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THE (k,s)-FRACTIONAL CALCULUS OF CLASS OF A FUNCTION

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ABSTRACT. In this present paper, we deal with the generalized (k,s)-fractional integral and differential operators recently defined by Nisar *et al.* and obtain some generalized (k,s)-fractional integral and differential formulas involving the class of a function as its kernels. Also, we investigate a certain number of their consequences containing the said function in their kernels.

1. INTRODUCTION

Fractional calculus has gained appreciable fame and significance due to its numerous and boundless applications in Science; more particularly in engineering. For the recent development in the field of fractional calculus, one can refer to [5, 13, 15] and [6, 7]. One more direction to such study was proposed by Atangana and Baleanu [1] by introducing derivatives which are based upon the generalized Mittag-Leffler function. Integral inequalities are considered to be of highly importance because they are useful in the research of different subjects of differential and integral equations (see [8]). We begin with the work of Diaz and Pariguan [3] which is defined as:

$$(p)_{n,k} = \begin{cases} p\,(p+k), ..., \,(p+(n-1)k) & (n \in N,\, p\, \in \mathbb{C}), \\ 1, \quad (n=0,\, p \in \mathbb{C}), \end{cases} \tag{1.1}$$

and

$$\Gamma_k(\eta) = \int_0^\infty t^{\eta-1} e^{-\frac{t^k}{k}} dt, \tag{1.2}$$

where $\eta \in \mathbb{C},\, k>0,\, \Re(z)>0$.
They also define the following relations:

$$\Gamma_k(\eta+k) = \eta\Gamma_k(\eta), \tag{1.3}$$

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and

$$\Gamma_k(\eta) = k^{\frac{\eta}{k}-1}\Gamma\left(\frac{\eta}{k}\right). \tag{1.4}$$

A new generalization of the k-fractional integral has been defined by Mubeen and Habibullah [11] as follows:

$$I_k^{\mu}(f(x)) = \frac{1}{k\Gamma_k(\mu)}\int_0^x (x-\tau)^{\frac{\mu}{k}-1}f(\tau)d\tau. \tag{1.5}$$

Clearly, it is observed that when $k=1$ (1.5) coincide with the result of Riemann-Liouville (R-L) fractional integration formula (see [5]). In fact, the following particular cases:

$$I^{\mu}(f(x)) = \frac{1}{\Gamma(\mu)}\int_0^x (x-\tau)^{\mu-1}f(\tau)d\tau. \tag{1.6}$$

In the same paper, they also define the following results:

$$I_k^{\rho}(x^{\frac{\beta}{k}-1}) = \frac{\Gamma_k(\beta)}{\Gamma_k(\rho+\beta)}x^{\frac{\rho}{k}+\frac{\beta}{k}-1}, \tag{1.7}$$

and

$$I_k^{\rho}((x-u)^{\frac{\beta}{k}-1}) = \frac{\Gamma_k(\beta)}{\Gamma_k(\rho+\beta)}(x-u)^{\frac{\rho}{k}+\frac{\beta}{k}-1}. \tag{1.8}$$

Sarikaya et al. [14] have developed the R-L (k,s)-fractional integral of order $\mu > 0$ is defined by:

$${}_k^sI_a^{\mu}f(x) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_0^x (x^{s+1}-t^{s+1})^{\frac{\mu}{k}-1}t^sf(t)dt, \tag{1.9}$$

where $s\in\Re\setminus\{-1\}, k>0$ and $x\in[a;b]$. Also, they define,

$${}_k^sI_a^{\mu}[(x^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}] = \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{\mu}{k}}\Gamma_k(\lambda+\mu)}(x^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}. \tag{1.10}$$

For more details about (k,s)-fractional integrals interesting readers can refer to [2, 12, 16].

