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SOME PROPERTIES OF h-MN-CONVEXITY AND JENSEN'S TYPE INEQUALITIES

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ABSTRACT. In this work, we introduce the class of h-MN-convex functions by generalizing the concept of MN-convexity and combining it with h-convexity. Namely, let $\mathrm{M} : [0,1] \to [a,b]$ be a Mean function given by $\mathrm{M}(t) = \mathrm{M}(t;a,b)$; where by $\mathrm{M}(t;a,b)$ we mean one of the following functions: $A_t(a,b) := (1-t)a + tb$, $G_t(a,b) = a^{1-t}b^t$ and $H_t(a,b) := \frac{ab}{ta+(1-t)b} = \frac{1}{A_t\left(\frac{1}{a},\frac{1}{b}\right)}$; with the property that $\mathrm{M}(0;a,b) = a$ and $\mathrm{M}(1;a,b) = b$.

Let I, J be two intervals subset of $(0,\infty)$ such that $(0,1) \subseteq J$ and $[a,b] \subseteq I$. Consider a non-negative function $h : J \to (0,\infty)$, a function $f : I \to (0,\infty)$ is said to be h-MN-convex (concave) if the inequality

$$f(\mathrm{M}(t;x,y)) \le (\ge)\, \mathrm{N}(h(t); f(x), f(y)),$$

holds for all $x, y \in I$ and $t \in [0,1]$. In this way, nine classes of h-MN-convex functions are established, and therefore some analytic properties for each class of functions are explored and investigated. Characterizations of each type are given. Various Jensen's type inequalities and their converses are proved.

1. Introduction

Let I be a real interval. A function $f : I \to \mathbb{R}$ is called convex iff

$$f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t)f(\beta), \tag{1.1}$$

for all points $\alpha, \beta \in I$ and all $t \in [0,1]$. If $-f$ is convex then we say that f is concave. Moreover, if f is both convex and concave, then f is said to be affine.

In 1978, Breckner [5] introduced the class of s-convex functions (in the second sense), as follows:

Definition 1. *Let $I \subseteq [0,\infty)$ and $s \in (0,1]$, a function $f : I \to [0,\infty)$ is s-convex function or that f belongs to the class $K_s^2(I)$ if for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y).$$

In [6], Breckner proved that every s-convex function satisfies the Hölder condition of order s. Another proof of this fact was given in [26]. For more properties regarding s-convexity see [7] and [15].

In 1985, E. K. Godnova and V. I. Levin (see [13] or [19], pp. 410-433) introduced the following class of functions:

Definition 2. *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \le \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

In the same work, the authors proved that all nonnegative monotonic and nonnegative convex functions belong to this class. For related works see [12] and [18].

In 1999, Pearce and Rubinov [24], established a new type of convex functions which is called P-functions.

Definition 3. *We say that $f : I \to \mathbb{R}$ is P-function or that f belongs to the class $P(I)$ if for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y).$$

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Indeed, $Q(I) \supseteq P(I)$ and for applications it is important to note that $P(I)$ also consists only of nonnegative monotonic, convex and quasi-convex functions. A related work was considered in [12] and [29].

In 2007, Varošanec [30] introduced the class of h-convex functions which generalize convex, s-convex, Godunova-Levin functions and P-functions. Namely, the h-convex function is defined as a non-negative function $f : I \to \mathbb{R}$ which satisfies

$$f\left(t\alpha+(1-t)\beta\right) \le h(t) f(\alpha) + h(1-t) f(\beta),$$

where h is a non-negative function, $t \in (0,1) \subseteq J$ and $x, y \in I$, where I and J are real intervals such that $(0,1) \subseteq J$. Accordingly, some properties of h-convex functions were discussed in the same work of Varošanec. For more results; generalization, counterparts and inequalities regarding h-convexity see [1],[3],[4], [8]–[10],[14],[16], [22] and [28].

We recall that, a function $\mathrm{M} : (0,\infty) \to (0,\infty)$ is called a Mean function if

(1) Symmetry: $\mathrm{M}(x,y) = \mathrm{M}(y,x)$.
(2) Reflexivity: $\mathrm{M}(x,x) = x$.
(3) Monotonicity: $\min\{x,y\} \le \mathrm{M}(x,y) \le \max\{x,y\}$.
(4) Homogeneity: $\mathrm{M}(\lambda x, \lambda y) = \lambda \mathrm{M}(x,y)$, for any positive scalar λ.

The most famous and old known mathematical means are listed as follows:

(1) The arithmetic mean :
$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}_+.$$
(2) The geometric mean :
$$G := G(\alpha,\beta) = \sqrt{\alpha\beta}, \quad \alpha,\beta \in \mathbb{R}_+$$
(3) The harmonic mean :
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \quad \alpha,\beta \in \mathbb{R}_+ - \{0\}.$$

In particular, we have the famous inequality $H \le G \le A$.

In 2007, Anderson et al. in [2] developed a systematic study to the classical theory of continuous and midconvex functions, by replacing a given mean instead of the arithmetic mean.

Definition 4. *Let $f : I \to (0,\infty)$ be a continuous function where $I \subseteq (0,\infty)$. Let M and N be any two Mean functions. We say f is MN-convex (concave) if*

$$f\left(\mathrm{M}(x,y)\right) \le (\ge)\, \mathrm{N}\left(f(x), f(y)\right), \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$.

In fact, the authors in [2] discussed the midconvexity of positive continuous real functions according to some Means. Hence, the usual midconvexity is a special case when both mean values are arithmetic means. Also, they studied the dependence of MN-convexity on M and N and give sufficient conditions for MN-convexity of functions defined by Maclaurin series. For other works regarding MN-convexity see [20] and [21].

The aim of this work, is to study the main properties of h-MN-convex functions, such as; addition, product, compositions and some functional type inequalities for some classes. Jensen inequality and its consequences with their converses play significant roles in (almost) all areas of Mathematics and Physics. For example, Jensen inequality used to prove some important inequalities such as AM, GM, HM inequalities and their consequences, moreover it can be used to generate some more ramified inequalities. All this happens using the classical concept of convex set and convex functions, but what happen when we replace these terms by another convexity terms such as h-MN-convexity?. The natural answer, is simply can change everything, e.g., discovering new Jensen type inequalities will help us to find, refine, and generate new inequalities of AM, GM, and HM type.

In this work, the class of h-MN-convex functions is introduced. Generalizing and extending some classes of convex functions are given. Some analytic properties for each class of functions are explored and investigated. Characterizations of each type of convexity are established. Some related Jensen's type inequalities and their converses are proved.

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2. The h-MN-convexity

Throughout this work, I and J are two intervals subset of $(0,\infty)$ such that $(0,1)\subseteq J$ and $[a,b]\subseteq I$. Let $0<a<b$. Define the function $\mathrm{M}:[0,1]\to[a,b]$ given by $\mathrm{M}(t)=\mathrm{M}(t;a,b)$; where by $\mathrm{M}(t;a,b)$ we mean one of the following functions:

(1) $A_t(a,b):=(1-t)a+tb$; The generalized Arithmetic Mean.

(2) $G_t(a,b)=a^{1-t}b^t$; The generalized Geometric Mean.

(3) $H_t(a,b):=\frac{ab}{ta+(1-t)b}=\frac{1}{A_t\left(\frac{1}{a},\frac{1}{b}\right)}$; The generalized Harmonic Mean.

Note that $\mathrm{M}(0;a,b)=a$ and $\mathrm{M}(1;a,b)=b$. Clearly, for $t=\frac{1}{2}$, the means $A_{\frac{1}{2}}$, $G_{\frac{1}{2}}$ and $H_{\frac{1}{2}}$, respectively; represents the midpoint of the A_t, G_t and H_t, respectively; which was discussed in [2] in viewing of Definition 4.

Also, we note that the above means are related with celebrated inequality

$$H_t(a,b)\le G_t(a,b)\le A_t(a,b),\qquad\forall\, t\in[0,1].$$

2.1. **Basic properties of h-MN-convex functions.** The Definition 4 can be extended according to the defined mean $\mathrm{M}(t;a,b)$, as follows: Let $f:I\to(0,\infty)$ be any function. Let M and N be any two Mean functions. We say f is MN-convex (concave) if

$$f(\mathrm{M}(t;x,y))\le(\ge)\,\mathrm{N}(t;f(x),f(y)),$$

for all $x,y\in I$ and $t\in[0,1]$.

Next, we introduce the class of h-$\mathrm{M_tN_t}$-convex functions by generalizing the concept of $\mathrm{M_tN_t}$-convexity and combining it with h-convexity.

Definition 5. *Let $h:J\to(0,\infty)$ be a positive function. Let $f:I\to(0,\infty)$ be any function. Let $\mathrm{M}:[0,1]\to[a,b]$ and $\mathrm{N}:(0,\infty)\to(0,\infty)$ be any two Mean functions. We say f is h-MN-convex (-concave) or that f belongs to the class $\overline{\mathcal{MN}}(h,I)$ ($\underline{\mathcal{MN}}(h,I)$) if*

$$f(\mathrm{M}(t;x,y))\le(\ge)\,\mathrm{N}(h(t);f(x),f(y)),\tag{2.1}$$

for all $x,y\in I$ and $t\in[0,1]$.

Clearly, if $\mathrm{M}(t;x,y)=A_t(x,y)=\mathrm{N}(t;x,y)$, then Definition 5 reduces to the original concept of h-convexity. Also, if we assume f is continuous, $h(t)=t$ and $t=\frac{1}{2}$ in (2.1), then the Definition 5 reduces to the Definition 4.

The cases of h-MN-convexity are given with respect to a certain mean, as follow:

(1) f is h-A_tG_t-convex iff

$$f(t\alpha+(1-t)\beta)\le[f(\alpha)]^{h(t)}[f(\beta)]^{h(1-t)},\qquad 0\le t\le1,\tag{2.2}$$

(2) f is h-A_tH_t-convex iff

$$f(t\alpha+(1-t)\beta)\le\frac{f(\alpha)f(\beta)}{h(1-t)f(\alpha)+h(t)f(\beta)},\qquad 0\le t\le1.\tag{2.3}$$

(3) f is h-G_tA_t-convex iff

$$f\left(\alpha^t\beta^{1-t}\right)\le h(t)f(\alpha)+h(1-t)f(\beta),\qquad 0\le t\le1.\tag{2.4}$$

(4) f is h-G_tG_t-convex iff

$$f\left(\alpha^t\beta^{1-t}\right)\le[f(\alpha)]^{h(t)}[f(\beta)]^{h(1-t)},\qquad 0\le t\le1.\tag{2.5}$$

(5) f is h-G_tH_t-convex iff

$$f\left(\alpha^t\beta^{1-t}\right)\le\frac{f(\alpha)f(\beta)}{h(1-t)f(\alpha)+h(t)f(\beta)},\qquad 0\le t\le1.\tag{2.6}$$

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(6) f is h-H_tA_t-convex iff

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\le h(1-t)\,f(\alpha)+h(t)\,f(\beta),\qquad 0\le t\le 1. \tag{2.7}$$

(7) f is h-H_tG_t-convex iff

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\le [f(\alpha)]^{h(1-t)}\,[f(\beta)]^{h(t)},\qquad 0\le t\le 1. \tag{2.8}$$

(8) f is H_tH_t-convex iff

$$f\left(\frac{\alpha\beta}{t\alpha+(1-t)\beta}\right)\le \frac{f(\alpha)\,f(\beta)}{h(t)\,f(\alpha)+h(1-t)\,f(\beta)},\qquad 0\le t\le 1. \tag{2.9}$$

Remark 1. *In all previous cases, $h(t)$ and $h(1-t)$ are not equal to zero at the same time. Therefore, if $h(0)=0$ and $h(1)=1$, then the Mean function N satisfying the conditions $\mathrm{N}(h(0),f(x),f(y))=f(x)$ and $\mathrm{N}(h(1),f(x),f(y))=f(y)$.*

Remark 2. *According to the Definition 5, we may extend the classes $Q(I),P(I)$ and K_s^2 by replacing the arithmetic mean by another given one. Let $\mathrm{M}:[0,1]\to[a,b]$ and $\mathrm{N}:(0,\infty)\to(0,\infty)$ be any two Mean functions.*

(1) Let $s\in(0,1]$, a function $f:I\to(0,\infty)$ is $\mathrm{M_tN_t}$-s-convex function or that f belongs to the class $K_s^2(I;\mathrm{M_t},\mathrm{N_t})$ if for all $x,y\in I$ and $t\in[0,1]$ we have

$$f(\mathrm{M}(t;x,y))\le \mathrm{N}(t^s;f(x),f(y)). \tag{2.10}$$

(2) We say that $f:I\to(0,\infty)$ is an extended Godunova-Levin function or that f belongs to the class $Q(I;\mathrm{M_t},\mathrm{N_t})$ if for all $x,y\in I$ and $t\in(0,1)$ we have

$$f(\mathrm{M}(t;x,y))\le \mathrm{N}\left(\frac{1}{t};f(x),f(y)\right). \tag{2.11}$$

(3) We say that $f:I\to(0,\infty)$ is P-$\mathrm{M_tN_t}$-function or that f belongs to the class $P(I;\mathrm{M_t},\mathrm{N_t})$ if for all $x,y\in I$ and $t\in[0,1]$ we have

$$f(\mathrm{M}(t;x,y))\le \mathrm{N}(1;f(x),f(y)). \tag{2.12}$$

In (2.10)–(2.12), setting $\mathrm{M}(t;x,y)=\mathrm{A_t}(x,y)=\mathrm{N}(t;x,y)$, we then refer to the original definitions of these class of convexities (see Definitions 1–3).

