Geometric Unification of Electromagnetism and Gravitation

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ABSTRACT

A recently proposed classical field theory comprised of four field equations that geometrically couple the Maxwell tensor to the Riemann-Christoffel curvature tensor in a fundamentally new way is reviewed and extended. The new theory's field equations show little resemblance to the field equations of classical physics, but both Maxwell's equations of electromagnetism and Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and dark energy are shown to be a consequence. Emphasized is the emergence of gravity and the unification brought to electromagnetic and gravitational phenomena as well as the consistency of solutions of the new theory with those of the classical Maxwell and Einstein field equations. Unique to the four field equations reviewed here and based on specific solutions to them are: the emergence of antimatter and its behavior in gravitational fields, the emergence of dark matter and dark energy mimicking terms in the context of General Relativity, an underlying relationship between electromagnetic and gravitational radiation, the impossibility of negative mass solutions that would generate repulsive gravitational fields or antigravity, and a method for quantizing the charge and mass of particle-like solutions.

Keywords: Maxwell's equations, General Relativity, unification of electromagnetism and gravitation, dark matter, dark energy

CONTENTS

1. INTRODUCTION

Electromagnetic and gravitational fields have long range interactions characterized by speed of light propagation; similarities that suggest these fields should be coupled together at the classical physics level. Although this coupling or unification is a well-worn problem with many potential solutions having been proposed, it is fair to say that there is still no generally accepted classical field theory that can explain both electromagnetism and gravitation in a coupled or unified framework.[1] Today, the existence of electromagnetic and gravitational fields are generally understood to be distinct and independent with electromagnetism described by Maxwell's field equations and gravitation described by Einstein's General Relativity. The purpose of this manuscript is to assess a recently proposed set of field equations that geometrically couple electromagnetism and gravitation in a fundamentally new way, putting both phenomena on an equal footing with both being intimately tied to the curvature of space-time.

Assuming the geometry of nature is Riemannian with four dimensions, the following four field equations provide a classical field description of physics at the level of the Maxwell and Einstein Field Equations (M&EFEs),[2] but then go further by geometrically coupling electromagnetism and gravity

$$F_{\mu\nu;\kappa} = a^{\lambda} R_{\lambda\kappa\mu\nu} \tag{1}$$

$$a^{\lambda} R_{\lambda}{}^{\nu} = \rho_c u^{\nu} \tag{2}$$

$$u^{\lambda} u_{\lambda} = -1 \tag{3}$$

$$\left(\rho_m u^{\mu} u^{\nu} + F^{\mu}{}_{\lambda} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)_{;\nu} = 0 \tag{4}$$

Both equations (1) and (2) are new, as is the vector field a^{λ} that appears in them and serves to couple electromagnetism to gravity and as will be seen in section 3.3 is also related to the 4-vector potential A^{λ} of classical electromagnetism. Equation (1) couples the derivatives of the Maxwell tensor $F_{\mu\nu}$ to the Riemann-Christoffel (R-C) tensor $R_{\lambda\kappa\mu\nu}$ in a way that is fundamentally new and unique by taking advantage of the indicial symmetries of the R-C tensor. Equation (2) couples charge density ρ_c to the Ricci tensor $R_{\lambda}{}^{\nu}$. Supplementing these first two equations are equations (3) and (4), both of which are well known. Equation (3) normalizes the four-velocity vector field u^{λ} that describes the motion of both the scalar charge density field ρ_c and the scalar mass density field ρ_m, which are assumed to be comoving.

Equation (4) describes the conservation of energy and momentum for a specific choice of the energy-momentum tensor. Much of the discussion that follows is focused on describing solutions to the fundamental field equations (1) through (4) and demonstrating that these solutions are consistent with those of the classical M&EFEs but then go further by unifying electromagnetic and gravitational phenomena.

Taken together, the fundamental field equations (1) through (4) are used to axiomatically build up a description of nature in terms of the six dynamic fields described in Table I.

Table I. Dynamic fields

Field	Description	Number of components
$g_{\mu\nu}$	Metric tensor	10
$F_{\mu\nu}$	Maxwell tensor	6
u^{λ}	Four-velocity vector field	4
a^{λ}	Four-vector field coupling electromagnetism to gravitation	4
ρ_c	Charge density scalar field	1
ρ_m	Mass density scalar field	1
Total number of independent field components		26

An outline of the paper is as follows:

First, a discussion of the consequences of fundamental field equations (1) through (4) is given. Dependent equations that are a consequence of fundamental field equations (1) through (4) are then derived. These dependent equations include Maxwell's field equations, Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and dark energy, a conservation of charge equation, a conservation of mass equation, and the Lorentz force law. The focus of this section is to show in a concise and axiomatic development how the classical M&EFEs which show little resemblance to the fundamental field equations (1) through (4) are in fact a consequence them.

Next, a discussion of the basic mathematical structure of the fundamental field equations (1) through (4) is given. Covered are the logical consistency of the equations from the standpoint of general covariance, the symmetries exhibited by the equations and the physical interpretation of those symmetries, the relationship of the vector field a^{λ} of equations (1) and (2) to the conventional 4-vector potential A^{λ} of classical electromagnetism $\left(F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right)$, and the solvability of the fundamental field equations (1) through (4).

Specific solutions to the fundamental field equations (1) through (4) are reviewed next. Covered are solutions for a non-rotating spherically symmetric charged particle, solutions for electromagnetic and gravitational waves, and a solution describing an isotropic and homogenous universe. The focus of this section is develop solutions of the fundamental field equations (1) through (4) that can be directly compared to well-known solutions of the classical M&EFEs.

Finally, a discussion of the consequences of fundamental field equations (1) through (4) that go beyond the classical M&EFEs is given. Covered in this section are the emergence of gravity described by Einstein's field equation of General Relativity augmented by a term that mimics the properties of dark matter and dark energy, the way in which antimatter naturally emerges from solutions of the fundamental field equations (1) through (4) and its interaction with electromagnetic and gravitational fields, the impossibility of negative mass solutions that would generate repulsive gravitational fields or antigravity, and a mechanism for quantizing the charge and mass of particle-like solutions.

In this manuscript geometric units are used throughout with a metric tensor having signature [+, +, +, -]. Spatial indices run from 1 to 3 with 4 the time index. The notation within uses commas and semicolons before tensor indices to indicate ordinary and covariant derivatives, respectively. For the definitions of the R-C curvature tensor and the Ricci tensor, the conventions used by Weinberg are followed.[3]

2. CONSEQUENCES OF FUNDAMENTAL FIELD EQUATIONS

2.1 Maxwell's equations and conservation of charge

Equations (1) and (2) relate the derivatives of the Maxwell tensor to the R-C tensor and the charge current density to the Ricci tensor, respectively, and are the fundamental relationships from which all of Maxwell's equations flow. Maxwell's homogenous equation is derived using the algebraic property of the R-C tensor

$$R_{\lambda\kappa\mu\nu} + R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} = 0 \ . \qquad (5)$$

Contracting (5) with a^{λ} gives

$$a^{\lambda}R_{\lambda\kappa\mu\nu} + a^{\lambda}R_{\lambda\mu\nu\kappa} + a^{\lambda}R_{\lambda\nu\kappa\mu} = 0 \ . \tag{6}$$

Using (1) to substitute $F_{\mu\nu;\kappa}$ for $a^{\lambda}R_{\lambda\kappa\mu\nu}$ in (6) leads to

$$F_{\mu\nu;\kappa} + F_{\nu\kappa;\mu} + F_{\kappa\mu;\nu} = 0, \tag{7}$$

which is Maxwell's homogenous equation.

Maxwell's inhomogeneous equation follows from (1) by contracting its μ and κ indices

$$F^{\mu\nu}{}_{;\mu} = -a^{\lambda}R_{\lambda}{}^{\nu} \ . \tag{8}$$

Making the connection between the RHS of (8) and the coulombic current density $\rho_c u^{\nu} \equiv J^{\nu}$ using (2) then gives the conventional Maxwell-Einstein version of Maxwell's inhomogeneous equation

$$F^{\mu\nu}{}_{;\mu} = -\rho_c u^{\nu} \ (\equiv -J^{\nu}) \ . \tag{9}$$

Because $F^{\mu\nu}$ is forced to be antisymmetric by (1), the identity $F^{\mu\nu}{}_{;\mu;\nu} = 0$ is forced which in turn forces the coulombic charge to be a conserved quantity

$$\left(\rho_c u^{\nu}\right)_{;\nu} = \left(a^{\lambda}R_{\lambda}{}^{\nu}\right)_{;\nu} = 0 \ . \tag{10}$$

Using equations (2) and (3), the coulombic charge density can be solved for in terms of a^{λ}, u^{λ} and the Ricci tensor,

$$\rho_c = \begin{cases} \pm\sqrt{-a^{\lambda}R_{\lambda}{}^{\nu}a^{\sigma}R_{\sigma\nu}} \\ or \\ -a^{\lambda}R_{\lambda}{}^{\nu}u_{\nu} \end{cases} \ . \tag{11}$$

In the forgoing development, only equations (1), (2) and (3) are fundamental to the new theory. Maxwell's equations (7) and (9), the conservation of the charge (10), and the solution for the coulombic charge density (11) are all consequences of (1), (2) , (3) and the properties of the R-C curvature tensor. These additional equations represent constraints; any solution of (1), (2) and (3) must also satisfy (7), (9) , (10) and (11).

2.2 Lorentz force law and conservation of mass

Now consider the last of the theory's fundamental equations, the energy-momentum conservation equation

$$\left(\rho_m u^\mu u^\nu + F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right)_{;\nu} = 0 \ . \tag{4}$$

The specific form of the energy-momentum tensor in (4) ensures that ρ_m is conserved and that there is a Lorentz force law. These two dependent equations are derived by first contracting (4) with u_μ which leads to the conservation of mass

$$\left(\rho_m u^\nu\right)_{;\nu} = 0 \ , \tag{12}$$

and then combining (4) and (12), which leads to the Lorentz force law

$$\rho_m \frac{Du^\mu}{D\tau} = \rho_c u^\lambda F^\mu{}_\lambda \tag{13}$$

where $\frac{Du^\mu}{D\tau} \equiv u^\mu{}_{;\sigma} u^\sigma$. A more detailed outline of the derivation of (12) and (13) is given in section 8.1 (Appendix I).

2.3 Is electromagnetism as it emerges in the new theory compatible with the classical Maxwell equations?

Equations (1) and (2) are the most important concepts being proposed in this manuscript, fundamentally new and tying the Maxwell tensor $F_{\mu\nu}$ and charge density ρ_c to the R-C curvature tensor and thereby unifying electromagnetic and gravitational phenomena. The cost of this unification is the introduction of a new field a^λ , a field that serves to couple electromagnetic and gravitational phenomena and as will be shown in section 3.3 is related to the 4-vector potential A^λ of classical electromagnetism. The unification that ensues puts electromagnetic and gravitational phenomena on the same footing, with both being intrinsically tied to nonzero curvatures.

