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Article

Behaviour of Charged Spinning Massless Particles

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Abstract: We review some properties of a relativistic classical massless charged particle with spin interacting with an external electromagnetic field. We give in particular a proper definition of kinetic energy and total energy, the latter being conserved when the external field is stationary. We find that the particle's velocity may differ from c as a result of the spin - electromagnetic field interaction, without jeopardizing Lorentz invariance.

Keywords: Lorentz symmetry, massless charged particle, spinning particle, relativistic particle.

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1. Introduction

Although charged, massless particles have never been observed in the world of real particles, electrons in two-dimensional materials such as graphene behave as massless quasi-particles, *i.e.*, , they show an approximately linear relativistic dispersion relation, of the type $E \sim v_{\mathrm{F}}|\mathbf{p}|$. The velocity $v_{\mathrm{F}} \sim 10^6$ m/s is the Fermi velocity, depending on the microscopic properties of the material. It plays the role of the velocity of light for such a "mini-relativistic theory". Moreover, beyond its (chiral) helicity, the quasi-electron possesses a quantum number, the "pseudo-spin", which makes its wave function to have four components and to obey a massless Dirac equation. An introduction to the physics of graphene may be found in the reviews [1,2].

The existence of this very special behaviour justifies a re-visitation of the somewhat old literature dedicated to the theory of the "relativistic spinning particle" [3]–[13]. We find that, in contrast with the case of the spinless particle, the behaviour of the particle with non-zero spin is drastically different of what is expected from a massless particle: its velocity may be lower or higher than the velocity of light c. Moreover, this happens without conflict with Special Relativity: Lorentz covariance of the equations is always preserved.

To the best of our knowledge, the only known way to derive the dynamics of a classical spinning particle, massive or not, from an action principle [3–6] is to describe the spin degrees of freedom by anticommuting Grassmann variables ψ_μ and impose a supersymmetry. This is the line we follow in the main part of the paper. However, in order to be able to give a realistic classical interpretation, in the final part, we introduce, following [8], spin variables $\Sigma_{\mu\nu}$ which are quadratic in the ψ's, treating the Σ's as real numbers. The resulting equations of motion are no more derivable from an action. Moreover, they suffer incompatibilities, excepted for special external field configurations, *e.g.*, for constant fields. It is for the latter configuration that we solve the equations, analytically or numerically, in order to gain some insight on the behaviours we mentioned above.

We restrict the scope of the present paper to the in interaction with an *external* electromagnetic field. The specific problem of the radiation field has been treated by the authors of Refs. [14–16].

The plan of the paper is the following. We begin with a complete study of the spinless case in order to make some basic points more transparent, in Section 2. In Section 3 we present the results for the spinning case, with a last subsection containing particular solutions of interest, and terminate with our conclusions.

2. The spinless relativistic particle

The action for a classical spinless particle of mass m of electric charge q interacting with an electromagnetic field given by the potential vector A_μ in 4-dimensional Minkowski space-time may be written as the following integral on a time-like curve $\mathcal{C}$ parametrized by λ [5,6,17]:

$$S[x,e] = -\int_{\mathcal{C}} d\lambda \left(\frac{1}{2e}\dot{x}(\lambda)^2 + \frac{e(\lambda)}{2}m^2 + q\,\dot{x}^\mu(\lambda)A_\mu(x(\lambda)) \right), \tag{2.1}$$

where[1] $x^\mu(\lambda)$ are the coordinates of the particle's position and $e(\lambda)$ a real function on the curve $\mathcal{C}$ parametrized by λ. Under (infinitesimal) reparametrizations $\lambda' = \lambda + \varepsilon(\lambda)$, the coordinates x^μ transform as scalars and e as a scalar density of weight 1:

$$\delta x^\mu = \varepsilon \dot{x}^\mu, \quad \delta e = \varepsilon \dot{e} + \dot{\varepsilon} e. \tag{2.2}$$

Under these transformations, the action is invariant, up to boundary terms, and the equations of motion following from the variation of $x^\mu(\lambda)$,

$$\frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{e}\right) - qF_{\mu\nu}\dot{x}^\nu = 0, \tag{2.3}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.4}$$

and the constraint following from the variation of $e(\lambda)$,

$$\frac{\dot{x}^\mu \dot{x}_\mu}{e^2} = m^2, \tag{2.5}$$

are covariant.

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1 The units are defined by $c = \hbar = 1$. The Minkowski metric is $(\eta_{\mu\nu}) = \text{diag}(1,-1,-1,-1)$. The dot means derivative with respect to λ and $\dot{x}^2$ stands for $\dot{x}^\mu \dot{x}_\mu$. Coordinates will be also denoted as $x^0 = t$ and $(x^i,\, i = 1,2,3) = (x,y,z)$.

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The propagation of a massless charged particle is described by the same action where m is set to zero.

We observe that the constraint (2.5) is not completely independent of the equations of motion (2.3). Indeed, multiplying the latter by $\dot{x}^\mu/e$, we find that the left-hand side of (2.5) is a constant:

$$\frac{d}{d\lambda}\left(\frac{\dot{x}^\mu \dot{x}_\mu}{e^2}\right) = 0. \tag{2.6}$$

This means that it will be sufficient to impose it at some initial value of the parameter λ.

The solution of the constraint (2.5) differs qualitatively in the massive and in the massless case. These cases will be therefore treated separately in the following subsections.