Remark 3. *Let h be a non-negative function such that $h(t)\ge t$ for $t\in(0,1)$. For instance $h_r(t)=t^r$, $t\in(0,1)$ has that property. In particular, for $r\le 1$, if f is a non-negative $\mathrm{M_tN_t}$-convex function on I, then for $x,y\in I$, $t\in(0,1)$ we have*

$$f(\mathrm{M}(t;x,y))\le \mathrm{N}(t;f(x),f(y))\le \mathrm{N}(t^r;f(x),f(y))=\mathrm{N}(h(t);f(x),f(y)),$$

for all $r\le 1$ and $t\in(0,1)$. So that f is h-$\mathrm{M_tN_t}$-convex. Similarly, if the function satisfies the property $h(t)\le t$ for $t\in(0,1)$, then f is a non-negative h-$\mathrm{M_tN_t}$-concave. In particular, for $r\ge 1$, the function $h_r(t)$ has that property for $t\in(0,1)$. So that if f is a non-negative $\mathrm{M_tN_t}$-concave function on I, then for $x,y\in I$, $t\in(0,1)$ we have

$$f(\mathrm{M}(t;x,y))\ge \mathrm{N}(t;f(x),f(y))\ge \mathrm{N}(t^r;f(x),f(y))=\mathrm{N}(h(t);f(x),f(y)),$$

for all $r\ge 1$ and $t\in(0,1)$, which means that f is h-$\mathrm{M_tN_t}$-concave.

Remark 4. *There exists an h-MN-convex function which is MN-convex. As shown by Varošanec (see Examples 6 and 7 in [30]), one can generate h-MN-convex functions but not MN-convex.*

Next, we give an extended generalization of Theorem 2.4 in [2]. This simply can help to illustrate the concept of h-MN-convex functions.

Theorem 1. *Let $h:J\to(0,\infty)$ be a positive function. $f:I\to(0,\infty)$ be any function. In parts (4)–(9), let $I=(0,\tau)$, $0<\tau<\infty$.*

(1) f is h-A_tA_t-convex (-concave) if and only if f is h-convex (h-concave).
(2) f is h-A_tG_t-convex (-concave) if and only if $\log f$ is h-convex (-concave).

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(3) *f is h-A_tH_t-convex (-concave) if and only if* $\frac{1}{f(x)}$ *is h-concave (-convex).*
(4) *f is h-G_tA_t-convex (-concave) on I if and only if* $f\left(\tau \mathrm{e}^{-t}\right)$ *is h-convex (-concave).*
(5) *f is h-G_tG_t-convex (-concave) if and only if* $\log f\left(\tau \mathrm{e}^{-t}\right)$ *is h-convex (-concave) on* $(0,\infty)$.
(6) *f is h-G_tH_t-convex (-concave) if and only if* $\frac{1}{f(\tau \mathrm{e}^{-t})}$ *is h-concave (-convex) on* $(0,\infty)$.
(7) *f is h-H_tA_t-convex (-concave) if and only if* $f\left(\frac{1}{x}\right)$ *is h-convex (-concave) on* $\left(\frac{1}{\tau},\infty\right)$.
(8) *f is h-H_tG_t-convex (-concave) if and only if* $\log f\left(\frac{1}{x}\right)$ *is h-convex (-concave) on* $\left(\frac{1}{\tau},\infty\right)$.
(9) *f is h-H_tH_t-convex (-concave) if and only if* $\frac{1}{f\left(\frac{1}{x}\right)}$ *is h-concave (-convex) on* $\left(\frac{1}{\tau},\infty\right)$.

Proof. (1) Follows by definition.
(2) Employing (2.2) in the Definition 5, we have

$$\begin{aligned}
&f\left(A_t\left(a,b\right)\right) \le (\ge)\, G\left(h(t); f(a), f(b)\right)\\
&\Leftrightarrow f\left(\left(1-t\right)a+tb\right) \le (\ge)\, \left[f\left(a\right)\right]^{h(1-t)}\left[f\left(b\right)\right]^{h(t)}\\
&\Leftrightarrow \log f\left(\left(1-t\right)a+tb\right) \le (\ge)\, h\left(1-t\right)\log\left[f\left(a\right)\right]+h\left(t\right)\log\left[f\left(b\right)\right],
\end{aligned}$$

which proves the result.
(3) Employing (2.3) in the Definition 5, we have

$$\begin{aligned}
&f\left(A_t\left(a,b\right)\right) \le (\ge)\, H\left(h(t); f(a), f(b)\right)\\
&\Leftrightarrow f\left(\left(1-t\right)a+tb\right) \le (\ge) \frac{f\left(a\right)f\left(b\right)}{h\left(t\right)f\left(a\right)+h\left(1-t\right)f\left(b\right)}\\
&\Leftrightarrow \frac{1}{f\left(\left(1-t\right)a+tb\right)} \ge (\le)\, \frac{h\left(1-t\right)}{f\left(a\right)}+\frac{h\left(t\right)}{f\left(b\right)},
\end{aligned}$$

which proves the result.
(4) Employing (2.4) in the Definition 5 and substituting $a=\tau e^{-r}$ and $b=\tau e^{-s}$, we have

$$\begin{aligned}
&f\left(G_t\left(a,b\right)\right) \le (\ge)\, A\left(h(t); f(a), f(b)\right)\\
&\Leftrightarrow f\left(a^{1-t}b^{t}\right) \le (\ge) h\left(1-t\right)f\left(a\right)+h\left(t\right)f\left(b\right)\\
&\Leftrightarrow f\left(\tau e^{-[r(1-t)+st]}\right) \le (\ge) h\left(1-t\right)f\left(\tau e^{-r}\right)+h\left(t\right)f\left(\tau e^{-s}\right),
\end{aligned}$$

which proves the result.
(5) Employing (2.5) in the Definition 5 and substituting $a=\tau e^{-r}$ and $b=\tau e^{-s}$, we have

$$\begin{aligned}
&f\left(G_t\left(a,b\right)\right) \le (\ge)\, G\left(h(t); f(a), f(b)\right)\\
&\Leftrightarrow f\left(a^{1-t}b^{t}\right) \le (\ge)\, \left[f\left(a\right)\right]^{h(1-t)}\left[f\left(b\right)\right]^{h(t)}\\
&\Leftrightarrow \log f\left(\tau e^{-[r(1-t)+st]}\right) \le (\ge) h\left(1-t\right)\log f\left(\tau e^{-r}\right)+h\left(t\right)\log f\left(\tau e^{-s}\right),
\end{aligned}$$

(6) Employing (2.6) in the Definition 5 and substituting $a=\tau e^{-r}$ and $b=\tau e^{-s}$, we have, we have

$$\begin{aligned}
&f\left(G_t\left(a,b\right)\right) \le (\ge)\, H\left(h(t); f(a), f(b)\right)\\
&\Leftrightarrow f\left(a^{1-t}b^{t}\right) \le (\ge) \frac{f\left(a\right)f\left(b\right)}{h\left(t\right)f\left(a\right)+h\left(1-t\right)f\left(b\right)}\\
&\Leftrightarrow \frac{1}{f\left(a^{1-t}b^{t}\right)} \ge (\le)\, \frac{h\left(1-t\right)}{f\left(a\right)}+\frac{h\left(t\right)}{f\left(b\right)}\\
&\Leftrightarrow \frac{1}{f\left(\tau e^{-[r(1-t)+st]}\right)} \ge (\le)\, \frac{h\left(1-t\right)}{f\left(\tau e^{-r}\right)}+\frac{h\left(t\right)}{f\left(\tau e^{-s}\right)},
\end{aligned}$$

which proves the result.

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(7) Let $g(x) = f\left(\frac{1}{x}\right)$ and let $a, b \in \left(\frac{1}{\tau}, \infty\right)$ with $a < b$, so that $a, b \in (0, \tau)$. Then f is h-H_tA_t-convex (-concave) on $(0, \tau)$ if and only if

$$f\left(\frac{1}{H_t(a,b)}\right) \le (\ge) A\left(h(t); \frac{1}{f(a)}, \frac{1}{f(b)}\right)$$
$$\Leftrightarrow f\left(\frac{1}{\frac{ab}{ta+(1-t)b}}\right) \le (\ge) h(t) f\left(\frac{1}{b}\right) + h(1-t) f\left(\frac{1}{a}\right)$$
$$\Leftrightarrow g\left(\frac{ab}{ta+(1-t)b}\right) \le (\ge) h(1-t) g(a) + h(t) g(b),$$

which proves the result.

(8) Let $g(x) = \log f\left(\frac{1}{x}\right)$ and let $a, b \in \left(\frac{1}{\tau}, \infty\right)$ with $a < b$, so that $a, b \in (0, \tau)$. Then f is h-H_tG_t-convex (-concave) on $(0, \tau)$ if and only if

$$f\left(\frac{1}{H_t(a,b)}\right) \le (\ge) G(h(t); f(a), f(b))$$
$$\Leftrightarrow f\left(\frac{t}{b} + \frac{(1-t)}{a}\right) \le (\ge) \left[f\left(\frac{1}{b}\right)\right]^{h(t)} \left[f\left(\frac{1}{a}\right)\right]^{h(1-t)}$$
$$\Leftrightarrow \log f\left(\frac{t}{b} + \frac{(1-t)}{a}\right) \le (\ge) h(t) \log f\left(\frac{1}{b}\right) + h(1-t) \log f\left(\frac{1}{a}\right)$$
$$\Leftrightarrow g\left(\frac{ab}{ta+(1-t)b}\right) \le (\ge) h(t) g(b) + h(1-t) g(a),$$

which proves the result.

(9) Let $g(x) = \frac{1}{f\left(\frac{1}{x}\right)}$ and let $a, b \in \left(\frac{1}{\tau}, \infty\right)$ with $a < b$, so that $a, b \in (0, \tau)$. Then f is h-H_tH_t-convex (-concave) on $(0, \tau)$ if and only if

$$f\left(\frac{1}{H_t(a,b)}\right) \le (\ge) H(h(t); f(a), f(b))$$
$$\Leftrightarrow f\left(\frac{t}{b} + \frac{(1-t)}{a}\right) \le (\ge) \frac{f\left(\frac{1}{a}\right) f\left(\frac{1}{b}\right)}{h(1-t) f\left(\frac{1}{b}\right) + h(t) f\left(\frac{1}{a}\right)}$$
$$\Leftrightarrow \frac{1}{f\left(\frac{t}{b} + \frac{(1-t)}{a}\right)} \ge (\le) \frac{h(1-t) f\left(\frac{1}{b}\right) + h(t) f\left(\frac{1}{a}\right)}{f\left(\frac{1}{a}\right) f\left(\frac{1}{b}\right)}$$
$$\Leftrightarrow \frac{1}{f\left(\frac{ta+(1-t)b}{ab}\right)} \ge (\le) \frac{h(1-t)}{f\left(\frac{1}{a}\right)} + \frac{h(t)}{f\left(\frac{1}{b}\right)}$$
$$\Leftrightarrow g\left(\frac{ab}{ta+(1-t)b}\right) \ge (\le) h(1-t) g(a) + h(t) g(b),$$

which proves the result.

□

Characterizations of each type of h-MN-convex functions using derivatives is given below. The next result can be considered as extended generalization of Corollary 2.5 in [2].

Corollary 1. *Let $h : J \to (0, \infty)$ be a non-negative function such that $h(\alpha) \ge (\le) \alpha$ for all $\alpha \in (0, 1)$. Let $f : I \to (0, \infty)$ be differentiable function. In parts (4)–(9), let $I = (0, \tau)$, $0 < \tau < \infty$.*

(1) f is h-A_tA_t-convex (-concave) if and only if $f'(x)$ is increasing (decreasing).
(2) f is h-A_tG_t-convex (-concave) if and only if $\frac{f'(x)}{f(x)}$ is increasing (decreasing).
(3) f is h-A_tH_t-convex (-concave) if and only if $\frac{f'(x)}{f^2(x)}$ is increasing (decreasing).
(4) f is h-G_tA_t-convex (-concave) on I if and only if $xf'(x)$ is increasing (decreasing).
(5) f is h-G_tG_t-convex (-concave) if and only if $\frac{xf'(x)}{f(x)}$ is increasing (decreasing).
(6) f is h-G_tH_t-convex (-concave) if and only if $\frac{xf'(x)}{f^2(x)}$ is increasing (decreasing).

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(7) *f is h-H_tA_t-convex (-concave) if and only if $x^2 f'(x)$ is increasing (decreasing).*
(8) *f is h-H_tG_t-convex (-concave) if and only if $\frac{x^2 f'(x)}{f(x)}$ is increasing (decreasing).*
(9) *f is h-H_tH_t-convex (-concave) if and only if $\frac{x^2 f'(x)}{f^2(x)}$ is increasing (decreasing).*

Proof. The proof follows from Theorem 1 and Remark 3. □

Proposition 1. *Let $h : J \to (0, \infty)$ be a non-negative function. Then*

$$\begin{array}{ccccc} f \text{ is } h\text{-}\mathrm{A_tH_t}\text{-convex} & \Longrightarrow & f \text{ is } h\text{-}\mathrm{A_tG_t}\text{-convex} & \Longrightarrow & f \text{ is } h\text{-}\mathrm{A_tA_t}\text{-convex} \\ \Downarrow f \nearrow & & \Downarrow f \nearrow & & \Downarrow f \nearrow \\ f \text{ is } h\text{-}\mathrm{G_tH_t}\text{-convex} & \Longrightarrow & f \text{ is } h\text{-}\mathrm{G_tG_t}\text{-convex} & \Longrightarrow & f \text{ is } h\text{-}\mathrm{G_tA_t}\text{-convex} \\ \Downarrow f \nearrow & & \Downarrow f \nearrow & & \Downarrow f \nearrow \\ f \text{ is } h\text{-}\mathrm{H_tH_t}\text{-convex} & \Longrightarrow & f \text{ is } h\text{-}\mathrm{H_tG_t}\text{-convex} & \Longrightarrow & f \text{ is } h\text{-}\mathrm{H_tA_t}\text{-convex.} \end{array}$$

By $f \nearrow$ we mean that f is increasing. For concavity and decreasing monotonicity, the implications are reversed.

