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Article

On Some New Properties of the Fundamental Solution to the Multi-Dimensional Space- and Time-Fractional Diffusion-Wave Equation

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Abstract: In this paper, some new properties of the fundamental solution to the multi-dimensional space- and time-fractional diffusion-wave equation are deduced. We start with the Mellin-Barnes representation of the fundamental solution that was derived in the previous publications of the author. The Mellin-Barnes integral is used to get two new representations of the fundamental solution in form of the Mellin convolution of the special functions of the Wright type. Moreover, some new closed form formulas for particular cases of the fundamental solution are derived. In particular, we solve an open problem of representation of the fundamental solution to the two-dimensional neutral-fractional diffusion-wave equation in terms of the known special functions.

Keywords: multi-dimensional diffusion-wave equation; neutral-fractional diffusion-wave equation; fundamental solution; Mellin-Barnes integral; integral representation; Wright function; generalized Wright function

MSC: 26A33, 35C05, 35E05, 35L05, 45K05, 60E99

1. Introduction

Partial fractional differential equations are nowadays both an important research subject and a popular modeling approach. Despite of importance of mathematical models in two- and three-dimensions for applications, most of the recent publications devoted to the fractional diffusion-wave equations dealt with the one-dimensional case. The literature dealing with the multi-dimensional partial fractional differential equations is still not numerous and can be divided into several groups as those devoted to the Cauchy problems on the whole space, the boundary-value problems on the bounded domains, and of course to different types of equations including the single-term and the multi-term equations as well as the equations of the distributed order. Because the focus of this paper is on the Cauchy problem for a model linear time- and space fractional diffusion-wave equation, we mention here only some important relevant publications.

The fundamental solution to the multi-dimensional time-fractional diffusion-wave equation with the Laplace operator was derived for the fist time by Kochubei in [13] and Schneider and Wyss in [29] independently from each other in terms of the Fox H-function. Let us note that in [13] the Cauchy problem for the general fractional diffusion equation with the regularized fractional derivative (the Caputo fractional derivative in the modern terminology) and the general second order spatial differential operator was investigated, too. In the series of publications [9]-[11], Hanyga considered mathematical, physical, and probabilistic properties of the fundamental solutions to the multi-dimensional time-, space- and space-time-fractional diffusion-wave equations, respectively. Recently, Luchko and his co-authors started to employ the method of the Mellin-Barnes integral representation to derive further properties of the multi-dimensional space-time-fractional diffusion-wave equation (see e.g. [1], [2],[17]-[19]). Still, the list of the properties, particular cases, integral and series representations, asymptotic formulas, etc. known for the fundamental solution to

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the one-dimensional diffusion-wave equation (see e.g. [25]) is essentially more expanded compared to the multi-dimensional case and thus further investigations of the multi-dimensional case are required.

In this paper, some new properties and particular cases of the fundamental solution to the multi-dimensional space- and time-fractional diffusion-wave equation are deduced. In the second section, we recall the Mellin-Barnes representations of the fundamental solution that were derived in the previous publications of the author and his co-authors. In the third section, the Mellin-Barnes integral is used to get two new representations of the fundamental solution in form of the Mellin convolution of the special functions of the Wright type. The fourth section is devoted to derivation of some new closed form formulas for the fundamental solution. In particular, an open problem of representation of the fundamental solution to the two-dimensional neutral-fractional diffusion-wave equation in terms of the known elementary or special functions is solved.

2. Problem formulation and auxiliary results

In this section we present a problem formulation and some auxiliary results that will be used in the rest of the paper.

2.1. Problem formulation

In this paper, we deal with the multi-dimensional space- and time-fractional diffusion-wave equation in the following form:

$$D_t^\beta u(\mathrm{x},t) = -(-\Delta)^{\frac{\alpha}{2}} u(\mathrm{x},t), \quad \mathrm{x} \in \mathbb{R}^n,\ t>0,\ 1<\alpha\le 2,\ 0<\beta\le 2, \tag{1}$$

where $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian and D_t^β is the Caputo time-fractional derivative of order β.

The Caputo time-fractional derivative of order $\beta > 0$ is defined by the formula

$$D_t^\beta u(\mathrm{x},t) = \left(I_t^{n-\beta}\frac{\partial^n u}{\partial t^n}\right)(t), \quad n-1<\beta\le n,\ n\in\mathbb{N}, \tag{2}$$

where I_t^γ is the Riemann-Liouville fractional integral:

$$(I_t^\gamma u)(t) = \begin{cases} \frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1} u(\mathrm{x},\tau)\,d\tau & \text{for } \gamma>0, \\ u(\mathrm{x},t) & \text{for } \gamma = 0. \end{cases}$$

The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is defined as a pseudo-differential operator with the symbol $|\kappa|^\alpha$ ([27,28]):

$$\left(\mathcal{F}\,(-\Delta)^{\frac{\alpha}{2}} f\right)(\kappa) = |\kappa|^\alpha (\mathcal{F}\,f)(\kappa), \tag{3}$$

where $(\mathcal{F}\,f)(\kappa)$ is the Fourier transform of a function f at the point $\kappa \in \mathbb{R}^n$ defined by

$$(\mathcal{F}\,f)(\kappa) = \hat{f}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot\mathrm{x}} f(\mathrm{x})\,d\mathrm{x}. \tag{4}$$

For $0<\alpha<m,\ m\in\mathbb{N}$ and $x\in\mathbb{R}^n$, the fractional Laplacian can be also represented as a hypersingular integral ([28]):

$$(-\Delta)^{\frac{\alpha}{2}} f(\mathrm{x}) = \frac{1}{d_{n,m}(\alpha)}\int_{\mathbb{R}^n} \frac{\left(\Delta_{\mathrm{h}}^m f\right)(\mathrm{x})}{|\mathrm{h}|^{n+\alpha}}\,d\mathrm{h} \tag{5}$$

with a suitably defined finite differences operator $\left(\Delta_{\mathrm{h}}^m f\right)(\mathrm{x})$ and a normalization constant $d_{n,m}(\alpha)$.

According to [28], the representation (5) of the fractional Laplacian in form of the hypersingular integral does not depend on $m,\ m\in\mathbb{N}$ provided $\alpha<m$.