The theory defined by the action (2.1) can be considered as a gauge theory in the one-dimensional space-time defined by the world line $\mathcal{C}$, the gauge invariance being that under reparametrizations (2.2) and the fields being the position coordinates $x^\mu(\lambda)$ and the "*einbein*" function $e(\lambda)$, the formers transforming as scalars and the latter as a density of weight 1.

One way to fix the gauge is to fix a value for the non-physical variable $e(\lambda)$. One equivalent way is to simply choose a particular parametrization, *e.g.*, proper time or coordinate time. Then $e(\lambda)$ will be determined by either the constraint (2.5) – if $\dot{x}^\mu \dot{x}_\mu \neq 0$, *i.e.*, in the massive case – or the equations of motion (2.3).

2.1. The massive case

Let us begin with the proper time parametrization $\lambda = \tau$. The 4-velocity $\dot{x}^\mu$ then satisfies

$$\dot{x}^2 = \dot{x}^\mu \dot{x}_\mu = 1,$$

so that the constraint (2.5) solves for $e(\tau)$ as

$$e = 1/m,$$

where we have chosen the positive solution. The equation of motion (2.3) then takes the familiar covariant form

$$m\ddot{x}_\mu - q\, F_{\mu\nu}\dot{x}^\nu = 0.$$

where the second term is the relativistic expression for the Lorentz force.

In the time coordinate parametrization (the dot meaning now a time derivative), the 4-velocity takes the form $(\gamma, \gamma\dot{\mathbf{x}})$, with $\dot{\mathbf{x}} = (\dot{x}^i,\, i = 1,2,3)$ and $\gamma = 1/\sqrt{1-\dot{\mathbf{x}}^2}$. The constraint (2.5) solves now as

$$\frac{1}{e} = m\gamma, \tag{2.7}$$

and the equations of motion (2.3) read

$$\begin{aligned} m\frac{d}{dt}\gamma &= q\mathbf{E}\cdot\dot{\mathbf{x}}, \\ m\frac{d}{dt}(\gamma\dot{\mathbf{x}}) &= q(\mathbf{E} + \dot{\mathbf{x}}\times\mathbf{B}), \end{aligned} \tag{2.8}$$

where we have identified the electric and magnetic fields as

$$\mathbf{E} = (F_{01}, F_{02}, F_{03}), \quad \mathbf{B} = (-F_{23}, -F_{31}, -F_{12}). \tag{2.9}$$

In the stationary case, defined by $\partial_t A_\mu = 0$, we have a conserved energy obtained by integrating the first of Eqs. (2.8):

$$\mathcal{E} = \frac{1}{e(t)} + qA_0(x(t)) = m\gamma(t) + qA_0(x(t)), \tag{2.10}$$

where the integration constant $\mathcal{E}$ is the total energy.

Example:

In the case of 4-dimensional space-time of coordinates t, x, y, z, with *constant fields* $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, B)$, we can perform a first integration of the equations (2.8), obtaining

$$\begin{aligned} &m\gamma\dot{x} - qBy + C_1 = 0, \\ &m\gamma\dot{y} + qBx - qEt + C_2 = 0, \\ &m\gamma\dot{z} + C_3 = 0, \end{aligned} \tag{2.11}$$

C_1, C_2 and C_3 being integration constants.

2.2. The massless case

2.2.1. Equations of motion

We are now going to investigate the main topics of this paper, *i.e.*, the motion of a massless charged particle in an electromagnetic field. The action is given in (2.1), with now $m = 0$. The main difference with respect to the massive one is in the constraint obtaining by varying the variable $e(\lambda)$ in the action: It takes now the form of the light-cone condition

$$\dot{x}^\mu \dot{x}_\mu = 0, \tag{2.12}$$

and we see that, to the contrary of the massive case, it does not determine $e(\lambda)$.

The equations of motion are given by (2.3) for a general parametrization. There is of course no proper time parametrization; we shall use the coordinate time as parameter, so that they take the form

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{e}\right) &= q\mathbf{E}\cdot\dot{\mathbf{x}}, \\ \frac{d}{dt}\left(\frac{\dot{\mathbf{x}}}{e}\right) &= q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}). \end{aligned} \tag{2.13}$$

The constraint (2.12) now reads

$$\sum_{i=1}^{3} (\dot{x}^i)^2 = 1. \tag{2.14}$$

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This constraint (2.14) *not* being taken into account, we have 4 differential equations for 4 functions, $x^i(t)$ and $e(t)$, of second order in the x's and first order in e. Its solutions depend therefore on 7 integration constants. 6 of them can by fixed by 6 boundary conditions, which may be chosen as 6 initial conditions at $t = 0$:

$$x^i(0) = 0,\ i = 1,2,3, \quad (\dot{x}^1(0), \dot{x}^2(0), \dot{x}^3(0)) = (v_{0x}, v_{0y}, v_{0z}). \tag{2.15}$$

Due to (2.6), the constraint (2.12) will be satisfied if it is verified at $t = 0$, *i.e.*,

$$(v_{0x})^2 + (v_{0y})^2 + (v_{0z})^2 = 1. \tag{2.16}$$

One of the integrations constants remain free and will be discussed in Subsection 2.2.2.