Proof. The proof of each statement follows from Definition 5 and by noting that $H_t(a, b) \le G_t(a, b) \le A_t(a, b)$, for all $t \in [0, 1]$. Furthermore, and for instance we note that if f is h-$\mathrm{A_tH_t}$-convex, therefore we have

$$\begin{aligned} f(\mathrm{A}_\alpha(x, y)) = f(\alpha x + (1 - \alpha) y) &\le \frac{f(x) f(y)}{h(1 - \alpha) f(x) + h(\alpha) f(y)} \\ &= \frac{1}{\frac{h(1-\alpha)}{f(y)} + \frac{h(\alpha)}{f(x)}} \\ &= \mathrm{H}(h(\alpha), f(x), f(y)), \end{aligned}$$

which is employing for $g(t) = \frac{1}{f(t)}$, i.e.,

$$\begin{aligned} g(\mathrm{A}_\alpha(x, y)) = g(\alpha x + (1 - \alpha) y) = \frac{1}{f(\alpha x + (1 - \alpha) y)} &\ge \frac{h(1 - \alpha)}{f(y)} + \frac{h(\alpha)}{f(x)} \\ &= h(1 - \alpha) g(y) + h(\alpha) g(x) \\ &= \mathrm{A}(h(\alpha), g(x), g(y)), \end{aligned}$$

and this shows that g is h-$\mathrm{A_tA_t}$-concave. □

These implications are strict, as shown by the examples below (see [2]).

Example 1. *Let h be a non-negative function such that $h(t) \ge t$ for all $t \in (0, 1)$. In particular, let $h(t) = h_k(t) = t^k$, $k \le 1$ and $t \in (0, 1)$. The functions*

(1) *$f(x) = \cosh(x)$ is t^k-$\mathrm{A_tG_t}$-convex, hence t^k-$\mathrm{G_tG_t}$-convex and t^k-$\mathrm{H_tG_t}$-convex, on $(0, \infty)$. But it is not t^k-$\mathrm{A_tH_t}$–convex, nor t^k-$\mathrm{G_tH_t}$–convex, nor t^k-$\mathrm{H_tH_t}$–convex.*
(2) *$f(x) = \arcsin(x)$ is t^k-$\mathrm{A_tA_t}$-convex but t^k-$\mathrm{A_tG_t}$-concave for all $0 \le x \le 1$.*
(3) *$f(x) = \mathrm{e}^x$ is t^k-$\mathrm{G_tG_t}$-convex and t^k-$\mathrm{H_tG_t}$-convex, but neither t^k-$\mathrm{G_tH_t}$-convex nor t^k-$\mathrm{H_tH_t}$-convex, for all $x > 0$.*
(4) *$f(x) = \log(1 + x)$ is t^k-$\mathrm{G_tA_t}$-convex but t^k-$\mathrm{G_tG_t}$-concave for all $0 < x < 1$.*
(5) *$f(x) = \mathrm{e}^{-x}$ is t^k-$\mathrm{H_tA_t}$-convex for $k \le \frac{1}{2}$ but not t^k-$\mathrm{H_tG_t}$-convex for all $0 < x < 1$. Also, f is t-$\mathrm{H_tA_t}$-convex but not t-$\mathrm{H_tG_t}$-convex for all $x > 1$.*

Proposition 2. *Let $h_1, h_2 : J \to (0, \infty)$ be two positive positive function with the property that $h_2(t) \le h_1(t)$ for all $t \in (0, 1)$. If f is h_2-MN-convex then h_1-MN-convex and if f is h_1-MN-concave then h_2-MN-concave.*

Proof. From Definition 5 we have

$$f(\mathrm{M}(t; x, y)) \le (\ge) \mathrm{N}(h_2(t); f(x), f(y)) \le (\ge) \mathrm{N}(h_1(t); f(x), f(y)),$$

which is required. □

Proposition 3. *If f and g are two h-MN-convex and $\lambda > 0$, then $f + g$, λf and $\max\{f, g\}$.*

Proof. The proof follows by Definition 5. □

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Proposition 4. *Let f and g be a similarly ordered functions. If f is h_1-$\mathrm{A_tA_t}$-convex (h_1-$\mathrm{G_tA_t}$-convex, h_1-$\mathrm{H_tA_t}$-convex), g is h_2-$\mathrm{A_tA_t}$-convex (h_2-$\mathrm{G_tA_t}$-convex, h_2-$\mathrm{H_tA_t}$-convex), respectively; and $h(t)+h(1-t)\le c$, where $h(t) := max\{h_1(t), h_2(t)\}$ and c is a fixed positive real number. Then the product (fg) is $(c\cdot h)$-$\mathrm{A_tA_t}$-convex ($-\mathrm{G_tA_t}$-convex, $-\mathrm{H_tA_t}$-convex), respectively.*

Proof. Since f and g are similarly ordered functions we have

$$f(x)g(x)+f(y)g(y)\ge f(x)g(y)+g(x)f(y).$$

Let t and s be positive numbers such that $t+s=1$. Then we obtain

$$\begin{aligned}
&(fg)(A_t(x,y))\\
&=(fg)(sx+ty)\\
&\le[h_1(s)f(x)+h_1(t)f(y)][h_2(s)g(x)+h_2(t)g(y)]\\
&\le h^2(s)f(x)g(x)+h(t)h(s)[f(y)g(x)+f(x)g(y)]+h^2(t)f(y)g(y)\\
&\le h^2(s)f(x)g(x)+h(t)h(s)[f(x)g(x)+f(y)g(y)]+h^2(t)f(y)g(y)\\
&=(h(s)+h(t))(h(s)(fg)(x)+h(t)(fg)(y))\\
&=c\cdot h(s)(fg)(x)+c\cdot h(t)(fg)(y)\\
&=A(c\cdot h(t);(fg)(x),(fg)(y)),
\end{aligned}$$

which shows that (fg) is $(c\cdot h)$-$\mathrm{A_tA_t}$-convex. The cases when fg is $(c\cdot h)$–$\mathrm{G_tA_t}$-convex or $(c\cdot h)$–$\mathrm{H_tA_t}$-convex, are follow in similar manner. □

Corollary 2. *Let f and g be an oppositely ordered functions. If f is h_1-$\mathrm{A_tA_t}$-concave (h_1-$\mathrm{G_tA_t}$-concave, h_1-$\mathrm{H_tA_t}$-concave), g is h_2-$\mathrm{A_tA_t}$-concave (h_2-$\mathrm{G_tA_t}$-concave, h_2-$\mathrm{H_tA_t}$-concave), respectively; and $h(t)+h(1-t)\ge c$, where $h(t) := min\{h_1(t), h_2(t)\}$ and c is a fixed positive real number. Then the product (fg) is $(c\cdot h)$-$\mathrm{A_tA_t}$-concave ($-\mathrm{G_tA_t}$-concave, $-\mathrm{H_tA_t}$-concave), respectively.*

Proposition 5. *If f is h_1-$\mathrm{A_tG_t}$-convex (h_1-$\mathrm{G_tG_t}$-convex, h_1-$\mathrm{H_tG_t}$-convex) and g is h_2-$\mathrm{A_tG_t}$-convex (h_2-$\mathrm{G_tG_t}$-convex, h_2-$\mathrm{H_tG_t}$-convex), respectively. Then the product (fg) is h-$\mathrm{A_tG_t}$-convex (h-$\mathrm{G_tG_t}$-convex, h-$\mathrm{H_tG_t}$-convex), respectively; where $h(t) := max\{h_1(t), h_2(t)\}$.*

Proof. let $t\in(0,1)\subseteq J$, then

$$\begin{aligned}
&(fg)(A_t(x,y))\\
&=(fg)((1-t)x+ty)\\
&\le\left\{[f(x)]^{h_1(1-t)}[f(y)]^{h_1(t)}\right\}\cdot\left\{[g(x)]^{h_2(1-t)}[g(y)]^{h_2(t)}\right\}\\
&=[f(x)]^{h_1(1-t)}[g(x)]^{h_2(1-t)}\cdot[f(y)]^{h_1(t)}[g(y)]^{h_2(t)}\\
&\le[(fg)(x)]^{h(1-t)}\cdot[(fg)(y)]^{h(t)}\\
&=G(h(t),(fg)(x),(fg)(y)),
\end{aligned}$$

which shows that (fg) is h-$\mathrm{A_tG_t}$-convex. The cases when fg is $(c\cdot h)$–$\mathrm{G_tG_t}$-convex or $(c\cdot h)$–$\mathrm{H_tG_t}$-convex, are follow in similar manner. □

Corollary 3. *If f is h_1-$\mathrm{A_tG_t}$-concave (h_1-$\mathrm{G_tG_t}$-concave, h_1-$\mathrm{H_tG_t}$-concave) and g is h_2-$\mathrm{A_tG_t}$-concave (h_2-$\mathrm{G_tG_t}$-concave, h_2-$\mathrm{H_tG_t}$-concave), respectively. Then the product (fg) is h-$\mathrm{A_tG_t}$-concave (h-$\mathrm{G_tG_t}$-concave, h-$\mathrm{H_tG_t}$-concave), respectively; where $h(t) := min\{h_1(t), h_2(t)\}$.*

Proposition 6. *Let f and g be an oppositely ordered functions. If f is h_1-$\mathrm{A_tH_t}$-convex (h_1-$\mathrm{G_tH_t}$-convex, h_1-$\mathrm{H_tH_t}$-convex), g is h_2-$\mathrm{A_tH_t}$-convex (h_2-$\mathrm{G_tH_t}$-convex, h_2-$\mathrm{H_tH_t}$-convex), respectively; and $h(t)+h(1-t)\ge c$, where $h(t) := min\{h_1(t), h_2(t)\}$ and c is a fixed positive real number. Then the product (fg) is $(c\cdot h)$-$\mathrm{A_tH_t}$-convex ($-\mathrm{G_tH_t}$-convex, $-\mathrm{H_tH_t}$-convex), respectively.*

Proof. Since f and g are oppositely ordered functions

$$f(x)g(x)+f(y)g(y)\le f(x)g(y)+g(x)f(y).$$

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Let t and s be positive numbers such that $t+s=1$. Then we obtain

$$\begin{aligned}
&(fg)(A_t(x,y))\\
&=(fg)(sx+ty)\\
&\leq \frac{f(x)f(y)}{h_1(t)f(x)+h_1(s)f(y)}\cdot\frac{g(x)g(y)}{h_2(t)g(x)+h_2(s)g(y)}\\
&\leq \frac{(fg)(x)(fg)(y)}{h_1(t)h_2(t)f(x)g(x)+h_1(s)h_2(t)f(y)g(x)+h_1(t)h_2(s)f(x)g(y)+h_1(s)h_2(s)f(y)g(y)}\\
&\leq \frac{(fg)(x)(fg)(y)}{h^2(t)f(x)g(x)+h(s)h(t)f(x)g(x)+h(t)h(s)f(y)g(y)+h^2(s)f(y)g(y)}\\
&=\frac{(fg)(x)(fg)(y)}{[h(t)+h(s)][h(t)(fg)(x)+h(s)(fg)(y)]}\\
&=\frac{(fg)(x)(fg)(y)}{c\cdot h(t)(fg)(x)+c\cdot h(s)(fg)(y)}\\
&=H(c\cdot h(t);(fg)(x),(fg)(y)),
\end{aligned}$$

which shows that (fg) is $(c\cdot h)$-$\mathrm{A_tH_t}$-convex. The cases when fg is $(c\cdot h)$–$\mathrm{G_tH_t}$-convex or $(c\cdot h)$–$\mathrm{H_tH_t}$-convex, are follow in similar manner. □

Corollary 4. *Let f and g be similarly ordered functions. If f is h_1-$\mathrm{A_tH_t}$-concave (h_1-$\mathrm{G_tH_t}$-concave, h_1-$\mathrm{H_tH_t}$-concave), g is h_2-$\mathrm{A_tH_t}$-concave (h_2-$\mathrm{G_tH_t}$-concave, h_2-$\mathrm{H_tH_t}$-concave), respectively; and $h(t)+h(1-t)\leq c$, where $h(t):=max\{h_1(t),h_2(t)\}$ and c is a fixed positive real number. Then the product (fg) is $(c\cdot h)$-$\mathrm{A_tH_t}$-concave (–$\mathrm{G_tH_t}$-concave, –$\mathrm{H_tH_t}$-concave), respectively.*

Sometimes we often use functional inequalities to describe and characterize all real functions that satisfy specific functional inequality. In [30], Varošanec proved a result regarding $\mathrm{A_tA_t}$-convex functions, following a similar approach; we next present some results of this type.

Theorem 2. *Let $I\subset\mathbb{R}$ with $0\in I$. Let h be a non-negative function on J.*

(1) Let f be h-$\mathrm{A_tG_t}$-convex and $f(0)=1$. If h is supermultiplicative, then the inequality

$$f(\alpha x+\beta y)\leq[f(x)]^{h(\alpha)}[f(y)]^{h(\beta)}, \tag{2.13}$$

holds for all $x,y\in I$ and all $\alpha,\beta>0$ such that $\alpha+\beta\leq 1$.

(2) Assume that $h(\alpha)<\frac{1}{2}$ for some $\alpha\in\left(0,\frac{1}{2}\right)$. If f is a non-negative function such that inequality (2.13) holds for all $x,y\in I$ and all $\alpha,\beta>0$ such that $\alpha+\beta\leq 1$, then $f(0)=1$.

(3) Let f be h-$\mathrm{A_tG_t}$-concave and $f(0)=1$. If h is submultiplicative, then the inequality

$$f(\alpha x+\beta y)\geq[f(x)]^{h(\alpha)}[f(y)]^{h(\beta)}, \tag{2.14}$$

holds for all $x,y\in I$ and all $\alpha,\beta>0$ such that $\alpha+\beta\leq 1$.

(4) Assume that $h(\alpha)>\frac{1}{2}$ for some $\alpha\in\left(0,\frac{1}{2}\right)$. If f is a non-negative function such that inequality (2.14) holds for all $x,y\in I$ and all $\alpha,\beta>0$ such that $\alpha+\beta\leq 1$, then $f(0)=1$.

Proof. Let $\alpha,\beta>0$ be positive real numbers such that $\alpha+\beta=\lambda\leq 1$.