On its surface, the central role for nonzero curvature in all electromagnetic phenomena imposed by fundamental field equations (1) through (4) might be construed as problematic due to the prevalent view today that electromagnetic phenomena exist in flat space-time. However, if one were not aware of the vector field a^λ , then the new theory directly morphs into the classical Maxwell's field equations. To see this consider equations (1) through (4) and all equations derived from them that contain a^{λ}. These equations are collected in Table II along with the equations that result after replacing all occurrences of a^{λ} using the substitutions $a^{\lambda}R_{\lambda\kappa\mu\nu} \to F_{\mu\nu;\kappa}$ by fundamental equation (1), and $a^{\lambda}R_{\lambda}^{\ \nu} \to J^{\nu}$ by fundamental equation (2). The updated equations with these substitutions do not reference a^{λ}, do not have an explicit connection to the R-C curvature tensor, and are exactly the classical Maxwell equations used in flat space-time and their consequences.

Table II. Equations containing a^{λ} updated with $a_{\lambda}R^{\lambda}_{\ \kappa\mu\nu} \to F_{\mu\nu;\kappa}$ and $a^{\lambda}R_{\lambda}^{\ \nu} \to J^{\nu}$

Original equation	Equation Number	Updated with Substitutions $a^{\lambda}R_{\lambda\kappa\mu\nu} \to F_{\mu\nu;\kappa}$ $a^{\lambda}R_{\lambda}^{\ \nu} \to J^{\nu}$	Comment
$a^{\lambda}R_{\lambda\kappa\mu\nu} + a^{\lambda}R_{\lambda\mu\nu\kappa} + a^{\lambda}R_{\lambda\nu\kappa\mu} = 0$	(6)	$F_{\mu\nu;\kappa} + F_{\nu\kappa;\mu} + F_{\kappa\mu;\nu} = 0$	Maxwell's homogenous equation
$F^{\mu\nu}_{\ \ ;\mu} = -a^{\lambda}R_{\lambda}^{\ \nu}$	(8)	$F^{\mu\nu}_{\ \ ;\mu} = -J^{\nu}$	Maxwell's inhomogeneous equation
$\left(\rho_c u^{\nu}\right)_{;\nu} = \left(a^{\lambda}R_{\lambda}^{\ \nu}\right)_{;\nu} = 0$	(10)	$J^{\nu}_{\ ;\nu} = 0$	Conservation of charge
$\rho_c = \begin{cases} \pm\sqrt{-a^{\lambda}R_{\lambda}^{\ \nu}a^{\sigma}R_{\sigma\nu}} \\ or \\ -a^{\lambda}R_{\lambda}^{\ \nu}u_{\nu} \end{cases}$	(11)	$\rho_c = \rho_c$	Trivial identity

The approach followed here relating the classical Maxwell field equations to the theory of electromagnetism valid in curved space-time that follows from equations (1) through (4) is fundamentally new and different than previously followed approaches, and in particular does not follow from the accepted flat space-time Maxwell's equations using the "minimal substitution rule".[4], [5] In the approach followed here, the classical Maxwell equations applicable in flat space-time are not recoverable from the fundamental field equations (1) through (4) using the usual principal of equivalence argument in which the flat space-time version of a physical law follows from its curved space-time version by replacing $g_{\mu\nu}$ with the Minkowski metric $\eta_{\mu\nu}$. This is not to say that the theory of electromagnetism being put forth here violates the principle of equivalence, it does not, but rather it implies that the classical Maxwell field equations applicable in flat space-time are at some level an approximation. In the context of the new theory, the view of the classical Maxwell field equations is that they are incomplete, there being a hidden field a^{λ} that has gone unrecognized. By not recognizing the existence of a^{λ} and writing the fundamental field equations and their consequences as is done using the substitutions outlined in Table II, all ties to curved space-time and the vector field a^{λ} are hidden and Maxwell's classical field equations emerge with the appearance of being applicable in flat space-time. However, the classical Maxwell field equations that describe electromagnetic phenomena in flat space-time are an approximation to the theory of electromagnetism that flows from fundamental field equations (1) through (4) and in which curved space-time is strictly required for a complete description of electromagnetic phenomena.

2.4 Is gravitation as it emerges in the new theory compatible with General Relativity?

While evident from the preceding discussion that Maxwell's field equations and the classical physics that flows from them are derivable from the fundamental field equations (1) through (4), at this point it is not obvious that the same can be said of Einstein's equation of General Relativity

$$G^{\mu\nu} = -8\pi\, T^{\mu\nu} \tag{14}$$

where $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is the Einstein tensor. The particle-like solution to be analyzed in section 4.1 demonstrates that the Reissner-Nordström metric is an exact solution of the fundamental field equations (1) through (4), thus establishing that the new theory and classical General Relativity (14) support the same metric field solutions, at least in the case of spherical symmetry, but one must go further to determine if Einstein's field equation is a derivable consequence of the fundamental equations of the new theory.

To investigate this question I start by considering the conserved energy-momentum tensor (4). An immediate consequence of $G^{\mu\nu}$ and $T^{\mu\nu}$ being both symmetric and independently conserved (independently conserved because ${G^{\mu\nu}}_{;\nu} = 0$ by the Bianchi identity and ${T^{\mu\nu}}_{;\nu} = 0$ by (4)) is that for any constant α one can define a tensor field $\Lambda^{\mu\nu}$ by

$$\Lambda^{\mu\nu} \equiv G^{\mu\nu} - \alpha T^{\mu\nu} \; . \tag{15}$$

where $\Lambda^{\mu\nu}$ is constrained to be both symmetric

$$\Lambda^{\mu\nu} = \Lambda^{\nu\mu} \tag{16}$$

and conserved

$$\Lambda^{\mu\nu}{}_{;\nu} = 0 \ . \tag{17}$$

The value of the constant α in (15) is completely arbitrary and without physical significance because $\Lambda^{\mu\nu}$ is defined so that any change in the value of α can be absorbed by a change to $\Lambda^{\mu\nu}$ such that (15) remains satisfied. Taking advantage of this arbitrariness and setting the value of the $\alpha = -8\pi$ gives with a slight rearrangement of (15)

$$G^{\mu\nu} = -8\pi\, T^{\mu\nu} + \Lambda^{\mu\nu} \tag{18}$$

which is recognized as Einstein's equation of General Relativity (14) augmented on its RHS by the term $\Lambda^{\mu\nu}$, a term that from the perspective of classical General Relativity mimics exactly the properties of dark matter and dark energy, *viz.*, it is a conserved and symmetric tensor field, it is a source of gravitational fields in addition to $T^{\mu\nu}$, and it has no interaction signature beyond the gravitational fields it sources.

To the questions posed in the title of this section, "Is Gravitation as it emerges in the new theory compatible with General Relativity?", the answer is that equation (18), Einstein's General Relativity field equation augmented on its RHS by the $\Lambda^{\mu\nu}$ term is a trivial result of fundamental field equations (1) through (4). In fact, the physical validity of (18) rests only on the existence of a conserved energy momentum tensor, as for example the one given in fundamental field equation (4), everything else in the derivation of (18) being just mathematics. This establishes that any solution of fundamental equations (1) through (4) must also be a solution of (18) for some choice of $\Lambda^{\mu\nu}$, a term that in the context of the new theory has no connection to dark matter or dark energy but rather is explained in terms of normal matter and normal energy through full solutions of the fundamental field equations. The emergence of $\Lambda^{\mu\nu}$ will be shown through several specific solutions to the fundamental field equations (1) through (4) investigated in section 4.

3. MATHEMATICAL STRUCTURE OF FUNDAMENTAL FIELD EQUATIONS

3.1 Logical consistency of fundamental field equations from the standpoint of General Covariance

Table III collects and summarizes the four fundamental equations of the new theory, along with the number of components of each equation.

Table III. Fundamental equations

Equation	Equation number in text	Number of components
$F_{\mu\nu;\kappa} = a^{\lambda} R_{\lambda\kappa\mu\nu}$	(1)	24
$a^{\lambda} R_{\lambda}{}^{\nu} = \rho_c u^{\nu}\ (\equiv J^{\nu})$	(2)	4
$u^{\lambda} u_{\lambda} = -1$	(3)	1
$\left(\rho_m u^{\mu} u^{\nu} + F^{\mu}{}_{\lambda} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right)_{;\nu} = 0$	(4)	4
Total number of equations		33

The total number of fundamental component equations listed in Table III is 33 which is greater than the 26 field components listed in Table I, 26 being the number of field components a logically consistent theory must be able to solve for. If all the fundamental component equations listed in Table III were independent, then the field components listed in Table I would be overdetermined and in general no solutions would be expected to exist. However, not all 33 component equations listed in Table III are independent. As already noted, dependent constraint equations were derived from the fundamental field equations (1) through (4) and the properties of the R-C curvature tensor. Table IV collects these dependent constraint equations along with a brief description of their derivation and number of components.

Table IV. Dependent equations

Equation	Equation number in text	Derivation	Number of components
$F_{\mu\nu;\kappa} + F_{\nu\kappa;\mu} + F_{\kappa\mu;\nu} = 0$	(7)	(1) and $R_{\lambda\kappa\mu\nu} + R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} = 0$	4

$\left(\rho_c u^\nu\right)_{;\nu} = \left(a^\lambda R_\lambda{}^\nu\right)_{;\nu} = 0$	(10)	(1) and (2)	1
$\rho_c = \begin{cases} \pm\sqrt{-a^\lambda R_\lambda{}^\nu a^\sigma R_{\sigma\nu}} \\ or \\ -a^\lambda R_\lambda{}^\nu u_\nu \end{cases}$	(11)	(2) and (3)	1
$\left(\rho_m u^\nu\right)_{;\nu} = 0$	(12)	(4), (3) and (7)	1
$\rho_m \frac{Du^\mu}{D\tau} = \rho_c u^\lambda F^\mu{}_\lambda$	(13)	(4), (3) and (12)	4
Total number of equations			11

The 11 dependent constraint equation components listed in Table IV reduce the number of independent fundamental equation components listed in Table III by 11, leaving 33-11=22 independent components of the fundamental field equations (1) through (4). These 22 independent component equations satisfy the requirements of general covariance for determining the 26 independent field components given in Table I. The remaining four degrees of freedom in the solution represent the four degrees of freedom in choosing a coordinate system. To further elucidate the mathematical content of the fundamental field equations (1) through (4), an outline of their solution when viewed as a Cauchy initial value problem is presented in section 8.2 (Appendix II).

3.2 Symmetries of fundamental field equations

Three important symmetries of the fundamental field equations (1) through (4) that are shared by all their solutions are reviewed here. The first of these symmetries corresponds to charge-conjugation

$$\begin{pmatrix} u^\lambda \\ a^\lambda \\ F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix} \rightarrow \begin{pmatrix} u^\lambda \\ -a^\lambda \\ -F^{\mu\nu} \\ g_{\mu\nu} \\ -\rho_c \\ \rho_m \end{pmatrix}, \tag{19}$$

the second corresponds to a matter-to-antimatter transformation as will be discussed in section 5.2

$$\begin{pmatrix} u^\lambda \\ a^\lambda \\ F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix} \to \begin{pmatrix} -u^\lambda \\ -a^\lambda \\ -F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix}, \tag{20}$$

and the third symmetry is the product of the first two

$$\begin{pmatrix} u^\lambda \\ a^\lambda \\ F^{\mu\nu} \\ g_{\mu\nu} \\ \rho_c \\ \rho_m \end{pmatrix} \to \begin{pmatrix} -u^\lambda \\ a^\lambda \\ F^{\mu\nu} \\ g_{\mu\nu} \\ -\rho_c \\ \rho_m \end{pmatrix}. \tag{21}$$

All three transformations (19) through (21) leave the fundamental equations (1) through (4) unchanged. Adding the identity transformation to these symmetries forms the Klein-4 group with the product of any two of the symmetries (19) through (21) giving the remaining symmetry. Note that among the fundamental fields of the theory only $g_{\mu\nu}$ and ρ_m are unchanged by all the symmetry transformations, a fact that will be useful in section 5.4 for defining boundary conditions that lead to quantized mass, charge and angular momentum of particle-like solutions, as well as for the treatment of antimatter.