2.2.2. Energy equation

The main difference with respect to the massive case is that the *einbein* function $e(t)$ is no more determined by the constraint (2.5). Let us try to interpret it. Its evolution is determined, up to an integration constant, by the first of the equations (2.13). Restricting ourselves to the stationary case where $\partial_t A_\mu = 0$, hence $\mathbf{E} = -\nabla \mathbf{A_0}$, we see that this equation is a total time derivative, which yields

$$\mathcal{E} = \frac{1}{e(t)} + qA_0(x(t)). \tag{2.17}$$

The second term being the potential energy, we interpret the integration constant $\mathcal{E}$ as the total energy of the particle, its "kinetic energy" being identified with $1/e(t)$. In order to understand better the physical meaning of this, let us normalize the electric potential – which in the stationary case is defined up to a constant – by

$$A_0(x(t)) = -\int_0^t dt'\, \mathbf{E}(x(t')\dot{\mathbf{x}}(t').$$

In this situation, $\mathcal{E} = 1/e(0)$, which may be interpreted as the kinetic energy accumulated until the time $t = 0$. We shall assume $\mathcal{E}$ to be positive:

$$\mathcal{E} > 0. \tag{2.18}$$

We may rewrite the second of Eqs. (2.13) as

$$\frac{d}{dt}\left((\mathcal{E} - qA_0)\dot{\mathbf{x}}\right) = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}). \tag{2.19}$$

and remark that the energy $\mathcal{E}$ – an arbitrary parameter – contributes to the inertia of the particle: increasing the value of $\mathcal{E}$ implies more inertia.

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2.2.3. Example of a constant electromagnetic field

In order to get more insight for the motion of the massless particle, let us consider *constant fields* $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, B)$, orthogonal to each other. We can perform a first integration of the equations (2.19), obtaining

$$\begin{aligned}
&(\mathcal{E} + qEy)\dot{x} - qBy + C_1 = 0, \\
&(\mathcal{E} + qEy)\dot{y} + qBx - qEt + C_2 = 0, \\
&(\mathcal{E} + qEy)\dot{z} + C_3 = 0,
\end{aligned} \tag{2.20}$$

where the energy (2.17) reads

$$\mathcal{E} = \frac{1}{e(t)} - qEy(t). \tag{2.21}$$

The integration constants C_1, C_2 and C_3 are fixed as

$$C_1 = -\mathcal{E}v_{0x}, \quad C_2 = -\mathcal{E}v_{0y}, \quad C_3 = -\mathcal{E}v_{0z}, \tag{2.22}$$

with $\sum_i (v_0^i)^2 = 1$, by the initial conditions(2.15).

A peculiar feature of the solutions of the equations (2.20) is a transition in their qualitative behaviour: for $|B| > |E|$, the trajectory is bounded in the y - direction, *i.e.*, the direction of the electric field, whereas it is unbounded in the case $|B| < |E|$. In order to show this, let us solve the system (2.20) for x and y:

$$\begin{aligned}
x(t) &= \frac{E}{B}t + \frac{\mathcal{E}v_{0y}}{qB} - \frac{\mathcal{E}\dot{y}(B - Ev_{0x})}{qB(B - E\dot{x})}, \\
y(t) &= \frac{\mathcal{E}(\dot{x} - v_{0x})}{q(B - E\dot{x})}.
\end{aligned} \tag{2.23}$$

Since the velocity components are all bounded by 1 in absolute value, it is clear that, if $|B| > |E|$, the denominator of the expression for $y(t)$ never vanishes, then $y(t)$ remains bounded. However $x(t)$ is asymptotically linear in t and thus is unbounded (unless the electric field vanishes).

Solutions with $y(t)$ unbounded are those for $|B| \leq |E|$. This set includes the limiting case $|B| = |E|$, where one explicitly checks that $y(t)$ and $x(t)$ go asymptotically as $x \sim t$ and $y \sim t^{2/3}$, respectively, as $t \to \infty$, unless the initial velocity is transverse to the electric field: $(v_{0x}, v_{0y}, v_{0z}) = (1, 0, 0)$, in which case the solution of the equations – with the given initial conditions (2.15) – is $x(t) = t, y(t) = 0$.

Analytic solutions are easy to find for pure electric field or pure magnetic field. The solution for $B = 0$ satisfying the boundary conditions (2.15),reads

$$\begin{aligned}
x(t) &= \frac{v_{0x}}{\omega} \log\left(\frac{\sqrt{(\omega t)^2 + 2v_{0y}\omega t + 1} + \omega t + v_{0y}}{(1 + v_{0y})}\right), \\
y(t) &= \frac{1}{\omega}\left(\sqrt{(\omega t)^2 + 2v_{0y}\omega t + 1} - 1\right), \\
z(t) &= \frac{v_{0z}}{v_{0x}} x(t),
\end{aligned} \tag{2.24}$$

with $\omega = qE/\mathcal{E}$, whereas the solution for $E = 0$ with the same boundary conditions reads

$$\begin{aligned} x(t) &= \frac{1}{\Omega}\Big(- v_{0y}(\cos(\Omega t) - 1) + v_{0x}\sin(\Omega t)\Big), \\ y(t) &= \frac{1}{\Omega}\Big(v_{0x}(\cos(\Omega t) - 1) + v_{0y}\sin(\Omega t)\Big), \\ z(t) &= v_{0z}t, \end{aligned} \tag{2.25}$$

where $\Omega = qB/\mathcal{E}$. We didn't find analytic solutions of the system (2.20) in the presence of both the electric and the magnetic fields, but a numerical analysis is summarized in Figures 1 and 2, where we have confined the movement to the plane (x, y) by setting to zero the initial velocity component v_{0z}. Figure 1 displays the particle trajectory for four values of the ratio B/E: As expected, the one for $B > E$ is bounded in the y direction – which is the direction of the electric field – and exhibits a drift in the orthogonal direction. On the other hand, the two trajectories for $B > E$ are unbounded in both directions. The dotted line corresponds to the limiting case $B = E$. These behaviours are similar[2]. Figure 2 displays the trajectories for three values of the total energy $\mathcal{E}$, showing clearly the increase of the inertia with increasing energy, for cases (a) of $B < E$ and (b) of $B > E$.