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Let us note that in the one-dimensional case the equation (1) is a particular case of a more general equation with the Caputo time-fractional derivative and the Riesz-Feller space-fractional derivative that was discussed in detail in [25]. For $\alpha = 2$, the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is just $-\Delta$ and thus the equation (1) is a particular case of the time-fractional diffusion-wave equation that was considered in many publications including, say, [3], [6], [11], [13], [14], [16], and [29]. For $\alpha = 2$ and $\beta = 1$, the equation (1) is reduced to the diffusion equation and for $\alpha = 2$ and $\beta = 2$ it is the wave equation that justifies its denotation as a fractional diffusion-wave equation.

In this paper, we deal with the Cauchy problem for the equation (1) with the Dirichlet initial conditions. If the order β of the time-derivative satisfies the condition $0 < \beta \le 1$, we pose an initial condition in the form

$$u(\mathrm{x}, 0) = \varphi(\mathrm{x}), \quad \mathrm{x} \in \mathbb{R}^n. \tag{6}$$

For the orders β satisfying the condition $1 < \beta \le 2$, the second initial condition in the form

$$\frac{\partial u}{\partial t}(\mathrm{x}, 0) = 0, \quad \mathrm{x} \in \mathbb{R}^n \tag{7}$$

is added to the Cauchy problem.

Because the initial-value problem (1), (6) (or (1), (6)-(7), respectively) is a linear one, its solution can be represented in the form

$$u(\mathrm{x}, t) = \int_{\mathbb{R}^n} G_{\alpha,\beta,n}(\mathrm{x} - \zeta, t)\varphi(\zeta)\, d\zeta,$$

where $G_{\alpha,\beta,n}$ is the first fundamental solution to the fractional diffusion-wave equation (1), i.e., the solution to the problem (1), (6) with the initial condition

$$u(\mathrm{x}, 0) = \prod_{i=1}^{n} \delta(x_i), \quad \mathrm{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

or to the problem (1), (6)-(7) with the initial conditions

$$u(\mathrm{x}, 0) = \prod_{i=1}^{n} \delta(x_i), \quad \mathrm{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

and

$$\frac{\partial u}{\partial t}(\mathrm{x}, 0) = 0, \quad \mathrm{x} \in \mathbb{R}^n,$$

for $0 < \beta \le 1$ or $1 < \beta \le 2$, respectively, with δ being the Dirac delta function.

Thus the behavior of the solutions to the problem (1), (6) (or (1), (6)-(7), respectively) is determined by the fundamental solution $G_{\alpha,\beta,n}(\mathrm{x}, t)$ and the focus of this paper is on derivation of the new properties of the fundamental solution.

2.2. Mellin-Barnes representations of the fundamental solution

A Mellin-Barnes representation of the fundamental solution to the multi-dimensional space- and time-fractional diffusion-wave equation (1) was derived for the first time in [18] for the case $\beta = \alpha$ (see also [19]), in [2] for the case $\beta = \alpha/2$, and in [1] for the general case. For the reader's convenience, we present here a short schema of its derivation.

Application of the multi-dimensional Fourier transform (4) with respect to the spatial variable $\mathrm{x} \in \mathbb{R}^n$ to the equation (1) and to the initial conditions (6) with $\varphi(\mathrm{x}) = \prod_{i=1}^{n} \delta(x_i)$ and (7) (the last condition is relevant only if $\beta > 1$) leads to the ordinary fractional differential equation in the Fourier domain

$$D_t^\beta \hat{G}_{\alpha,\beta,n}(\kappa,t) + |\kappa|^\alpha \hat{G}_{\alpha,\beta,n}(\kappa,t) = 0, \tag{8}$$

along with the initial conditions

$$\hat{G}_{\alpha,\beta,n}(\kappa,0) = 1 \tag{9}$$

in the case $0 < \beta \le 1$ or with the initial conditions

$$\hat{G}_{\alpha,\beta,n}(\kappa,0) = 1,\ \frac{\partial}{\partial t}\hat{G}_{\alpha,\beta,n}(\kappa,0) = 0 \tag{10}$$

in the case $1 < \beta \le 2$.

In both cases, the unique solution of (8) with the initial conditions (9) or (9) and (10), respectively, has the following form (see e.g. [15]):

$$\hat{G}_{\alpha,\beta,n}(\kappa,t) = E_\beta\left(-|\kappa|^\alpha t^\beta\right) \tag{11}$$

in terms of the Mittag-Leffler function $E_\beta(z)$ that is defined by a convergent series

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}, \quad \beta > 0,\ z \in \mathbb{C}. \tag{12}$$

Because of the asymptotic formula (see e.g. [5])

$$E_\beta(-x) = -\sum_{k=1}^{m} \frac{(-x)^{-k}}{\Gamma(1-\beta k)} + O(|x|^{-1-m}),\ m \in \mathbb{N},\ x \to +\infty,\ 0 < \beta < 2, \tag{13}$$

we have the inclusion $\hat{G}_{\alpha,\beta,n} \in L_1(\mathbb{R}^n)$ under the condition $\alpha > 1$ and thus the inverse Fourier transform of (11) can be represented as follows

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\kappa\cdot\mathrm{x}} E_\beta\left(-|\kappa|^\alpha t^\beta\right) d\kappa\,, \quad \mathrm{x} \in \mathbb{R}^n\,, t > 0\,. \tag{14}$$

Because $E_\beta\left(-|\kappa|^\alpha t^\beta\right)$ is a radial function, the known formula (see e.g. [28])

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\kappa\cdot\mathrm{x}} \varphi(|\kappa|)\, d\kappa = \frac{|\mathrm{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^\infty \varphi(\tau)\tau^{\frac{n}{2}} J_{\frac{n}{2}-1}(\tau|\mathrm{x}|)\, d\tau \tag{15}$$

for the Fourier transform of the radial functions can be applied, where J_ν denotes the Bessel function with the index ν (for the properties of the the Bessel function see e.g. [4]), and we arrive at the representation

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{|\mathrm{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^\infty E_\beta\left(-\tau^\alpha t^\beta\right) \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}(\tau|\mathrm{x}|)\, d\tau\,, \tag{16}$$

whenever the integral in (16) converges absolutely or at least conditionally.

The representation (16) can be transformed to a Mellin-Barnes integral.