3.3 Relationship of a^λ to the conventional electromagnetic 4-vector potential A^λ

One of the new pieces of physics in the foregoing development is the introduction of the vector field a^λ, a vector field that has no counterpart in the conventionally accepted development of classical physics but here serves to couple the derivatives of the Maxwell tensor to the R-C tensor through (1) and the charge density to the Ricci tensor through (2). In some respects a^λ also appears to play a role similar to the conventional 4-vector potential A^λ of classical electromagnetism

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}\,. \tag{22}$$

The mathematical connection between a^λ and A^λ can be seen by differentiating (22), giving

$$F_{\mu\nu;\kappa} = A_{\nu;\mu;\kappa} - A_{\mu;\nu;\kappa} \tag{23}$$

and comparing it to equation (1) rewritten as

$$F_{\mu\nu;\kappa} = a^{\lambda} R_{\lambda\kappa\mu\nu} = -a_{\kappa;\mu;\nu} + a_{\kappa;\nu;\mu} \tag{24}$$

where the RHS follows from the commutation property of covariant derivatives. Equating the RHSs of equations (23) and (24) gives

$$a_{\kappa;\nu;\mu} - a_{\kappa;\mu;\nu} = A_{\nu;\mu;\kappa} - A_{\mu;\nu;\kappa} \tag{25}$$

establishing the connection between a^{λ} and A^{λ}.

3.4 A system of first order partial differential equations – Do solutions exist?

Equation (1) represents a mixed system of first order partial differential equations for $F_{\mu\nu}$ and illustrates one of the complexities of the fundamental field equations (1) through (4) that must be dealt with when attempting to find solutions. Specifically, mixed systems of first order partial differential equations must satisfy integrability conditions if solutions are to exist.[6] Although there are several ways of stating what these integrability conditions are, perhaps the simplest is given by,

$$F_{\mu\nu;\kappa;\lambda} - F_{\mu\nu;\lambda;\kappa} = -F_{\mu\sigma} R^{\sigma}{}_{\nu k\lambda} - F_{\sigma\nu} R^{\sigma}{}_{\mu\kappa\lambda} \tag{26}$$

This condition can be derived using the commutation relations for covariant derivatives, the analogue of the commutation property for ordinary derivatives. Using (1) to substitute for $F_{\mu\nu;\kappa}$ in (26) gives,

$$\left(a^{\rho} R_{\rho\kappa\mu\nu}\right)_{;\lambda} - \left(a^{\rho} R_{\rho\lambda\mu\nu}\right)_{;k} = -F_{\mu\sigma} R^{\sigma}{}_{\nu k\lambda} - F_{\sigma\nu} R^{\sigma}{}_{\mu\kappa\lambda} \tag{27}$$

which can be interpreted as conditions that are automatically satisfied by any solution consisting of expressions for $g_{\mu\nu}$, a^{λ} and $F_{\mu\nu}$ that satisfy (1). With (27) as integrability conditions that must be satisfied by any solution of (1), the question that naturally arises is this: Are these integrability conditions so restrictive that perhaps no solution exists to the proposed theory? Although this view could be construed as making the proposed field theory based on the fundamental field equations (1) through (4) uninteresting because perhaps no solutions exist, it will be shown that solutions that are consistent with known solutions of the M&EFEs can be found. Finally, to further elucidate this and other questions regarding solutions of the fundamental field equations (1) through (4), and to outline how they could be solved numerically, section 8.2 (Appendix II) presents an analysis of (1) through (4) in terms of a Cauchy initial value problem.

4. SOLUTIONS TO FUNDAMENTAL FIELD EQUATIONS

4.1 Spherically symmetric charged particle

In this section a solution representing a non-rotating, spherically symmetric charged mass is investigated. It is demonstrated that the Reissner-Nordström metric with an appropriate choice for the fields $F_{\mu\nu}, a^{\lambda}, u^{\lambda}, \rho_c$ and ρ_m satisfy the fundamental field equations (1) through (4). Although the presentation in this section is purely formal, it is included here for several reasons. First, if the theory could not describe the asymptotic electric and gravitational fields of a charged particle it would be of no interest on physical grounds. Second, as already discussed the presented theory requires the solution of a mixed system of first order partial differential equations, a system that may be so restrictive that no solutions exist and so at least a mechanical outline of one methodology to a solution is warranted. Finally, having an exact solution to field equations (1) through (4) that corresponds to a known solution of the M&EFEs enables a direct comparison of the two solutions to be made.

To proceed, I draw on a solution for a spherically symmetric charged particle that was previously derived.[7] Starting with the Reissner-Nordström metric,[8]

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{1+\frac{q^2}{r^2}-\frac{2m}{r}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\mathrm{Sin}[\theta]^2 & 0 \\ 0 & 0 & 0 & -1-\frac{q^2}{r^2}+\frac{2m}{r} \end{pmatrix}, \tag{28}$$

and a guess for a^{λ},

$$a^{\lambda} = (0,0,0,c_1), \tag{29}$$

where c_1 is a yet to be determined constant, ρ_c is determined from the first form of (11), $\rho_c = \pm\sqrt{-a^{\lambda}R_{\lambda}^{\ \nu}a^{\sigma}R_{\sigma\nu}}$, to be

$$\rho_c = \pm \frac{q^2\sqrt{q^2 + r(r-2m)}}{r^5}\left|c_1\right|. \tag{30}$$

Using equation (2), $a^{\lambda}R_{\lambda}^{\ \nu} = \rho_c u^{\nu}$, u^{λ} is then found to be

$$u^{\lambda} = \left(0,0,0,\pm\frac{r}{\sqrt{q^2 + r(r-2m)}}\frac{c_1}{\left|c_1\right|}\right). \tag{31}$$

The next step is to satisfy (1) by solving for $F_{\mu\nu}$. Rather than tackle this head on by directly trying to find a solution to the mixed system of first order partial differential equations (1), I solve the integrability equations (27) which are linear in $F_{\mu\nu}$ for $F_{\mu\nu}$. Proceeding in this manner I find all the integrability equations are satisfied for $F_{\mu\nu}$ given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_{\phi} & -B_{\theta} & E_r \\ -B_{\phi} & 0 & B_r & E_{\theta} \\ B_{\theta} & -B_r & 0 & E_{\phi} \\ -E_r & -E_{\theta} & -E_{\phi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{(mr-q^2)}{r^3}c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{(mr-q^2)}{r^3}c_1 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

By direct substitution it is easily verified that $F_{\mu\nu}$ as given in (32) is a solution of (1). Choosing $c_1 = q/m$ then gives an electric field which agrees with the Coulomb field of a point charge to leading order in $1/r$. Finally, the remaining unknown field, the scalar mass density field ρ_m is found using (4) the conservation of energy-momentum equation. Substituting the known fields into (4) and solving for ρ_m gives

$$\rho_m = \frac{q^4(q^2 - 2mr + r^2)}{m^2 r^6}. \tag{33}$$

To summarize, the following expressions for $g_{\mu\nu}, F_{\mu\nu}, a^{\lambda}, u^{\lambda}, \rho_c$ and ρ_m are an exact solution to the fundamental field equations (1) through (4)

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{1-\frac{2m}{r}+\frac{q^2}{r^2}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\mathrm{Sin}[\theta]^2 & 0 \\ 0 & 0 & 0 & -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right) \end{pmatrix}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_\phi & -B_\theta & E_r \\ -B_\phi & 0 & B_r & E_\theta \\ B_\theta & -B_r & 0 & E_\phi \\ -E_r & -E_\theta & -E_\phi & 0 \end{pmatrix} = s\begin{pmatrix} 0 & 0 & 0 & \frac{q}{r^2}-\frac{q^3/m}{r^3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{q}{r^2}+\frac{q^3/m}{r^3} & 0 & 0 & 0 \end{pmatrix}$$

$$u^\lambda = s\left(0,0,0,\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right)$$

$$a^\lambda = s\left(0,0,0,\frac{q}{m}\right)$$

$$\rho_c = \frac{q^3}{m}\frac{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{r^4} \qquad (34)$$

$$\rho_m = \frac{q^4}{m^2}\frac{(1-\frac{2m}{r}+\frac{q^2}{r^2})}{r^4}$$

In (34) I've introduced a parameter s where $s=\pm1$, which as will be explained in section 5.2 distinguishes between matter ($s=+1$) and antimatter ($s=-1$). Except for this possibility of both matter and antimatter solutions, the physical interpretation of solution (34) is identical to that of the M&EFEs, *i.e.*, a non-rotating, spherically symmetric particle having charge q and mass m. Of note is the metric tensor which is identical to the Reissner-Nordström metric, establishing that the new theory predicts a gravitational field identical to the prediction of Einstein's General Relativity.

Also of note in the solution (34) is the form of the radial electric field

$$E_r = \frac{q}{r^2}-\frac{q^3/m}{r^3} = \frac{q}{r^2}\left(1-\frac{q^2/m}{r}\right), \qquad (35)$$

which agrees with the coulomb field q/r^2 to leading order in $1/r$. Going beyond the leading order coulomb field, the next term in the radial electric field depends on both the charge and mass of the particle. Taking an electron as an example, its electric field as given by (35) is

$$E_r = \frac{q_e}{r^2}\left(1-\frac{r_e}{r}\right) \approx \frac{q_e}{r^2}\left(1-\frac{2.82x10^{-15}m}{r}\right), \tag{36}$$

where $r_e = q_e^{\,2}/m_e$ is recognized as the classical radius of an electron (~2.82x10^{-15}m). Although the correction term to the coulomb field is small, only 53 ppm at a distance of a Bohr radius it may have interesting consequences in various situations because it depends on both the charge and the mass of the particle. For example, considering a hydrogen atom the correction term to the electron's coulomb field would be larger than that of the proton's by $m_{proton}/m_{electron} \approx 1836$, generating a slight imbalance (non-canceling) in the asymptotic electric fields originating from the hydrogen's electron and proton.