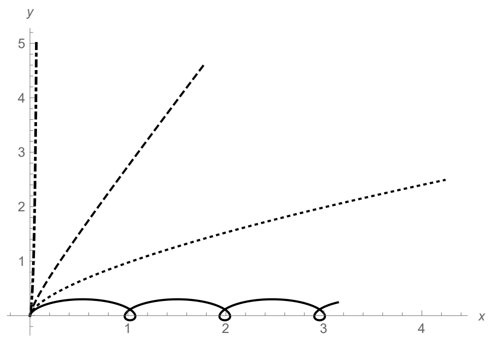

Figure 1. Particle trajectories in the $z = 0$ plane for $0 \le t \le 5$. Charge $q = 1$, constant electric field E in the positive y direction, constant magnetic field B in the positive z direction. Energy $\mathcal{E} = 0.2$, initial velocity $\mathbf{v_0} = (0.1, 0.995, 0)$. Solid line: $B = 1.6$, $E = 1$; dotted line: $B = E = 1$; dashed line: $B = 0.4$, $E = 1$; dotted-dashed line: $B = 0$, $E = 1$. The $B = 0$ trajectory would be on the upper vertical axis in case of $\mathbf{v}_0 = (0, 1, 0)$.

3. The spinning charged and massless particle

We turn now to the case of a spinning particle [5,6,8,20,21], completing the action (2.1) by terms involving the spin degrees of freedom. The latter are described by Grassmann odd (*i.e.*, anti-commuting) variables: a Lorentz vector $\psi^\mu(\lambda)$ and a scalar $\chi(\lambda)$. We restrict here to the less

[2] In the case of $B = 0$, we have the trajectory equation

$$y(x) = -\frac{1}{\omega} + \frac{v_{0x}}{\omega}\cosh\Big(\frac{x}{v_{0x}/\omega} + \mathrm{sech}^{-1}(v_{0x})\Big),$$

which is not the catenary curved observed in the case of a massive particle [18,19], excepted if $v_{0x} = 1$. Se in partcular p.55 of [18]

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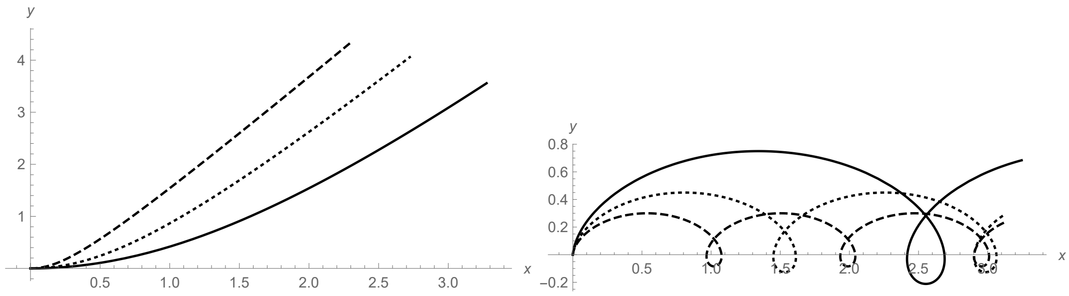

Figure 2. (a) Particle trajectories in the $z = 0$ plane for $0 \leq t \leq 5$. Charge $q = 1$, constant electric field $E = 1$ in the positive y direction, constant magnetic field $B = 0.4$ in the positive z direction. Initial velocity $\mathbf{v}_0 = (1, 0, 0)$. Dashed line: $\mathcal{E} = 0.1$; dotted line: $\mathcal{E} = 0.3$; solid line: $\mathcal{E} = 0.7$; **(b)** Particle trajectories in the $z = 0$ plane for $0 \leq t \leq 5$. Charge $q = 1$, constant electric field $E = 1$ in the positive y direction, constant magnetic field $B = 1.6$ in the positive z direction. Initial velocity $\mathbf{v}_0 = (0.1, 0.995, 0)$. Dashed line: $\mathcal{E} = 0.2$; dotted line: $\mathcal{E} = 0.3$; solid line: $\mathcal{E} = 0.5$.

well established case of a massless particle. Recent accounts for the massive spinning particle may be found in [17,22]

The manifestly Lorentz invariant action reads, as an integral along a curve $\mathcal{C}$ parametrized by λ[3]:

$$S = -\int_{\mathcal{C}} d\lambda \left(\frac{1}{2e} \dot{x}^{\mu} \left(\dot{x}_{\mu} - i\chi\psi_{\mu} \right) + \frac{i}{2}\psi^{\mu}\dot{\psi}_{\mu} + qA_{\mu}\dot{x}^{\mu} - \frac{iq}{2} e\psi^{\mu} F_{\mu\nu} \psi^{\nu} \right), \tag{3.1}$$

where a dot means a derivative with respect to λ.