(1) Define numbers a and b such as $a=\frac{\alpha}{\lambda}$ and $b=\frac{\beta}{\lambda}$. Then $a+b=1$ and we have the following:

$$\begin{aligned}
f(\alpha x+\beta y)&=f(\lambda ax+\lambda by)\\
&\leq[f(\lambda x)]^{h(a)}[f(\lambda y)]^{h(b)}\\
&=[f(\lambda x+(1-\lambda)\cdot 0)]^{h(a)}[f(\lambda y+(1-\lambda)\cdot 0)]^{h(b)}\\
&\leq\left\{[f(x)]^{h(\lambda)}[f(0)]^{h(1-\lambda)}\right\}^{h(a)}\left\{[f(y)]^{h(\lambda)}[f(0)]^{h(1-\lambda)}\right\}^{h(b)}\\
&=[f(x)]^{h(a)h(\lambda)}[f(y)]^{h(b)h(\lambda)}\\
&=[f(x)]^{h(\lambda a)}[f(y)]^{h(\lambda b)}\\
&=[f(x)]^{h(\alpha)}[f(y)]^{h(\beta)},
\end{aligned}$$

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where we use that f is A_tG_t, $f(0) = 1$ and h is supermultiplicative, respectively.

(2) Suppose that $f(0) \neq 1$. Putting $x = y = 0$ in (2.13) we get
$$f(0) \le [f(0)]^{h(\alpha)+h(\beta)}, \qquad \text{for all } \alpha, \beta > 0,\ \alpha + \beta \le 1.$$
Setting $\beta = \alpha$, $\alpha \in \left(0, \frac{1}{2}\right)$, then $0 \le (2h(\alpha) - 1)\log f(0)$, it follows that $h(\alpha) \ge \frac{1}{2}$, since $f(0) \neq 1$, which contradicts the assumption of theorem. So that $f(0) = 1$.

The proofs for cases (3) and (4) are similar to the previous. Hence, the proof is completely established. □

Theorem 3. *Let $a, b \in \left(\frac{1}{\tau}, \infty\right)$ with $a < b$, so that $a, b \in I$ where $I = (0, \tau)$. Let h be a non-negative function on J.*

(1) Let f be h-$\mathrm{G_tA_t}$-convex and $f(1) = 0$. If h is supermultiplicative, then the inequality
$$f\left(x^\alpha y^\beta\right) \le h(\alpha) f(x) + h(\beta) f(y), \tag{2.15}$$
holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$.

(2) Assume that $h(\alpha) < \frac{1}{2}$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. If f is a non-negative function such that inequality (2.15) holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$, then $f(1) = 0$.

(3) Let f be h-$\mathrm{G_tA_t}$-concave and $f(1) = 0$. If h is submultiplicative, then the inequality
$$f\left(x^\alpha y^\beta\right) \ge h(\alpha) f(x) + h(\beta) f(y), \tag{2.16}$$
holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$.

(4) Assume that $h(\alpha) > \frac{1}{2}$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. If f is a non-negative function such that inequality (2.16) holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$, then $f(1) = 0$.

Proof. Let $\alpha, \beta > 0$ be positive real numbers such that $\alpha + \beta = \lambda \le 1$.

(1) Define numbers a and b such as $a = \frac{\alpha}{\lambda}$ and $b = \frac{\beta}{\lambda}$. Then $a + b = 1$ and we have the following:
$$\begin{aligned}
f\left(x^\alpha y^\beta\right) &= f\left(x^{\lambda a} y^{\lambda b}\right)\\
&\le h(a) f\left(x^\lambda\right) + h(b) f\left(y^\lambda\right)\\
&= h(a) f\left(x^\lambda \cdot 1^{1-\lambda}\right) + h(b) f\left(y^\lambda \cdot 1^{1-\lambda}\right)\\
&\le h(a)\left[h(\lambda) f(x) + h(1-\lambda) f(1)\right] + h(b)\left[h(\lambda) f(y) + h(1-\lambda) f(1)\right]\\
&= h(a) h(\lambda) f(x) + h(b) h(\lambda) f(y)\\
&\le h(\alpha) f(x) + h(\beta) f(y),
\end{aligned}$$
where we use that f is G_tA_t, $f(1) = 0$ and h is supermultiplicative, respectively.

(2) Suppose that $f(1) \neq 0$, since f is non-negative then $f(1) > 0$. Putting $x = y = 1$ in (2.15) we get
$$f(1) \le h(\alpha) f(1) + h(\beta) f(1), \qquad \text{for all } \alpha, \beta > 0,\ \alpha + \beta \le 1.$$
Setting $\beta = \alpha$, $\alpha \in \left(0, \frac{1}{2}\right)$, then $0 \le (2h(\alpha) - 1) f(1)$, it follows that $h(\alpha) \ge \frac{1}{2}$, which contradicts the assumption of theorem. So that $f(1) = 0$.

The proofs for cases (3) and (4) are similar to the previous. Hence, the proof is completely established. □

Theorem 4. *Let $a, b \in \left(\frac{1}{\tau}, \infty\right)$ with $a < b$, so that $a, b \in I$ where $I = (0, \tau)$. Let h be a non-negative function on J.*

(1) Let f be h-$\mathrm{G_tG_t}$-convex and $f(1) = 1$. If h is supermultiplicative, then the inequality
$$f\left(x^\alpha y^\beta\right) \le [f(x)]^{h(\alpha)} [f(y)]^{h(\beta)}, \tag{2.17}$$
holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$.

(2) Assume that $h(\alpha) < \frac{1}{2}$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. If f is a non-negative function such that inequality (2.17) holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$, then $f(1) = 1$.

(3) Let f be h-$\mathrm{G_tG_t}$-concave and $f(1) = 1$. If h is submultiplicative, then the inequality
$$f\left(x^\alpha y^\beta\right) \ge [f(x)]^{h(\alpha)} [f(y)]^{h(\beta)}, \tag{2.18}$$
holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \le 1$.

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(4) *Assume that* $h(\alpha) > \frac{1}{2}$ *for some* $\alpha \in \left(0, \frac{1}{2}\right)$. *If* f *is a non-negative function such that inequality* (2.18) *holds for all* $x, y \in I$ *and all* $\alpha, \beta > 0$ *such that* $\alpha + \beta \le 1$, *then* $f(1) = 1$.

Proof. Let $\alpha, \beta > 0$ be positive real numbers such that $\alpha + \beta = \lambda \le 1$.

(1) Define numbers a and b such as $a = \frac{\alpha}{\lambda}$ and $b = \frac{\beta}{\lambda}$. Then $a + b = 1$ and we have the following:

$$\begin{aligned} f\left(x^{\alpha} y^{\beta}\right) &= f\left(x^{\lambda a} y^{\lambda b}\right) \\ &\le \left[f\left(x^{\lambda}\right)\right]^{h(a)} \left[f\left(y^{\lambda}\right)\right]^{h(b)} \\ &= \left[f\left(x^{\lambda} \cdot 1^{1-\lambda}\right)\right]^{h(a)} \left[f\left(y^{\lambda} \cdot 1^{1-\lambda}\right)\right]^{h(b)} \\ &\le \left\{[f(x)]^{h(\lambda)} [f(1)]^{h(1-\lambda)}\right\}^{h(a)} \left\{[f(y)]^{h(\lambda)} [f(1)]^{h(1-\lambda)}\right\}^{h(b)} \\ &= [f(x)]^{h(a)h(\lambda)} [f(y)]^{h(b)h(\lambda)} \\ &= [f(x)]^{h(\lambda a)} [f(y)]^{h(\lambda b)} \\ &= [f(x)]^{h(\alpha)} [f(y)]^{h(\beta)}, \end{aligned}$$

where we use that f is G_tG_t, $f(1) = 1$ and h is supermultiplicative, respectively.

(2) Suppose that $f(1) \neq 1$. Putting $x = y = 1$ in (2.17) we get

$$f(1) \le [f(1)]^{h(\alpha)} [f(1)]^{h(\beta)}, \qquad \text{for all } \alpha, \beta > 0,\ \alpha + \beta \le 1.$$

Setting $\beta = \alpha$, $\alpha \in \left(0, \frac{1}{2}\right)$, then $1 \le [f(1)]^{(2h(\alpha)-1)}$, it follows that $h(\alpha) \ge \frac{1}{2}$, which contradicts the assumption of theorem. So that $f(1) = 1$.

The proofs for cases (3) and (4) are similar to the previous. Hence, the proof is completely established. □

2.2. **Composition of h-MN-convex functions.** In the next three results, we assume the g $h_i : J_i \to (0, \infty)$, $i = 1, 2$, $h_2(J_2) \subseteq J_1$ are non-negative functions such that $h_2(\alpha) + h_2(1-\alpha) \le 1$, for $\alpha\,(0,1) \subseteq J_2$, let $f : I_1 \to [0, \infty)$, $g : I_2 \to [0, \infty)$, be functions with $g(I_2) \subseteq I_1$.

Theorem 5. *Let* $f(1) = 0$. *If* h_1 *is a supermultiplicative function,* f *is* h_1-$\mathrm{G_tA_t}$*-convex and increasing (decreasing) on* I_1*, while* g *is* h_2-$\mathrm{A_tG_t}$*-convex (-concave) on* I_2*, then the composition* $f \circ g$ *is* $(h_1 \circ h_2)$-$\mathrm{A_tA_t}$*-convex on* I_2.

If h_1 *is a submultiplicative function,* f *is* h_1-$\mathrm{G_tA_t}$*-concave and increasing (decreasing) on* I_1*, while* g *is* h_2-$\mathrm{A_tG_t}$*-convex (-convex) on* I_2*, then the composition* $f \circ g$ *is* $(h_1 \circ h_2)$-$\mathrm{A_tA_t}$*-concave on* I_2.

Proof. If g is h_2-$\mathrm{A_tG_t}$-convex on I_2 and f increasing then

$$f \circ g(\alpha x + (1-\alpha) y) \le f\left([g(x)]^{h_2(\alpha)} [g(y)]^{h_2(1-\alpha)}\right),$$

for all $x, y \in I_2$ and $\alpha \in (0,1)$. Using Theorem 3(1), we obtain that

$$\begin{aligned} f\left([g(x)]^{h_2(\alpha)} [g(y)]^{h_2(1-\alpha)}\right) &\le h_1(h_2(\alpha)) f(g(x)) + h_1(h_2(1-\alpha)) f(g(y)) \\ &= (h_1 \circ h_2)(\alpha)(f \circ g)(x) + (h_1 \circ h_2)(1-\alpha)(f \circ g)(y), \end{aligned}$$

which means that $f \circ g$ is $(h_1 \circ h_2)$-$\mathrm{A_tA_t}$-convex on I_2. □

Theorem 6. *Let* $0 \in I_1$ *and* $f(0) = 1$. *If* h_1 *is a supermultiplicative function,* f *is* h_1-$\mathrm{A_tG_t}$*-convex and increasing (decreasing) on* I_1*, while* g *is* h_2-$\mathrm{G_tA_t}$*-convex (-concave) on* I_2*, then the composition* $f \circ g$ *is* $(h_1 \circ h_2)$-$\mathrm{G_tG_t}$*-convex on* I_2.

If h_1 *is a submultiplicative function,* f *is* h_1-$\mathrm{A_tG_t}$*-concave and increasing (decreasing) on* I_1*, while* g *is* h_2-$\mathrm{G_tA_t}$*-convex (-convex) on* I_2*, then the composition* $f \circ g$ *is* $(h_1 \circ h_2)$-$\mathrm{G_tG_t}$*-concave on* I_2.

Proof. The proof is similar to the proof of Theorem 5 and using Theorem 2(1). □

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Theorem 7. *Let $f(1)=1$. If h_1 is a supermultiplicative function, f is h_1-$\mathrm{G_tG_t}$-convex and increasing (decreasing) on I_1, while g is h_2-$\mathrm{G_tG_t}$-convex (-concave) on I_2, then the composition $f\circ g$ is $(h_1\circ h_2)$-$\mathrm{G_tG_t}$-convex on I_2.*

If h_1 is a submultiplicative function, f is h_1-$\mathrm{G_tG_t}$-concave and increasing (decreasing) on I_1, while g is h_2-$\mathrm{G_tG_t}$-convex (-convex) on I_2, then the composition $f\circ g$ is $(h_1\circ h_2)$-$\mathrm{G_tG_t}$-concave on I_2.

Proof. The proof is similar to the proof of Theorem 5 and using Theorem 4(1). □

Next, we examine functions compositions, one of them is of type h_1-$\mathrm{M_tK_t}$-convex while the other is h_2-$\mathrm{K_tN_t}$-convex.