We start with the case $|\mathrm{x}| = 0$ ($\mathrm{x} = (0,\ldots,0)$) and get the formula

$$G_{\alpha,\beta,n}(0,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} E_\beta(-|\kappa|^\alpha t^\beta) d\kappa$$

that can be represented in the form

$$G_{\alpha,\beta,n}(0,t) = \frac{1}{(2\pi)^n}\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^\infty E_\beta(-\tau^\alpha t^\beta)\, \tau^{n-1}\, d\tau \tag{17}$$

due to the known formula (see e.g. [28])

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$$\int_{\mathbb{R}^n} f(|\mathrm{x}|)d\mathrm{x} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^\infty \tau^{n-1} f(\tau)d\tau. \tag{18}$$

The asymptotics of the Mittag-Leffler function ensures convergence of the integral in (17) under the condition $0 < n < \alpha$ and thus for $1 < \alpha \le 2$ the fundamental solution $G_{\alpha,\beta,n}$ is finite at $|x| = 0$ only in the one-dimensional case and we get the formula

$$G_{\alpha,\beta,1}(0,t) = \frac{t^{-\frac{\beta}{\alpha}}}{\alpha\pi}\int_0^\infty E_\beta(-u)\, u^{\frac{1}{\alpha}-1}\, du \;=\; \frac{t^{-\frac{\beta}{\alpha}}}{\alpha\pi}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}\right)}$$

that is valid for $\alpha > 1$ if $0 < \beta < 2$ and for $\alpha > 2$ if $\beta = 2$. This formula is nothing else as an easy consequence from the known Mellin integral transform of the Mittag-Leffler function (see e.g. [21], [26]):

$$\int_0^\infty E_\beta(-u)\, u^{s-1}\, du \;=\; \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\beta s)} \;\text{ if }\; \begin{cases} 0 < \Re(s) < 1 \;\text{ for }\; 0 < \beta < 2, \\ 0 < \Re(s) < 1/2 \;\text{ for }\; \beta = 2. \end{cases} \tag{19}$$

The Mellin integral transform plays an important role in Fractional Calculus in general and for derivation of the results of this paper in particular, so let us recall the definitions of the Mellin ternasform and the inverse Mellin transform, respectively:

$$f^*(s) = (\mathcal{M}f(\tau))(s) = \int_0^\infty f(\tau)\tau^{s-1}\, d\tau\,, \quad \gamma_1 < \Re(s) < \gamma_2, \tag{20}$$

$$f(\tau) = (\mathcal{M}^{-1}f^*(s))(\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)\tau^{-s}\, ds\,, \quad \tau > 0, \gamma_1 < \Re(s) = \gamma < \gamma_2\,. \tag{21}$$

The Mellin integral transform exists in particular for the functions continuous on the intervals $(0,\varepsilon]$ and $[E,+\infty)$ and integrable on the interval (ϵ, E) with any $\varepsilon,\ E, 0 < \varepsilon < E < +\infty$ that satisfy the estimates $|f(\tau)| \le M_1\tau^{-\gamma_1}$ for $0 < \tau < \varepsilon$ and $|f(\tau)| \le M_2\tau^{-\gamma_2}$ for $\tau > E$ with $\gamma_1 < \gamma_2$ and some constants $M_1,\ M_2$. In this case the Mellin integral transform $f^*(s)$ is analytic in the vertical strip $\gamma_1 < \Re(s) = \gamma < \gamma_2$.

If f is piecewise differentiable and $\tau^{\gamma-1}f(\tau) \in L^c(0,\infty)$, then the formula (21) holds at all points of continuity for f. The integral in the formula (21) has to be considered in the sense of the Cauchy principal value.

For the general theory of the Mellin integral transform we refer the reader to [26]. Several applications of the Mellin integral transform in fractional calculus are discussed in [18,21].

If the dimension n of the equation (1) is greater that one, the fundamental solution $G_{\alpha,\beta,n}(x,t)$ has an integrable singularity at the point $|x| = 0$.

Now we proceed with the case $\mathrm{x} \neq 0$ and first discuss convergence of the integral in the integral representation (16). It follows from the asymptotic formulas for the Mittag-Leffler function and the known asymptotic behavior of the Bessel function (see e.g. [4]) that the integral in (16) converges conditionally in the case $n < 2\alpha + 1$ and absolute in the case $n < 2\alpha - 1$. Thus for $1 < \alpha \le 2$ and $n = 1, 2, 3$ the integral in (16) is at least conditionally convergent.

Now the technique of the Mellin integral transform is applied to deduce a Mellin-Barnes representation of the fundamental solution $G_{\alpha,\beta,n}(\mathrm{x},t)$. In particular, we use the convolution theorem for the Mellin integral transform that reads as

$$\int_0^\infty f_1(\tau)f_2\left(\frac{y}{\tau}\right)\frac{d\tau}{\tau} \;\stackrel{\mathcal{M}}{\longleftrightarrow}\; f_1^*(s)f_2^*(s)\,, \tag{22}$$

where by $\stackrel{\mathcal{M}}{\longleftrightarrow}$ the juxtaposition of a function f with its Mellin transform f^* is denoted.

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It can be easily seen that for $x \neq 0$ the integral at the right-hand side of the formula (16) is nothing else as the Mellin convolution of the functions

$$f_1(\tau) = E_\beta(-\tau^\alpha t^\beta) \quad \text{and} \quad f_2(\tau) = \frac{|x|^{-n}}{(2\pi)^{\frac{n}{2}}} \tau^{-\frac{n}{2}-1} J_{\frac{n}{2}-1}\left(\frac{1}{\tau}\right)$$

at the point $y = \frac{1}{|x|}$.