Recently Nisar *et al.*, [9, 10] defined the following R-L left and right sided (k,s)-fractional integral and differential operators of order μ as:

$$\begin{aligned} {}_{a,k}^{s}D_x^{-\mu}f(x) &= {}_{a,k}^{s}I_x^{\mu}f(x) = {}_k^sI_{a+}^{\mu}f(x) \\ &= ({}_k^sI_{a+}^{\mu}f)(x) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x \frac{f(t)}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}t^sdt, (x>a), \end{aligned} \tag{1.11}$$

$$\begin{aligned} {}_{\rho,k}^{s}D_\beta^{-\mu}f(x) &= {}_{\rho,k}^{s}I_\beta^{\mu}f(x) = {}_k^sI_{a-}^{\mu}f(x) \\ &= ({}_k^sI_{a-}^{\mu}f)(x) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_x^\beta \frac{f(t)}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}t^sdt, (x<\beta), \end{aligned} \tag{1.12}$$

$$({}_k^sD_{a+}^{\mu}f)(x) = \left(\frac{1}{x^s}\frac{d}{dx}\right)^n\left(k^n\ {}_k^sI_{a+}^{nk-\mu}f\right)(x), \tag{1.13}$$

and

$$({}_k^sD_{a-}^{\mu}f)(x)=\Big(-\frac{1}{x^s}\frac{d}{dx}\Big)^n\Big(k^n\ {}_k^sI_{a-}^{nk-\mu}f\Big)(x), \tag{1.14}$$

respectively. Substituting $k=1$ and $s=0$, then the above relation will coincide to the R-L left and right sided (k,s)-fractional integral and derivatives see ([5],[7]).

The R-L left and right sided (k,s)-fractional derivative operator ${}_k^sD_{a+}^{\mu}$ defined in (1.12) is generalized by (k,s)-fractional derivative operator is denoted by ${}_k^sD_{a+}^{\mu,\nu}$, where μ is the order such that $0<\nu<1$, we define as

$$({}_k^sD_{a+}^{\mu,\nu}f)(x)=\left[{}_k^sI_{\rho+}^{\nu(k-\mu)}\left(\frac{1}{x^s}\frac{d}{dx}\right)\right](k^n\ {}_k^sI_{a+}^{(1-\nu)(k-\mu)}f)(x), \tag{1.15}$$

Obviously, when $v=0$ then (1.15) approaches to the R-L (k,s)-fractional derivatives operator ${}_k^sD_{a+}^{\mu}$ (1.13).
They [9] also defined the following lemma as:

Lemma 1.1. *For $k>0$, with $x>\rho$, $0<\nu<1$ and $\Re(\lambda)>0$, then the following result for (k,s)-fractional derivatives operator ${}_k^sD_{a+}^{\mu}$ in (1.15) hold true:*

$$({}_k^sD_{a+}^{\mu,\nu}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})(x)=\frac{\Gamma_k(\lambda)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\lambda-\mu)}(x^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}. \tag{1.16}$$

For the present investigation, we consider the following class of a function recently defined by Tunc [17]

$$\digamma_{\rho,\lambda}^{\sigma,k}(x)=\sum_{n=0}^{\infty}\frac{\sigma(m)}{k\Gamma_k(\rho km+\lambda)}x^m,(\rho,\lambda>0;|x|<\mathbb{R}), \tag{1.17}$$

where the coefficients $\sigma(m)(m\in N_0=N\cup\{0\}$ is a bounded sequence of positive real numbers $\mathbb{R}$ is the set of real numbers.

2. Generalized (k,s)-fractional integrals and differentials formulas

In this section, we present the generalized (k,s)-fractional integrals and differentials formulas involving a class of a function $\digamma_{\rho,\lambda}^{\sigma,k}(x)$ as defined in (1.17). In this continuation of the study of generalized k-fractional calculus, we define the following fractional integral operator.