Theorem 8. *Let M,N and K be three mean functions. Let $h_1:J_1\to(0,\infty)$ and $h_1:J_2\to(0,1)$, $h_2(J_2)\subseteq(0,1)\subseteq J_1$ are non-negative functions for $\alpha\in(0,1)\subseteq J_2$ and $h_2(\alpha)\in(0,1)\subseteq J_1$, let $f:I_1\to[0,\infty)$, $g:I_2\to[0,\infty)$, be functions with $g(I_2)\subseteq I_1$. If f is h_1-$\mathrm{K_tN_t}$-convex and increasing (decreasing) on I_1, while g is h_2-$\mathrm{M_tK_t}$-convex (-concave) on I_2, then the composition $f\circ g$ is $(h_1\circ h_2)$-$\mathrm{M_tN_t}$-convex on I_2. Namely, we explore this corollary in the table below.*

Proof. We select to prove one of the mentioned cases and the others follow in similar fashion. For example, if g is h_2-$\mathrm{H_tA_t}$-convex on I_2 and f is increasing then

$$f\circ g\left(\frac{xy}{\alpha x+(1-\alpha)y}\right)\le f\left(h_2(1-\alpha)g(x)+h_2(\alpha)g(y)\right),$$

for all $x,y\in I_2$ and $\alpha\in(0,1)$. Using Definition 5, we obtain that

$$\begin{aligned}f\left(h_2(1-\alpha)g(x)+h_2(\alpha)g(y)\right)&\le\frac{f(g(x))f(g(y))}{h_1(h_2(\alpha))f(g(x))+h_1(h_2(1-\alpha))f(g(y))}\\&=\frac{(f\circ g)(x)(f\circ g)(y)}{(h_1\circ h_2)(\alpha)(f\circ g)(x)+(h_1\circ h_2)(1-\alpha)(f\circ g)(y)},\end{aligned}$$

for $h_2(\alpha)\in(0,1)$, which shows that $f\circ g$ is $(h_1\circ h_2)$-$\mathrm{H_tH_t}$-convex on I_2. □

f	g	$f\circ g$
h_1-$\mathrm{A_tA_t}$-convex h_1-$\mathrm{G_tA_t}$-convex h_1-$\mathrm{H_tA_t}$-convex	h_2-$\mathrm{A_tA_t}$-convex h_2-$\mathrm{A_tG_t}$-convex h_2-$\mathrm{A_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{A_tA_t}$-convex
h_1-$\mathrm{A_tG_t}$-convex h_1-$\mathrm{G_tG_t}$-convex h_1-$\mathrm{H_tG_t}$-convex	h_2-$\mathrm{A_tA_t}$-convex h_2-$\mathrm{A_tG_t}$-convex h_2-$\mathrm{A_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{A_tG_t}$-convex
h_1-$\mathrm{A_tH_t}$-convex h_1-$\mathrm{G_tH_t}$-convex h_1-$\mathrm{H_tH_t}$-convex	h_2-$\mathrm{A_tA_t}$-convex h_2-$\mathrm{A_tG_t}$-convex h_2-$\mathrm{A_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{A_tH_t}$-convex
h_1-$\mathrm{A_tA_t}$-convex h_1-$\mathrm{G_tA_t}$-convex h_1-$\mathrm{H_tA_t}$-convex	h_2-$\mathrm{G_tA_t}$-convex h_2-$\mathrm{G_tG_t}$-convex h_2-$\mathrm{G_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{G_tA_t}$-convex
h_1-$\mathrm{G_tG_t}$-convex h_1-$\mathrm{A_tG_t}$-convex h_1-$\mathrm{H_tG_t}$-convex	h_2-$\mathrm{G_tG_t}$-convex h_2-$\mathrm{G_tA_t}$-convex h_2-$\mathrm{G_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{G_tG_t}$-convex
h_1-$\mathrm{A_tH_t}$-convex h_1-$\mathrm{G_tH_t}$-convex h_1-$\mathrm{H_tH_t}$-convex	h_2-$\mathrm{G_tA_t}$-convex h_2-$\mathrm{G_tG_t}$-convex h_2-$\mathrm{G_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{G_tH_t}$-convex
h_1-$\mathrm{A_tA_t}$-convex h_1-$\mathrm{G_tA_t}$-convex h_1-$\mathrm{H_tA_t}$-convex	h_2-$\mathrm{H_tA_t}$-convex h_2-$\mathrm{H_tG_t}$-convex h_2-$\mathrm{H_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{H_tA_t}$-convex
h_1-$\mathrm{A_tG_t}$-convex h_1-$\mathrm{G_tG_t}$-convex h_1-$\mathrm{H_tG_t}$-convex	h_2-$\mathrm{H_tA_t}$-convex h_2-$\mathrm{H_tG_t}$-convex h_2-$\mathrm{H_tH_t}$-convex	$h_1\circ h_2$-$\mathrm{H_tG_t}$-convex

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f	g	$f\circ g$
h_1-$\mathrm{H_tH_t}$-convex	h_2-$\mathrm{H_tH_t}$-convex	
h_1-$\mathrm{A_tH_t}$-convex	h_2-$\mathrm{H_tA_t}$-convex	$h_1\circ h_2$-$\mathrm{H_tH_t}$-convex
h_1-$\mathrm{G_tH_t}$-convex	h_2-$\mathrm{H_tG_t}$-convex	

3. Characterization of h-$\mathrm{M_tN_t}$-convexity

Let $h:J\to[0,\infty)$ be a non-negative function and let $f:I\to\mathbb{R}$ be a function. For all points $x_1,x_2,x_3\in I$, $x_1<x_2<x_3$ such that x_2-x_1, x_3-x_2 and x_3-x_1 in J. In [30], Varošanec proved that if h is supermultiplicative, and f is h-$\mathrm{A_tA_t}$-convex function, then the inequality

$$h(x_3-x_2)\,f(x_1)+h(x_2-x_1)\,f(x_3)\ge h(x_3-x_1)\,f(x_2),$$

holds. Also, if h is submultiplicative, and f is h-$\mathrm{A_tA_t}$-convex function, then the above inequality is reversed. In what follows, similar results for $\mathrm{M_tN_t}$-convex functions are proved.

Theorem 9. *Let $h:J\to[0,\infty)$ be a non-negative function and let $f:I\to\mathbb{R}$ be a function. For all points $x_1,x_2,x_3\in I$, $x_1<x_2<x_3$ such that x_2-x_1, x_3-x_2 and x_3-x_1 in J,*

(1) If h is supermultiplicative, and f is h-$\mathrm{A_tG_t}$-convex function, then the following inequality hold:

$$[f(x_1)]^{h(x_3-x_2)}\,[f(x_3)]^{h(x_2-x_1)}\ge[f(x_2)]^{h(x_3-x_1)},$$

(2) If h is submultiplicative, and f is h-$\mathrm{A_tH_t}$-convex function, then the following inequality hold:

$$h(x_3-x_1)\,f(x_1)\,f(x_3)\ge h(x_2-x_1)\,f(x_1)\,f(x_2)+h(x_3-x_2)\,f(x_3)\,f(x_2).$$

In case of h-$\mathrm{A_tN_t}$-concavity the inequalities are reversed.

Proof. Let $x_1,x_2,x_3\in I$ with $x_1<x_2<x_3$, such that x_2-x_1, x_3-x_2 and x_3-x_1 in J. Consequently, $\frac{x_2-x_1}{x_3-x_1},\frac{x_3-x_2}{x_3-x_1}\in(0,1)\subseteq J$ and $\frac{x_2-x_1}{x_3-x_1}+\frac{x_3-x_2}{x_3-x_1}=1$. Also, since h is super(sub)multiplicative then for all $p,q\in J$ we have

$$h(p)=h\left(\frac{p}{q}\cdot q\right)\ge(\le)h\left(\frac{p}{q}\right)h(q),$$

and this yield that

$$\frac{h(p)}{h(q)}\ge(\le)h\left(\frac{p}{q}\right).$$

Setting $t=\frac{x_3-x_2}{x_3-x_1}$, $\alpha=x_1$, $\beta=x_3$, therefore we have the following cases:

(1) For $x_2=t\alpha+(1-t)\beta$ and since f is A_tG_t-convex, then by (2.2)

$$\begin{aligned} f(x_2)&\le[f(x_1)]^{h\left(\frac{x_3-x_2}{x_3-x_1}\right)}\,[f(x_3)]^{h\left(\frac{x_2-x_1}{x_3-x_1}\right)}\\ &\le[f(x_1)]^{\frac{h(x_3-x_2)}{h(x_3-x_1)}}\,[f(x_3)]^{\frac{h(x_2-x_1)}{h(x_3-x_1)}},\end{aligned} \tag{3.1}$$

since f is positive, then the above inequality equivalent to

$$h(x_3-x_1)\log f(x_2)\le h(x_3-x_2)\log f(x_1)+h(x_2-x_1)\log f(x_3).$$

Rearranging the terms again we get

$$[f(x_1)]^{h(x_3-x_2)}\,[f(x_3)]^{h(x_2-x_1)}\ge[f(x_2)]^{h(x_3-x_1)},$$

as desired.

(2) For $x_2=t\alpha+(1-t)\beta$ and since f is A_tH_t-convex then by (2.3)

$$\begin{aligned} f(x_2)&\le\frac{f(x_1)\,f(x_3)}{h\left(\frac{x_2-x_1}{x_3-x_1}\right)f(x_1)+h\left(\frac{x_3-x_2}{x_3-x_1}\right)f(x_3)}\\ &\le\frac{h(x_3-x_1)\,f(x_1)\,f(x_3)}{h(x_2-x_1)\,f(x_1)+h(x_3-x_2)\,f(x_3)},\end{aligned} \tag{3.2}$$

and this is equivalent to write

$$h(x_3-x_1)\,f(x_1)\,f(x_3)\ge h(x_2-x_1)\,f(x_1)\,f(x_2)+h(x_3-x_2)\,f(x_3)\,f(x_2),$$

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as desired.

Thus, the proof is completely established. □

Corollary 5. *Let $h : (0, 1) \to [0, \infty)$ be a non-negative function and let $f : (0, 1) \to (0, \infty)$ be a function. For all points $x_1, x_2, x_3 \in (0, 1)$, $x_1 < x_2 < x_3$ such that $x_2 - x_1$, $x_3 - x_2$ and $x_3 - x_1$ in $(0, 1)$. Let $h_r(t) = t^r$, $r \in (-\infty, -1] \cup [0, 1]$.*

(1) *If f is h-$\mathrm{A_tG_t}$-convex function, then the following inequality hold:*
$$[f(x_1)]^{(x_3-x_2)^r}[f(x_3)]^{(x_2-x_1)^r} \ge [f(x_2)]^{(x_3-x_1)^r}.$$
Furthermore, if $f(x) = x^\lambda$ $(\lambda < 0)$ we get several Schur type inequalities.

(2) *If f is h-$\mathrm{A_tH_t}$-convex function, then the following inequality hold:*
$$(x_3 - x_1)^r f(x_1) f(x_3) \ge (x_2 - x_1)^r f(x_1) f(x_2) + (x_3 - x_2)^r f(x_3) f(x_2).$$
Furthermore, if $f(x) = x^\lambda$ $(-1 < \lambda < 0)$ we get several Schur type inequalities.

In case of h-$\mathrm{A_tN_t}$-concavity the inequalities are reversed.

Theorem 10. *Let $h : J \to [0, \infty)$ be a non-negative function and let $f : I \to \mathbb{R}$ be a function. For all points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ such that $\ln\left(\frac{x_3}{x_2}\right)$, $\ln\left(\frac{x_2}{x_1}\right)$ and $\ln\left(\frac{x_3}{x_1}\right)$ in J.*

(1) *If h is supermultiplicative, and f is h-$\mathrm{G_tA_t}$-convex function, then the following inequality hold:*
$$h\left(\ln\left(\frac{x_3}{x_2}\right)\right) \cdot f(x_1) + h\left(\ln\left(\frac{x_2}{x_1}\right)\right) \cdot f(x_3) \ge h\left(\ln\left(\frac{x_3}{x_1}\right)\right) f(x_2).$$

(2) *If h is supermultiplicative, and f is h-$\mathrm{G_tG_t}$-convex function, then the following inequality hold:*
$$[f(x_1)]^{h\left(\ln\left(\frac{x_3}{x_1}\right)\right)}[f(x_3)]^{h\left(\ln\left(\frac{x_2}{x_1}\right)\right)} \ge f(x_2)^{h\left(\ln\left(\frac{x_3}{x_1}\right)\right)}.$$

(3) *If h is submultiplicative, and f is h-$\mathrm{G_tH_t}$-convex function, then the following inequality hold:*
$$h\left(\ln\left(\frac{x_3}{x_1}\right)\right) f(x_1) f(x_3) + \ge h\left(\ln\left(\frac{x_2}{x_1}\right)\right) f(x_1) f(x_2) + h\left(\ln\left(\frac{x_3}{x_2}\right)\right) f(x_3) f(x_2).$$

In case of h-$\mathrm{G_tN_t}$-concavity the inequalities are reversed.

Proof. Let $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$, such that $\ln\left(\frac{x_3}{x_2}\right)$, $\ln\left(\frac{x_2}{x_1}\right)$ and $\ln\left(\frac{x_3}{x_1}\right)$ in J. Consequently, $\frac{\ln x_2 - \ln x_1}{\ln x_3 - \ln x_1}, \frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1} \in (0, 1) \subseteq J$ and $\frac{\ln x_2 - \ln x_1}{\ln x_3 - \ln x_1} + \frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1} = 1$. Setting $t = \frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1}$, $\alpha = x_1$, $\beta = x_3$, therefore we have the following cases:

(1) For $x_2 = \alpha^t\beta^{1-t}$ and since f is G_tA_t-convex then by (2.4)
$$\begin{aligned} f(x_2) &\le h\left(\frac{\ln(x_3) - \ln(x_2)}{\ln(x_3) - \ln(x_1)}\right) \cdot f(x_1) + h\left(\frac{\ln(x_2) - \ln(x_1)}{\ln(x_3) - \ln(x_1)}\right) \cdot f(x_3) \\ &\le \frac{h(\ln(x_3) - \ln(x_1))}{h(\ln(x_3) - \ln(x_1))} \cdot f(x_1) + \frac{h(\ln(x_2) - \ln(x_1))}{h(\ln(x_3) - \ln(x_1))} \cdot f(x_3), \end{aligned} \tag{3.3}$$
and this is equivalent to write
$$h\left(\ln\left(\frac{x_3}{x_2}\right)\right) \cdot f(x_1) + h\left(\ln\left(\frac{x_2}{x_1}\right)\right) \cdot f(x_3) \ge h\left(\ln\left(\frac{x_3}{x_1}\right)\right) f(x_2),$$
as desired.