The action (3.1) is invariant, up to boundary terms, under arbitrary reparametrizations of λ and local supersymmetric transformations. With $\varepsilon(\lambda)$ (even) and $\alpha(\lambda)$ (odd) as infinitesimal parameters, these transformations read, respectively[4],

$$\begin{aligned}
\delta_{\varepsilon} x^{\mu} &= \varepsilon \dot{x}^{\mu}, & \delta_{\alpha} x^{\mu} &= i\alpha\psi^{\mu}, \\
\delta_{\varepsilon} e &= \dot{\varepsilon} e + \varepsilon \dot{e}, & \delta_{\alpha} e &= -i\alpha\chi, \\
\delta_{\varepsilon} \psi^{\mu} &= \varepsilon \dot{\psi}^{\mu}, & \delta_{\alpha} \psi^{\mu} &= -\alpha \left(\dot{x}^{\mu} - \tfrac{i}{2}\chi\psi^{\mu} \right) / e, \\
\delta_{\varepsilon} \chi &= \dot{\varepsilon}\chi + \varepsilon\dot{\chi}, & \delta_{\alpha} \chi &= 2\dot{\alpha}.
\end{aligned} \tag{3.2}$$

The electromagnetic potentials and fields then transform as

$$\begin{aligned}
\delta_{\varepsilon} A_{\mu} &= \varepsilon \dot{A}_{\mu}, & \delta_{\alpha} A_{\mu} &= i\alpha \partial_{\nu} A_{\mu} \psi^{\nu}, \\
\delta_{\varepsilon} F_{\mu\nu} &= \varepsilon \dot{F}_{\mu\nu}, & \delta_{\alpha} F_{\mu\nu} &= i\alpha \partial_{\rho} F_{\mu\nu} \psi^{\rho}.
\end{aligned} \tag{3.3}$$

3 We follow the conventions of [8].
4 The second term in the transformation of ψ_5, which does not appear in [8], is necessary and may be found in Eq. (6.2) of [6].

The commutator of two supersymmetry transformation yields the combination of a reparametrization and of a supersymmetry transformation:

$$[\delta_\alpha, \delta_\beta] = \delta_{\tilde{\varepsilon}} + \delta_{\tilde{\alpha}}, \tag{3.4}$$

with their infinitesimal parameters

$$\tilde{\varepsilon} = \frac{2i}{e}\alpha\beta, \qquad \tilde{\alpha} = -\frac{i\chi}{e}\alpha\beta.$$

$$C_1(\lambda) := \frac{\dot{x}^\mu \dot{x}_\mu}{e^2} - i\frac{\chi \dot{x}^\mu \psi_\mu}{e^2} + iq\psi^\mu F_{\mu\nu}\psi^\nu = 0, \tag{3.5}$$

$$C_2(\lambda) := \dot{x}^\mu \psi_\mu = 0. \tag{3.6}$$

The dynamical equations are obtained by varying the action with respect to $x^\mu(\lambda)$ and $\psi^\mu(\lambda)$:

$$\begin{aligned} &\frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{e} - i\frac{\chi\psi_\mu}{2e}\right) - q\left(\dot{x}^\nu F_{\mu\nu} - \frac{ie}{2}\psi^\rho \partial_\mu F_{\rho\sigma}\psi^\sigma\right) = 0, \\ &\dot{\psi}_\mu + \frac{\dot{x}_\mu \chi}{2e} - eqF_{\mu\nu}\psi^\nu = 0. \end{aligned} \tag{3.7}$$

The local supersymmetry transformation (3.2) for χ shows that the latter is a pure gauge of freedom, which will be set from now on to zero:

$$\chi = 0. \tag{3.8}$$

This fixes the supersymmetry invariance. Later on we will also fix reparametrization invariance by choosing a specific parametrization, namely $\lambda = t$, instead of attributing a value to the einbein e as it often done in the literature [3]–[13].

Using the dynamical equations (3.7), the anticomutativity of the ψ^μ's and the equation (2.4), one shows that the left-hand sides of the constraints obey the equations

$$\dot{C}_1 = 0, \qquad \frac{\dot{C}_2}{C_2} = \frac{\dot{e}}{e}. \tag{3.9}$$

These are consistency conditions which show that the constraints (3.5) and (3.6) are automatic consequences of the equations of motion (3.7) if they are satisfied for some initial value λ_0 of the evolution parameter.

The theory with Grassmann parameters just described is the appropriate one for an Hamiltonian formulation and a subsequent quantization, as it has been done for the free particle in [6,10] and in [12,13] with electromagnetic interaction, but only in the massive case. A theory easier to interpret as the one of a classical spinning particle may be obtained introducing the spin tensor Σ, whose components are even Grassmann numbers [8,17]:

$$\Sigma_{\mu\nu} = -i\psi_\mu\psi_\nu = -\Sigma_{\nu\mu}. \tag{3.10}$$

This formulation is the one which is suitable as an effective theory which ought to describe the semi-classical limit of the quantum theory in terms of expectation values.

