangle_k & \cdots & \langle x_{2n}+y_{2n}I\rangle_k \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_{m1}+y_{m1}I\rangle_k & \langle x_{m2}+y_{m2}I\rangle_k & & \langle x_{mn}+y_{mn}I\rangle_k \end{pmatrix}$$

Here, $\langle x_{ij}+y_{ij}I\rangle_k$ represents the NN rating value of the alternative L_i with respect to the criterion C_j for the decision maker DM_k.

Step 2. Using Eq. (3), determine the aggregation values ($DM^{aggr}_k(L_i)$), (i = 1, 2, …, n) for all decision matrices.

Step 3. To fuse all the aggregation values ($DM^{aggr}_k(L_i)$), corresponding to alternatives L_i, we define the averaging function as follows.

$$DM^{aggr}(L_i) = \frac{\sum_{t=1}^{k} DM^{aggr}_k(L_i)}{k} \quad (i = 1, 2, \ldots, n) \tag{7}$$

Step 4. Determine the preference ranking order

Using Eq. (1), determine the score values $S(z_i)$ (accuracy degrees $A(z_i)$, if necessary) (i = 1, 2, . . . , m) of all alternatives L_i. All the score values are arranged in descending order. The alternative corresponding to the highest score value (accuracy values) reflects the best choice.

Step 5. Select the best alternative from the preference ranking order.

Step 6. End.

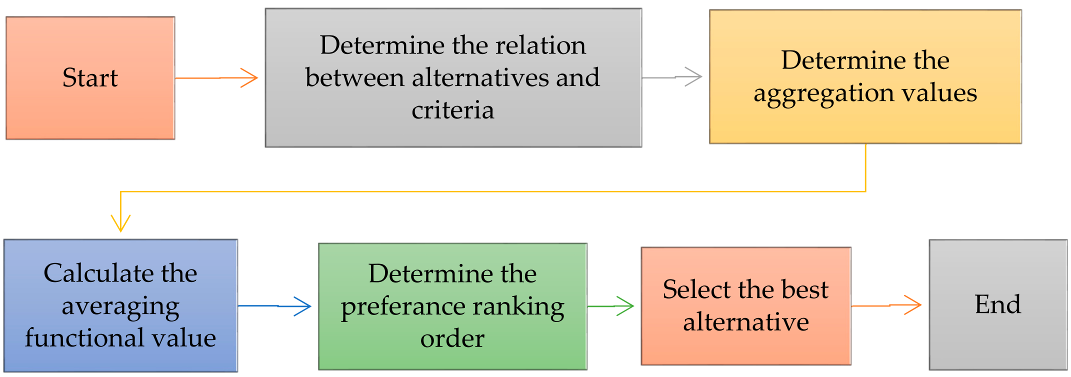

Figure 1. Steps of MCGDM strategy 1 based on NNHMO.

5.2. MCGDM Strategy 2 (based on NNWHMO)

The strategy 2 is presented (see Figure 2) using the following seven steps.

Step 1. This step is similar to the first step of strategy 1.

Step 2. Determine the criteria weights

Using Eq. (6), determine the criteria weights from decision matrices ($DM_t[L \mid C]$), (t = 1, 2, ..., k).

Step 3. Determine the weighted aggregation values ($DM^{waggr}_k(L_i)$)

Using Eq. (4), determine the weighted aggregation values ($DM^{waggr}_k(L_i)$), (i = 1, 2, …, n) for all decision matrices.

Step 4. Determine the averaging values

To fuse all the weighted aggregation values ($DM^{waggr}_k(L_i)$), corresponding to alternatives L_i, we define the averaging function as follows.

$$DM^{aggr}(L_i) = \frac{\sum_{t=1}^{k} DM^{waggr}_k(L_i)}{k} \quad (i = 1, 2, \ldots, n) \tag{8}$$

Step 5. Determine the ranking order

Using Eq. (1), determine the score values $S(z_i)$ (accuracy degrees $A(z_i)$, if necessary) (i = 1, 2, . . . , m) of all alternatives L_i. All the score values are arranged in descending order. The alternative corresponding to the highest score value (accuracy values) reflects the best choice.

Step 6 Select the best alternative from the preference ranking order.

Step 7 End.