(2) For $x_2 = \alpha^t\beta^{1-t}$ and since f is G_tG_t-convex then by (2.5)
$$\begin{aligned} f(x_2) &\le [f(x_1)]^{h\left(\frac{\ln(x_3)-\ln(x_2)}{\ln(x_3)-\ln(x_1)}\right)}[f(x_3)]^{h\left(\frac{\ln(x_2)-\ln(x_1)}{\ln(x_3)-\ln(x_1)}\right)} \\ &\le [f(x_1)]^{\frac{h(\ln(x_3)-\ln(x_2))}{h(\ln(x_3)-\ln(x_1))}}[f(x_3)]^{\frac{h(\ln(x_2)-\ln(x_1))}{h(\ln(x_3)-\ln(x_1))}}, \end{aligned} \tag{3.4}$$
since f is positive therefore
$$h\left(\ln\left(\frac{x_3}{x_1}\right)\right) \log f(x_2) \le h\left(\ln\left(\frac{x_3}{x_1}\right)\right) \log[f(x_1)] + h\left(\ln\left(\frac{x_2}{x_1}\right)\right) \log[f(x_3)],$$

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and this equivalent to write

$$[f(x_1)]^{h\left(\ln\left(\frac{x_3}{x_1}\right)\right)}[f(x_3)]^{h\left(\ln\left(\frac{x_2}{x_1}\right)\right)} \ge f(x_2)^{h\left(\ln\left(\frac{x_3}{x_1}\right)\right)},$$

as desired.

(3) For $x_2 = \alpha^t\beta^{1-t}$ and since f is G_tH_t-convex then by (2.6)

$$f(x_2) \le \frac{f(x_1)f(x_3)}{h\left(\frac{\ln x_2 - \ln x_1}{\ln x_3 - \ln x_1}\right)f(x_1) + h\left(\frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1}\right)f(x_3)}$$

$$\le \frac{h(\ln x_3 - \ln x_1)f(x_1)f(x_3)}{h(\ln x_2 - \ln x_1)f(x_1) + h(\ln x_3 - \ln x_2)f(x_3)}, \tag{3.5}$$

which is equivalent to write

$$h\left(\ln\left(\frac{x_3}{x_1}\right)\right)f(x_1)f(x_3) - h\left(\ln\left(\frac{x_2}{x_1}\right)\right)f(x_1)f(x_2) \ge h\left(\ln\left(\frac{x_3}{x_2}\right)\right)f(x_3)f(x_2),$$

as desired.

Thus, the proof is completely established. □

Corollary 6. *Let $h:(0,1)\to[0,\infty)$ be a non-negative function and let $f:(0,1)\to\mathbb{R}$ be a function. For all points $x_1,x_2,x_3\in(0,1)$, $x_1<x_2<x_3$ such that $\ln\left(\frac{x_3}{x_2}\right)$, $\ln\left(\frac{x_2}{x_1}\right)$ and $\ln\left(\frac{x_3}{x_1}\right)$ in $(0,1)$. For $h_r(t)=t^r$, $r\in(-\infty,-1]\cup[0,1]$.*

(1) If f is h-$\mathrm{G_tA_t}$-convex function, then the following inequality hold:

$$\left(\ln\left(\frac{x_3}{x_2}\right)\right)^r\cdot f(x_1) + \left(\ln\left(\frac{x_2}{x_1}\right)\right)^r\cdot f(x_3) \ge \left(\ln\left(\frac{x_3}{x_1}\right)\right)^r f(x_2).$$

Furthermore, if $f(x)=x^\lambda$ ($\lambda\in\mathbb{R}$) we get several Schur type inequalities.

(2) If f is h-$\mathrm{G_tG_t}$-convex function, then the following inequality hold:

$$[f(x_1)]^{\left(\ln\left(\frac{x_3}{x_1}\right)\right)^r}[f(x_3)]^{\left(\ln\left(\frac{x_2}{x_1}\right)\right)^r} \ge f(x_2)^{\left(\ln\left(\frac{x_3}{x_1}\right)\right)^r}.$$

(3) If f is h-$\mathrm{G_tH_t}$-convex function, then the following inequality hold:

$$\left(\ln\left(\frac{x_3}{x_1}\right)\right)^r f(x_1)f(x_3) + \ge \left(\ln\left(\frac{x_2}{x_1}\right)\right)^r f(x_1)f(x_2) + \left(\ln\left(\frac{x_3}{x_2}\right)\right)^r f(x_3)f(x_2).$$

In case of h-$\mathrm{G_tN_t}$-concavity the inequalities are reversed.

Theorem 11. *Let $h:J\to[0,\infty)$ be a non-negative function and let $f:I\to\mathbb{R}$ be a function. For all points $x_1,x_2,x_3\in I$, $x_1<x_2<x_3$ such that $x_1(x_3-x_2)$, $x_3(x_2-x_1)$ and $x_2(x_3-x_1)$ in J,*

(1) If h is supermultiplicative, and f is h-$\mathrm{H_tA_t}$-convex function, then the following inequality hold:

$$h(x_1(x_3-x_2))f(x_1) + h(x_3(x_2-x_1))f(x_3) \ge h(x_2(x_3-x_1))f(x_2),$$

(2) If h is supermultiplicative, and f is h-$\mathrm{H_tG_t}$-convex function, then the following inequality hold:

$$[f(x_1)]^{h(x_1(x_3-x_2))}\cdot[f(x_3)]^{h(x_3(x_2-x_1))} \ge [f(x_2)]^{h(x_2(x_3-x_1))},$$

(3) If h is submultiplicative, and f is h-$\mathrm{H_tH_t}$-convex function, then the following inequality hold:

$$h(x_3(x_2-x_1))f(x_1)f(x_2) + h(x_1(x_3-x_2))f(x_2)f(x_3) \le h(x_2(x_3-x_1))f(x_1)f(x_3),$$

In case of h-$\mathrm{H_tN_t}$-concavity the inequalities are reversed.

Proof. Let $x_1,x_2,x_3\in I$ with $x_1<x_2<x_3$, such that $x_1(x_3-x_2), x_3(x_2-x_1), x_2(x_3-x_1)\in J$. And $\frac{x_1(x_3-x_2)}{x_2(x_3-x_1)}, \frac{x_3(x_2-x_1)}{x_2(x_3-x_1)}\in(0,1)\subseteq J$, so that $\frac{x_1(x_3-x_2)}{x_2(x_3-x_1)}+\frac{x_3(x_2-x_1)}{x_2(x_3-x_1)}=1$. Setting $t=\frac{x_3(x_2-x_1)}{x_2(x_3-x_1)}$, $\alpha=x_1$, $\beta=x_3$, therefore we have the following cases:

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(1) For $x_2=\frac{\alpha\beta}{t\alpha+(1-t)\beta}$ and since f is H_tA_t-convex then by (2.7)

$$\begin{aligned} f\left(x_2\right) &\le h\left(\frac{x_1\left(x_3-x_2\right)}{x_2\left(x_3-x_1\right)}\right)f\left(x_1\right)+h\left(\frac{x_3\left(x_2-x_1\right)}{x_2\left(x_3-x_1\right)}\right)f\left(x_3\right) \\ &\le \frac{h\left(x_1\left(x_3-x_2\right)\right)}{h\left(x_2\left(x_3-x_1\right)\right)}f\left(x_1\right)+\frac{h\left(x_3\left(x_2-x_1\right)\right)}{h\left(x_2\left(x_3-x_1\right)\right)}f\left(x_3\right), \end{aligned} \tag{3.6}$$

which is equivalent to write

$$h\left(x_1\left(x_3-x_2\right)\right)f\left(x_1\right)+h\left(x_3\left(x_2-x_1\right)\right)f\left(x_3\right)\ge h\left(x_2\left(x_3-x_1\right)\right)f\left(x_2\right),$$

as desired.

(2) For $x_2=\frac{\alpha\beta}{t\alpha+(1-t)\beta}$ and since f is H_tG_t-convex then by (2.8)

$$\begin{aligned} f\left(x_2\right) &\le \left[f\left(x_1\right)\right]^{h\left(\frac{x_1\left(x_3-x_2\right)}{x_2\left(x_3-x_1\right)}\right)}\left[f\left(x_3\right)\right]^{h\left(\frac{x_3\left(x_2-x_1\right)}{x_2\left(x_3-x_1\right)}\right)} \\ &\le \left[f\left(x_1\right)\right]^{\frac{h\left(x_1\left(x_3-x_2\right)\right)}{h\left(x_2\left(x_3-x_1\right)\right)}}\left[f\left(x_3\right)\right]^{\frac{h\left(x_3\left(x_2-x_1\right)\right)}{h\left(x_2\left(x_3-x_1\right)\right)}}, \end{aligned} \tag{3.7}$$

and this equivalent to write

$$\left[f\left(x_1\right)\right]^{h\left(x_1\left(x_3-x_2\right)\right)}\cdot\left[f\left(x_3\right)\right]^{h\left(x_3\left(x_2-x_1\right)\right)}\ge\left[f\left(x_2\right)\right]^{h\left(x_2\left(x_3-x_1\right)\right)},$$

as desired.

(3) For $x_2=\frac{\alpha\beta}{t\alpha+(1-t)\beta}$ and since f is H_tH_t-convex then by (2.9)

$$\begin{aligned} f\left(x_2\right) &\le \frac{f\left(x_1\right)f\left(x_3\right)}{h\left(\frac{x_3\left(x_2-x_1\right)}{x_2\left(x_3-x_1\right)}\right)f\left(x_1\right)+h\left(\frac{x_1\left(x_3-x_2\right)}{x_2\left(x_3-x_1\right)}\right)f\left(x_3\right)} \\ &\le \frac{h\left(x_2\left(x_3-x_1\right)\right)f\left(x_1\right)f\left(x_3\right)}{h\left(x_3\left(x_2-x_1\right)\right)f\left(x_1\right)+h\left(x_1\left(x_3-x_2\right)\right)f\left(x_3\right)}, \end{aligned} \tag{3.8}$$

and this equivalent to write

$$h\left(x_3\left(x_2-x_1\right)\right)f\left(x_1\right)f\left(x_2\right)+h\left(x_1\left(x_3-x_2\right)\right)f\left(x_2\right)f\left(x_3\right)\le h\left(x_2\left(x_3-x_1\right)\right)f\left(x_1\right)f\left(x_3\right),$$

as desired.

Thus, the proof is completely established. □

Corollary 7. *Let $h:(0,1)\to[0,\infty)$ be a non-negative function and let $f:(0,1)\to\mathbb{R}$ be a function. For all points $x_1,x_2,x_3\in(0,1)$, $x_1<x_2<x_3$ such that $x_1\left(x_3-x_2\right)$, $x_3\left(x_2-x_1\right)$ and $x_2\left(x_3-x_1\right)$ in $(0,1)$. For $h_r(t)=t^r$, $r\in(-\infty,-1]\cup[0,1]$.*

(1) If f is h-$\mathrm{H_tA_t}$-convex function, then the following inequality hold:

$$\left(x_1\left(x_3-x_2\right)\right)^r f\left(x_1\right)+\left(x_3\left(x_2-x_1\right)\right)^r f\left(x_3\right)\ge\left(x_2\left(x_3-x_1\right)\right)^r f\left(x_2\right).$$

Furthermore, if $f(x)=x^\lambda$ $(\lambda>0)$ we get several Schur type inequalities.

(2) If f is h-$\mathrm{H_tG_t}$-convex function, then the following inequality hold:

$$\left[f\left(x_1\right)\right]^{\left(x_1\left(x_3-x_2\right)\right)^r}\cdot\left[f\left(x_3\right)\right]^{\left(x_3\left(x_2-x_1\right)\right)^r}\ge\left[f\left(x_2\right)\right]^{\left(x_2\left(x_3-x_1\right)\right)^r}.$$

(3) If f is h-$\mathrm{H_tH_t}$-convex function, then the following inequality hold:

$$\left(x_3\left(x_2-x_1\right)\right)^r f\left(x_1\right)f\left(x_2\right)+\left(x_1\left(x_3-x_2\right)\right)^r f\left(x_2\right)f\left(x_3\right)\le\left(x_2\left(x_3-x_1\right)\right)^r f\left(x_1\right)f\left(x_3\right).$$

Furthermore, if $f(x)=x^\lambda$ $(1>\lambda>0)$ we get several Schur type inequalities.

In case of h-$\mathrm{H_tN_t}$-concavity the inequalities are reversed.

Remark 5. *In [18], Mitrinović and Pečarić proved the validity of the inequality*

$$\left(x_1-x_2\right)\left(x_1-x_3\right)f\left(x_1\right)+\left(x_2-x_1\right)\left(x_2-x_3\right)f\left(x_2\right)+\left(x_3-x_1\right)(,x_3-x_2)f\left(x_3\right)\ge 0$$

for all $x_1,x_2,x_3\in(0,1)$ and $f\in Q(I)$. Moreover, if $f(x)=x^\lambda$ $(\lambda\in\mathbb{R})$, then the inequality is of Schur type, see ([19], p.117). A similar inequality for monotone convex functions was proved by Wright in [31]. A generalization to h-convex type functions was also presented in [30].

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In Corollaries 5–7, if we choose $r=-1$, i.e., $h(x)=x^{-1}$, then several inequalities for $\mathrm{M_tN_t}$*-convex functions can be deduced. For inequalities of Schur type choose $f(x)=x^{\lambda}$ $(\lambda\in\mathbb{R})$, taking into account that some additional assumption on λ have to be made to guarantee the* $\mathrm{M_tN_t}$*-convexity of f.*

4. Jensen's type inequalities

The weighted Arithmetic, Geometric, and Harmonic Means for n-points $x_1,x_2,\cdots,x_n$ $(n\ge 2)$ are defined respectively, to be

$$A(x_1,x_2,\ldots,x_n)=\sum_{k=1}^{n}t_kx_k$$
$$G(x_1,x_2,\ldots,x_n)=\prod_{k=1}^{n}(x_k)^{t_k}$$
$$H(x_1,x_2,\ldots,x_n)=\frac{1}{A_{t_k}\left(\frac{t_1}{x_1},\frac{t_2}{x_2},\ldots,\frac{t_n}{x_n}\right)}=\frac{1}{\sum\limits_{k=1}^{n}\frac{t_k}{x_k}},$$

where $t_k\in[0,1]$ such that $\sum\limits_{k=1}^{n}t_k=1$ and $(x_1,x_2,\ldots,x_n)\in(0,\infty)^n$. The weighted form of the HM–GM–AM inequality is known as ([21], p. 11):

$$H(x_1,x_2,\ldots,x_n)\le G(x_1,x_2,\ldots,x_n)\le A(x_1,x_2,\ldots,x_n).$$

Let $w_1,w_2,\cdots,w_n$ be positive real numbers $(n\ge 2)$ and $h:J\to\mathbb{R}$ be a non-negative supermultiplicative function. In [30], Varošanec discussed the case that, if f is a non-negative h-$\mathrm{A_tA_t}$-convex on I, then for $x_1,x_2,\cdots,x_n\in I$ the following inequality holds

$$f\left(\frac{1}{W_n}\sum_{k=1}^{n}w_kx_k\right)\le\sum_{k=1}^{n}h\left(\frac{w_k}{W_n}\right)f(x_k),$$

where $W_n=\sum\limits_{k=1}^{n}w_k$. If h is submultiplicative function and f is an h-$\mathrm{A_tA_t}$-concave then inequality is reversed. A converse result was also given in [30]. For more new results see [10], [11], [17], [23], [25] and [32].