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The constraints (3.5), (3.6) now read

$$\frac{\dot{x}_\mu \dot{x}^\mu}{e^2} - qF^{\mu\nu}\Sigma_{\mu\nu} = 0, \tag{3.11}$$

$$\dot{x}^\mu \Sigma_{\mu\nu} = 0, \tag{3.12}$$

and the equations of motion (3.7) take now the form

$$\frac{d}{d\lambda}\left(\frac{\dot{x}_\mu}{e}\right) - q\left(F_{\mu\nu}\dot{x}^\nu + \frac{e}{2}\partial_\mu F^{\rho\sigma}\Sigma_{\rho\sigma}\right) = 0, \tag{3.13}$$

$$\dot{\Sigma}_{\mu\nu} - q\,e(F_\mu{}^\sigma\Sigma_{\sigma\nu} - F_\nu{}^\sigma\Sigma_{\sigma\mu}) = 0. \tag{3.14}$$

3.1. Time parametrization

Choosing now the time parametrization, $\lambda = t$, we see that the spin constraint (3.6) can be solved for the component ψ_0 in terms of the ψ_i ($i = 1, 2, 3$):

$$\psi_0 = -\dot{x}^i\psi_i, \tag{3.15}$$

where a dot now means the time derivative.

Instead of working with the odd Grassmann variables ψ_μ, we shall use the even Grassmann spin tensor Σ defined in (3.10). Its components can be written as components of the two 3-vectors

$$\mathbf{n} = (\Sigma_{01}, \Sigma_{02}, \Sigma_{03}), \quad \mathbf{s} = (\Sigma_{23}, \Sigma_{31}, \Sigma_{12}), \tag{3.16}$$

so that the spin constraint (3.12) can be solved for $\mathbf{n}$ in term of $\mathbf{s}$:

$$\mathbf{n} = \dot{\mathbf{x}} \times \mathbf{s}, \tag{3.17}$$

and we observe that the vector $\mathbf{n}$ is orthogonal to the velocity. We remark that (3.12) or (3.17) is identical to the covariant version of the Frenkel condition [17,23] – introduced for the massive case! – that the 3-vector $\mathbf{n}$ vanishes in the rest frame of the particle.

The constraint (3.11) and the dynamical equations for the position (3.13) read, respectively,

$$\frac{1 - \dot{\mathbf{x}}^2}{e^2} + 2q(\mathbf{sB} + \mathbf{nE}) = 0, \tag{3.18}$$

and

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{e}\right) - q\mathbf{E}\cdot\dot{\mathbf{x}} + q\,e(\mathbf{s}\partial_t\mathbf{B} + \mathbf{n}\partial_t\mathbf{E}) = 0, \\ &\frac{d}{dt}\left(\frac{\dot{\mathbf{x}}}{e}\right) - q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) - q\,e\textstyle\sum_{i=1}^3(s_i\nabla B_i + n_i\nabla E_i) = 0. \end{aligned} \tag{3.19}$$

In the same way, the equation (3.14) for the spin vector $\mathbf{s}$ reads

$$\dot{\mathbf{s}} + q\,e(\mathbf{E} \times \mathbf{n} + \mathbf{B} \times \mathbf{s}) = 0, \tag{3.20}$$

the one for $\mathbf{n}$, $\dot{\mathbf{n}} + q\,e(\mathbf{B} \times \mathbf{n} - \mathbf{E} \times \mathbf{s}) = 0$, following from (3.17) and (3.19).

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In the case of a stationary exterior field, $\partial_t \mathbf{E} = \partial_t \mathbf{B} = 0$, integration of the first of Eqs. (3.19) leads to the same conserved energy $\mathcal{E}$ as in the spinless case:

$$\mathcal{E} = \frac{1}{e(t)} + qA_0(\mathbf{x}(\mathbf{t})). \tag{3.21}$$

117 We observe from (3.18) that the particle's velocity may be different from that of the light. This
118 feature is a peculiarity of the massless theory. We will show some concrete examples in Subsection
119 3.3.

120 *3.2. Physical interpretation of the classical theory*

In order to be able to interpret the theory as a truly classical one, in terms of real numbers, on should forget about the Grassmann character of the spin variables $\Sigma_{\mu\nu}$ (or **s** and **n**) and consider them as real number quantities. The theory would still be defined by the set of equations (3.11-3.14), or (3.17-3.20) in the 3D notation. These equations do no more derive from an action principle, so that their consistency must be checked. Unfortunately, it happens that the spin constraint (3.12) is incompatible with the rest of the equations. Indeed, deriving it with respect to the evolution parameter λ,

$$\frac{d}{d\lambda}\left(\dot{x}^\mu \Sigma_{\mu\nu}\right) = \frac{q}{2}\partial^\mu F^{\rho\sigma}\Sigma_{\mu\nu}\Sigma_{\rho\sigma}, \tag{3.22}$$

121 which only vanishes for special field configurations, such as, *e.g.*, a constant electromagnetic
122 field[5], which we shall consider in Subsection 3.3.

123 An alternative could be to use the constraint and equations of motion (3.11), the
124 spin-constraint (3.12) or (3.17)on one hand, and the spin equation (3.14) only for $\mu\nu = ij$, *i.e.*, for
125 the spin 3-vector **s**, on the other hand. However, such a choice would break Lorentz covariance.

126 *3.3. Constant electromagnetic field*

127 As we saw in the last Subsection, the restriction to a constant electromagnetic field preserves
128 the full set of the Lorentz covariant constraints and dynamical equations.