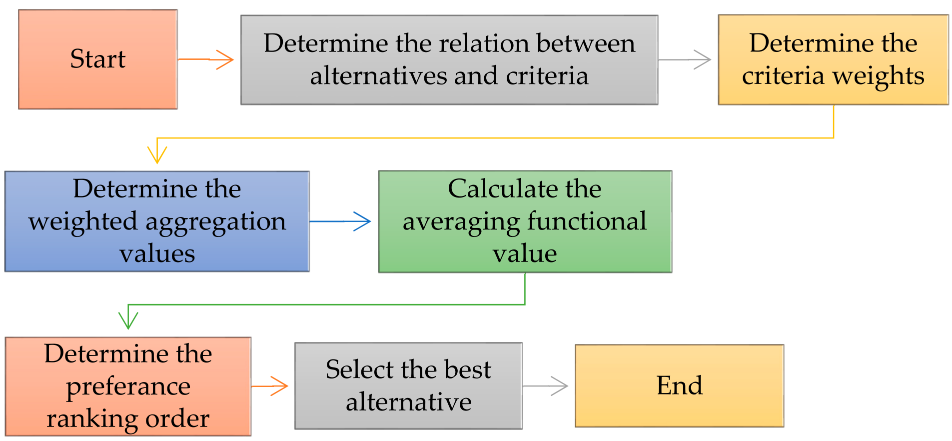

Figure 2. Steps of MCGDM strategy based on NNWHMO.

6. Simulation results

We solve a numerical example studied by Zheng et al. [13]. An investment company desires to invest a sum of money in the best investment fund. There are four possible selection options to invest the money. Feasible selection options are namely, L_1: Car company (CARC), L_2: Food company (FOODC), L_2: Computer company (COMC), L_4: Arms company (ARMC). Decision making must be based on the three criteria namely, Risk analysis (C_1), Growth analysis (C_2), Environmental impact analysis (C_3). The four possible selection options/alternatives are to be selected under the criteria by the NN assessments provided by the three decision makers DM_1, DM_2, DM_3.

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6.1. Solution using MCGDM Strategy 1

Step 1. Determine the relation between alternatives and criteria.

All assessment values are provided by the following three NN based decision matrices (shown in Tables 2, Table 3, and Table 4).

Table 2. NN based decision matrix for DM_1

$$DM_1[L \mid C] = \begin{matrix} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \begin{pmatrix} C_1 & C_2 & C_3 \\ 4+I & 5 & 3+I \\ 6 & 6 & 5 \\ 3 & 5+I & 6 \\ 7 & 6 & 4+I \end{pmatrix}$$

Table 3. NN based decision matrix for DM_2

$$DM_2[L \mid C] = \begin{matrix} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \begin{pmatrix} C_1 & C_2 & C_3 \\ 5 & 4 & 4 \\ 5+I & 6 & 6 \\ 4 & 5 & 5+I \\ 6+I & 6 & 5 \end{pmatrix}$$

Table 4. NN based decision matrix for DM_3

$$DM_3[L \mid C] = \begin{matrix} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \end{matrix} \begin{pmatrix} C_1 & C_2 & C_3 \\ 4 & 5+I & 4 \\ 6 & 7 & 5+I \\ 4+I & 5 & 6 \\ 8 & 6 & 4+I \end{pmatrix}$$

Step 2. Determine the weighted aggregation values ($DM_k^{aggr}(L_i)$)

Using Eq. (3), we calculate the aggregation values ($DM_k^{aggr}(L_i)$) as follows:

$DM_1^{aggr}(L_1)=3.829+0.785I$; $DM_1^{aggr}(L_2)=5.625$; $DM_1^{aggr}(L_3)=4.285+0.214I$; $DM_1^{aggr}(L_4)=5.362+0.514I$;

$DM_2^{aggr}(L_1)=4.285$; $DM_2^{aggr}(L_2)=5.206+0.415I$; $DM_2^{aggr}(L_3)=4.196+0.532I$; $DM_2^{aggr}(L_4)=5.234+0.618I$;

$DM_3^{aggr}(L_1)=4.019+0.605I$; $DM_3^{aggr}(L_2)=5.817+0.433I$; $DM_3^{aggr}(L_3)=4.876+0.387I$

$DM_3^{aggr}(L_4)=6.023+0.257I$.

Step 3. Determine the averaging values

Using Eq. (7), we calculate the averaging values to fuse all the aggregation values corresponding to the alternative L_i.

$DM^{aggr}(L_1) = 4.044 + 0.463I$; $DM^{aggr}(L_2) = 5.549 + 0.282I$; $DM^{aggr}(L_3) = 4.452 + 0.378I$;

$DM^{aggr}(L_4) = 5.539 + 0.463I$.