At least for the case of spherical symmetry, (34) demonstrates that the fundamental field equations (1) through (4) and the M&EFEs have consistent metric field solutions. However, the new theory's solution (34) goes further than the M&EFE's solution by specifying the spatial dependence of both the charge density ρ_c and the mass density ρ_m, solutions which themselves then completely define the spatial dependence of the energy-momentum tensor

$$T^{\mu\nu} = \rho_m u^\mu u^\nu + F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \,. \tag{37}$$

By direct substitution it can be verified that the resulting spatial dependence of the energy momentum tensor (37) along with the Reissner-Nordström metric (28) do not satisfy Einstein's equation of General Relativity (14). However, Einstein's equation of General Relativity augmented by the $\Lambda^{\mu\nu}$ term on its RHS (18) is trivially satisfied. For completeness the values of $G^{\mu\nu}$, $T^{\mu\nu}$ and $\Lambda^{\mu\nu}$ that go with solution (34) are given here

$$
\begin{gathered}
G^{\mu\nu} = \begin{pmatrix} \frac{q^2\left(q^2+r(-2m+r)\right)}{r^6} & 0 & 0 & 0 \\ 0 & -\frac{q^2}{r^6} & 0 & 0 \\ 0 & 0 & -\frac{q^2\csc[\theta]^2}{r^6} & 0 \\ 0 & 0 & 0 & -\frac{q^2}{r^2\left(q^2+r(-2m+r)\right)} \end{pmatrix} \\
T^{\mu\nu} = \begin{pmatrix} -\frac{q^2\left(q^2-mr\right)^2\left(q^2+r(-2m+r)\right)}{2m^2r^8} & 0 & 0 & 0 \\ 0 & \frac{q^2\left(q^2-mr\right)^2}{2m^2r^8} & 0 & 0 \\ 0 & 0 & \frac{q^2\left(q^2-mr\right)^2\csc[\theta]^2}{2m^2r^8} & 0 \\ 0 & 0 & 0 & \frac{3q^6+m^2q^2r^2+2q^4r(-3m+r)}{2m^2r^4\left(q^2+r(-2m+r)\right)} \end{pmatrix} \\
\Lambda^{\mu\nu} = \begin{pmatrix} -\frac{q^2\left(q^2+r(-2m+r)\right)\left(-m^2r^2+4\pi\left(q^2-mr\right)^2\right)}{m^2r^8} & 0 & 0 & 0 \\ 0 & -\frac{q^2}{r^6}+\frac{4\pi q^2\left(q^2-mr\right)^2}{m^2r^8} & 0 & 0 \\ 0 & 0 & \frac{q^2\left(-m^2r^2+4\pi\left(q^2-mr\right)^2\right)\csc[\theta]^2}{m^2r^8} & 0 \\ 0 & 0 & 0 & -\frac{q^2\left(m^2r^2-4\pi\left(3q^4+m^2r^2+2q^2r(-3m+r)\right)\right)}{m^2r^4\left(q^2+r(-2m+r)\right)} \end{pmatrix}
\end{gathered}
\tag{38}
$$

In the context of classical General Relativity (14), the interpretation of $\Lambda^{\mu\nu}$ in (38) is that of dark matter and dark energy, a source term for gravitational fields in addition $T^{\mu\nu}$. However, in the context of the new theory, the value of $\Lambda^{\mu\nu}$ depends only on the existence of normal matter and normal energy and is a consequence of fundamental field equations (1) through (4). This description of what would naturally be viewed as dark matter and dark energy in the context of General Relativity but is defined in terms of normal matter and normal energy in the new theory will be discussed more fully in section 5.1.

Another feature of the spatial distributions for both the charge and mass density being specified as part of the solution to the fundamental field equations (1) through (4) is that it gives a mechanism for quantizing both the mass m and charge q parameters of the Reisner-Nordstrom metric (28). By requiring that both the mass m and charge q be consistent with the spatially integrated mass density and charge density, respectively, self-consistency equations can be derived that put additional requirements on the physically allowable values these parameters can have, a topic that will be picked up in section 5.4.

Finally, an interesting constraint on particle-like solutions with metrics that depend explicitly on a charge parameter q such as the Reissner-Nordström metric used in (34) follows from the charge-conjugation symmetry (19) of fundamental field equations (1) through (4). The charge conjugation symmetry transformation (19) takes $g_{\mu\nu} \to g_{\mu\nu}$ and $\rho_c \to -\rho_c$, or equivalently $q \to -q$ as will be justified in section 5.4. This forces the conclusion that the sign of q has no impact on the metric, *i.e.*, the metric can only depend on the absolute value of q (or the absolute value of ρ_c) since it is unchanged by the transformation $q \to -q$. This result, which is applicable to all solutions of fundamental field equations (1) through (4) is also in-line with known charge containing solutions of Einstein's field equation such as the Reissner-Nordström and Kerr-Newman metrics, both of which depend on q^2.

4.2 Electromagnetic radiation

Working in the weak field limit, derived here are expressions for a propagating electromagnetic plane wave in terms of the vector field a^λ and the metric tensor $g_{\mu\nu}$. This example is useful because it establishes a fundamental relationship between electromagnetic and gravitational radiation imposed by the field equations (1) through (4) and predicts that electromagnetic and gravitational waves are both manifestations of wave propagation of the underlying metric $g_{\mu\nu}$. To begin, consider an electromagnetic plane wave having frequency ω, propagating in the +z-direction and polarized in the x-direction. The Maxwell tensor for this field is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & -B_y & E_x \\ 0 & 0 & 0 & 0 \\ B_y & 0 & 0 & 0 \\ -E_x & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} \tag{39}$$

where E_x and B_y are the constant field amplitudes. Assume a near-Minkowski weak field metric

$$\begin{gathered} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} e^{i\omega(t-z)} \\ \left| h_{\mu\nu} \right| \ll 1 \end{gathered} \tag{40}$$

where $\eta_{\mu\nu} = diag[1,1,1,-1]$, the $h_{\mu\nu}$ are complex constants, and the vector field a^λ is also assumed to be constant

$$a^{\lambda} = \left(a^1, a^2, a^3, a^4\right) \quad . \tag{41}$$

I proceed by substituting for $F_{\mu\nu}$ from (39), $g_{\mu\nu}$ from (40) and a^{λ} from (41) into (1), and then only retain terms to first order in the fields $h_{\mu\nu}$ and $F_{\mu\nu}$ which are both assumed to be small and of the same order.[9] Doing this leads to a set of 8 independent linear equations for the 16 unknown constants: $h_{\mu\nu}$, a^{λ}, E_x and B_y. Solving these 8 independent equations, the 8 field components E_x, B_y, h_{13}, h_{22}, h_{23}, h_{34}, a^2 and a^3 can be solved for in terms of 8 free constants a^1, a^4, h_{11}, h_{12}, h_{14}, h_{24}, h_{33}, and h_{44}

$$\begin{aligned} E_x &= i\,\omega \frac{\left(h_{11}{}^2 + h_{12}{}^2\right)}{2\,h_{11}} a^1 \quad , \\ B_y &= E_x \end{aligned} \tag{42}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & -h_{14} & h_{14} \\ h_{12} & -h_{11} & -h_{24} & h_{24} \\ -h_{14} & -h_{24} & h_{33} & -\frac{1}{2}\left(h_{33}+h_{44}\right) \\ h_{14} & h_{24} & -\frac{1}{2}\left(h_{33}+h_{44}\right) & h_{44} \end{pmatrix} e^{i\omega(t-z)} \quad , \tag{43}$$

and

$$a^{\lambda} = \left(a^1, a^1\frac{h_{12}}{h_{11}}, a^4, a^4\right) \quad . \tag{44}$$

This solution illustrates several ways in which the new theory departs from the traditional view of electromagnetic radiation. Of most significance, the undulations in the electromagnetic field are due to undulations in the metric field $g_{\mu\nu}$ (43) via the coupling defined in (1). This result also underscores that the existence of electromagnetic radiation is forbidden in strictly flat space-time. An interesting aspect of this solution is that while electromagnetic radiation necessitates the presence of an underlying gravitational radiation field, the underlying gravitational radiation is not completely defined by the electromagnetic radiation. The supporting gravitational radiation has 6 undetermined constants $\left(h_{11}, h_{12}, h_{14}, h_{24}, h_{33}, h_{44}\right)$ with the only restriction being $\left|h_{\mu\nu}\right| \ll 1$ and $h_{11} \neq 0$ as required by (42).

Further insight into the content of the metric (43) is evident after making the infinitesimal coordinate transformation from $x^{\mu} \rightarrow x'^{\mu}$ given by

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x + \dfrac{i}{\omega} h_{14}\, e^{i\omega(t-z)} \\ y + \dfrac{i}{\omega} h_{24}\, e^{i\omega(t-z)} \\ z - \dfrac{i}{2\omega} h_{33}\, e^{i\omega(t-z)} \\ t - \dfrac{i}{2\omega} h_{44}\, e^{i\omega(t-z)} \end{pmatrix} \tag{45}$$

and only retaining terms to first order in the *h's*. Doing this, the metric (43) is transformed to

$$g'_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & -h_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} \ , \tag{46}$$

while E'_x and B'_y, the transformed electric and magnetic field amplitudes, respectively, are identical to E_x and B_y given in (42). Note, only the h_{11} and h_{12} components of the metric (46) have an absolute physical significance, and $h_{22} = -h_{11}$ which makes the plane wave solution (46) identical to the gravitational plane wave solution of the classical Einstein field equations.[10], [11]

Because the underlying gravitational wave couples to both charged and uncharged matter, one consequence of the solution here is that there will be an uncertainty when describing the interaction of electromagnetic radiation with matter if the gravitational wave component of the problem is ignored. However, for nonrelativistic matter, this gravitational interaction (46) vanishes to first order in the *h's*. To see this, consider the following expansion of the Lorentz force law

$$\begin{gathered} \rho_m \frac{Du^{\mu}}{D\tau} = \rho_c u^{\lambda} F^{\mu}{}_{\lambda} \\ \downarrow \\ \rho_m \frac{du^{\mu}}{d\tau} = -\rho_m u^{\nu} u^{\lambda} \Gamma^{\mu}{}_{\nu\lambda} + \rho_c u^{\lambda} F^{u}{}_{\lambda} \end{gathered} \quad . \tag{47}$$

The first term on the RHS in the line above represents the gravitational interaction. This gravitational interaction term vanishes for nonrelativistic matter with $u^\lambda \approx (0,0,0,1)$ because for the metric (46) all the $\Gamma^\mu{}_{44}$ vanish to first order in the *h's*.

4.3 Gravitational radiation

The forgoing analysis demonstrates the necessity of having an underlying gravitational wave to support the presence of an electromagnetic wave, but the converse is not true and gravitational radiation can exist independent of any electromagnetic radiation. The following analysis demonstrates this by solving for the structure of gravitational radiation in the absence of electromagnetic radiation. Following the same weak field formalism for the unknown fields $h_{\mu\nu}$ given in (40), but this time zeroing out E_x and B_y in (39), leads to the following solutions for $g_{\mu\nu}$ and a^λ

$$g_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & -h_{14} & h_{14} \\ h_{12} & \dfrac{h_{12}^{\ 2}}{h_{11}} & -h_{24} & h_{24} \\ -h_{14} & -h_{24} & h_{33} & -\dfrac{h_{33}+h_{44}}{2} \\ h_{14} & h_{24} & -\dfrac{h_{33}+h_{44}}{2} & h_{44} \end{pmatrix} e^{i\omega(t-z)} \tag{48}$$

and

$$a^\lambda = \left(a^1, -a^1 \frac{h_{11}}{h_{12}}, a^4, a^4 \right) . \tag{49}$$

Both $g_{\mu\nu}$ given by (48) and a^λ given by (49) are modified from their solutions in the presence of an electromagnetic wave as given by (43) and (44), respectively. Performing a transformation to the same primed coordinate system as given in (45) here gives the metric field

$$g'_{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & \dfrac{h_{12}^{\ 2}}{h_{11}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} \tag{50}$$

illustrating again that only the h_{11} and h_{12} components have an absolute physical significance. The interaction of nonrelativistic matter with the gravitational wave (50) vanishes to first order in the *h's* for the same reason that it vanished for the gravitational wave (46) that accompanies electromagnetic radiation. Of particular note is the change in the value of the h_{22} component depending on whether the gravitational wave supports an electromagnetic wave as in (46) or is standalone as in (50).