Definition 2.1. *If $k>0$, ρ, δ, $\omega\in\mathbb{C}$; with $R(\rho)>0$, $R(\beta)>0, R(\delta)>0$, and $x>\rho$, then*

$$({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega;\sigma}f)(x)=\frac{1}{k}\int_0^x(x^{s+1}-\tau^{s+1})^{\frac{\lambda}{k}-1}\digamma_{\rho,\lambda}^{\sigma,k}(w(x^{s+1}-\tau^{s+1})^{\rho})\tau^sf(\tau)d\tau, \tag{2.1}$$

By putting $s=0$, then (2.1) can be written as

$$({}_k\varepsilon_{a+;\rho,\lambda}^{\omega;\sigma}f)(x)=\frac{1}{k}\int_0^x(x-\tau)^{\frac{\lambda}{k}-1}\digamma_{\rho,\lambda}^{\sigma,k}(w(x^{s+1}-\tau^{s+1})^{\rho})\tau^sf(\tau)d\tau, \tag{2.2}$$

see [17]. Similarly, when $\omega = 0$ and $k = 1$ then (2.2) turns to:

$$(I^{\lambda}_{a+}f)(x) = \frac{1}{\Gamma(\lambda)}\int_0^x \frac{f(\tau)}{(x-\tau)^{1-\lambda}}d\tau, (R(\mu) > 0). \tag{2.3}$$

To prove the generalized (k, s)-fractional integral and differential formulas of a class of a function, we first prove the following result.

Lemma 2.1. *For $k > 0$, the following result holds true:*

$$\begin{aligned}&\left(\frac{1}{x^s}\frac{d}{dx}\right)^m \left\{(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}F^{\alpha,k}_{\rho,\lambda}[w(x^{s+1} - a^{s+1})]^{\rho}\right\}\\ &= k^{-m}(s+1)^m(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}-m-1}F^{\alpha,k}_{\rho,\lambda-mk}[w(x^{s+1} - a^{s+1})]^{\rho}\end{aligned} \tag{2.4}$$

where $s \in \mathbb{R}\setminus\{-1\}$; $\omega,\ \alpha,\ \lambda,\ \rho \in \mathbb{C}$; $R(\omega) > 0,\ R(\lambda) > 0,\ R(\rho) > 0$ and $R(\alpha) > 0$.

Proof. Let S_1 be the L. H. S. of (2.4) then

$$\begin{aligned}S_1 &= \left(\frac{1}{x^s}\frac{d}{dx}\right)^m \left\{(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}F^{\alpha,k}_{\rho,\lambda}[w(x^{s+1} - a^{s+1})]^{\rho}\right\}\\ &= \left(\frac{1}{x^s}\frac{d}{dx}\right)^m \left\{(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}-1}\sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(\rho kn+\lambda)}[w(x^{s+1} - a^{s+1})^{\rho}]^n\right\}.\end{aligned}$$

Changing the order of summation and differentiation, we have

$$S_1 = \sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(\rho kn+\lambda)}[w]^n\left(\frac{1}{x^s}\frac{d}{dx}\right)^m(\tau^{s+1} - a^{s+1})^{\frac{\lambda}{k}+\rho n-1}. \tag{2.5}$$

Now we find

$$\begin{aligned}&\left(\frac{1}{x^s}\frac{d}{dx}\right)^m(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}+\rho n-1}\\ &= (s+1)^m(\rho n+\frac{\lambda}{k}-1)...(\rho n+\frac{\lambda}{k}-m)(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}+\rho n-m-1}\\ &= (s+1)^m\frac{\Gamma(\rho n+\frac{\lambda}{k})}{\Gamma(\rho n+\frac{\lambda}{k}-m)}(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}+\rho n-m-1}\end{aligned} \tag{2.6}$$

Using (1.4), we have

$$\frac{\Gamma(\rho n+\frac{\lambda}{k})}{\Gamma(\rho n+\frac{\lambda}{k}-m)} = \frac{\Gamma_k(n\rho k+\lambda)}{k^m\Gamma_k(n\rho k+\lambda-mk)}. \tag{2.7}$$