The Mellin transform of the Mittag-Leffler function (19), the known Mellin integral transform of the Bessel function ([26])

$$J_\nu(2\sqrt{\tau}) \overset{\mathcal{M}}{\longleftrightarrow} \frac{\Gamma(\nu/2+s)}{\Gamma(\nu/2+1-s)}, \; -\Re(\nu/2) < \Re(s) < 3/4,$$

and some elementary properties of the Mellin integral transform (see e.g. [21,26]) lead to the Mellin transform formulas:

$$f_1^*(s) = \frac{t^{-\frac{\beta}{\alpha}s}}{\alpha} \frac{\Gamma(\frac{s}{\alpha})\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha}s)}, \quad 0 < \Re(s) < \alpha,$$

$$f_2^*(s) = \frac{|x|^{-n}}{(2\pi)^{\frac{n}{2}}} \left(\frac{1}{2}\right)^{-\frac{n}{2}+s} \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \quad \frac{n}{2}-\frac{1}{2} < \Re(s) < n.$$

These two formulas, the convolution theorem (22) for the Mellin transform, and the inverse Mellin transform formula (21) result in the following Mellin-Barnes integral representation of the fundamental solution $G_{\alpha,\beta,n}$:

$$G_{\alpha,\beta,n}(x,t) = \frac{1}{\alpha}\frac{|x|^{-n}}{\pi^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)\Gamma\left(\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{s}{2}\right)} \left(\frac{2t^{\frac{\beta}{\alpha}}}{|x|}\right)^{-s} ds, \tag{23}$$

where $\frac{n}{2}-\frac{1}{2} < \gamma < \min(\alpha, n)$. Starting with this representation and using simple linear variables substitutions, we can easily derive some other forms of this representation that will be useful for further discussions. Say, the substitutions $s \to -s$ and then $s \to s-n$ in the Mellin-Barnes representation (23) result in two other equivalent representations

$$G_{\alpha,\beta,n}(x,t) = \frac{1}{\alpha}\frac{|x|^{-n}}{\pi^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{n}{2}+\frac{s}{2}\right)\Gamma\left(-\frac{s}{\alpha}\right)\Gamma\left(1+\frac{s}{\alpha}\right)}{\Gamma\left(1+\frac{\beta}{\alpha}s\right)\Gamma\left(-\frac{s}{2}\right)} \left(\frac{|x|}{2t^{\frac{\beta}{\alpha}}}\right)^{-s} ds \tag{24}$$

and

$$G_{\alpha,\beta,n}(x,t) = \frac{1}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)} \left(\frac{|x|}{2t^{\frac{\beta}{\alpha}}}\right)^{-s} ds \tag{25}$$

under the conditions $-\min(\alpha, n) < \gamma < \frac{1}{2}-\frac{n}{2}$ or $\max(n-\alpha, 0) < \gamma < n$, respectively.

Finally, let us demonstrate how these integral representations can be used, say, for deriving some series representations of $G_{\alpha,\beta,n}(x,t)$ and then its representations in terms of elementary or special functions of the hypergeometric type. To this end, we consider a simple example. In the case $\beta = 1$ and $\alpha = 2$ (standard diffusion equation), the representation (25) takes the form (two pairs of the Gamma-functions in the integral at the right-hand side of (25) are canceled):

$$G_{2,1,n}(x,t) = \frac{t^{-\frac{n}{2}}}{2\,(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma\left(\frac{s}{2}\right)\left(\frac{z}{2}\right)^{-s} ds, \; z = \frac{|x|}{\sqrt{t}}.$$

Substitution of the variables $s \to 2s$ leads to an even simpler representation

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$$G_{2,1,n}(\mathrm{x},t) = \frac{t^{-\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\left(\frac{z}{2}\right)^{-2s}ds,\ z = \frac{|x|}{\sqrt{t}}.$$

According to the Cauchy theorem, the contour of integration in the integral at the right-hand side of the last formula can be transformed to the loop $L_{-\infty}$ starting and ending at $-\infty$ and encircling all poles $s_k = -k,\ k = 0,1,2,\ldots$ of the function $\Gamma(s)$. Taking into account the Jordan lemma, the formula

$$\mathrm{res}_{s=-k}\Gamma(s) = \frac{(-1)^k}{k!},\ k = 0,1,2,\ldots$$

and the Cauchy residue theorem lead to a series representation of $G_{2,1,n}(\mathrm{x},t)$:

$$G_{2,1,n}(\mathrm{x},t) = \frac{t^{-\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)\left(\frac{z}{2}\right)^{-2s}ds = \frac{t^{-\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{z}{2}\right)^{2k},\ z = \frac{|x|}{\sqrt{t}}.$$

Thus the fundamental solution $G_{2,1,n}$ to the n-dimensional diffusion equation takes its standard form:

$$G_{2,1,n}(\mathrm{x},t) = \frac{1}{(\sqrt{4\pi t})^n}\exp\left(-\frac{|\mathrm{x}|^2}{4t}\right). \tag{26}$$

2.3. Special functions of the Wright type

The fundamental solutions to different time-, space, or time- and space-fractional partial differential equations are closely connected to the special functions of the hypergeometric type. In the general situation, some particular cases of the Fox H-function are often involved (see e.g. [13] and [29]). However, for particular cases of the orders of the fractional derivatives, the H-function can be sometimes reduced to some simpler special functions, mainly of the Wright-type (see e.g. [22] for the one-dimensional case of the time-fractional diffusion-wave equation). Because the Fox H-function is still not investigated in all details and in particular, no packages for its numerical calculation are available, this reduction is very welcome. In this paper, some new reduction formulas for the fundamental solution to the multi-dimensional time- and space-fractional diffusion-wave equation (1) will be derived. In this subsection, we shortly discuss the special functions of the Wright type that appear in these derivations.

We start with the Wright function

$$W_{a,\mu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{k!\Gamma(a+\mu k)},\ \mu > -1,\ a,\ z \in \mathbb{C} \tag{27}$$

that was introduced for the first time in [30] in the case $\mu > 0$. In particular, in [30] and [31], Wright investigated some elementary properties and asymptotic behavior of the function (27) in connection with his research in the asymptotic theory of partitions.

Because of the relation

$$J_\nu(z) = \left(\frac{z}{2}\right)^{\nu} W_{1+\nu,1}\left(-\frac{1}{4}z^2\right), \tag{28}$$

the Wright function can be considered as a generalization of the Bessel function $J_\nu(z)$. In its turn, the Wright function is a particular case of the Fox H-function (see e.g. [8] or [12]):

$$W_{a,\mu}(-z) = H^{1,0}_{0,2}\left[z\,\middle|\,\begin{array}{c} - \\ (0,1),(1-a,\mu)\end{array}\right]. \tag{29}$$

The Wright function is an entire function for all real values of the parameter μ (both positive and negative) under the condition $-1 < \mu$, but its asymptotic behavior is different in the cases $\mu > 0$, $\mu = 0$, and $\mu < 0$ (see [32] for details).