In what follows, Jensen's type inequalities for h-$\mathrm{M_tN_t}$-convex functions are introduced.

Theorem 12. *Let $w_1,w_2,\cdots,w_n$ be positive real numbers $(n\ge 2)$, and $W_n=\sum\limits_{k=1}^{n}w_k$.*

(1) If h is a non-negative supermultiplicative function and f is a non-negative h-$\mathrm{A_tG_t}$-convex on I, then for $x_1,x_2,\cdots,x_n\in I$ the following inequality holds

$$f\left(\frac{1}{W_n}\sum_{k=1}^{n}w_kx_k\right)\le\prod_{k=1}^{n}\left\{[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}\right\}. \tag{4.1}$$

If h is submultiplicative function and f is an h-$\mathrm{A_tG_t}$-concave then inequality is reversed.

(2) If h is a non-negative submultiplicative function and f is a non-negative h-$\mathrm{A_tH_t}$-convex on I, then for $x_1,x_2,\cdots,x_n\in I$ the following inequality holds

$$f\left(\frac{1}{W_n}\sum_{k=1}^{n}w_kx_k\right)\le\left(\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\right)^{-1}. \tag{4.2}$$

If h is supermultiplicative function and f is an h-$\mathrm{A_tH_t}$-concave then inequality is reversed.

Proof. Our proof carries by induction. In case $n=2$, the both results hold.

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(1) Assume (4.1) holds for $n-1$ and we are going to prove it for n.

$$\begin{aligned}
f\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k x_k\right) &= f\left(\frac{w_n}{W_n}x_n + \sum_{k=1}^{n-1}\frac{w_k}{W_n}x_k\right)\\
&= f\left(\frac{w_n}{W_n}x_n + \frac{W_{n-1}}{W_n}\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)\\
&\le \left[f\left(x_n\right)\right]^{h\left(\frac{w_n}{W_n}\right)}\left[f\left(\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)\right]^{h\left(\frac{W_{n-1}}{W_n}\right)}\\
&\le \left[f\left(x_n\right)\right]^{h\left(\frac{w_n}{W_n}\right)}\cdot\prod_{k=1}^{n-1}\left\{\left[f\left(x_k\right)\right]^{h\left(\frac{W_{n-1}}{W_n}\right)h\left(\frac{w_k}{W_{n-1}}\right)}\right\}\\
&= \prod_{k=1}^{n}\left\{\left[f\left(x_k\right)\right]^{h\left(\frac{w_k}{W_n}\right)}\right\},
\end{aligned}$$

and this proves the desired result in (4.1).

(2) Assume (4.2) holds for $n-1$ and we are going to prove it for n.

$$\begin{aligned}
f\left(\frac{1}{W_n}\sum_{k=1}^{n} w_k x_k\right) = f\left(\frac{w_n}{W_n}x_n + \sum_{k=1}^{n-1}\frac{w_k}{W_n}x_k\right) &= f\left(\frac{w_n}{W_n}x_n + \frac{W_{n-1}}{W_n}\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)\\
&\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)} + \frac{h\left(\frac{W_{n-1}}{W_n}\right)}{f\left(\sum\limits_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)}}\\
&\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)} + h\left(\frac{W_{n-1}}{W_n}\right)\sum\limits_{k=1}^{n-1}\frac{h\left(\frac{w_k}{W_{n-1}}\right)}{f(x_k)}}\\
&\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)} + \sum\limits_{k=1}^{n-1}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}} \le \frac{1}{\sum\limits_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}},
\end{aligned}$$

which proves the desired result in (4.2).

Hence, by Mathematical Induction both statements are hold for all $n \ge 2$, and therefore the proof is completely established. □

The corresponding converse versions of Jensen inequality for h-$\mathrm{A_tG_t}$-convex and h-$\mathrm{A_tH_t}$-convex are incorporated in the following theorem.

Theorem 13. *Let $w_1, w_2, \cdots, w_n$ be positive real numbers ($n \ge 2$), and $(m, M) \subseteq I$.*

(1) If $h : (0,\infty) \to (0,\infty)$ is a non-negative supermultiplicative function and f is positive h-$\mathrm{A_tG_t}$-convex, then for every finite sequence of points $x_1, \cdots, x_n \in (m, M) \subseteq I$ we have

$$\prod_{k=1}^{n}\left[f\left(x_k\right)\right]^{h\left(\frac{w_k}{W_n}\right)} \le \prod_{k=1}^{n}\left\{\left[f\left(m\right)\right]^{h\left(\frac{M-x_k}{M-m}\cdot\frac{w_k}{W_n}\right)}\cdot\left[f\left(M\right)\right]^{h\left(\frac{x_k-m}{M-m}\cdot\frac{w_k}{W_n}\right)}\right\}, \tag{4.3}$$

If h is submultiplicative function and f is an h-$\mathrm{A_tG_t}$-concave then inequality is reversed.

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(2) *If $h:(0,\infty)\to(0,\infty)$ is a non-negative submultiplicative function and f is positive h-$\mathrm{A_tH_t}$-convex, then for every finite sequence of points $x_1,\cdots,x_n\in(m,M)\subseteq I$ we have*

$$\left(\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\right)^{-1}\le\left(\sum_{k=1}^{n}\frac{h\left(\frac{x_k-m}{M-m}\right)f(m)+h\left(\frac{M-x_k}{M-m}\right)f(M)}{f(m)f(M)}h\left(\frac{w_k}{W_n}\right)\right)^{-1},\tag{4.4}$$

If h is supermultiplicative function and f is an h-$\mathrm{A_tH_t}$-concave then inequality is reversed.

Proof. (1) In (2.2), setting $m=x_1$, $x_2=x_k$ and $x_3=M$ we get

$$f(x_k)\le[f(m)]^{h\left(\frac{M-x_k}{M-m}\right)}[f(M)]^{h\left(\frac{x_k-m}{M-m}\right)}.$$

Since f is positive therefore we have

$$\begin{aligned}[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}&\le[f(m)]^{h\left(\frac{M-x_k}{M-m}\right)\cdot h\left(\frac{w_k}{W_n}\right)}[f(M)]^{h\left(\frac{x_k-m}{M-m}\right)\cdot h\left(\frac{w_k}{W_n}\right)}\\&\le[f(m)]^{h\left(\frac{M-x_k}{M-m}\cdot\frac{w_k}{W_n}\right)\cdot}[f(M)]^{h\left(\frac{x_k-m}{M-m}\cdot\frac{w_k}{W_n}\right)},\end{aligned}$$

Multiplying the above inequality up to n we get the required results in (4.3).

(2) Setting $m=x_1$, $x_2=x_k$ and $x_3=M$ in the reverse of (2.3) we get

$$f(x_k)\le\frac{f(m)f(M)}{h\left(\frac{x_k-m}{M-m}\right)f(m)+h\left(\frac{M-x_k}{M-m}\right)f(M)}.$$

Reversing the inequality and then multiplying the above inequality by $h\left(\frac{w_k}{W_n}\right)$ we get

$$\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\ge\frac{h\left(\frac{x_k-m}{M-m}\right)f(m)+h\left(\frac{M-x_k}{M-m}\right)f(M)}{f(m)f(M)}h\left(\frac{w_k}{W_n}\right).$$

Summing up to n and then reverse the above inequality, we get the required result in (4.4). □

Theorem 14. *Let $w_1,w_2,\cdots,w_n$ be positive real numbers $(n\ge2)$, and $W_n=\sum_{k=1}^{n}w_k$.*

(1) *If h is a non-negative supermultiplicative function and f is positive h-$\mathrm{G_tA_t}$-convex on I, then for $x_1,x_2,\cdots,x_n\in I$ the following inequality holds*

$$f\left(\prod_{k=1}^{n}(x_k)^{\frac{w_k}{W_n}}\right)\le\sum_{k=1}^{n}h\left(\frac{w_k}{W_n}\right)f(x_k).\tag{4.5}$$

If h is submultiplicative function and f is an h-$\mathrm{G_tA_t}$-concave then inequality is reversed.

(2) *If h is a non-negative supermultiplicative function and f is positive h-$\mathrm{G_tG_t}$-convex on I, then for $x_1,x_2,\cdots,x_n\in I$ the following inequality holds*

$$f\left(\prod_{k=1}^{n}(x_k)^{\frac{w_k}{W_n}}\right)\le\prod_{k=1}^{n}[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}.\tag{4.6}$$

If h is submultiplicative function and f is an h-$\mathrm{G_tG_t}$-concave then inequality is reversed.

(3) *If h is a non-negative submultiplicative function and f is positive h-$\mathrm{G_tH_t}$-convex on I, then for $x_1,x_2,\cdots,x_n\in I$ the following inequality holds*

$$f\left(\prod_{k=1}^{n}(x_k)^{\frac{w_k}{W_n}}\right)\le\frac{1}{\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}}.\tag{4.7}$$

If h is supermultiplicative function and f is an h-$\mathrm{G_tH_t}$-concave then inequality is reversed.

Proof. Our proof carries by induction. In case $n=2$, the results hold by definition.

(1) Assume (4.5) holds for $n-1$ and we are going to prove it for n.

$$\begin{aligned}
f\left(\prod_{k=1}^{n}(x_k)^{\frac{w_k}{W_n}}\right) &= f\left((x_n)^{\frac{w_n}{W_n}}\cdot\prod_{k=1}^{n-1}(x_k)^{\frac{w_k}{W_n}}\right)\\
&= f\left((x_n)^{\frac{w_n}{W_n}}\cdot\prod_{k=1}^{n-1}(x_k)^{\frac{w_k}{W_{n-1}}\cdot\frac{W_{n-1}}{W_n}}\right)\\
&\le h\left(\frac{w_n}{W_n}\right)f(x_n)+h\left(\frac{W_{n-1}}{W_n}\right)f\left(\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)\\
&\le h\left(\frac{w_n}{W_n}\right)f(x_n)+h\left(\frac{W_{n-1}}{W_n}\right)\sum_{k=1}^{n-1}h\left(\frac{w_k}{W_{n-1}}\right)f(x_k)\\
&\le h\left(\frac{w_n}{W_n}\right)f(x_n)+\sum_{k=1}^{n-1}h\left(\frac{w_k}{W_n}\right)f(x_k)=\sum_{k=1}^{n}h\left(\frac{w_k}{W_n}\right)f(x_k),
\end{aligned}$$

which proves the desired result in (4.5).

(2) Assume (4.6) holds for $n-1$ and we are going to prove it for n.

$$\begin{aligned}
f\left(\prod_{k=1}^{n}(x_k)^{\frac{w_k}{W_n}}\right) &\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[f\left(\prod_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)\right]^{h\left(\frac{W_{n-1}}{W_n}\right)}\\
&\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[\prod_{k=1}^{n-1}(f(x_k))^{h\left(\frac{w_k}{W_{n-1}}\right)}\right]^{h\left(\frac{W_{n-1}}{W_n}\right)}\\
&\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[\prod_{k=1}^{n-1}(f(x_k))^{h\left(\frac{w_k}{W_{n-1}}\right)h\left(\frac{W_{n-1}}{W_n}\right)}\right]\\
&\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[\prod_{k=1}^{n-1}(f(x_k))^{h\left(\frac{w_k}{W_n}\right)}\right]=\prod_{k=1}^{n}[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)},
\end{aligned}$$

which proves the desired result in (4.6).

(3) Assume (4.7) holds for $n-1$ and we are going to prove it for n.

$$\begin{aligned}
f\left(\prod_{k=1}^{n}(x_k)^{\frac{w_k}{W_n}}\right) &\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)}+\frac{h\left(\frac{W_{n-1}}{W_n}\right)}{f\left(\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)}}\\
&\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)}+h\left(\frac{W_{n-1}}{W_n}\right)\sum_{k=1}^{n-1}\frac{h\left(\frac{w_k}{W_{n-1}}\right)}{f(x_k)}}\\
&\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)}+\sum_{k=1}^{n-1}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}}\le\frac{1}{\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}},
\end{aligned}$$

which proves the desired result in (4.7).

Hence, by Mathematical Induction both statements are hold for all $n\ge 2$, and therefore the proof is completely established. □

The corresponding converse versions of Jensen inequality for h-$\mathrm{G_tA_t}$-convex, h-$\mathrm{G_tG_t}$-convex and h-$\mathrm{G_tH_t}$-convex are incorporated in the following theorem.

Theorem 15. *Let $w_1, w_2, \cdots, w_n$ be positive real numbers $(n\ge 2)$, and $(m,M)\subseteq I$.*

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(1) *If $h:(m,M)\to[m,M)$ is a non-negative supermultiplicative function and f is positive h-$\mathrm{G_tA_t}$-convex, then for every finite sequence of points $x_1,\cdots,x_n\in(m,M)$ $(x_k<x_{k+1})$ we have*

$$\sum_{k=1}^{n}h\left(\frac{w_k}{W_n}\right)f(x_k) \leq \sum_{k=1}^{n}h\left(\frac{w_k}{W_n}\right)\cdot\left[h\left(\frac{\ln(M)-\ln(x_k)}{\ln(M)-\ln(m)}\right)\cdot f(m)+h\left(\frac{\ln(x_k)-\ln(m)}{\ln(M)-\ln(m)}\right)\cdot f(M)\right]. \tag{4.8}$$

If h is submultiplicative function and f is an h-$\mathrm{G_tA_t}$-concave then inequality is reversed.