Using (2.7) in (2.6), we get

$$\begin{aligned}&\left(\frac{1}{x^s}\frac{d}{dx}\right)^m(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}+\rho n-1}\\ &= (s+1)^m\frac{\Gamma_k(n\rho k+\lambda)}{k^m\Gamma_k(n\rho k+\lambda-mk)}(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}+\rho n-m-1}.\end{aligned} \tag{2.8}$$

Thus by using (2.8) in (2.5), we get the following required result.

$$S_1 = k^{-m}(s+1)^m(x^{s+1} - a^{s+1})^{\frac{\lambda}{k}-m-1}F^{\alpha,k}_{\rho,\lambda-mk}[w(x^{s+1} - a^{s+1})]^{\rho}. \tag{2.9}$$

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$\square$

Theorem 2.1. *For $k>0$, the following integral formulas holds true:*

$$\begin{aligned}&{}_k^sI^{\mu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\\ &=\frac{(x^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}}F^{\alpha,k}_{\rho,\lambda+\mu}[w(x^{s+1}-a^{s+1})]^{\rho}.\end{aligned} \tag{2.10}$$

where $x>a(a\in\mathbb{R}_+=[0,\infty))$; $\omega,\alpha,\lambda,\rho\in\mathbb{C}$; $R(\omega)>0$, $R(\lambda)>0$, $R(\rho)>0$, *and* $R(\mu)>0$.

Proof. Let S_2 be L. H. S. side of (2.10) then

$$\begin{aligned}S_2&={}_k^sI^{\mu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\\ &=\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x\frac{(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)}{(x^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}d\tau\\ &=\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x\frac{(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}}{(x^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}\sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(n\rho k+\lambda)}[w(\tau^{s+1}-a^{s+1})^{\rho}]^n d\tau\end{aligned}$$

Changing the order of summation and integration, we have

$$S_2=\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(n\rho k+\lambda)}[w]^n\int_a^x\frac{(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}+\rho n-1}}{(x^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}d\tau \tag{2.11}$$

Substituting $\tau^{s+1}=a^{s+1}+y(x^{s+1}-a^{s+1})$ in (2.11), then this implies $\tau^s d\tau=\frac{(x^{s+1}-a^{s+1})}{s+1}dy$, when $\tau\to a$, $\Rightarrow$ $y\to 0$ and $\tau\to x$, $\Rightarrow$ $y\to 1$, we have

$$\begin{aligned}S_2&=\sum_{n=0}^{\infty}\frac{\sigma(n)[w]^n}{k\Gamma_k(n\rho k+\lambda)}\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\\ &\times\int_0^1\frac{(\alpha^{s+1}+y(x^{s+1}-\alpha^{s+1})-a^{s+1})^{\frac{\lambda}{k}+\rho n-1}}{(x^{s+1}-\alpha^{s+1}-y(x^{s+1}-\alpha^{s+1}))^{1-\frac{\mu}{k}}}\frac{(x^{s+1}-\alpha^{s+1})}{s+1}dy\\ &=\sum_{n=0}^{\infty}\frac{\sigma(n)[w]^n}{k\Gamma_k(n\rho k+\lambda)}\frac{(x^{s+1}-\alpha^{s+1})^{\frac{\mu+\lambda+\rho nk}{k}-1}}{(s+1)^{\frac{\mu}{k}}k\Gamma_k(\mu)}\int_0^1 y^{\frac{\lambda}{k}+\rho n-1}(1-y)^{\frac{\mu}{k}-1}dy\\ &=\sum_{n=0}^{\infty}\frac{\sigma(n)[w]^n}{k\Gamma_k(n\rho k+\lambda)}\frac{(x^{s+1}-\alpha^{s+1})^{\frac{\mu+\lambda+\rho nk}{k}-1}}{(s+1)^{\frac{\mu}{k}}k\Gamma_k(\mu)}\frac{\Gamma_k(\lambda+\rho nk)\Gamma_k(\mu)}{\Gamma_k(\lambda+\rho nk+\mu)}\\ &=\frac{(x^{s+1}-\alpha^{s+1})^{\frac{\mu+\lambda}{k}-1}}{(s+1)^{\frac{\mu}{k}}}\sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(n\rho k+\lambda+\mu)}[w(\tau^{s+1}-a^{s+1})^{\rho}]^n\\ &=\frac{(x^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}}F^{\alpha,k}_{\rho,\lambda+\mu}[w(x^{s+1}-a^{s+1})]^{\rho}.\end{aligned}$$