One of the successes of fundamental field equations (1) through (4) is the existence of solutions describing both electromagnetic and gravitational radiation, unifying both phenomena as undulations of the underlying metric field $g_{\mu\nu}$. In some respects this is not too surprising, equation (1) with $F_{\mu\nu}=0$ is a system of second order partial differential equations $a^{\lambda}R_{\lambda\kappa\mu\nu}=0$ in the metric field components $g_{\mu\nu}$ just as Einstein's field equation is, so the fact that both sets of field equations give similar solutions for gravitational waves is not to be completely unexpected. Finally, because both gravitational and electromagnetic radiation are due to undulations of the metric field $g_{\mu\nu}$ in the new theory, the speed of propagation of these waves is predicted to be identical, a result that has recently been refined experimentally with observations made during a binary neutron star merger in NGC 4993, 130 million light years from Earth.[12] The nearly simultaneous detection, within 2 seconds of each other, of gravity waves[13] and a burst of gamma rays[14] from this event constrain the propagation speed of electromagnetic and gravitational radiation to be the same to better than 1 part in 10^{15}.

4.4 Isotropic and homogenous universe

As shown in a previous section, the M&EFEs and fundamental field equations (1) through (4) share particle-like solutions having similar character. However, when considering non-static metrics, differences between the predictions of the two theories start to emerge. To illustrate some of these differences, here I investigate the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

$$g_{\mu\nu}=\begin{pmatrix} \frac{R_{cs}{}^{2}(t)}{1-kr^{2}} & 0 & 0 & 0 \\ 0 & R_{cs}{}^{2}(t)r^{2} & 0 & 0 \\ 0 & 0 & R_{cs}{}^{2}(t)r^{2}\,\mathrm{Sin}^{2}(\theta) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{51}$$

where *k* equals +1, 0 or -1 depending on whether the spatial curvature is positive, zero or negative, respectively, and $R_{cs}(t)$ is the cosmic scale factor. Just as in the case of Einstein's equation of General

Relativity where the FLRW metric is a cosmological solution representing a homogenous and isotropic universe, it is the same for fundamental field equations (1) through (4) with an appropriate choice for the time development of the cosmic scale parameter $R_{cs}(t)$. To derive the time dependence of the cosmic scale factor I start by noting that the 3-dimensional spatial subspace of (51) is maximally symmetric and so any tensor fields that inhabit that subspace must also be maximally symmetric.[15] Specifically, this restricts the form of a^{μ} to be

$$a^{\mu}=\left(0,0,0,a^{4}\left(t\right)\right), \tag{52}$$

and forces the antisymmetric Maxwell tensor to vanish,

$$F_{\mu\nu}=0\ . \tag{53}$$

Because $F_{\mu\nu}$ vanishes so must $F_{\mu\nu;\kappa}$

$$F_{\mu\nu;\kappa}=0\ , \tag{54}$$

which on substitution in (1) forces

$$a^{\lambda}R_{\lambda\kappa\mu\nu}=0\,. \tag{55}$$

This in turn forces

$$a^{\lambda}R_{\lambda}^{\ \nu}=0, \tag{56}$$

which is just equation (2) with $\rho_c=0$. Substituting a^{μ} given by (52), and the FLRW metric given by (51) into (55) then leads to the following set of equations to be satisfied

$$\begin{aligned}
a^{4}(t)R_{4114} &= a^{4}(t)\left(\frac{R_{cs}(t)}{k\,r^{2}-1}\frac{d^{2}R_{cs}(t)}{dt^{2}}\right)=0\\
a^{4}(t)R_{4224} &= a^{4}(t)\left(-r^{2}\,R_{cs}(t)\frac{d^{2}R_{cs}(t)}{dt^{2}}\right)=0\\
a^{4}(t)R_{4334} &= a^{4}(t)\ \left(-r^{2}\,R_{cs}(t)\ \mathrm{Sin}^{2}(\theta)\frac{d^{2}R_{cs}(t)}{dt^{2}}\right)=0
\end{aligned} \tag{57}$$

with all other components of (55) not listed in (57) being trivially satisfied, *i.e.*, $0=0$. The nontrivial component equations (57) are all satisfied if

$$\frac{d^2 R_{cs}(t)}{dt^2} = 0 \qquad (58)$$

or

$$R_{cs}(t) = R_{cs\,0} + v_{cs}\, t \ , \qquad (59)$$

where $R_{cs\,0}$ is the cosmic scale factor at *t=0* and v_{cs} is the rate of change of the cosmic scale factor. The solution for $R_{cs}(t)$ given in (59) ensures that the metric (51) satisfies both (55) and (56) for all values of *k*. Based on this solution, the predictions of the new theory for a homogenous and isotropic universe are:

1. It must be charge neutral, *i.e.,* $\rho_c = 0$.
2. The cosmic scale factor changes linearly with cosmic time.

The second prediction above runs counter to results of the Friedmann models of classical General Relativity in which the growth of the cosmic scale factor is divided into three regimes: the radiation dominated regime with the scale factor growing as $t^{1/2}$, the matter dominated regime with the scale factor growing as $t^{2/3}$, and the dark energy dominated regime with the scale factor growing exponentially with time. This emphasizes one of the challenges facing the classical field theory based on fundamental equations (1) through (4), that of finding additional solutions that are in agreement with the interpretation of recent observations and analyses indicating an accelerating universe.

Just as in the case of the spherically symmetric particle-like solution analyzed in section 4.1, the cosmological solution of fundamental field equations (1) through (4) analyzed here must satisfy Einstein's equation of General Relativity augmented by a $\Lambda^{\mu\nu}$ term on its RHS (18). For the cosmological solution considered here it is straight forward to calculate the $G^{\mu\nu}, T^{\mu\nu}$ and $\Lambda^{\mu\nu}$ by taking the energy-momentum tensor to be

$$T^{\mu\nu} = (p + \rho_m) u^{\mu} u^{\nu} + p\, g^{\mu\nu} \qquad (60)$$

and the four-velocity vector field to be

$$u^{\mu} = (0,0,0,1) \ , \qquad (61)$$

where the form of the energy-momentum tensor (60) as a perfect fluid with pressure p and mass-density ρ_m and the form of the u^μ (61) are both dictated by the requirement that they be maximally symmetric in the 3-dimensional spatial subspace of (51). For completeness the values of $G^{\mu\nu}, T^{\mu\nu}$ and $\Lambda^{\mu\nu}$ that go with (59), (60) and (61) are given here

$$G^{\mu\nu}=\begin{pmatrix} \frac{k-k^2r^2+v_{CS}{}^2-kr^2v_{CS}{}^2}{(R_{CS0}+t\,v_{CS})^4} & 0 & 0 & 0 \\ 0 & \frac{k+v_{CS}{}^2}{r^2(R_{CS0}+t\,v_{CS})^4} & 0 & 0 \\ 0 & 0 & \frac{(k+v_{CS}{}^2)\csc[\theta]^2}{r^2(R_{CS0}+t\,v_{CS})^4} & 0 \\ 0 & 0 & 0 & -\frac{3(k+v_{CS}{}^2)}{(R_{CS0}+t\,v_{CS})^2} \end{pmatrix}$$

$$T^{\mu\nu}=\begin{pmatrix} \frac{(1-kr^2)\,p(t)}{(R_{CS0}+t\,v_{CS})^2} & 0 & 0 & 0 \\ 0 & \frac{p(t)}{r^2(R_{CS0}+t\,v_{CS})^2} & 0 & 0 \\ 0 & 0 & \frac{\csc[\theta]^2\,p(t)}{r^2(R_{CS0}+t\,v_{CS})^2} & 0 \\ 0 & 0 & 0 & \rho_m(t) \end{pmatrix}$$

$$\Lambda^{\mu\nu}=\begin{pmatrix} -\frac{(-1+kr^2)\left(k+v_{CS}{}^2+8\pi(R_{CS0}+t\,v_{CS})^2\,p(t)\right)}{(R_{CS0}+t\,v_{CS})^4} & 0 & 0 & 0 \\ 0 & \frac{k+v_{CS}{}^2+8\pi(R_{CS0}+t\,v_{CS})^2\,p(t)}{r^2(R_{CS0}+t\,v_{CS})^4} & 0 & 0 \\ 0 & 0 & \frac{\csc[\theta]^2\left(k+v_{CS}{}^2+8\pi(R_{CS0}+t\,v_{CS})^2\,p(t)\right)}{r^2(R_{CS0}+t\,v_{CS})^4} & 0 \\ 0 & 0 & 0 & -\frac{3(k+v_{CS}{}^2)}{(R_{CS0}+t\,v_{CS})^2}+8\pi\rho_m(t) \end{pmatrix} \quad (62)$$

Just as was the case for the spherically symmetric particle-like solution studied in section 4.1, in the context of classical General Relativity (14) the interpretation of $\Lambda^{\mu\nu}$ in (62) is that of dark matter and dark energy, a source of gravitational fields in addition to those generated by $T^{\mu\nu}$. However, in the context of the new theory, the value of $\Lambda^{\mu\nu}$ depends only on the existence of normal matter and normal energy and is a consequence of fundamental field equations (1) through (4).

5. DISCUSSION

5.1 Dark matter and dark energy

Dark matter and dark energy are postulated to exist because of the many galactic and cosmological scale observations that cannot be understood using General Relativity with normal matter and normal energy alone. Most notably, observations of some of the large-scale gravitational features of galaxies and galactic clusters dating back to Zwicky's observations in the 1930's have been explained using dark matter[16], and the acceleration of the universe discovered in the 1990's has been explained using dark energy[17]. One of the vexing problems facing these dark matter and dark energy-based explanations is an ongoing inability to directly detect such forms of matter and energy. This has led to dark matter and dark energy distributions being defined to justify observations that are not explainable using normal matter and normal energy alone, a development amounting to what is essentially an ad hoc correction to General Relativity's predictions based on only normal matter and normal energy. For example, the cosmological constant term $\lambda g_{\mu\nu}$ that Einstein added to the RHS of his original field equation

$$G_{\mu\nu} = -8\pi T_{\mu\nu} + \lambda g_{\mu\nu} \tag{63}$$

to enable a solution for a static universe but then dropped after it was discovered that the universe was expanding is today seen as a possible representation of dark energy.

Fundamental field equations (1) through (4) offer the prospect that dark matter and dark energy effects can be explained in terms of normal matter and normal energy, *i.e.*, the $\Lambda^{\mu\nu}$ term representing dark matter and dark energy in the context of General Relativity is provided with a mechanism for directly calculating its structure using fundamental field equations (1) through (4) and only normal matter and normal energy. The already investigated spherically symmetric particle-like solution which assumed a Reissner-Nordström metric and the cosmological solution which assumed an FLRW metric are two accessible examples that outline such a direct calculation of $\Lambda^{\mu\nu}$. With questions today regarding the validity of classical General Relativity beyond the confines of our own solar system,[18] the possible explanation of the $\Lambda^{\mu\nu}$ term in (18) is an enticing feature of fundamental field equations (1) through (4).

Finally, it must be acknowledged that one of the challenging tasks facing the field theory based on fundamental equations (1) through (4), and one well beyond the analysis presented in this manuscript, is that of finding additional solutions that could be interpreted as being in agreement with the rapidly developing observational understanding of galactic and cosmological structures that are presently explained by invoking the existence of dark matter and dark energy.