Step 4. Using Eq. (1), we calculate the score values $S(L_i)$ (i = 1, 2, 3, 4). Score values and ranking of alternatives for different values of "I" are shown in Table 5.

Table 5. Ranking order with variation of "*I*" on NNs for strategy 1

I	$S(L_i)$	Ranking order
I =[0, 0]	$S(L_1)$ = 0.4988, $S(L_2)$ = 0.4993, $S(L_3)$ = 0.4982, $S(L_4)$ = 0.4983	$L_2 \succ L_1 \succ L_4 \succ L_3$
$I \in$ [0, 0.2]	$S(L_1)$ = 0.5081, $S(L_2)$ = 0.5144, $S(L_3)$ = 0.5067, $S(L_4)$ = 0.5056	$L_2 \succ L_1 \succ L_3 \succ L_4$
$I \in$ [0, 0.4]	$S(L_1)$ = 0.5182, $S(L_2)$ = 0.5195, $S(L_3)$ = 0.5151, $S(L_4)$ = 0.5249	$L_2 \succ L_1 \succ L_4 \succ L_3$
$I \in$ [0, 0.6]	$S(L_1)$ = 0.5289, $S(L_2)$ = 0.5346, $S(L_3)$ = 0.5236, $S(L_4)$ = 0.5233	$L_2 \succ L_1 \succ L_3 \succ L_4$
$I \in$ [0, 0.8]	$S(L_1)$ = 0.5396, $S(L_2)$ = 0.5497, $S(L_3)$ = 0.5320, $S(L_4)$ = 0.5316	$L_2 \succ L_1 \succ L_3 \succ L_4$
$I \in$ [0, 1]	$S(L_1)$ = 0.5503, $S(L_2)$ = 0.5547, $S(L_3)$ = 0.5405, $S(L_4)$ = 0.5399	$L_2 \succ L_1 \succ L_3 \succ L_4$

Step 5. Food company (FOODC) is the best alternative for investment.

Step 6. End

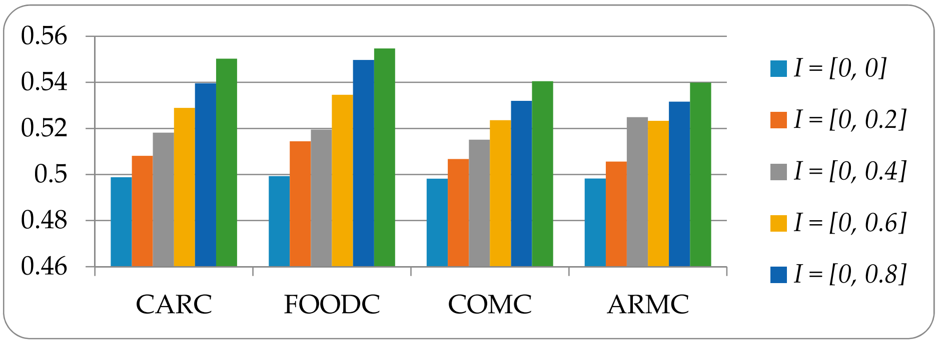

Figure 3. Ranking order with variation of '*I*' on NNs for strategy 1

6.2. Solution using MCGDM Strategy 2

Step 1. Determine the relation between alternatives and criteria

This step is similar to the first step of strategy 1.

Step 2. Determine the criteria weights

Using Eqs. (5) and (6), criteria weights are calculated as follows:

[w_1 = 0.3265, w_2 = 0.3430, w_3 = 0.3305] for DM_1,

[w_1 = 0.3332, w_2 = 0.3334, w_3 = 0.3334] for DM_2,

[w_1 = 0.3333, w_2 = 0.3335, w_3 = 0.3332] for DM_3.

Step 3. Determine the weighted aggregation values ($DM^{waggr}_k(L_i)$)

Using Eq. (4), we calculate the aggregation values ($DM^{aggr}_k(L_i)$) as follows:

$DM^{aggr}_1(L_1)=3.861+0.774I$; $DM^{aggr}_1(L_2)=6.006$; $DM^{aggr}_1(L_3)=4.307+0.234I$; $DM^{aggr}_1(L_4)=5.399+0.541I$;

$DM^{aggr}_2(L_1)=4.288$; $DM^{aggr}_2(L_2)=5.219+0.429I$; $DM^{aggr}_2(L_3)=4.206+0.541I$; $DM^{aggr}_2(L_4)=5.251+0.629I$;

$DM^{aggr}_3(L_1)=4.024+0.616I$; $DM^{aggr}_3(L_2)=5.824+0.445I$; $DM^{aggr}_3(L_3)=4.889+0.393I$; $DM^{aggr}_3(L_4)=6.029+0.265I$.