$\square$

Theorem 2.2. *For $k>0$, the following result holds true:*

$$\begin{aligned}&{}_k^sD^{\mu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\\&=\frac{(x^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}}F^{\alpha,k}_{\rho,\lambda-\mu}[w(x^{s+1}-a^{s+1})]^{\rho}.\end{aligned} \tag{2.12}$$

where $x>a(a\in\mathbb{R}_+=[0,\infty))$; $\omega,\alpha,\lambda,\rho\in\mathbb{C}$; $R(\omega)>0$, $R(\lambda)>0$, $R(\rho)>0$, *and* $R(\mu)>0$.

Proof. Let S_3 be L. H. S. of (2.12) then

$$S_3 = \left(\frac{1}{x^s}\frac{d}{dx}\right)^n\left\{k^n\,{}_k^sI^{nk-\mu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1}F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\right\}$$

Applying (2.10), this takes the following form

$$\begin{aligned}&{}_k^sD^{\mu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\\&=k^n\left(\frac{1}{x^s}\frac{d}{dx}\right)^n\frac{(x^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}+n-1}}{(s+1)^{-\frac{\mu}{k}+n}}F^{\alpha,k}_{\rho,\lambda-\mu+nk}[w(x^{s+1}-a^{s+1})]^{\rho}.\end{aligned}$$

Applying Lemma 2.4, we have

$$S_3 = \frac{(x^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}}F^{\alpha,k}_{\rho,\lambda-\mu}[w(x^{s+1}-a^{s+1})]^{\rho}.$$

□

Theorem 2.3. *For $k>0$, the following result holds true:*

$$\begin{aligned}&{}_k^sD^{\mu,\nu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\\&=\frac{(x^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{\frac{-\mu}{k}}}F^{\alpha,k}_{\rho,\lambda-\mu}[w(x^{s+1}-a^{s+1})]^{\rho}.\end{aligned} \tag{2.13}$$

where $x>a(a\in\mathbb{R}_+=[0,\infty))$; $\omega,\alpha,\lambda,\rho\in\mathbb{C}$; $R(\omega)>0$, $R(\lambda)>0$, $R(\rho)>0$, *and* $R(\mu)>0$.

Proof.

$$\begin{aligned}&{}_k^sD^{\mu,\nu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)\\&={}_k^s\,D^{\mu,\nu}_{a+}\left[\sum_{n=0}^{\infty}\frac{\sigma(n)}{k\Gamma_k(n\rho k+\lambda)}[w(\tau^{s+1}-a^{s+1})^{\frac{\lambda-1}{nk}+\rho}]^n\right].\end{aligned}$$

This can be written as

$${}_k^sD^{\mu,\nu}_{a+}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F^{\alpha,k}_{\rho,\lambda}[w(\tau^{s+1}-a^{s+1})^{\rho}](x)$$

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$$= \sum_{n=0}^{\infty} \frac{\sigma(n)[w]^n}{k\Gamma_k(n\rho k+\lambda)} \left[{}_k^sD_{a+}^{\mu,\nu}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}+\rho n-1}\right].$$