5.2 Emergence of antimatter from solutions of fundamental field equations and behavior in electromagnetic and gravitational fields

One of the unique features of fundamental field equations (1) through (4) as a classical field theory is that the properties of matter and antimatter emerge naturally in solutions. For example, every matter containing solution to equations (1) through (4) has a corresponding antimatter solution generated by the symmetry transformation (20). This is evident in the spherically symmetric particle-like solution (34) where the multiplicative factor s in the expressions for $F_{\mu\nu}$, a^{λ} and u^{λ} is defined by

$$s = \begin{cases} +1 \ for\ matter \\ -1 \ for\ antimatter \end{cases} \tag{64}$$

and accounts for the matter-antimatter symmetry expressed in (20). The physical interpretation is the $s=-1$ solution represents a particle having the same mass but opposite charge and four-velocity as the $s=+1$ solution. This is equivalent to the view today that a particle's antiparticle is the particle moving backwards through time.[19] Said another way, the time-like component of the four-velocity is positive for matter and negative for antimatter

$$u^4 \begin{cases} >0 \ for\ matter \\ <0 \ for\ antimatter \end{cases} . \tag{65}$$

With these definitions for the four-velocity of matter and antimatter, charged mass density can annihilate similarly charged anti-mass density and satisfy both the local conservation of charge (10) and local conservation of mass (12). Additionally, such annihilation reactions must conserve total energy by (4).

Building on the distinction between matter and antimatter, their behavior in electromagnetic and gravitational fields is now investigated. As already mentioned, antimatter can be viewed as matter moving backwards through time. To see this more rigorously consider the four-velocity associated with a fixed quantity of charge and mass density

$$u^{\lambda} = \frac{dx^{\lambda}}{d\tau} . \tag{66}$$

Under the matter-antimatter transformation (20), $u^{\lambda} \to -u^{\lambda}$, or equivalently $d\tau \to -d\tau$. This motivates the following expression for the four-velocity in terms of the coordinate time

$$u^{\lambda} = \frac{dx^{\lambda}}{d\tau} = s\gamma \frac{dx^{\lambda}}{dt} = s\gamma \begin{pmatrix} \vec{v} \\ 1 \end{pmatrix} \tag{67}$$

where s is the matter-antimatter parameter defined in (64), $\vec{v} = (v_x, v_y, v_z)$ is the ordinary 3-space velocity of the charge and mass density, and $\gamma = 1/\sqrt{1-\vec{v}^2}$. Equation (67) establishes that corresponding matter and antimatter solutions travel in opposite time directions relative to each other. One of the unusual aspects of the matter-antimatter transformation (20) is that ρ_c does not change sign under the transformation. To see that this is consistent with the usual view in which antiparticles have the opposite charge of their corresponding particles, I use (67) to illustrate the behavior of a charged matter and antimatter density in an electromagnetic field. Consider a region with an externally defined electromagnetic field

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \tag{68}$$

but no, or at least a very weak gravitational field so that $g_{\mu\nu} \approx \eta_{\mu\nu}$ and $\Gamma^{\lambda}{}_{\mu\nu} \approx 0$. Starting with the Lorentz force law (13) and expanding

$$\rho_p \frac{Du^{\mu}}{D\tau} = \rho_c u^{\lambda} F^{\mu}{}_{\lambda}$$

$$\downarrow$$

$$\rho_p s\gamma \frac{du_{\mu}}{dt} = \rho_c F_{\mu\lambda} u^{\lambda}$$

$$\downarrow$$

$$\rho_p s\gamma \frac{d}{dt}\begin{pmatrix} s\gamma\vec{v} \\ -s\gamma \end{pmatrix} = \rho_c \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix} \begin{pmatrix} s\gamma v_x \\ s\gamma v_y \\ s\gamma v_z \\ s\gamma \end{pmatrix}$$

$$\downarrow$$

$$\rho_p \frac{d}{dt}\begin{pmatrix} \gamma\vec{v} \\ \gamma \end{pmatrix} = s\rho_c \begin{pmatrix} \vec{E} + \vec{v}\times\vec{B} \\ \vec{v}\cdot\vec{E} \end{pmatrix} \tag{69}$$

which on the last line above ends up at the conventional form of the Lorentz force law except for the extra factor of s on the RHS. This factor of s in (69) gives the product $s\rho_c$ the appearance that antimatter charge density has the opposite sign to that of matter charge density when interacting with an electromagnetic field.

Next, I investigate the behavior of antimatter in a gravitational field. There is no question about the gravitational fields generated by matter and antimatter, they are identical under the matter-antimatter symmetry (20) as $g_{\mu\nu}$ is unchanged by that transformation. To understand whether antimatter is attracted or repelled by a gravitational field I again go to the Lorentz force law (13), but this time assume there is no electromagnetic field present, just a gravitational field given by a Schwarzschild metric generated by a central mass $m > 0$ that is composed of either matter or antimatter. I explicitly call out $m > 0$ because I am endeavoring to develop a physical theory that axiomatically flows from fundamental field equations (1) through (4) and at this point in the development there is nothing precluding the existence of negative mass density $\rho_m < 0$, a consideration I will return to in section 5.3. Placing a test particle having mass m_{test} composed of either matter or antimatter a distance r from the center of the gravitational field and assuming it to be initially at rest, the trajectory of the test particle is that of a geodesic given by the following development

$$
\begin{gathered}
m_{test}\frac{Du^{\mu}}{D\tau}=0 \\
\downarrow \\
s\gamma\frac{du^{\mu}}{dt}=-\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho} \\
\downarrow \\
s\gamma\frac{d}{dt}\left(s\gamma\frac{d}{dt}\begin{pmatrix} r \\ \theta \\ \phi \\ t \end{pmatrix}\right)=-\Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho}\approx-\Gamma^{\mu}{}_{44}u^{4}u^{4}=-\begin{pmatrix} \left(1-\frac{2m}{r}\right)\frac{m}{r^{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}\left(\frac{s}{\sqrt{1-\frac{2m}{r}}}\right)^{2}=-\begin{pmatrix} \frac{m}{r^{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}s^{2}
\end{gathered}
\tag{70}
$$

where $s=\pm1$ references whether the test particle is composed of matter or antimatter (64). In the last line of (70) I have approximated the RHS using the initial at rest value of the test particle's four-velocity $u^{\mu}=\left(0,0,0,s/\sqrt{1-2m/r}\right)$, and additionally used the fact that the only nonzero $\Gamma^{\mu}{}_{44}$ in a

Schwarzschild metric is $\Gamma^1{}_{44} = \left(1 - \frac{2m}{r}\right) m / r^2$. Simplifying the LHS of the last line in (70) by noting that initially $\gamma = 1$ gives

$$\frac{d^2 r}{dt^2} \approx -\frac{m}{r^2} \tag{71}$$

independent of s, demonstrating that the proposed theory predicts both matter and antimatter test particles will be attracted by the source of the gravitational field, and this regardless of whether the source of the gravitational field is matter or antimatter. The result that the test particle is attracted toward the source of the gravitational field is also independent of whether the test particle's mass m_{test} is positive or negative, this because equation the geodesic trajectory (71) is independent m_{test}.

5.3 Possibility of negative mass solutions and antigravity

As already noted, there appears to be nothing in the fundamental equations (1) through (4) that preclude the possibility of negative mass density $\rho_m < 0$. The existence of negative mass density is equivalent to the existence of antigravity because negative mass density would generate gravitational fields that are repulsive, *viz*., equation (71) with $m < 0$. However, logical inconsistencies are introduced if negative mass density can exist. As just shown, equation (71) with $m > 0$ predicts a test particle at some distance from the origin will feel an attractive gravitational force regardless of whether it is comprised of matter or antimatter and regardless of whether the test particle's mass is positive or negative. Now consider equation (71) with $m < 0$, *i.e.*, the gravitational field is generated by negative mass. Using the same argument as in the previous section, the test particle in this case will feel a repulsive gravitational force regardless of whether the test particle's composition is matter or antimatter and regardless of whether the test particle's mass is positive or negative. These two situations directly contradict each other, making fundamental equations (1) through (4) logically inconsistent if negative mass density were allowed to exist. The only way to avoid this logical contradiction is to require mass density be non-negative always. This condition that mass density ρ_m be non-negative always is also consistent with the symmetry transformations (19) through (21) where it was noted that the field ρ_m does not change sign under any of the symmetry transformations.

It is interesting to note that the existence of negative mass in the context of classical General Relativity has been studied extensively[20], [21] and invoked particularly when trying to find stable particle-like solutions using the conventional Einstein field equations.[22], [23], [24] However, in the context of the present theory the existence of negative mass density leads to a logical contradiction that can only be resolved by requiring mass density be non-negative always, *i.e.*, $\rho_m \geq 0$.

5.4 Proposal for quantizing charge and mass of particle-like solutions

Consider particle-like solutions such as (34), because the mass density and charge density are specified as part of the solution of fundamental field equations (1) through (4), a self-consistency constraint exists on physically allowed solutions that provides a mechanism for quantizing the charge and mass of the solutions. For example, in solution (34) the particle's total charge q and total mass m are parameters in the Reissner-Nordström metric that must agree with the spatially integrated charge and mass density, respectively, for the solution to be self-consistent. For the charge, this amounts to requiring the asymptotic value of the electric field be consistent with the spatially integrated charge density

$$q = \lim_{r \to \infty} r^2 F_{14} = \int \rho_c u^4 \sqrt{\gamma_{sp}}\, d^3x \tag{72}$$

where q is the total charge of the particle and given by the asymptotic value of $r^2 F_{14}$, F_{14} being the radial electric field component of the Maxwell tensor, and γ_{sp} is the determinant of the spatial metric defined by[25]

$$\gamma_{sp\, i\, j} = g_{i\, j} - \frac{g_{i4}\, g_{j4}}{g_{44}} \tag{73}$$

where *i* and *j* run over the spatial dimensions 1, 2 and 3. An analogous quantizing boundary condition for the mass of the particle is arrived at by requiring the asymptotic value of its gravitational field be consistent with the spatially integrated mass density of the solution

$$m = \lim_{r \to \infty} r \frac{1 + g_{44}}{2} = \int \rho_m \left| u^4 \right| \sqrt{\gamma_{sp}}\, d^3x \ . \tag{74}$$

The reason for the absolute value of u^4 in the mass boundary condition (74) but not in the charge boundary condition (72) is the symmetry (20) exhibited by the theory's fundamental field equations (1) through (4) and the requirement that the boundary conditions exhibit that same symmetry. The boundary conditions (72) and (74) represent self-consistency constraints on the charge and the mass, respectively, of any particle-like solution. The proposal here is that these boundary or self-consistency conditions represent additional constraints on physically allowable solutions, conditions beyond the fundamental field equations (1) through (4) which must be satisfied for solutions to be physically realizable. Finally, defining the particle's total charge q and total mass m in terms of the asymptotic expression in (72) and (74), respectively, allows the self-consistency constraints to be applied to metrics with complicated charge and mass distributions, *e.g.*, metrics that cannot be written in terms of a simple total charge q or total mass m parameter.

For the spherically-symmetric solution investigated in (34), the RHS of both (72) and (74) diverge leaving no hope for satisfying these quantization/boundary conditions. The upshot of this observation is that while (34) represents a solution that describes the gravitational and electrical fields of a particle-like solution that formally satisfy the fundamental field equations (1) through (4), (34) cannot represent a physically allowed solution. The possibility of finding solutions that satisfy both the fundamental field equations (1) through (4) and the charge and mass boundary conditions (72) and (74) remains an open question at this point. However, interesting possibilities exist beyond the spherically symmetric solution based on the Reissner-Nordström metric investigated within (34). For example, the modified Reissner-Nordström and modified Kerr-Newman metrics developed by S.M. Blinder[26] give finite values for the LHS of both (72) and (74). Finally, when considering metrics that include nonzero angular momentum, as for example would be required for particles having an intrinsic magnetic field, the same methodology used here to quantize the particle's mass and charge can be used to quantize its angular momentum.