Two particular cases of the Wright function, namely, the functions $M(z;\beta) = W_{1-\beta,-\beta}(-z)$ and $F(z;\beta) = W_{0,-\beta}(-z)$ with the parameter β between zero and one have been introduced and investigated in detail in [23,24]. These functions play an important role as fundamental solutions of the Cauchy and signaling problems to the one-dimensional time-fractional diffusion-wave equation ([22]).

In this paper, a four parameters Wright function in the form

$$W_{(a,\mu),(b,\nu)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)}, \quad \mu, \nu \in \mathbb{R},\ a,\ b,\ z \in \mathbb{C} \tag{30}$$

will be used, too. Wright himself investigated this function in [33] in the case $\mu > 0$, $\nu > 0$. For $a = \mu = 1$ or $b = \nu = 1$, respectively, the four parameters Wright function is reduced to the Wright function (27). In [20], Luchko and Gorenflo investigated the four parameters Wright function for the first time in the case when one of the parameters μ or ν is negative. In particular, they proved that the function $W_{(a,\mu),(b,\nu)}(z)$ is an entire function provided that $0 < \mu + \nu$, $a, b \in \mathbb{C}$.

It is important to emphasize that the function $W_{(a,\mu),(b,\nu)}(z)$ can have an algebraic asymptotic expansion on the positive real semi-axis in the case of suitably restricted parameters (see [20] for details):

$$W_{(a,\mu),(b,\nu)}(x) = \sum_{l=0}^{L-1} \frac{x^{(a-1-l)/(-\mu)}}{(-\mu)\Gamma(l+1)\Gamma(b+\nu(a-l-1)/(-\mu))} \tag{31}$$

$$-\sum_{k=1}^{P} \frac{x^{-k}}{\Gamma(b-\nu k)\Gamma(a-\mu k)} + O(x^{(a-1-L)/(-\mu)}) + O(x^{-1-P}),\ \ x \to +\infty$$

when $0 < \nu/3 < -\mu < \nu \le 2,\ L, P \in \mathbb{N}$.

In the important case $\mu + \nu = 0$, the four parameters Wright function is not en entire function anymore. Indeed, in this case the convergence radius of the series from (30) is equal to one, not to infinity, as can be seen from the asymptotics of the series terms as $k \to \infty$:

$$\left|\frac{1}{\Gamma(a-\nu k)\Gamma(b+\nu k)}\right| = \left|\frac{\sin(\pi(a-\nu k))}{\pi}\frac{\Gamma(1-a+\nu k)}{\Gamma(b+\nu k)}\right| =$$

$$= \left|\frac{\cosh(\pi\Im(a))}{\pi}(\nu k)^{1-a-b}\left[1+O(k^{-1})\right]\right|,\ k \to +\infty.$$

In the chain of the equalities above, the following known formulas for the Gamma-function were employed:

$$\frac{\Gamma(z)}{\Gamma(1-z)} = \frac{\pi}{\sin(\pi z)},$$

$$\frac{\Gamma(s+a)}{\Gamma(s+b)} = s^{a-b}\left[1+O(s^{-1})\right],\ |s| \to +\infty,\ |\arg(s)| < \pi.$$

Finally, we mention here the generalized Wright function that is defined by the following series (in the case of its convergence):

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1)\ldots(b_q,B_q)\end{matrix};z\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{i=1}^{q}\Gamma(b_i+B_ik)}\frac{z^k}{k!}. \tag{32}$$

This function was introduced and investigated by Wright in [33]. For details regarding the generalized Wright function we refer the readers to the recent book [7].

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3. New integral representations of the fundamental solution

In the previous section, we derived the following integral representation of the fundamental solution

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{|\mathrm{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^\infty E_\beta\left(-\tau^\alpha t^\beta\right) \tau^{\frac{n}{2}} J_{\frac{n}{2}-1}(\tau|\mathrm{x}|)\, d\tau\,. \tag{33}$$

In this section, we demonstrate how the Mellin-Barnes representations of the fundamental solution can be employed to obtain other integral representations of the same type. The idea is very simple. Say, let us start with the Mellin-Barnes representation (25) and consider the kernel function

$$L_{\alpha,\beta,n}(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}. \tag{34}$$

When the kernel function is represented as a product of two factors, the convolution theorem for the Mellin integral transform can be applied and we get an integral representation of $G_{\alpha,\beta,n}$ of the type (33). Say, we got the integral representation (33) by employing the Mellin integral transform formulas for the Mittag-Leffler function and for the Bessel function, i.e., by representing the kernel function $L_{\alpha,\beta,n}(s)$ as the following product:

$$L_{\alpha,\beta,n}(s) = \frac{\Gamma\left(\frac{n}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{\beta}{\alpha}s\right)} \times \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}. \tag{35}$$

Let us consider other possibilities of representation of the kernel function $L_{\alpha,\beta,n}(s)$ as a product of two factors. Of course, these factors should be chosen in a way that makes it possible to easily obtain the inverse Melling integral transform of these factors in terms of the known elementary or special functions. In the following theorem, two possible representations are given.

Theorem 1. *Let the inequalities $1 < \alpha \le 2,\ 0 < \beta \le 2$ hold true. Then the first fundamental solution $G_{\alpha,\beta,n}$ of the multi-dimensional space- and time-fractional diffusion-wave equation (1) has the following integral representations of the Mellin convolution type:*

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{(\sqrt{\pi}|\mathrm{x}|)^n} \int_0^\infty e^{-\tau}\tau^{\frac{n}{2}-1} W_{(1,\beta),(0,-\alpha/2)}\left(-\frac{\tau^{\alpha/2}t^\beta}{(|\mathrm{x}|/2)^\alpha}\right) d\tau \quad \text{if } \beta > \alpha/2, \tag{36}$$

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{1}{(\sqrt{\pi}|\mathrm{x}|)^n} \int_0^\infty W_{\frac{\alpha}{2},\frac{\alpha}{2}}(-\tau)\ {}_1\Psi_1\left[\begin{matrix}\left(\frac{n}{2},\frac{\alpha}{2}\right)\\ (1,\beta)\end{matrix};-\frac{\tau\, t^\beta}{(|\mathrm{x}|/2)^\alpha}\right] d\tau. \tag{37}$$