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Step 4. Determine the averaging values

Using Eq. (7), we calculate the averaging values to fuse all the aggregation values corresponding to the alternative L_i.

$DM^{aggr}(L_1) = 4.057 + 0.463I$; $DM^{aggr}(L_2) = 5.568 + 0.291I$; $DM^{aggr}(L_3) = 4.467 + 0.389I$;

$DM^{aggr}(L_4) = 5.559 + 0.478I$

Step 5. Determine the ranking order

Using Eq. (1), we calculate the score values $S(L_i)$ (i = 1, 2, 3, 4). Since scores values are different, accuracy values are not required. Ranking of alternatives are shown in Table 6.

Table 6. Ranking order with variation of "*I*" on NNs for strategy 2

I	S(L_i)	Ranking order
$I = 0$	$S(L_1) = 0.4968$, $S(L_2) = 0.4993$, $S(L_3) = 0.4981$, $S(L_4) = 0.4982$	$L_2 \succ L_4 \succ L_3 \succ L_1$
$I \in [0, 0.2]$	$S(L_1) = 0.5081$, $S(L_2) = 0.5095$, $S(L_3) = 0.5068$, $S(L_4) = 0.5067$	$L_2 \succ L_1 \succ L_4 \succ L_3$
$I \in [0, 0.4]$	$S(L_1) = 0.5195$, $S(L_2) = 0.5198$, $S(L_3) = 0.5155$, $S(L_4) = 0.5153$	$L_2 \succ L_1 \succ L_3 \succ L_4$
$I \in [0, 0.6]$	$S(L_1) = 0.5308$, $S(L_2) = 0.5350$, $S(L_3) = 0.5241$, $S(L_4) = 0.5239$	$L_2 \succ L_1 \succ L_3 \succ L_4$
$I \in [0, 0.8]$	$S(L_1) = 0.5421$, $S(L_2) = 0.5502$, $S(L_3) = 0.5328$, $S(L_4) = 0.5324$	$L_2 \succ L_1 \succ L_3 \succ L_4$
$I \in [0, 1]$	$S(L_1) = 0.5535$, $S(L_2) = 0.5654$, $S(L_3) = 0.5415$, $S(L_4) = 0.5410$	$L_2 \succ L_1 \succ L_3 \succ L_4$

Step 6. Food company (FOODC) is the best alternative for investment.

Step 7. End

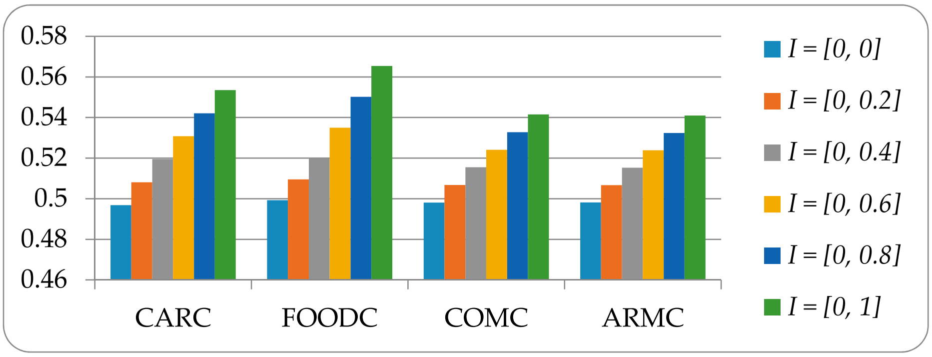

Figure 4. Ranking order with variation of '*I*' on NNs for strategy 2

7. Comparison analysis and contributions of the proposed approach

7.1. Comparison analysis

In this subsection, a comparison analysis is conducted between the proposed MCGDM strategies and other existing strategies in NN environment. Table 5 reflects that L_2 is the best alternative for $I = 0$ and $I \neq 0$. Table 6 reflects that L_2 is the best alternative for any values of *I*. The ranking results obtained from the existing strategies [12, 13, 17] are furnished in Table 7.