(2) *If $h:(0,\infty)\to(0,\infty)$ is a non-negative supermultiplicative function and f is positive h-$\mathrm{G_tG_t}$-convex, then for every finite sequence of points $x_1,\cdots,x_n\in(m,M)\subseteq I$ we have*

$$\prod_{k=1}^{n}[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}\leq\prod_{k=1}^{n}\left\{[f(m)]^{h\left(\frac{\ln(M)-\ln(x_k)}{\ln(M)-\ln(m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}[f(M)]^{h\left(\frac{\ln(x_k)-\ln(m)}{\ln(M)-\ln(m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}\right\}. \tag{4.9}$$

If h is submultiplicative function and f is an h-$\mathrm{G_tG_t}$-concave then inequality is reversed.

(3) *If $h:(0,\infty)\to(0,\infty)$ is a non-negative submultiplicative function and f is positive h-$\mathrm{G_tH_t}$-convex, then for every finite sequence of points $x_1,\cdots,x_n\in(m,M)\subseteq I$ we have*

$$\left(\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\right)^{-1}\leq\left(\sum_{k=1}^{n}\frac{h\left(\frac{\ln x_k-\ln m}{\ln M-\ln m}\right)f(m)+h\left(\frac{\ln M-\ln x_k}{\ln M-\ln m}\right)f(M)}{f(m)f(M)}h\left(\frac{w_k}{W_n}\right)\right)^{-1}. \tag{4.10}$$

If h is supermultiplicative function and f is an h-$\mathrm{G_tH_t}$-concave then inequality is reversed.

Proof. (1) In (2.4), setting $m=x_1$, $x_2=x_k$ and $x_3=M$ we get

$$f(x_k)\leq h\left(\frac{\ln(M)-\ln(x_k)}{\ln(M)-\ln(m)}\right)\cdot f(m)+h\left(\frac{\ln(x_k)-\ln(m)}{\ln(M)-\ln(m)}\right)\cdot f(M)$$

Multiplying the above inequality by $h\left(\frac{w_k}{W_n}\right)$ and summing up to n we get the required results in (4.8).

(2) Setting $m=x_1$, $x_2=x_k$ and $x_3=M$ in (2.5) we get

$$f(x_k)\leq[f(m)]^{h\left(\frac{\ln(M)-\ln(x_k)}{\ln(M)-\ln(m)}\right)}[f(M)]^{h\left(\frac{\ln(x_k)-\ln(m)}{\ln(M)-\ln(m)}\right)}.$$

Since f is positive, the above inequality implies that

$$[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}\leq[f(m)]^{h\left(\frac{\ln(M)-\ln(x_k)}{\ln(M)-\ln(m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}[f(M)]^{h\left(\frac{\ln(x_k)-\ln(m)}{\ln(M)-\ln(m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}.$$

Multiplying the above inequality up to n we get the required result in (4.9).

(3) Since f is h-$\mathrm{G_tH_t}$-convex, then (2.6) holds.

$$f(x_k)\leq\frac{f(m)f(M)}{h\left(\frac{\ln x_k-\ln m}{\ln M-\ln m}\right)f(m)+h\left(\frac{\ln M-\ln x_k}{\ln M-\ln m}\right)f(M)}.$$

Reversing the order in the inequality we get

$$\frac{1}{f(x_k)}\geq\frac{h\left(\frac{\ln x_k-\ln m}{\ln M-\ln m}\right)f(m)+h\left(\frac{\ln M-\ln x_k}{\ln M-\ln m}\right)f(M)}{f(m)f(M)}.$$

Multiplying both sides by $h\left(\frac{w_k}{W_n}\right)$ and summing up to n we get

$$\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\geq\sum_{k=1}^{n}\frac{h\left(\frac{\ln x_k-\ln m}{\ln M-\ln m}\right)f(m)+h\left(\frac{\ln M-\ln x_k}{\ln M-\ln m}\right)f(M)}{f(m)f(M)}h\left(\frac{w_k}{W_n}\right).$$

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Reversing the order in the inequality again we get the required result in (4.10).

□

Theorem 16. *Let $w_1, w_2, \cdots, w_n$ be positive real numbers $(n \geq 2)$, and $W_n = \sum_{k=1}^{n} w_k$.*

(1) *If h is a non-negative supermultiplicative function and f is positive h-$\mathrm{H_tA_t}$-convex on I, then for $x_1, x_2, \cdots, x_n \in I$ the following inequality holds*

$$f\left(\left(\frac{1}{W_n}\sum_{k=1}^{n}\frac{w_k}{x_k}\right)^{-1}\right) \leq \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f(x_k). \tag{4.11}$$

If h is submultiplicative function and f is an h-$\mathrm{H_tA_t}$-concave then inequality is reversed.

(2) *If h is a non-negative supermultiplicative function and f is positive h-$\mathrm{H_tG_t}$-convex on I, then for $x_1, x_2, \cdots, x_n \in I$ the following inequality holds*

$$f\left(\left(\frac{1}{W_n}\sum_{k=1}^{n}\frac{w_k}{x_k}\right)^{-1}\right) \leq \prod_{k=1}^{n} [f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}. \tag{4.12}$$

If h is submultiplicative function and f is an h-$\mathrm{H_tG_t}$-concave then inequality is reversed.

(3) *If h is a non-negative submultiplicative function and f is positive h-$\mathrm{H_tH_t}$-convex on I, then for $x_1, x_2, \cdots, x_n \in I$ the following inequality holds*

$$f\left(\left(\frac{1}{W_n}\sum_{k=1}^{n}\frac{w_k}{x_k}\right)^{-1}\right) \leq \left(\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\right)^{-1}. \tag{4.13}$$

If h is supermultiplicative function and f is an h-$\mathrm{H_tH_t}$-concave then inequality is reversed.

Proof. Our proof carries by induction. In case $n = 2$, both results hold.

(1) Assume (2.7) holds for $n-1$ and we are going to prove it for n.

$$\begin{aligned}
f\left(\frac{1}{\sum_{k=1}^{n}\frac{w_k}{W_n}\frac{1}{x_k}}\right) &= f\left(\frac{1}{\frac{w_n}{W_n}\frac{1}{x_n} + \sum_{k=1}^{n-1}\frac{w_k}{W_n}\frac{1}{x_k}}\right) \\
&= f\left(\frac{1}{\frac{w_n}{W_n}\frac{1}{x_n} + \frac{W_{n-1}}{W_n}\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}\frac{1}{x_k}}\right) \\
&\leq h\left(\frac{w_n}{W_n}\right) f(x_n) + h\left(\frac{W_{n-1}}{W_n}\right) f\left(\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right) \\
&\leq h\left(\frac{w_n}{W_n}\right) f(x_n) + h\left(\frac{W_{n-1}}{W_n}\right)\sum_{k=1}^{n-1} h\left(\frac{w_k}{W_{n-1}}\right) f(x_k) \\
&\leq h\left(\frac{w_n}{W_n}\right) f(x_n) + \sum_{k=1}^{n-1} h\left(\frac{w_k}{W_n}\right) f(x_k) \\
&= \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f(x_k),
\end{aligned}$$

which proves the desired result in (4.11).

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(2) Assume (2.8) holds for $n-1$ and we are going to prove it for n.

$$f\left(\frac{1}{\sum_{k=1}^{n}\frac{w_k}{W_n}\frac{1}{x_k}}\right) \le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[f\left(\prod_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)\right]^{h\left(\frac{W_{n-1}}{W_n}\right)}$$
$$\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[\prod_{k=1}^{n-1}(f(x_k))^{h\left(\frac{w_k}{W_{n-1}}\right)}\right]^{h\left(\frac{W_{n-1}}{W_n}\right)}$$
$$\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[\prod_{k=1}^{n-1}(f(x_k))^{h\left(\frac{w_k}{W_{n-1}}\right)h\left(\frac{W_{n-1}}{W_n}\right)}\right]$$
$$\le [f(x_n)]^{h\left(\frac{w_n}{W_n}\right)}\left[\prod_{k=1}^{n-1}(f(x_k))^{h\left(\frac{w_k}{W_n}\right)}\right] = \prod_{k=1}^{n}[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)},$$

which proves the desired result in (4.12).

(3) Assume (2.9) holds for $n-1$ and we are going to prove it for n.

$$f\left(\frac{1}{\frac{1}{W_n}\sum_{k=1}^{n}\frac{w_k}{x_k}}\right) \le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)}+\frac{h\left(\frac{W_{n-1}}{W_n}\right)}{f\left(\sum_{k=1}^{n-1}\frac{w_k}{W_{n-1}}x_k\right)}}$$
$$\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)}+h\left(\frac{W_{n-1}}{W_n}\right)\sum_{k=1}^{n-1}\frac{h\left(\frac{w_k}{W_{n-1}}\right)}{f(x_k)}}$$
$$\le \frac{1}{\frac{h\left(\frac{w_n}{W_n}\right)}{f(x_n)}+\sum_{k=1}^{n-1}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}} \le \frac{1}{\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}},$$

which proves the desired result in (4.13).

Hence, by Mathematical Induction the three statements are hold for all $n \ge 2$, and therefore the proof is completely established. □

The corresponding converse versions of Jensen inequality for h-$\mathrm{H_tA_t}$-convex, h-$\mathrm{H_tG_t}$-convex and h-$\mathrm{H_tH_t}$-convex are incorporated in the following theorem.

Theorem 17. *Let $w_1, w_2, \cdots, w_n$ be positive real numbers $(n \ge 2)$, and $(m, M) \subseteq I$.*

(1) If $h : (0,\infty) \to (0,\infty)$ is a non-negative supermultiplicative function and f is positive h-$\mathrm{H_tA_t}$-convex, then for every finite sequence of points $x_1, \cdots, x_n \in (m, M) \subseteq I$ we have

$$\sum_{k=1}^{n}h\left(\frac{w_k}{W_n}\right)f(x_k) \le \sum_{k=1}^{n}\left[h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(m)+h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(M)\right]h\left(\frac{w_k}{W_n}\right). \tag{4.14}$$

If h is submultiplicative function and f is an h-$\mathrm{H_tA_t}$-concave then inequality is reversed.

(2) If $h : (0,\infty) \to (0,\infty)$ is a non-negative supermultiplicative function and f is positive h-$\mathrm{H_tG_t}$-convex, then for every finite sequence of points $x_1, \cdots, x_n \in (m, M) \subseteq I$ we have

$$\prod_{k=1}^{n}[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)} \le \prod_{k=1}^{n}\left\{[f(m)]^{h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}[f(M)]^{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}\right\}. \tag{4.15}$$

If h is submultiplicative function and f is an h-$\mathrm{H_tG_t}$-concave then inequality is reversed.

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(3) *If $h:(0,\infty)\to(0,\infty)$ is a non-negative submultiplicative function and f is positive h-$\mathrm{H_t H_t}$-convex, then for every finite sequence of points $x_1,\cdots,x_n\in(m,M)\subseteq I$ we have*

$$\left(\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\right)^{-1}\le\left(\sum_{k=1}^{n}\frac{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(m)+h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(M)}{f(m)f(M)}h\left(\frac{w_k}{W_n}\right)\right)^{-1}. \tag{4.16}$$

If h is supermultiplicative function and f is an h-$\mathrm{H_t H_t}$-concave then inequality is reversed.

Proof. (1) In (2.7), setting $m=x_1$, $x_2=x_k$ and $x_3=M$ we get

$$f(x_k)\le h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(m)+h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(M)$$

Multiplying the above inequality by $h\left(\frac{w_k}{W_n}\right)$ and summing up to n we get the required results in (4.14).

(2) Setting $m=x_1$, $x_2=x_k$ and $x_3=M$ in (2.8) we get

$$f(x_k)\le[f(m)]^{h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)}[f(M)]^{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)}.$$

Since f is positive, the above inequality implies that

$$[f(x_k)]^{h\left(\frac{w_k}{W_n}\right)}\le[f(m)]^{h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}[f(M)]^{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)\cdot h\left(\frac{w_k}{W_n}\right)}.$$

Multiplying the above inequality up to n we get the required result in (4.15).

(3) Setting $m=x_1$, $x_2=x_k$ and $x_3=M$ in (2.9) we get

$$f(x_k)\le\frac{f(x_1)f(x_3)}{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(x_1)+h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(x_3)}.$$

Reversing the order in the inequality we get

$$\frac{1}{f(x_k)}\ge\frac{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(x_1)+h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(x_3)}{f(x_1)f(x_3)}.$$

Multiplying both sides by $h\left(\frac{w_k}{W_n}\right)$ and summing up to n we get

$$\sum_{k=1}^{n}\frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)}\ge\sum_{k=1}^{n}\frac{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(m)+h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(M)}{f(m)f(M)}h\left(\frac{w_k}{W_n}\right).$$

Reversing the order in the inequality again we get the required result in (4.16). □

Remark 6. *Theorem 22 and Corollary 23 in [30], can be extended to h-$\mathrm{M_t N_t}$-convexity in similar manner, we omit the details.*

Remark 7. *We note that, in this work, all results are valid for*

(1) *the class $\overline{\mathcal{MN}}(h,I)$, whenever $h(t)=t$, $t\in[0,1]$*
(2) *the class $Q(I;\mathrm{M_t},\mathrm{N_t})$, whenever $h(t)=\frac{1}{t}$, $t\in(0,1)$*
(3) *the class $P(I;\mathrm{M_t},\mathrm{N_t})$, whenever $h(t)=1$, $t\in[0,1]$*
(4) *the class $K_s^2(I;\mathrm{M_t},\mathrm{N_t})$, whenever $h(t)=t^s$, $s\in(0,1]$ and $t\in[0,1]$.*

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