Proof. To make calculations easier, let us first perform the variables substitution $s \to 2s$ in the integral representation (25). We get

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{2}{\alpha}\frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma\left(\frac{n}{\alpha}-\frac{2}{\alpha}s\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{2}{\alpha}s\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{2\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-s\right)}\left(z^2\right)^{-s} ds\,,\quad z = \frac{|\mathrm{x}|}{2t^{\frac{\beta}{\alpha}}}. \tag{38}$$

Now we represent the kernel function of the last integral as follows:

$$L_{\alpha,\beta,n}(s) = \Gamma(s) \times \frac{\Gamma\left(\frac{n}{\alpha}-\frac{2}{\alpha}s\right)\Gamma\left(1-\frac{n}{\alpha}+\frac{2}{\alpha}s\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\frac{2\beta}{\alpha}s\right)\Gamma\left(\frac{n}{2}-s\right)}. \tag{39}$$

The inverse Mellin integral transform of $\Gamma(s)$ is just the exponential function $exp(-\tau)$ ([26]):

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$$f_1(\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s)\,\tau^{-s}\,ds = e^{-\tau}. \tag{40}$$

To calculate the inverse Mellin transform of the second factor, the variables substitution $s \to \frac{\alpha}{2}s$ is first applied. We then get the formula

$$f_2(\tau) = \frac{\alpha}{2}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{n}{\alpha}-s\right)\Gamma\left(1-\frac{n}{\alpha}+s\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\beta s\right)\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}s\right)}\left(\tau^{\frac{\alpha}{2}}\right)^{-s}\,ds. \tag{41}$$

To get a series representation of the function f_2, we employ the standard technique for the Mellin-Barnes integrals. According to the Cauchy theorem, the contour of integration in the integral at the right-hand side of the last formula can be transformed to the loop $L_{+\infty}$ starting and ending at $+\infty$ and encircling all poles $s_k = k + \frac{n}{\alpha}$, $k = 0, 1, 2, \ldots$ of the function $\Gamma\left(\frac{n}{\alpha}-s\right)$. Taking into account the Jordan lemma and the formula for the residual of the Gamma-function, the Cauchy residue theorem leads to a series representation of f_2:

$$f_2(\tau) = \frac{\alpha}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(k+1)}{\Gamma(1+\beta k)\,\Gamma\left(-\frac{\alpha}{2}k\right)}\left(\tau^{\frac{\alpha}{2}}\right)^{-k-\frac{n}{\alpha}}. \tag{42}$$

We thus got a representation of f_2 in terms of the four parametric Wright function (30):

$$f_2(\tau) = \frac{\alpha}{2}\,\tau^{-n/2}\,W_{(1,\beta),(0,-\alpha/2)}\left(-\tau^{-\alpha/2}\right) \tag{43}$$

that is valid under condition $\beta > \alpha/2$.

Now we take into consideration the Mellin-Barnes integral (38), the formulas (40) and (43) and the Mellin transform convolution theorem and thus get the integral representation (36) of the fundamental solution.

The same procedure can be applied for other representations of the kernel function $L_{\alpha,\beta,n}(s)$ as a product of two factors. Let us again start with the Mellin-Barnes integral (25) and perform the variables substitution $s \to \alpha s$. Then we get the representation

$$G_{\alpha,\beta,n}(\mathrm{x},t) = \frac{t^{-\frac{\beta n}{\alpha}}}{(4\pi)^{\frac{n}{2}}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{\alpha}{2}s\right)\Gamma\left(\frac{n}{\alpha}-s\right)\Gamma\left(1-\frac{n}{\alpha}+s\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\beta s\right)\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}s\right)}(z^\alpha)^{-s}\,ds\,,\quad z = \frac{|\mathrm{x}|}{2t^{\frac{\beta}{\alpha}}}. \tag{44}$$

The next step is a representation of the kernel function of the last integral as a product of two factors:

$$L_{\alpha,\beta,n}(s) = \frac{\Gamma\left(1-\frac{n}{\alpha}+s\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}s\right)} \times \frac{\Gamma\left(\frac{\alpha}{2}s\right)\Gamma\left(\frac{n}{\alpha}-s\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\beta s\right)}. \tag{45}$$

Now let us calculate the inverse Mellin integral transforms of the factors. For the first factor we employ the same technique as above and get the series representation

$$f_1(\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{n}{\alpha}+s\right)}{\Gamma\left(\frac{n}{2}-\frac{\alpha}{2}s\right)}\tau^{-s}\,ds = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{1}{\Gamma\left(\frac{\alpha}{2}+\frac{\alpha}{2}k\right)}\tau^{k+1-\frac{n}{\alpha}}. \tag{46}$$

Thus the function f_1 can be represented in terms of the Wright function (27):

$$f_1(\tau) = \tau^{1-\frac{n}{\alpha}}\,W_{\frac{\alpha}{2},\frac{\alpha}{2}}(-\tau). \tag{47}$$

As to the second factor, we first get the series representation

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$$f_2(\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{\alpha}{2}s\right)\Gamma\left(\frac{n}{\alpha}-s\right)}{\Gamma\left(1-\frac{\beta}{\alpha}n+\beta s\right)}\,ds = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma\left(\frac{n}{2}+\frac{\alpha}{2}k\right)}{\Gamma(1+\beta k)}\,\tau^{-k-\frac{n}{\alpha}} \tag{48}$$

and then its representation in terms of the generalized Wright function (32)

$$f_2(\tau) = \tau^{-\frac{n}{\alpha}}\,{}_1\Psi_1\left[\begin{matrix}(\frac{n}{2},\frac{\alpha}{2})\\(1,\beta)\end{matrix};-\frac{1}{\tau}\right]. \tag{49}$$

Putting the formulas (44), (47), and (49) together and applying the Mellin convolution theorem, we finally arrive at the integral representation (37) of the fundamental solution in terms of the Wright function and the generalized Wright function. □

4. New closed form formulas for particular cases of the fundamental solution

In the paper [1], the Mellin-Barnes representations of the fundamental solution to the multi-dimensional time- and space-fractional diffusion-wave equation were employed to derive some new particular cases of the solution in terms of the elementary functions and the special functions of the Wright type. In particular, the closed form formulas for the fundamental solution to the neutral-fractional diffusion equation ($\beta = \alpha$ in the equation (1)) in terms of elementary functions were deduced for the odd-dimensional case ($n = 1, 3, \ldots$). In this section, we derive among other things a representation of the fundamental solution to the neutral-fractional diffusion equation in the two-dimensional case in terms of the four parameters Wright function (30).