We shall consider the same configuration with a constant electromagnetic field as discussed in the spinless case at the end of Section 2.2.3, *i.e.*, with $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, B)$, with mass zero and with the time parametrization, $\lambda = t$. The equations of motion for e, x, y and z take the same form (2.13), or there integrated form (2.20), as in the spinless case. The conserved energy is given by (2.21). The constraint equation and the spin equations read

$$\frac{1-\dot{\mathbf{x}}^2}{e^2} + 2q\left(Bs_z + E\dot{z}s_x - E\dot{x}s_z\right)) = 0, \tag{3.23}$$

$$\begin{aligned}
\dot{s}_x &= q\,e\left(E\dot{y}s_x - E\dot{x}s_y + Bs_y\right), \\
\dot{s}_y &= -q\,eBs_x, \\
\dot{s}_z &= q\,e\left(E\dot{y}s_z - E\dot{z}s_y\right).
\end{aligned} \tag{3.24}$$

---

5 If the $\Sigma^{\mu\nu}$'s still were even Grassmann numbers, products of odd elements as in (3.10), then the expression $\Sigma^{\mu\nu}\Sigma^{\rho\sigma}$ would be antisymmetric in the three indices μ, ρ, σ and the right-hand side of (3.22) would vanish due to $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$.

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Solutions for $B = 0$ or $E = 0$ are easy to find. We use again the notations

$$\omega = \frac{qE}{\mathcal{E}}, \qquad \Omega = \frac{qB}{\mathcal{E}}, \tag{3.25}$$

of Subsection 2.2.3 and the boundary conditions (2.15).

For $B = 0$, the evolution of the position coordinates is given by (2.24), but with differences in the initial velocity components due to the modified constraint (see (3.27)). The evolution of the spin components is given by

$$\begin{aligned} s_x(t) &= -\frac{s_{0y}v_{0x}(v_{0y}+\omega t)}{1-v_{0y}^2} + s_{0x}\sqrt{1+2v_{0y}\omega t+\omega^2 t^2}, \\ s_y(t) &= s_{0y}, \\ s_z(t) &= -\frac{s_{0y}v_{0z}(v_{0y}+\omega t)}{1-v_{0y}^2} + s_{0z}\sqrt{1+2v_{0y}\omega t+\omega^2 t^2}. \end{aligned} \tag{3.26}$$

The constraint reads

$$\mathcal{E}(1 - v_{0x}^2 - v_{0y}^2 - v_{0z}^2) + 2\omega(s_{0x}v_{0z} - s_{0z}v_{0x}) = 0. \tag{3.27}$$

In view of its constancy (see the first of Eqs. (3.9)), we have taken it at $t = 0$: it is thus a constraint on the initial parameters v_{0i} and s_{0i}. On sees that the particle's velocity is not constrained to be equal to the velocity of light $c = 1$ – excepted for very peculiar initial spin/velocity configurations in which the spin part of (3.27) vanishes. The particle's velocity can even exceed c. We note that Lorentz invariance remains nevertheless unbroken. This feature, peculiar to the present approach of the classical massless spinning particle, will be encountered in various other examples, as we will see.

For the purely magnetic case, $E = 0$, the position coordinates are given by (2.25), and the spin components by the precession equations

$$\begin{aligned} s_x(t) &= s_{0x}\cos(\Omega t) + s_{0y}\sin(\Omega t), \\ s_y(t) &= s_{0y}\cos(\Omega t) - s_{0x}\sin(\Omega t), \\ s_z(t) &= s_{0z}, \end{aligned} \tag{3.28}$$

where (s_{0x}, s_{0y}, s_{0z}) is the spin vector at $t = 0$. The constraint reads

$$\mathcal{E}(1 - \dot{\mathbf{x}}^2) + 2\Omega s_{0z} = 0, \tag{3.29}$$

On sees that, the magnitude of the particle's velocity – which here is constant due to the first of Eqs. (3.9) – can be higher or lower than the velocity of light, depending on the sign of $\Omega s_{0z}/\mathcal{E}$.

In the case of both E and B being non-zero, one has first to solve the constraint (3.23) for one of the velocity components, let us say the x-component. In view of the constancy of the constraint (see the first of Eqs. (3.9)), it is sufficient to do it at the initial time $t = 0$ for the initial velocity

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component v_{0x}. It is thus a quadratic equation for the initial value of the velocity component, v_{0x}. In order to obtain real solutions, the discriminant

$$\Delta := (1 - v_{0y}^2 - v_{0z}^2)\mathcal{E}^2 + 2(s_{0x}v_{0z}\omega + s_{0z}\Omega)\mathcal{E} + s_{0z}^2\omega^2, \tag{3.30}$$

must be non-negative, hence the reality condition:

$$\Delta \geq 0 \tag{3.31}$$

must hold.