By applying (1.16), we get

$$\begin{aligned} &{}_k^sD_{a+}^{\mu,\nu}(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}-1})F_{\rho,\lambda}^{\alpha,k}[w(\tau^{s+1}-a^{s+1})^{\rho}](x) \\ &= \sum_{n=0}^{\infty} \frac{\sigma(n)[w]^n}{k\Gamma_k(n\rho k+\lambda)} \frac{\Gamma_k(\lambda+\rho nk)}{\Gamma_k(\lambda+\rho nk-\mu)} \frac{(\tau^{s+1}-a^{s+1})^{\frac{\lambda}{k}+\rho n-1}}{(s+1)^{-\frac{\mu}{k}}} \\ &= \frac{(x^{s+1}-a^{s+1})^{\frac{\lambda-\mu}{k}-1}}{(s+1)^{\frac{-\mu}{k}}} F_{\rho,\lambda-\mu}^{\alpha,k}[w(x^{s+1}-a^{s+1})]^{\rho}. \end{aligned}$$

Which complete the desired proof.

□

3. Properties of the operator $\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x)$

Theorem 3.1. *For $k>0$, $\omega,\sigma,\lambda,\rho\in\mathbb{C}$; $R(\omega)>0$, $R(\lambda)>0$, $R(\rho)>0$, and $R(\mu)>0$, then the following result holds:*

$$\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1}\right)(x) = \frac{(x^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)} F_{\rho,\lambda+\mu}^{\sigma,k}[w\left(x^{s+1}-a^{s+1}\right)^{\rho}]. \tag{3.1}$$

Proof. From (2.1), we have

$$\begin{aligned} &\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1}\right)(x) \\ &= \frac{1}{k}\int_a^x \left(\tau^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1}\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\lambda}{k}-1}F_{\rho,\lambda}^{\sigma,k}[\omega\left(x^{s+1}-\tau^{s+1}\right)^{\rho}]\tau^s f(\tau)d\tau. \end{aligned}$$

This can be written as

$$\begin{aligned} &\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1}\right)(x) \\ &= \sum_{n=0}^{\infty} \frac{\sigma(n)[w]^n}{k\Gamma_k(n\rho k+\lambda)} \frac{1}{k}\int_a^x \left(x^{s+1}-\tau^{s+1}\right)^{\frac{\lambda+\rho kn}{k}-1}\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\mu}{k}-1}\tau^s d\tau \\ &= \frac{(x^{s+1}-\alpha^{s+1})^{\frac{\lambda+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)} \sum_{n=0}^{\infty} \frac{\sigma(n)[\omega\left(x^{s+1}-\tau^{s+1}\right)^{\rho}]^n}{k\Gamma_k(n\rho k+\lambda)} \frac{\Gamma_k(n\rho k+\lambda)}{\Gamma_k(n\rho k+\lambda+\mu)} \\ &= \frac{(x^{s+1}-a^{s+1})^{\frac{\lambda+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)} F_{\rho,\lambda+\mu}^{\sigma,k}[w\left(x^{s+1}-a^{s+1}\right)^{\rho}]. \end{aligned}$$

Which complete the desired proof.

□

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Theorem 3.2. *Let $s \in \mathbb{R} \setminus \{-1\}$; $k > 0$; $\omega, \lambda, \rho \in \mathbb{C}$; $R(\omega) > 0$, $R(\omega) > 0$, $R(\lambda) > 0$, $R(\rho) > 0$, $R(\mu) > 0$ and $x > a$, then*

$$\left({}_k^sI_{a+}^{\mu}\right)\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x) = \frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}_k^s\varepsilon_{a+;\rho,\lambda+\mu}^{\omega,\sigma}f\right)(x) = \left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left[{}_k^sI_{a+}^{\mu}f\right]\right)(x), \tag{3.2}$$

hold for any $f \in L[\rho, \lambda]$.