Theorem 2. *The first fundamental solution to the multi-dimensional space- and time-fractional diffusion equation* (1) *can be represented in terms of the Wright type functions*

a) for $\beta = \alpha$ and $n = 2$ under the condition $1 < \alpha \le 2$:

$$G_{\alpha,\alpha,2}(\mathrm{x},t) = \begin{cases} \frac{|\mathrm{x}|^{\alpha-2}}{\sqrt{\pi}t^{\alpha}}\,W_{(\frac{1}{2}-\frac{\alpha}{2},-\frac{\alpha}{2}),(\frac{\alpha}{2},\frac{\alpha}{2})}\left(-\left(\frac{|\mathrm{x}|}{t}\right)^{\alpha}\right) & \text{if } |\mathrm{x}| < t,\\ \frac{|\mathrm{x}|^{-2}}{\sqrt{\pi}}\,W_{(0,-\frac{\alpha}{2}),(\frac{1}{2},\frac{\alpha}{2})}\left(-\left(\frac{t}{|\mathrm{x}|}\right)^{\alpha}\right) & \text{if } |\mathrm{x}| > t.\end{cases} \tag{50}$$

b) for $\beta = \frac{3}{2}\alpha$ and $n = 2$ under the condition $1 < \alpha \le \frac{4}{3}$:

$$G_{\alpha,\frac{3}{2}\alpha,2}(\mathrm{x},t) = \frac{\sqrt{3}}{2\pi^2|\mathrm{x}|^2}\,{}_1\Psi_3\left[\begin{matrix}(1,1)\\ \left(\frac{1}{3},\frac{\alpha}{2}\right),\left(\frac{2}{3},\frac{\alpha}{2}\right),\left(0,-\frac{\alpha}{2}\right)\end{matrix};-\left(\frac{|\mathrm{x}|}{2(3t)^{\frac{3}{2}}}\right)^{\alpha}\right]. \tag{51}$$

Proof. Once again we start with the Mellin-Barnes integral representation (25) that for $\beta = \alpha$ and $n = 2$ takes the following form

$$G_{\alpha,\alpha,2}(\mathrm{x},t) = \frac{1}{\alpha}\frac{t^{-2}}{4\pi}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma(-1+s)\,\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|\mathrm{x}|}{2t}\right)^{-s}ds\,. \tag{52}$$

The general theory of the Mellin-Barnes integrals (see e.g. [26]) says that for $|\mathrm{x}| \le 2t$ a series representation of (52) can be obtained by transforming the contour of integration in the integral at the right-hand side of (52) to the loop $L_{-\infty}$ starting and ending at $-\infty$ and encircling all poles of the functions $\Gamma\left(\frac{s}{2}\right)$ and $\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)$. The problem is that in this case we have to take into consideration the cases where some of the poles of $\Gamma\left(\frac{s}{2}\right)$ coincide with the poles $\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)$ and then the series representation becomes to be very complicated.

To avoid this problem let us try to "eliminate" one of this Gamma-functions. Application of the duplication formula for the Gamma-function

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$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$

to the function $\Gamma\left(-1+s\right)$ (one of the Gamma-functions in the denominator of the kernel function from the integral in (52)) results in the following representation:

$$\Gamma(1-s) = \Gamma\left(2\left(-\frac{1}{2}+\frac{s}{2}\right)\right) = \frac{2^{s-2}}{\sqrt{\pi}}\Gamma\left(-\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right).$$

Now we substitute the last formula into the integral in (52) and get another Mellin-Barnes representation

$$G_{\alpha,\alpha,2}(x,t) = \frac{1}{\alpha}\frac{t^{-2}}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(-\frac{1}{2}+\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|x|}{t}\right)^{-s} ds\,. \tag{53}$$

In contrast to the representation (52), the numerator of the kernel function in (53) has just one Gamma-function with the poles tending to minus infinity and one Gamma-function with the poles tending to plus infinity and thus this representation is very suitable for derivation of a series representation of $G_{\alpha,\alpha,2}$.

To proceed, the variables substitution $s \to \alpha s$ is first employed in the integral from (53). We get then the representation

$$G_{\alpha,\alpha,2}(x,t) = \frac{t^{-2}}{\sqrt{\pi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(\frac{2}{\alpha}-s\right)\Gamma\left(1-\frac{2}{\alpha}+s\right)}{\Gamma\left(-\frac{1}{2}+\frac{\alpha}{2}s\right)\Gamma\left(1-\frac{\alpha}{2}s\right)}\left(\left(\frac{|x|}{t}\right)^{\alpha}\right)^{-s} ds\,. \tag{54}$$

To get the series representation of the Mellin-Barnes integral (54), we have to consider two cases:

i) $|x| < t$,

ii) $|x| > t$.

In the first case, the contour of integration in the integral at the right-hand side of (54) can be transformed to the loop $L_{-\infty}$ starting and ending at $-\infty$ and encircling all poles of the functions $\Gamma\left(1-\frac{2}{\alpha}+s\right)$. Taking into account the Jordan lemma and the formula for the residuals of the Gamma-function, the Cauchy residue theorem leads to the following series representation of $G_{\alpha,\alpha,2}$:

$$G_{\alpha,\alpha,2}(x,t) = \frac{t^{-2}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{k!\left(\left(\frac{|x|}{t}\right)^{\alpha}\right)^{1+k-\frac{2}{\alpha}}}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}-\frac{\alpha}{2}k\right)\Gamma\left(\frac{\alpha}{2}+\frac{\alpha}{2}k\right)}. \tag{55}$$

We thus arrive at the closed form formula

$$G_{\alpha,\alpha,2}(x,t) = \frac{|x|^{\alpha-2}}{\sqrt{\pi}t^{\alpha}}W_{\left(\frac{1}{2}-\frac{\alpha}{2},-\frac{\alpha}{2}\right),\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)}\left(-\left(\frac{|x|}{t}\right)^{\alpha}\right) \tag{56}$$

in terms of the four parameters Wright function (30) that is valid for $|x| < t$.