In strategy [17], deneutrosophication process is analyzed. It does not recognize the importance of the aggregation information. MCGDM due to Liu and Liu [12] is based on NN generalized weighted power averaging operator. This strategy cannot deal the situation when larger value other than arithmetic mean, geometric mean and harmonic mean is necessary for experiment purpose. In Zheng et al. [13], the NN general hybrid weighted averaging operators are proposed. The strategy proposed

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by Zheng et al. [13] cannot be used when few observations contribute disproportionate amount to the arithmetic mean. To overcome the problem, we have proposed two NN harmonic aggregation operators for MCGDM.

Table 7. Comparison of ranking results with variation of '*I*' on NNs for different strategies

I	Ye [17]	Zheng et al. [13]	Liu and Liu [12]	Proposed Strategy 1	Proposed Strategy 2
[0, 0]	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_4 \succ L_1 \succ L_3$	$L_2 \succ L_1 \succ L_4 \succ L_3$	$L_2 \succ L_4 \succ L_3 \succ L_1$
[0, 0.2]	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_3 \succ L_1 \succ L_4$	$L_2 \succ L_1 \succ L_3 \succ L_4$	$L_2 \succ L_1 \succ L_4 \succ L_3$
[0, 0.4]	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_3 \succ L_4 \succ L_1$	$L_2 \succ L_1 \succ L_4 \succ L_3$	$L_2 \succ L_1 \succ L_3 \succ L_4$
[0, 0.6]	$L_4 \succ L_2 \succ L_3 \succ L_1$	$L_4 \succ L_2 \succ L_3 \succ L_1$	$L_2 \succ L_3 \succ L_4 \succ L_1$	$L_2 \succ L_1 \succ L_3 \succ L_4$	$L_2 \succ L_1 \succ L_3 \succ L_4$
[0, 0.8]	$L_4 \succ L_2 \succ L_3 \succ L_1$	$L_4 \succ L_2 \succ L_3 \succ L_1$	$L_2 \succ L_3 \succ L_4 \succ L_1$	$L_2 \succ L_1 \succ L_3 \succ L_4$	$L_2 \succ L_1 \succ L_3 \succ L_4$
[0, 1]	$L_4 \succ L_2 \succ L_3 \succ L_1$	$L_4 \succ L_2 \succ L_3 \succ L_1$	$L_2 \succ L_4 \succ L_3 \succ L_1$	$L_2 \succ L_1 \succ L_3 \succ L_4$	$L_2 \succ L_1 \succ L_3 \succ L_4$

7.2. *Contributions of the proposed approach*

- NNHMO and NNWHMO in NN environment are firstly defined in the literature. We also proved their basic properties.
- We proposed score and accuracy functions of NN numbers. If two score values are same, then accuracy function can be used for ranking purpose.
- The proposed two strategies can also used when observations/experiments contribute is disproportionate amount to the arithmetic mean. The harmonic mean is used when sample values contain fractions and/or extreme values (either too small or too big).
- To calculate unknown weights structure in NN environment, cosine function is proposed.
- Steps and calculations of the proposed strategies are easy to use.
- We have solved a numerical example to show the feasibility, applicability, and effectiveness of the proposed two strategies.

8. Conclusions

In the study, we have proposed NNHMO and NNWHMO. We have developed two strategies of ranking NNs based on proposed score function and accuracy function. We have proposed cosine function to determine unknown criteria weights in NN environment. We have developed two novel MCGDM strategies based on the proposed aggregation operators. We have solved a hypothetical case study and compared the obtained results with other existing strategies to demonstrate the effectiveness of the proposed MCGDM strategies. Sensitivity analysis for different values of *I* is also conducted to show the influence of *I* in preference ranking of the alternatives. The significance of the paper is that we combine NNs with harmonic aggregation operators to cope with MCGDM problems. We hope that the proposed two MCGDM strategies will open up new avenue of research in NN environment for dealing with MCGDM problems and its practical implementation to real world decision making problems in the current neutrosophic decision making arena.

Author Contributions: Kalyan Mondal and Surapati Pramanik conceived and designed the experiments; Kalyan Mondal, and Surapati Pramanik performed the experiments; Surapati Pramanik and B. C. Giri analyzed the data; Kalyan Mondal, Surapati Pramanik, and B. C. Giri contributed to analysis tools; Kalyan Mondal and Surapati Pramanik wrote the paper.

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Conflicts of Interest: The authors declare no conflict of interest.

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