Proof. From equations (1.11) and (2.1), we have

$$\begin{aligned}
&\left({}_k^sI_{a+}^{\mu}\right)\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x)\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x \frac{\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(\tau)}{\left(x^{s+1}-\tau^{s+1}\right)^{\frac{1-\mu}{k}}}\tau^s d\tau\\
&= \frac{(s+1)^{1-\frac{\mu}{k}}}{k^2\Gamma_k(\mu)}\int_a^x \left(x^{s+1}-\tau^{s+1}\right)^{\frac{\mu}{k}-1}\\
&\times\left[\int_a^\tau \left(\tau^{s+1}-u^{s+1}\right)^{\frac{\lambda}{k}-1} F_{\rho,\lambda}^{\sigma,k}\left(w\left(\tau^{s+1}-u^{s+1}\right)^{\rho}\right)f(u)u^s du\right]\tau^s d\tau.
\end{aligned}$$

By changing the order of integration, we obtain

$$\begin{aligned}
&\left({}_k^sI_{a+}^{\mu}\right)\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x)\\
&= \frac{1}{k}\int_a^x \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\left[\int_u^x \left(x^{s+1}-\tau^{s+1}\right)^{\frac{\mu}{k}-1}\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\lambda}{k}-1}\right.\\
&\left.F_{\rho,\lambda}^{\sigma,k}\left(w\left(\tau^{s+1}-u^{s+1}\right)^{\rho}\right)\tau^s d\tau\right] f(u)u^s du.
\end{aligned} \tag{3.3}$$

Substituting $\tau^{s+1} - u^{s+1} = t^{s+1}$, this implies $\tau^s d\tau = t^s dt$. Therefore (3.3) can be written as

$$\begin{aligned}
&\left({}_k^sI_{a+}^{\mu}\right)\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x)\\
&= \frac{1}{k}\int_a^x \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\left[\int_0^{x^{s+1}-u^{s+1}} \left(x^{s+1}-t^{s+1}-u^{s+1}\right)^{\frac{\mu}{k}-1}\left(t^{s+1}\right)^{\frac{\lambda}{k}-1}\right.\\
&\left.F_{\rho,\lambda}^{\sigma,k}\left(w\left(t^{s+1}\right)^{\rho}\right)t^s dt\right] f(u)u^s du.
\end{aligned}$$

By the use of (1.11) and applying (2.10), we have

$$\begin{aligned}
&\left({}_k^sI_{a+}^{\mu}\right)\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x)\\
&= \frac{1}{k(s+1)^{\frac{\mu}{k}}}\int_a^x \left(x^{s+1}-u^{s+1}\right)^{\frac{\mu+\lambda}{k}-1} F_{\rho,\lambda+\mu}^{\sigma,k}\left(w\left(x^{s+1}-u^{s+1}\right)^{\rho}\right)f(u)u^s du.
\end{aligned}$$

thus, we get

$$\left({}_k^sI_{a+}^{\mu}\right)\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}f\right)(x) = \frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}_k^s\varepsilon_{a+;\rho,\lambda+\mu}^{\omega,\sigma}f\right)(x).$$

To prove the second part, consider the right part of (3.2), we have

$$\begin{aligned}
&\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left[{}_k^s I_{a+}^{\mu} f\right]\right)(x)\\
&=\frac{1}{k}\int_a^x\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\lambda}{k}-1} F_{\rho,\lambda}^{\sigma,k}\left(w\left(x^{s+1}-\tau^{s+1}\right)^{\rho}\right)\left[{}_k^s I_{a+}^{\mu} f\right]\tau^s d\tau\\
&=\frac{1}{k}\int_a^x\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\lambda}{k}-1} F_{\rho,\lambda}^{\sigma,k}\left(w\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\rho}{k}}\right)\\
&\left(\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^\tau \frac{f(u)}{\left(\tau^{s+1}-u^{s+1}\right)^{1-\frac{\mu}{k}}}u^s du\right)d\tau
\end{aligned}$$

By changing the order of integration, we obtain

$$\begin{aligned}
&\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left[{}_k^s I_{a+}^{\mu} f\right]\right)(x)\\
&=\frac{1}{k}\int_a^x\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\left[\int_u^x\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\mu}{k}-1}\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\lambda}{k}-1}\right.\\
&\left. F_{\rho,\lambda}^{\sigma,k}\left(w\left(x^{s+1}-\tau^{s+1}\right)^{\rho}\right)\tau^s d\tau\right] f(u)u^s du.
\end{aligned}$$