In the case $|x| > t$, the contour of integration in the integral at the right-hand side of (54) can be transformed to the loop $L_{+\infty}$ starting and ending at $+\infty$ and encircling all poles of the functions $\Gamma\left(\frac{2}{\alpha}-s\right)$. Proceeding as in the case i), we first get a series representation of $G_{\alpha,\alpha,2}$ in the form

$$G_{\alpha,\alpha,2}(x,t) = \frac{t^{-2}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{k!\left(\left(\frac{|x|}{t}\right)^{\alpha}\right)^{-k-\frac{2}{\alpha}}}{\Gamma\left(-\frac{\alpha}{2}k\right)\Gamma\left(\frac{1}{2}+\frac{\alpha}{2}k\right)} \tag{57}$$

and then the closed form formula

$$G_{\alpha,\alpha,2}(\mathrm{x},t) = \frac{|\mathrm{x}|^{-2}}{\sqrt{\pi}} W_{(0,-\frac{\alpha}{2}),(\frac{1}{2},\frac{\alpha}{2})}\left(-\left(\frac{t}{|\mathrm{x}|}\right)^{\alpha}\right) \tag{58}$$

in terms of the four parameters Wright function that is valid for $|\mathrm{x}| > t$.

Combining (56) and (58), the get the representation (50) of the fundamental solution $G_{\alpha,\alpha,2}$ in terms of the four parameters Wright function.

In the case $|\mathrm{x}| = t$ both series are divergent and the problem of determining of a series representation of $G_{\alpha,\alpha,2}$ is more complicated and will be considered elsewhere.

The method described above can be used for derivation of other closed form formulas for particular cases of the fundamental solution $G_{\alpha,\beta,n}$ in terms of the Wright type functions. Say, let us consider the case $\beta = \frac{3}{2}\alpha$ and $n = 2$ (because of the condition $\beta \le 2$, in this case the inequalities $1 < \alpha \le \frac{4}{3}$ have to be satisfied). The Mellin-Barnes representation of $G_{\alpha,\frac{3}{2}\alpha,2}$ is as follows:

$$G_{\alpha,\frac{3}{2}\alpha,2}(\mathrm{x},t) = \frac{1}{\alpha}\frac{t^{-3}}{4\pi}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{2}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}+\frac{s}{\alpha}\right)}{\Gamma\left(-2+\frac{3}{2}s\right)\Gamma\left(1-\frac{s}{2}\right)}\left(\frac{|\mathrm{x}|}{2t^{\frac{3}{2}}}\right)^{-s} ds\,. \tag{59}$$

To proceed, let us apply the multiplication formula for the Gamma-function

$$\Gamma(ms) = m^{ms-\frac{1}{2}}(2\pi)^{\frac{1-m}{2}}\prod_{k=0}^{m-1}\Gamma\left(s+\frac{k}{m}\right),\ m = 2,3,4,\ldots$$

with $m = 3$ to the Gamma-function $\Gamma\left(-2+\frac{3}{2}s\right)$ from the denominator of the kernel function from the Mellin-Barnes representation (59). We thus get the representation

$$\Gamma\left(-2+\frac{3}{2}s\right) = \Gamma\left(3\left(-\frac{2}{3}+\frac{1}{2}s\right)\right) = 3^{-\frac{5}{2}+\frac{3}{2}s}(2\pi)^{-1}\Gamma\left(-\frac{2}{3}+\frac{1}{2}s\right)\Gamma\left(-\frac{1}{3}+\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s\right).$$

By applying this formula to (59) and by variables substitution $s \to \alpha s$ we arrive at the following Mellin-Barnes representation:

$$G_{\alpha,\frac{3}{2}\alpha,2}(\mathrm{x},t) = \frac{t^{-3}}{4\pi}\frac{3^{-\frac{5}{2}}}{2\pi}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\frac{2}{\alpha}-s\right)\Gamma\left(1-\frac{2}{\alpha}+s\right)}{\Gamma\left(-\frac{2}{3}+\frac{\alpha}{2}s\right)\Gamma\left(-\frac{1}{3}+\frac{\alpha}{2}s\right)\Gamma\left(1-\frac{\alpha}{2}s\right)}\left(\left(\frac{|\mathrm{x}|}{2(3t)^{\frac{3}{2}}}\right)^{\alpha}\right)^{-s} ds\,. \tag{60}$$

Using the technique presented above, the representation (60) leads first to a series representation of $G_{\alpha,\frac{3}{2}\alpha,2}$ in form

$$G_{\alpha,\frac{3}{2}\alpha,2}(\mathrm{x},t) = \frac{\sqrt{3}}{2\pi^2|\mathrm{x}|^2}\sum_{k=0}^{\infty}\frac{\left(-\left(\frac{|\mathrm{x}|}{2(3t)^{\frac{3}{2}}}\right)^{\alpha}\right)^{k}}{\Gamma\left(\frac{1}{3}+\frac{\alpha}{2}k\right)\Gamma\left(\frac{2}{3}+\frac{\alpha}{2}k\right)\Gamma\left(-\frac{\alpha}{2}k\right)}$$

that can be represented as a particular case of the generalized Wright function (51). □

5. Discussion

This paper is devoted to some applications of the Mellin-Barnes integral representations of the fundamental solution to the multi-dimensional space- and time-fractional diffusion-wave equation for analysis of its properties. In particular, this representation is used to get two new representations of the fundamental solution in form of the Mellin convolution of the special functions of the Wright type and for derivation of some new closed form formulas for particular cases of the fundamental solution. Among other things, an open problem of representation of the fundamental solution to the two-dimensional neutral-fractional diffusion-wave equation in terms of the known special functions is solved. The potential of the Mellin-Integral representation of the fundamental solution to the multi-dimensional space- and time-fractional diffusion-wave equation is of course not yet ladled. It can be used among other things for derivation of the new closed form formulas for its particular cases, asymptotical formulas for the fundamental solution, and relationships between the fundamental solutions for different values of the derivatives orders α and β. These problems will be considered elsewhere in the further publications.

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