In order to be more explicit, we specialise from now on to the case of trajectories in the (x, y)-plane with the spin pointing to the z-direction, which is guaranteed by the initial conditions

$$v_{0z} = 0, \qquad s_{0x} = s_{0y} = 0. \tag{3.32}$$

The reality condition holds if and only if

$$v_{0y}^2 \leq 1 + \frac{2\Omega}{\mathcal{E}}s_{0z} + \frac{\omega^2}{\mathcal{E}^2}s_{0z}^2. \tag{3.33}$$

A necessary condition for this inequality is the positivity of the right-hand side, which holds in the following three cases:

$$\begin{aligned}
&\text{(a) } |\omega| \geq |\Omega|, && \forall\, s_{0z}, \\
&\text{(b) } |\omega| < |\Omega|, && s_{0z} < -\frac{\mathcal{E}\Omega}{\omega^2} - \frac{\mathcal{E}^2}{\omega^2}\sqrt{\frac{\Omega^2 - \omega^2}{\mathcal{E}^2}}, \\
&\text{(c) } |\omega| < |\Omega|, && s_{0z} > -\frac{\mathcal{E}\Omega}{\omega^2} + \frac{\mathcal{E}^2}{\omega^2}\sqrt{\frac{\Omega^2 - \omega^2}{\mathcal{E}^2}}.
\end{aligned} \tag{3.34}$$

Figure 3 shows some characteristic solutions. One observes the same behaviour as in the spinless case for the trajectories (Figure 3a): bounded in the electric field direction (component $\dot{y}(t)$ for $|B| > |E|$, unbounded for $|B| \leq |E|$. A similar behaviour happens for the spin, as shown in Figure 3b: $s_z(t)$ is unbounded for $|B| \leq |E|$. In fact, all numerical examples investigated show this transition between bounded and unbounded behaviour happening both for $\dot{y}$ and s_z at $|B| = |E|$. In Figure 3c one sees the variation of the velocity's absolute value in function of t. This velocity turns out to be always bounded.

4. Discussion

We have presented a complete treatment of the massless charged particle in interaction with an external electromagnetic field. One of our new results is the proper definition of energy given in (2.17) for the spinless particle and in (3.21) for the spinning one. The inverse of the einbein function, $1/e(t)$, plays the role of the "kinetic energy".

We have considered both the pseudo-classical supersymmetric theory with odd Grassmann parameters, suitable for a canonical quantization, and the classical theory with spin described by real valued functions, which we have argued to better describe the classical limit of the quantum

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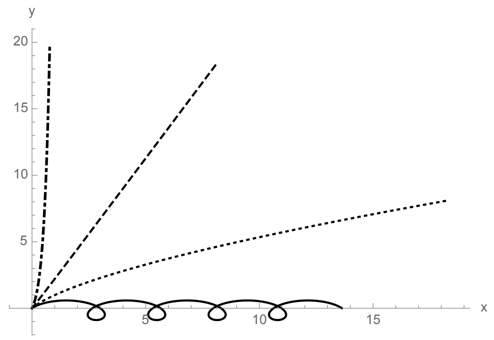
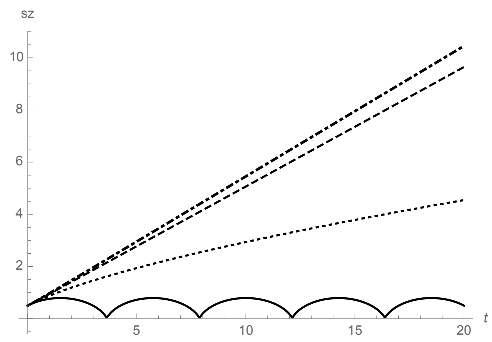
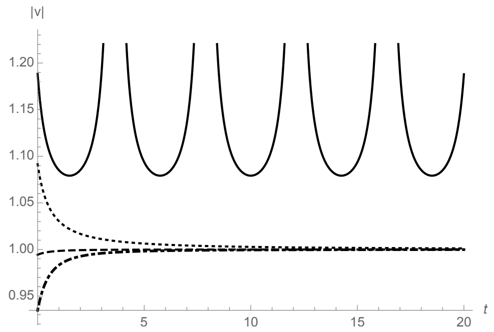


(a) Trajectories in the $z = 0$ plane.



(b) Spin $s_z(t)$.



(c) Velocity $|\mathbf{v}(\mathbf{t})|$.

Figure 3. Particle trajectories in the $z = 0$ plane, spin $s_z(t)$ and velocity $|\mathbf{v}| = |\dot{\mathbf{x}}(t)|$ for a constant electric field E in the positive y direction and a constant magnetic field B in the positive z direction. Parameters' values are chosen : charge $q = 1$, energy $\mathcal{E} = 2$, initial spin $\mathbf{s}(\mathbf{0}) = (0, 0, 0.5)$ and initial velocity $\mathbf{v}(\mathbf{0}) = (v_{0x}, 0.9, 0)$, v_{0x} being the largest of the solutions of the constraint (3.23). The following field configurations have been chosen: $B = 3.2$, $E = 2$ (solid lines); $B = E = 2$ (dotted lines); $B = 0.8$, $E = 2$ (dashed lines); $B = 0$, $E = 2$ (dotted-dashed lines). The $B = 0$ trajectory would be on the upper vertical axis in case of $\mathbf{v}(0) = (v_{0x}, 1, 0)$.

theory in terms of expectation values. The drawback of the latter description is the absence of an action principle and the incompatibility of the full system of equations excepted for special external field configurations, such as a constant one.

It is for a constant field configuration that we have calculated explicit solutions showing characteristic behaviours of the particle, in particular the fact that due to the interaction of the spin with the external field its velocity is in general different from the velocity of light, without contradiction with Lorentz invariance.

This latter result would of course generate conflict with causality, as tachyons do, and may constitute an argument explaining the absence of such particles in the realm of fundamental physics. On the other hand, there would be no such problem in application in condensed matter physics, such as graphene, where the critical velocity which plays the role of the "velocity of light" is far smaller than c [1,2].

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