Again making the use of (1.11) and applying (2.10), we have

$$\begin{aligned}
&\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left[{}_k^s I_{a+}^{\mu} f\right]\right)(x)\\
&=\frac{1}{k(s+1)^{\frac{\mu}{k}}}\int_a^x\left(x^{s+1}-u^{s+1}\right)^{\frac{\mu+\lambda}{k}-1} F_{\rho,\lambda+\mu}^{\sigma,k}\left(w\left(x^{s+1}-u^{s+1}\right)^{\rho}\right)f(u)u^s.
\end{aligned}$$

thus, we get

$$\left({}_k^s\varepsilon_{a+;\rho,\lambda}^{\omega,\sigma}\left[{}_k^s I_{a+}^{\mu} f\right]\right)(x)=\frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}_k^s\varepsilon_{a+;\rho,\lambda+\mu}^{\omega,\sigma} f\right)(x).$$

Which complete the desired proof. □

References

[1] A. Atangana, D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, Thermal Science 20, 763-769, doi:10.2298/tsci160111018a (2016).

[2] P. Agarwal, M. Jaleli, M. Tomar, *Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals*, Journal of Inequalities and Applications 2017:55 (2017).

[3] R. Diaz, E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat. 17, 179-192 (2007).

[4] A. A. Kilbas, M. Saigo, R. K. Saxena, *Generalized Mittag-Lefler function and generalized fractional calculus operators*, Integral Transforms and Special Functions 15, 31-49, doi:10.1080/10652460310001600717 (2004).

[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Volume 204 (North-Holland Mathematics Studies). (Elsevier Science Inc., 2006).

[6] V. Kiryakova, *Generalized fractional calculus and applications*, Vol. 38 (Longman Scientific and Technical, 1994).

[7] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, (John Wiley and Sons, 1993).

[8] D. S. Mitrinovic, J. E. Pecaric, A. M. Fink, *Classical and new inequalities in analysis, Mathematics and its Applications*, (Kluwer Academic Publishers Group, 1993).

[9] K. S. Nisar, G. Rahman, D. Baleanu, S. Mubeen, M. Arshad, *The (k, s)-fractional calculus of k-Mittag-Leffler function*, Advances in Difference Equations (2017) 2017:118.

[10] K. S. Nisar, G. Rahman, J. Choi, S. Mubeen, M. Arshad, *Generalized hypergeometric k-functions via (k, s)-fractional calculus*, J. Nonlinear Sci. Appl., 10 (2017), 17911800.

[11] S.Mubeen, G. M. Habibullah, *k-Fractional Integrals and Application* Int. J. Contemp. Math. Sciences 7, 89 - 94 (2012).

[12] S. Mubeen, S. Iqbal, *Gruss type integral inequalities for generalized Riemann-Liouville k-fractional integrals*, Journal of Inequalities and Applications 2016:109 (2016).

[13] S. D. Purohit, S. L. Kalla, *On fractional partial differential equations related to quantum mechanics*, Journal of Physics A: Mathematical and Theoretical 44, 045202 (2011).

[14] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, F. Ahmad, *(k; s)-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat. 45, 77-89 (2016).

[15] H. M. Srivastava, Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Lefler function in the kernel*, Applied Mathematics and Computation 211 (2009), 198-210.

[16] M. Tomar, S. Maden, E. Set, *(k, s)-Riemann-Liouville fractional integral inequalities for continuous random variables*, Arab. J. Math. 6:55-63 (2017).

[17] T. Tunc, H. Budak, F. Usta, M. Z. Sarikaya, *On new generalized fractional integral operators and related fractional inequalities*, ResearchGate, : https://www.researchgate.net/publication/313650587.

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