6. CONCLUSION

Based on equations (1) and (2) which couple the Maxwell tensor and charge density to the Riemann-Christoffel curvature tensor in a fundamentally new way, a set of four fundamental field equations has been assembled that encompass classical physics at the level of the M&EFEs but then go further by unifying electromagnetic and gravitational phenomena. The cost of this unification is the introduction of a new vector field a^{λ} that is mathematically related to the 4-vector potential A^{λ} of classical electromagnetism but also serves to couple electromagnetic and gravitational phenomena. A compelling aspect of the classical field theory based on the fundamental field equations (1) through (4) is that both Maxwell's equations of electromagnetism and Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and dark energy are shown to be consequences. The unification that emerges between electromagnetic and gravitational phenomena is demonstrated through several specific solutions: the electric and gravitational fields of a spherically symmetric charged particle, radiative solutions representing both electromagnetic and gravitational waves, and a cosmological solution representing a symmetric and homogeneous universe. Of note to the solutions investigated within are:

1. The emergence of antimatter and its behavior in electromagnetic and gravitational fields.
2. The emergence of an underlying unification of electromagnetic and gravitational radiation in terms of undulations of the metric field.
3. A potential explanation for dark matter and dark energy in the context of classical General Relativity in terms of normal matter and normal energy.
4. The impossibility of negative mass solutions that would generate repulsive gravitational fields or antigravity.
5. A proposed mechanism for the quantization of a particle's charge and mass via self-consistency requirements that emerge for physically realizable solutions.

One of the strengths of the new theory's fundamental field equations (1) through (4), and in fact a guiding principle in their development is that they be logically consistent and satisfy the requirements of general covariance. Another strength of the new theory is the reductionism brought to electromagnetic and gravitational phenomena by treating their sources as dynamic variables rather than external entities as is often done in classical physics, a development which provides a mechanism for the quantization of the mass, charge and angular momentum of particle-like solutions in the context of a classical field theory. Finally, to elucidate the mathematical completeness of the new theory's fundamental field equations, an outline for their numerical solution in the form of a Cauchy initial value problem is given.

The genesis of the work presented within was reported in a preliminary form in reference [2]. The same fundamental field equations and quantizing boundary conditions reviewed here were first reported there. New to this manuscript is the discussion of the symmetries of the fundamental field equations (1) through (4), and based on these symmetries the interpretation of the particle-like solution has been advanced. The derivation of the Einstein's equation of General Relativity augmented by a term that can mimic the properties of dark matter and dark energy is also new to this manuscript as is the discussion of the cosmological solution based on the FLRW metric. The present manuscript also corrects an error in the weak field analysis of reference [2] leading to the expanded discussion of electromagnetic radiation and its underlying gravitational radiation. The discussion of the impossibility of negative mass solutions and antigravity is also new to this manuscript. Finally, the analysis of the Cauchy initial value problem as it relates to the theory's fundamental field equations (1) through (4) is new.

7. ACKNOWLEDGEMENTS

For his penetrating insights and suggestions offered during the early phases of this work[27] I would like to express my appreciation and thanks to David Eimerl, and for their many useful conversations and critical comments during the preparation of this manuscript I would like to thank my colleagues Charles Boley and Alexander Rubenchik. This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

8. APPENDICES

8.1 Appendix I – Derivation of the Lorentz force law and the conservation of mass

The conservation of mass (12) and the Lorentz force law (13) follow from the fundamental field equations (1) through (4) and their consequences. An outline of the derivation of these equations is given here. To derive the conservation of mass equation (12), I begin with equation (4) contracted with u_μ

$$u_\mu\left(\rho_m u^\mu u^\nu + F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right)_{;\nu} = 0 \ . \qquad (75)$$

Expanding (75) and then simplifying per the following development leads to the conservation of mass equation (12) on the last line of (76) below.

$$
\begin{gathered}
u_\mu\left(\left(\rho_m u^\nu\right)_{;\nu} u^\mu+\rho_m u^\nu u^\mu{}_{;\nu}+F^\mu{}_\lambda F^{\nu\lambda}{}_{;\nu}+F^\mu{}_{\lambda;\nu} F^{\nu\lambda}-\frac{1}{2} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma;\nu}\right)=0 \\
\downarrow \\
\left(\rho_m u^\nu\right)_{;\nu}\left(u^\mu u_\mu\right)+\rho_m u^\nu\left(u_\mu u^\mu{}_{;\nu}\right)+u_\mu F^\mu{}_\lambda\left(F^{\nu\lambda}{}_{;\nu}\right)+u_\mu F^\mu{}_{\lambda;\nu} F^{\nu\lambda}-\frac{1}{2} u_\mu g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma;\nu}=0 \\
\downarrow \\
\left(\rho_m u^\nu\right)_{;\nu}(-1)+\rho_m u^\nu(0)+u_\mu F^\mu{}_\lambda\left(-\rho_c u^\lambda\right)+u_\mu F^\mu{}_{\sigma;\rho} F^{\rho\sigma}-\frac{1}{2} u_\mu F^{\rho\sigma} F_{\rho\sigma}{}^{;\mu}=0 \\
\downarrow \\
-\left(\rho_m u^\nu\right)_{;\nu}-\rho_c\left(u^\mu u^\lambda F_{\mu\lambda}\right)+u_\mu F^{\rho\sigma}\left(F^\mu{}_{\sigma;\rho}-\frac{1}{2} F_{\rho\sigma}{}^{;\mu}\right)=0 \\
\downarrow \\
-\left(\rho_m u^\nu\right)_{;\nu}-\rho_c(0)+u_\mu F^{\rho\sigma}\left(F^\mu{}_{\sigma;\rho}+\frac{1}{2} F_\sigma{}^\mu{}_{;\rho}+\frac{1}{2} F^\mu{}_{\rho;\sigma}\right)=0 \\
\downarrow \\
-\left(\rho_m u^\nu\right)_{;\nu}+u_\mu F^{\rho\sigma}\left(\frac{1}{2} F^\mu{}_{\sigma;\rho}+\frac{1}{2} F^\mu{}_{\rho;\sigma}\right)=0 \\
\downarrow \\
-\left(\rho_m u^\nu\right)_{;\nu}+0=0 \\
\downarrow \\
\left(\rho_m u^\nu\right)_{;\nu}=0
\end{gathered} \tag{76}
$$

To go from the 2[nd] to the 3[rd] line I use $u_\mu u^\mu{}_{;\nu}=0$ as forced by fundamental equation (3) and $F^{\nu\lambda}{}_{;\nu}=-\rho_c u^\lambda$ Maxwell's inhomogeneous equation (9). To go from the 4[th] to the 5[th] line I use $F_{\rho\sigma}{}^{;\mu}+F_\sigma{}^\mu{}_{;\rho}+F^\mu{}_{\rho;\sigma}=0$ which follows from Maxwell's homogenous equation (7).

The Lorentz force law (13) is now derived using the conservation of mass result (76) and equation (4). Expanding and then simplifying equation (4) per the following development leads to the Lorentz force law (13) on the last line of (77) below.

$$\left(\rho_m u^\mu u^\nu + F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right)_{;\nu} = 0$$
$$\downarrow$$
$$\left(\rho_m u^\nu\right)_{;\nu} u^\mu + \rho_m u^\nu u^\mu{}_{;\nu} + F^\mu{}_\lambda F^{\nu\lambda}{}_{;\nu} + F^\mu{}_{\lambda;\nu} F^{\nu\lambda} - \frac{1}{2} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma;\nu} = 0$$
$$\downarrow$$
$$(0) u^\mu + \rho_m u^\nu u^\mu{}_{;\nu} + F^\mu{}_\lambda \left(F^{\nu\lambda}{}_{;\nu}\right) + F^\mu{}_{\lambda;\nu} F^{\nu\lambda} - \frac{1}{2} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma;\nu} = 0$$
$$\downarrow$$
$$\rho_m \frac{Du^\mu}{D\tau} + F^\mu{}_\lambda \left(-\rho_c u^\lambda\right) + F^\mu{}_{\sigma;\rho} F^{\rho\sigma} - \frac{1}{2} F^{\rho\sigma} F_{\rho\sigma}{}^{;\mu} = 0$$
$$\downarrow$$
$$\rho_m \frac{Du^\mu}{D\tau} - \rho_c F^\mu{}_\lambda u^\lambda + F^{\rho\sigma} \left(F^\mu{}_{\sigma;\rho} - \frac{1}{2} F_{\rho\sigma}{}^{;\mu}\right) = 0$$
$$\downarrow$$
$$\rho_m \frac{Du^\mu}{D\tau} - \rho_c F^\mu{}_\lambda u^\lambda + F^{\rho\sigma} \left(F^\mu{}_{\sigma;\rho} + \frac{1}{2} F_\sigma{}^\mu{}_{;\rho} + \frac{1}{2} F^\mu{}_{\rho;\sigma}\right) = 0$$
$$\downarrow$$
$$\rho_m \frac{Du^\mu}{D\tau} - \rho_c F^\mu{}_\lambda u^\lambda + F^{\rho\sigma} \left(\frac{1}{2} F^\mu{}_{\sigma;\rho} + \frac{1}{2} F^\mu{}_{\rho;\sigma}\right) = 0$$
$$\rho_m \frac{Du^\mu}{D\tau} - \rho_c F^\mu{}_\lambda u^\lambda + 0 = 0 \qquad (77)$$
$$\downarrow$$
$$\rho_m \frac{Du^\mu}{D\tau} = \rho_c F^\mu{}_\lambda u^\lambda$$

To go from the 2nd to the 3rd line I use $\left(\rho_m u^\nu\right)_{;\nu} = 0$ the conservation of mass equation (76) just derived. To go from the 3rd to the 4th line I use $F^{\nu\lambda}{}_{;\nu} = -\rho_c u^\lambda$ Maxwell's inhomogeneous equation (9). To go from the 5th to the 6th line I use $F_{\rho\sigma}{}^{;\mu} + F_\sigma{}^\mu{}_{;\rho} + F^\mu{}_{\rho;\sigma} = 0$ which follows from Maxwell's homogenous equation (7).

8.2 Appendix II – The Cauchy problem applied to the fundamental field equations

One of the unusual features of fundamental field equations (1) through (4) is the lack of any explicit derivatives of the vector field a^λ, a situation which raises questions about the time dependent development of a^{λ}. To further elucidate this and other questions regarding solutions of the fundamental field equations, and to outline how they could be solved numerically, they are here analyzed in terms of a Cauchy initial value problem.

Given a set of initial conditions comprising the values of the fundamental fields in Table I at all spatial locations, a procedure is outlined that propagates those fields to any other time. To begin, assume $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ are known at all spatial coordinates at some initial coordinate time t_0. Note that the initial values for the field a^{λ} are not required, rather they will be solved for using equation (1) as described below. Also note that in addition to $g_{\mu\nu}$ the initial values of $\frac{\partial g_{\mu\nu}}{\partial t}$ must be specified because the fundamental field equations are second order in the time derivatives of $g_{\mu\nu}$, a situation analogous to classical General Relativity. The goal of the Cauchy method as it applies here is to start with specified initial conditions for $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ at t_0, and then using the fundamental field equations (1) through (4) solve for $a^{\lambda}, R_{\lambda\kappa\mu\nu}, \frac{\partial F_{\mu\nu}}{\partial t}, \frac{\partial u^{\lambda}}{\partial t}, \frac{\partial \rho_m}{\partial t}, \frac{\partial \rho_c}{\partial t}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ at t_0. Armed with these values at t_0, it is straight forward to propagate the fields $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ from their initial conditions at t_0 to $t_0 + dt$ and then solve for $a^{\lambda}, R_{\lambda\kappa\mu\nu}, \frac{\partial F_{\mu\nu}}{\partial t}, \frac{\partial u^{\lambda}}{\partial t}, \frac{\partial \rho_m}{\partial t}, \frac{\partial \rho_c}{\partial t}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ at $t_0 + dt$ using the same procedure that was used to find them at t_0. Repeating, values for the fundamental fields of the theory can then be found at all times. One additional requirement on the field values specified by initial conditions is that they must be self-consistent with the fundamental field equations (1) through (4), *i.e.*, the specified initial conditions must be consistent with a solution to the fundamental field equations (1) through (4).

In what follows, Greek indices (μ, ν, κ, …) take on the usual space-time coordinates 1-4 but Latin indices (i, j, k, …) are restricted to spatial coordinates, 1-3 only. Since the values of $g_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ are known at all spatial coordinates at time t_0, the values of $\frac{\partial g_{\mu\nu}}{\partial x^i}, \frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial x^j}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial t}$ can be calculated at all spatial coordinates at time t_0. This leaves the ten quantities $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ as the only second derivatives of $g_{\mu\nu}$ not known at t_0. To find the values of $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ at t_0 proceed as follows. First find the values of the six $\frac{\partial^2 g_{ij}}{\partial t^2}$ at t_0 using a subset of equations from (1), the subset containing only those equations having spatial derivatives of $F_{\mu\nu}$ on the LHS and at most one time-index in each occurrence of the R-C tensor on the RHS. These equations will be used to solve for the values of a^λ at time t_0. In all there are 12 such equations out of the 24 that comprise (1), as listed here

$$\begin{aligned}
F_{12;1} &= a^\lambda R_{\lambda 112} \\
F_{13;1} &= a^\lambda R_{\lambda 113} \\
F_{23;1} &= a^\lambda R_{\lambda 123} \\
F_{12;2} &= a^\lambda R_{\lambda 212} \\
F_{13;2} &= a^\lambda R_{\lambda 213} \\
F_{23;2} &= a^\lambda R_{\lambda 223} \\
F_{12;3} &= a^\lambda R_{\lambda 312} \\
F_{13;3} &= a^\lambda R_{\lambda 313} \\
F_{23;3} &= a^\lambda R_{\lambda 323} \\
F_{12;4} &= -F_{24;1} - F_{41;2} = a^\lambda R_{\lambda 412} \\
F_{13;4} &= -F_{34;1} - F_{41;3} = a^\lambda R_{\lambda 413} \\
F_{23;4} &= -F_{34;2} - F_{42;3} = a^\lambda R_{\lambda 423}
\end{aligned} \quad . \tag{78}$$

The last three equations in (78) use (7), Maxwell's homogenous equation to express the time derivative of a Maxwell tensor component on the LHS as the sum of the spatial derivatives of two Maxwell tensor components. The importance of having only spatial derivatives of the Maxwell tensor components on the LHS of (78) is that they are all known quantities at time t_0, *i.e.*, since all the $F_{\mu\nu}$ are known at time t_0, all $\frac{\partial F_{\mu\nu}}{\partial x^i}$ and $F_{\mu\nu;i}$ can be calculated at time t_0. Equally important is that the RHS of the 12 equations that comprise (78) contain at most a single time index in each occurrence of their R-C tensor and so are also known at time t_0. To see that this is so I examine the general form of the R-C tensor in a locally inertial coordinate system where all first derivatives of $g_{\mu\nu}$ vanish, *i.e.*,

$$R_{\lambda\kappa\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\lambda}}{\partial x^\nu \partial x^\kappa} - \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^\mu \partial x^\kappa} + \frac{\partial^2 g_{\kappa\nu}}{\partial x^\mu \partial x^\lambda} \right). \tag{79}$$

Note, having at most a single time index on the RHS of (79) means that the R-C tensor is made up entirely of terms from $\frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial x^j}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial x^i \partial t}$, all of which are known at time t_0. Examining the set of equations (78) there are 12 equations for 4 unknowns, the unknowns being the components of a^λ. These 12 equations can be solved for a^λ at time t_0 if the initial conditions were chosen self-consistently with the fundamental field equations (1) through (4), *i.e.*, chosen such that a solution to the field equations is indeed possible.

Knowing the R-C tensor components with at most one time-index at t_0, I now proceed to determine the R-C tensor components with two time indices. Going back to the 24 equations that comprise the set of equations (1), here I collect the subset of those equations in which the LHS is known at time t_0, *i.e.*, contains only spatial derivatives of the Maxwell tensor, and the RHS has an R-C tensor component that contain two time indices

$$\begin{aligned}
F_{14;1} &= a^\lambda R_{\lambda 114} \\
F_{24;1} &= a^\lambda R_{\lambda 124} \\
F_{34;1} &= a^\lambda R_{\lambda 134} \\
F_{14;2} &= a^\lambda R_{\lambda 214} \\
F_{24;2} &= a^\lambda R_{\lambda 224} \\
F_{34;2} &= a^\lambda R_{\lambda 234} \\
F_{14;3} &= a^\lambda R_{\lambda 314} \\
F_{24;3} &= a^\lambda R_{\lambda 324} \\
F_{34;3} &= a^\lambda R_{\lambda 334}
\end{aligned} \quad . \tag{80}$$

Each of the equations in (80) contains only one unknown, the R-C component having two time indices. In total, there are six such independent R-C tensor components,

$$\begin{array}{l} R_{1414} \\ R_{1424} \\ R_{1434} \\ R_{2424} \\ R_{2434} \\ R_{3434} \end{array} \tag{81}$$

so the system of nine equations (80) can be algebraically solved for the these six unknown R-C components at time t_0. With this I now know the value of all components of the R-C tensor at time t_0. From the t_0 values of the R-C tensor components listed in (81), the values of the six unknown $\frac{\partial^2 g_{ij}}{\partial t^2}$ at t_0 can be found.

There are three remaining equations from the set of equations (1) that have not yet been addressed

$$\begin{array}{l} F_{14;4} = a^{\lambda} R_{\lambda 414} \\ F_{24;4} = a^{\lambda} R_{\lambda 424} \\ F_{34;4} = a^{\lambda} R_{\lambda 434} \end{array} . \tag{82}$$

These are the equations for which the temporal derivatives of the Maxwell tensor components are not yet known. Because all values of the R-C tensor and a^{λ} are now known at t_0, these three remaining time-differentiated components of the Maxwell tensor can now be solved for directly using (82), giving complete knowledge of $\frac{\partial F_{\mu\nu}}{\partial t}$ at time t_0.

If the values of the four $\frac{\partial^2 g_{\mu 4}}{\partial t^2}$ could be calculated then all $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ would be known and all $\frac{\partial g_{\mu\nu}}{\partial t}$ could be propagated from t_0 to $t_0 + dt$. Just as is the case with classical General Relativity, the four $\frac{\partial^2 g_{\mu 4}}{\partial t^2}$ can be determined from the four coordinate conditions that are fixed by the choice of coordinate system.[28] Recapping, at t_0 the following quantities are now known: $g_{\mu\nu}$, $F_{\mu\nu}$, u^{λ}, ρ_c, ρ_m and $\frac{\partial g_{\mu\nu}}{\partial t}$ are defined by initial conditions, and a^{λ}, $\frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa} \partial x^{\lambda}}$, $R_{\lambda\kappa\mu\nu}$, and $\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}}$ are solved for using those initial conditions, the fundamental field equations, and the four coordinate conditions that are fixed by the choice of coordinate system. Still needed to propagate the initial conditions in time from t_0 to $t_0 + dt$ are $\frac{\partial u^\mu}{\partial t}$, $\frac{\partial \rho_m}{\partial t}$ and $\frac{\partial \rho_c}{\partial t}$. Using the Lorentz force law (13), the following progression,

$$\rho_m \frac{Du^\mu}{D\tau} = \rho_c u^\lambda F^\mu{}_\lambda$$
$$\downarrow$$
$$\rho_m u^\mu{}_{;\nu} u^\nu = \rho_c u^\lambda F^\mu{}_\lambda$$
$$\downarrow \tag{83}$$
$$\rho_m u^\mu{}_{;4} u^4 = -\rho_m u^\mu{}_{;i} u^i + \rho_c u^\lambda F^\mu{}_\lambda$$
$$\downarrow$$
$$\rho_m \left(\frac{\partial u^\mu}{\partial t} + \Gamma^\mu{}_{4\sigma} u^\sigma \right) u^4 = -\rho_m u^\mu{}_{;i} u^i + \rho_c u^\lambda F^\mu{}_\lambda$$

shows on the last line above that $\frac{\partial u^\mu}{\partial t}$ can be solved for at t_0 in terms of knowns at t_0. Using the conservation of mass (12) and knowing $\frac{\partial u^\mu}{\partial t}$ at t_0, the following progression

$$\left(\rho_m u^\nu \right)_{;\nu} = 0$$
$$\downarrow$$
$$\left(\rho_m u^4 \right)_{;4} = -\left(\rho_m u^i \right)_{;i} \tag{84}$$
$$\downarrow$$
$$\frac{\partial \rho_m}{\partial t} u^4 = -\rho_m u^4{}_{;4} - \left(\rho_m u^i \right)_{;i}$$

shows on the last line above that $\frac{\partial \rho_m}{\partial t}$ can be solved for at t_0 in terms of knowns at t_0. Following an analogous progression for ρ_c using the charge conservation equation (10), $\frac{\partial \rho_c}{\partial t}$ can be solved for at t_0 in terms of knowns at t_0. With these, the values of a^λ, $R_{\lambda\kappa\mu\nu}$, $\frac{\partial F_{\mu\nu}}{\partial t}$, $\frac{\partial u^\lambda}{\partial t}$, $\frac{\partial \rho_m}{\partial t}$, $\frac{\partial \rho_c}{\partial t}$ and $\frac{\partial^2 g_{\mu\nu}}{\partial t^2}$ are all known at t_0 and can be used to propagate the initial conditions $g_{\mu\nu}, F_{\mu\nu}, u^{\lambda}, \rho_c, \rho_m$ and $\frac{\partial g_{\mu\nu}}{\partial t}$ at t_0 to time $t_0 + dt$. Iterating the process, the values of the fundamental fields can be determined at all times.

9. REFERENCES AND END NOTES

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[3] S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, New York, NY 1972. The definition of the Riemann-Christoffel curvature tensor is $R^{\lambda}{}_{\mu\nu\kappa} \equiv \partial_{\kappa}\Gamma^{\lambda}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\nu}\Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa}\Gamma^{\lambda}{}_{\nu\eta}$, and the definition of the Ricci tensor is $R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$.

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[7] The interpretation of the solution here is different than in the derivation in reference [2]. Specifically, in reference [2] the charge density was restricted to be positive, a restriction that is lifted here.

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